Commun. Math. Phys. 157, 119–137 (1993)

Communications in
Mathematical
Physics

Free Boson Representation of q-Vertex Operators and their Correlation Functions

Akishi Kato[1]*, Yas-Hiro Quano[2], and Jun'ichi Shiraishi[2]**

[1] Department of Mathematical Sciences, University of Tokyo, Meguro-ku, Tokyo 153, Japan
[2] Department of Physics, University of Tokyo, Bunkyo-ku, Tokyo 113, Japan

Received September 7, 1992

Abstract. A bosonization scheme of the q-vertex operators of $U_q(\widehat{\mathfrak{sl}}_2)$ for arbitrary level is obtained. They act as intertwiners among the highest weight modules constructed in a bosonic Fock space. An integral formula is proposed for N-point functions and explicit calculation for two-point function is presented.

1. Introduction

One of the central subjects of mathematical physics has been studies on exactly solvable models in two dimensions for many years. Infinite dimensional symmetries such as conformal and current algebra give powerful tools to investigate systems just on the critical point [1]. It is now a very important problem how to extend the method developed in the critical theories to massive field theories and lattice models.

A breakthrough was brought by Frenkel and Reshetikhin [2] who studied the q-deformation of the vertex operator as an intertwiner between certain modules of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$. They showed that the correlation functions satisfy a q-difference equation, the q-deformed Knizhnik–Zamolodchikov equation, and that the resulting connection matrices give rise to the elliptic solution to the Yang–Baxter equation of RSOS models [3, 4]. Using the q-vertex operators people in the Kyoto school [5] succeeded in diagonalization of the XXZ spin chain and showed that the spectra of the XXZ model is completely determined in terms of the representation theory of $U_q(\widehat{\mathfrak{sl}}_2)$. Furthermore, they found an integral formula for correlation functions of the local operators of the XXZ model [6] by utilizing bosonization of $U_q(\widehat{\mathfrak{sl}}_2)$ of level one [7] and the bosonized q-vertex operators.

* Partly supported by the Grant-in-Aid for Scientific Research from the Ministry of Education, Science and Culture (No. 04245206)
** A Fellow of the Japan Society of the Promotion of Science for Japanese Junior Scientists. Partly supported by the Grant-in-Aid for Scientific Research from the Ministry of Education, Science and Culture (No. 04-2297)

In the previous paper [8], one of the authors construct the bosonization of $U_q'(\widehat{\mathfrak{sl}}_2)$ currents for arbitrary level à la Wakimoto [9]. In this paper we shall introduce a bosonization of the "elementary" q-vertex operators, which have exactly the same commutation relations with the generators of $U_q'(\widehat{\mathfrak{sl}}_2)$ as the bona-fide q-vertex operators have. They are well-defined operators acting on a bosonic Fock space, in which all the integrable highest weight modules of a given level can be embedded. Finally q-vertex operators as intertwiners among these modules are obtained in terms of the elementary q-vertex operators dressed with the screening charges. This technique provides a natural framework to write down an integral formula for correlation functions of the q-vertex operators. Our formula will be useful to examine the higher spin chain [10].

The present article is organized as follows. In Sect. 2 we construct the currents which give Drinfeld realization of $U_q'(\widehat{\mathfrak{sl}}_2)$ [11] in terms of free bosons [8]. In Sect. 3 we construct the "elementary" q-vertex operator. In Sect. 4 we define the Fock space on which the currents and the elementary q-vertex operators act. We also introduce the screening change, which is necessary to calculate correlation functions. Furthermore we give the expression of the N-point function in terms of the bosonized operators. In Sect. 5 we calculate the two-point function in a simple case and show the relevance of our formulation. In Sect. 6 we summarize our results and give some remarks.

Three appendices are devoted to the details of the calculation in Sect. 5. In Appendix A OPE formulae among the bosonized operators are listed. In Appendix B we give the normalization of the elementary vertex operators. In Appendix C we discuss the response of Jackson integrals to p-shift of variables.

2. Free Boson Realization of $U_q'(\widehat{\mathfrak{sl}}_2)$

In this section we briefly recall the bosonization of $U_q'(\widehat{\mathfrak{sl}}_2)$ [8].

2.1. Definition of $U_q'(\widehat{\mathfrak{sl}}_2)$. To begin with, let us fix notation concerning the affine Lie algebra $\widehat{\mathfrak{sl}}_2$ [12]. Let $P = \mathbf{Z}\Lambda_0 \oplus \mathbf{Z}\Lambda_1 \oplus \mathbf{Z}\delta$ be the weight lattice and $Q = \mathbf{Z}\alpha_0 \oplus \mathbf{Z}\alpha_1$ be the root lattice endowed with the symmetric bilinear form $(,)$ defined by

$$(\Lambda_0, \Lambda_0) = 0, \quad (\Lambda_0, \alpha_1) = 0, \quad (\Lambda_0, \delta) = 1, \quad (\alpha_1, \alpha_1) = 2, \quad (\alpha_1, \delta) = 0, \quad (\delta, \delta) = 0\,,$$

where $\Lambda_1 = \Lambda_0 + \alpha_1/2$, $\delta = \alpha_0 + \alpha_1$. As we set $\rho = \Lambda_0 + \Lambda_1$. We define $P^* = \mathbf{Z}h_0 \oplus \mathbf{Z}h_1 \oplus \mathbf{Z}d$ as the dual space of P. The dual pairing $\langle , \rangle$ is defined by

$$\langle h_i, \lambda \rangle := (\alpha_i, \lambda), \quad (i = 0, 1) \quad \text{for } \lambda \in P\,.$$

We denote by $P_k = \{(k-i)\Lambda_0 + i\Lambda_1 \,|\, i = 0, 1, \ldots, k\}$ the set of dominant integral weights of level k. For simplicity, we set $\lambda_i = (k-i)\Lambda_0 + i\Lambda_1$.

Throughout this paper let q be transcendental over $\mathbf{Q}$ with $|q| < 1$. We use the following standard notation:

$$[m] = \frac{q^m - q^{-m}}{q - q^{-1}}\,,$$

for $m \in \mathbf{Z}$.

The quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$ is an associative algebra over $\mathbf{Q}(q)$ with 1, generated by e_0, e_1, f_0, f_1 and $q^h (h \in P^*)$. The defining relations are as follows [13, 14, 15]:

$$
\begin{aligned}
&q^h q^{h'} = q^{h+h'}, \quad q^0 = 1 ,\\
&q^h e_i q^{-h} = q^{\langle h, \alpha_i \rangle} e_i ,\\
&q^h f_i q^{-h} = q^{-\langle h, \alpha_i \rangle} e_i ,\\
&[e_i, f_j] = \delta_{i,j} \frac{t_i - t_i^{-1}}{q - q^{-1}} \quad (t_i = q^{h_i}) ,\\
&e_i^3 e_j - [3] e_i^2 e_j e_i + [3] e_i e_j e_i^2 - e_j e_i^3 = 0 \quad (i \neq j) ,\\
&f_i^3 f_j - [3] f_i^2 f_j f_i + [3] f_i f_j f_i^2 - f_j f_i^3 = 0 \quad (i \neq j) .
\end{aligned} \tag{2.1}
$$

The algebra $U_q'(\widehat{\mathfrak{sl}}_2)$ is the subalgebra of $U_q(\widehat{\mathfrak{sl}}_2)$ generated by $\{e_i, f_i, t_i \, (i = 0, 1)\}$.

The algebra $U_q(\widehat{\mathfrak{sl}}_2)$ has a Hopf algebra structure with the following coproduct $\Delta: U_q(\widehat{\mathfrak{sl}}_2) \to U_q(\widehat{\mathfrak{sl}}_2) \otimes U_q(\widehat{\mathfrak{sl}}_2)$

$$
\begin{aligned}
&\Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i. \quad (i = 0, 1) ,\\
&\Delta(q^h) = q^h \otimes q^h \quad h \in P^* .
\end{aligned}
$$

2.2. Drinfeld Realization of $U_q'(\widehat{\mathfrak{sl}}_2)$. The Chevalley generators e_i, f_i, t_i are not convenient for considering the bosonization. We recall here the Drinfeld realization of $U_q'(\widehat{\mathfrak{sl}}_2)$ [11] which we will bosonize. The Drinfeld realization of $U_q(\widehat{\mathfrak{sl}}_2)$ is an associative algebra generated by the letters $\{J_n^\pm \mid n \in \mathbf{Z}\}$, $\{J_n^3 \mid n \in \mathbf{Z}_{\neq 0}\}$, $\gamma^{\pm 1/2}$ and K, satisfying the following relations:

$$
\begin{aligned}
&\gamma^{\pm 1/2} \in \text{the center of the algebra} ,\\
&[J_n^3, J_m^3] = \delta_{n+m,0} \frac{1}{n} [2n] \frac{\gamma^n - \gamma^{-n}}{q - q^{-1}} ,\\
&[J_n^3, K] = 0 ,\\
&K J_n^\pm K^{-1} = q^{\pm 2} J_n^\pm ,\\
&[J_n^3, J_m^\pm] = \pm \frac{1}{n} [2n] \gamma^{\mp |n|/2} J_{n+m}^\pm ,\\
&J_{n+1}^\pm J_m^\pm - q^{\pm 2} J_m^\pm J_{n+1}^\pm = q^{\pm 2} J_n^\pm J_{m+1}^\pm - J_{m+1}^\pm J_n^\pm ,\\
&[J_n^+, J_m^-] = \frac{1}{q - q^{-1}} (\gamma^{(n-m)/2} \psi_{n+m} - \gamma^{(m-n)/2} \varphi_{n+m}) ,
\end{aligned}
$$

where $\{\psi_r, \varphi_s \mid s \in \mathbf{Z}\}$ are related to $\{J_l^3 \mid l \in \mathbf{Z}_{\neq 0}\}$ by

$$
\begin{aligned}
&\sum_{n \in \mathbf{Z}} \psi_n z^{-n} = K \exp\left\{(q - q^{-1}) \sum_{k=1}^{\infty} J_k^3 z^{-k}\right\} ,\\
&\sum_{n \in \mathbf{Z}} \varphi_n z^{-n} = K^{-1} \exp\left\{ -(q - q^{-1}) \sum_{k=1}^{\infty} J_{-k}^3 z^{k}\right\} .
\end{aligned} \tag{2.3}
$$

The standard Chevalley generators $\{e_i, f_i, t_i\}$ are given by the identification

$$t_0 = \gamma K^{-1}, \quad t_1 = K, \quad e_1 = J_0^+, \quad f_1 = J_0^-, \quad e_0 t_1 = J_1^-, \quad t_1^{-1} f_0 = J_{-1}^+ . \tag{2.4}$$

2.3. Bosonization of $U_q'(\widehat{\mathfrak{sl}}_2)$. Let k be a non-negative integer. Let $\{a_n, b_n, c_n, Q_a, Q_b, Q_c \mid n \in \mathbf{Z}\}$ be a set of operators satisfying the following commutation relations:

$$\begin{aligned}
&[a_n, a_m] = \delta_{n+m,0}\frac{[2n][(k+2)n]}{n}, \quad &&[\tilde{a}_0, Q_a] = 2(k+2), \\
&[b_n, b_m] = -\delta_{n+m,0}\frac{[2n][2n]}{n}, \quad &&[\tilde{b}_0, Q_b] = -4, \\
&[c_n, c_m] = \delta_{n+m,0}\frac{[2n][2n]}{n}, \quad &&[\tilde{c}_0, Q_c] = 4,
\end{aligned} \tag{2.5}$$

where

$$\tilde{a}_0 = \frac{q - q^{-1}}{2\log q} a_0, \quad \tilde{b}_0 = \frac{q - q^{-1}}{2\log q} b_0, \quad \tilde{c}_0 = \frac{q - q^{-1}}{2\log q} c_0,$$

and others commute.

Let us introduce the free bosonic fields a, b, and c carrying parameters $L, M, N \in \mathbf{Z}_{>0}$, $\alpha \in \mathbf{R}$. Define $a(L; M, N \mid z; \alpha)$ by

$$a(L; M, N \mid z; \alpha) = -\sum_{n \neq 0} \frac{[Ln] a_n}{[Mn][Nn]} z^{-n} q^{|n|\alpha} + \frac{L\tilde{a}_0}{MN}\log z + \frac{LQ_a}{MN}. \tag{2.6}$$

$b(L; M, N \mid z; \alpha)$, $c(L; M, N \mid z; \alpha)$ are defined in the same way. In the case $L = M$ we also write

$$\begin{aligned}
a(N \mid z; \alpha) &= a(L; L, N \mid z; \alpha) \\
&= -\sum_{n \neq 0} \frac{a_n}{[Nn]} z^{-n} q^{|n|\alpha} + \frac{\tilde{a}_0}{N}\log z + \frac{Q_a}{N},
\end{aligned} \tag{2.7}$$

and likewise for $b(N \mid z; \alpha)$, $c(N \mid z; \alpha)$.

Let $\{a_n, b_n, c_n \mid n \in \mathbf{Z}_{\geqq 0}\}$ be annihilation operators, and $\{a_n, b_n, c_n, Q_a, Q_b, Q_c \mid n \in \mathbf{Z}_{<0}\}$ creation operators. We denote by $:\mathcal{O}(z):$ the normal ordering of $\mathcal{O}(z)$. For example,

$$:\exp\{b(2 \mid z; \alpha)\}: = \exp\left\{-\sum_{n<0}\frac{b_n}{[2n]} z^{-n} q^{|n|\alpha}\right\} \exp\left\{-\sum_{n>0}\frac{b_n}{[2n]} z^{-n} q^{|n|\alpha}\right\} e^{Q_b/2} z^{\tilde{b}_0/2}.$$

Now we define the currents $J^3(z)$, $J^\pm(z)$ as follows:

$$\begin{aligned}
J^3(z) &= {}_{k+2}\partial_z a(k+2 \mid q^{-2} z; -1) + {}_2\partial_z b\left(2 \mid q^{-k-2} z; -\frac{k+2}{2}\right), \\
J^+(z) &= -:[{}_1\partial_z \exp\{-c(2 \mid q^{-k-2} z; 0)\}] \times \exp\{-b(2 \mid q^{-k-2} z; 1)\}:, \\
J^-(z) &= :\left[{}_{k+2}\partial_z \exp\left\{a\left(k+2 \mid q^{-2} z; -\frac{k+2}{2}\right) + b(2 \mid q^{-k-2} z; -1)\right.\right. \\
&\qquad \left.\left. + c(k+1; 2, k+2 \mid q^{-k-2} z; 0)\right\}\right] \\
&\qquad \times \exp\left\{-a\left(k+2 \mid q^{-2} z; \frac{k+2}{2}\right) + c(1; 2, k+2 \mid q^{-k-2} z; 0)\right\}: .
\end{aligned} \tag{2.8}$$

Here the q-difference operator with parameter $n \in \mathbf{Z}_{>0}$ is defined by

$$ {}_n\partial_z f(z) \equiv \frac{f(q^n z) + f(q^{-n} z)}{(q - q^{-1})z} \, . $$

Define further the auxiliary fields $\psi(z)$, $\varphi(z)$ as

$$ \psi(z) = :\exp\left\{(q - q^{-1}) \sum_{n>0} (q^n a_n + q^{(k+2)n/2} b_n) z^{-n} + (\tilde{a}_0 + \tilde{b}_0) \log q\right\}: , $$

$$ \varphi(z) = :\exp\left\{-(q - q^{-1}) \sum_{n<0} (q^{3n} a_n + q^{3(k+2)n/2} b_n) z^{-n} - (\tilde{a}_0 + \tilde{b}_0) \log q\right\}: . \tag{2.9} $$

We give the mode expansions of these fields as

$$ \sum_{n\in\mathbf{Z}} J_n^3 z^{-n-1} = J^3(z), \quad \sum_{n\in\mathbf{Z}} J_n^{\pm} z^{-n-1} = J^{\pm}(z) , $$

$$ \sum_{n\in\mathbf{Z}} \psi_n z^{-n} = \psi(z), \quad \sum_{n\in\mathbf{Z}} \varphi_n z^{-n} = \varphi(z) , \tag{2.10} $$

and let

$$ K = q^{\tilde{a}_0 + \tilde{b}_0}, \quad \gamma = q^k . \tag{2.11} $$

Then we get the following [8]:

Proposition 2.1. $\{J_n^3 | n \in \mathbf{Z}_{\neq 0}\}$, $\{J_n^{\pm} | n \in \mathbf{Z}\}$, $\{\varphi_n, \psi_n | n \in \mathbf{Z}\}$, K, *and* γ *defined by* (2.8), (2.9), (2.10) *and* (2.11) *satisfy the relations* (2.2).

2.4. Finite Dimensional $U_q(\widehat{\mathfrak{sl}}_2)$ *Module.* For $l \in \mathbf{Z}_{\geq 0}$ let $V^{(l)}$ denote the $(l+1)$-dimensional $U_q'(\widehat{\mathfrak{sl}}_2)$-module (spin $l/2$ representation) with basis $\{v_m^{(l)} | 0 \leq m \leq l\}$ given by

$$ e_1 v_m^{(l)} = [m] v_{m-1}^{(l)}, \quad f_1 v_m^{(l)} = [l-m] v_{m+1}^{(l)}, \quad t_1 v_m^{(l)} = q^{l-2m} v_m^{(l)} , $$

$$ e_0 = f_1, \quad f_0 = e_1, \quad t_0 = t_1^{-1} \quad \text{on} \quad V^{(l)} . $$

Here $v_m^{(l)}$ with $m < 0$ or $m > l$ is understood to be 0. In the case $l = 1$ we also write $v_0^{(1)} = v_+$ and $v_1^{(1)} = v_-$.

We equip $V_z^{(l)} = V^{(l)} \otimes \mathbf{Q}(q)[z, z^{-1}]$ with a $U_q(\widehat{\mathfrak{sl}}_2)$-module structure via

$$ e_i(v_m^{(l)} \otimes z^n) = e_i v_m^{(l)} \otimes z^{n+\delta_{i0}}, \quad f_i(v_m^{(l)} \otimes z^n) = f_i v_m^{(l)} \otimes z^{n-\delta_{i0}} , $$

$$ \mathrm{wt}(v_m^{(l)} \otimes z^n) = n\delta + (l-2m)(\Lambda_1 - \Lambda_0) . $$

We also need the representation of Drinfeld generators on level 0 modules.

Proposition 2.2. *Spin* $l/2$ *representation of* $U_q(\widehat{\mathfrak{sl}}_2)$ *is given in terms of the Drinfeld generators by*

$$ \begin{aligned} \gamma^{\pm 1/2} v_m^{(l)} &= v_m^{(l)} , \\ K v_m^{(l)} &= q^{l-2m} v_m^{(l)} , \\ J_n^+ v_m^{(l)} &= z^n q^{n(l-2m+2)} [m] v_{m-1}^{(l)} , \\ J_n^- v_m^{(l)} &= z^n q^{n(l-2m)} [l-m] v_{m+1}^{(l)} , \\ J_n^3 v_m^{(l)} &= \frac{z^n}{n} \{[nl] - q^{n(l+1-m)}(q^n + q^{-n})[nm]\} , \end{aligned} \tag{2.12} $$

where $v_m^{(l)} = 0$ *if* $m > l$ *or* $m < 0$.

3. Elementary q-Vertex Operators

In this section we construct the operators in terms of free fields which have exactly the same commutation relations with the generators of $U_q'(\widehat{\mathfrak{sl}}_2)$ as the bona-fide q-vertex operators have. For that purpose we review the definition and properties of the q-vertex operators (q-VOs) [2, 5, 16] in Subsect. 3.1. Furthermore, we give a free field realization of q-VOs in Subsect. 3.2.

3.1. Definition of q-Vertex Operators. We recall below the properties of the q-vertex operators (q-VOs) relevant to the subsequent discussions. A vector $|\lambda\rangle$ is called a highest weight vector of weight λ if it satisfies the highest weight condition

$$e_i|\lambda\rangle = 0, \quad t_i|\lambda\rangle = q^{\langle h_i, \lambda\rangle}|\lambda\rangle, \quad f_i^{\langle h_i, \lambda\rangle+1}|\lambda\rangle = 0, \quad i = 0, 1 .$$

The left highest weight module $V(\lambda)$ with the highest weight vector $|\lambda\rangle$ is defined by

$$V(\lambda) := U_q(\widehat{\mathfrak{sl}}_2)|\lambda\rangle .$$

The right highest weight module is defined in a similar manner.

The left (resp. right) highest weight module with highest weight $\lambda \in P_k$ will be denoted by $V(\lambda)$ (resp. $V^r(\lambda)$). We fix a highest weight vector $|\lambda\rangle \in V(\lambda)$ (resp. $\langle\lambda| \in V^r(\lambda)$) once and for all. There is a unique symmetric bilinear pairing $V^r(\lambda) \times V(\lambda) \to \mathbf{Q}(q)$ such that

$$\langle\lambda|\lambda\rangle = 1, \quad \langle ux|u'\rangle = \langle u|xu'\rangle ,$$
$$\forall x \in U_q(\widehat{\mathfrak{sl}}_2), \quad \forall \langle u| \in V^r(\lambda), \quad \forall |u'\rangle \in V(\lambda) .$$

For positive integers k, l and let λ, $\mu \in P_k$. We set $\Delta_\lambda = (\lambda, \lambda + 2\rho)/2(k+2)$.
We shall use the following type of q-VO[1]

$$\Phi_\lambda^{\mu V^{(l)}}(z) = z^{\Delta_\mu - \Delta_\lambda} \tilde{\Phi}_\lambda^{\mu V^{(l)}}(z) ,$$
$$\tilde{\Phi}_\lambda^{\mu V^{(l)}}(z)\colon\ V(\lambda) \to V(\mu) \hat{\otimes} V_z^{(l)} . \tag{3.1}$$

The map (3.1) means a formal series of the form

$$\tilde{\Phi}_\lambda^{\mu V^{(l)}}(z) = \sum_{n\in\mathbf{Z}} \sum_{m=0}^{l} \tilde{\Phi}_{m,n} \otimes v_m^{(l)} z^{-n} ,$$
$$\tilde{\Phi}_{m,n}\colon\ V(\lambda)_\nu \to V(\mu)_{\nu - \mathrm{wt}(v_m^{(l)}) + n\delta} ,$$

where $\mathrm{wt}(v_m^{(l)}) = (l - 2m)(\Lambda_1 - \Lambda_0)$, $\delta = \alpha_0 + \alpha_1$.

By definition, the q-VO satisfies the intertwining relations

$$\tilde{\Phi}_\lambda^{\mu V^{(l)}}(z) \circ x = \Delta(x) \circ \tilde{\Phi}_\lambda^{\mu V^{(l)}}(z), \quad \forall x \in U_q(\widehat{\mathfrak{sl}}_2) . \tag{3.2}$$

From the general arguments on q-VOs [16], in the $U_q(\widehat{\mathfrak{sl}}_2)$ case there exists at most one q-VO up to proportionality. We normalize $\tilde{\Phi}_\lambda^{\mu V^{(l)}}(z)$ such that the leading term is $|\mu\rangle \otimes v_m^{(l)}$:

$$\tilde{\Phi}_\lambda^{\mu V^{(l)}}(z)|\lambda\rangle = |\mu\rangle \otimes v_m^{(l)} + \cdots , \tag{3.3}$$

where $\cdots$ means terms of the form $u \otimes v$, $\mathrm{wt}\, u \neq \mu$.

---

[1] This q-VO is called "type I" in ref. [5]

Proposition 3.1. *If we write*

$$\tilde{\Phi}_{\lambda}^{\mu V^{(l)}}(z) = \sum_{m=0}^{l} \Phi_{\lambda m}^{\mu V^{(l)}}(z) \otimes v_m^{(l)} ,$$

then

$$\tilde{\Phi}_{\lambda m-1}^{\mu V^{(l)}}(z) = \frac{1}{[l-m+1]} \{\tilde{\Phi}_{\lambda m}^{\mu V^{(l)}}(z) f_1 - q^{2m-l} f_1 \tilde{\Phi}_{\lambda m}^{\mu V^{(l)}}(z)\}, \quad m = 1, 2, \ldots, l . \tag{3.4}$$

This is easily checked by evaluating both sides of (3.2) for $x = f_1$.

For two vertex operators

$$\tilde{\Phi}_{\mu_0}^{\mu_1 V^{(l_1)}}(z_1) = \sum_{n \in Z} \sum_{m=0}^{l} \tilde{\Phi}_{m,n} \otimes v_m^{(l_1)} z_1^{-n} ,$$

$$\tilde{\Phi}_{\mu_1}^{\mu_2 V^{(l_2)}}(z_2) = \sum_{n \in Z} \sum_{m=0}^{l} \tilde{\Phi}_{m,n} \otimes v_m^{(l_2)} z_2^{-n} ,$$

the composition of these two is defined as a formal series in z_1, z_2:

$$\tilde{\Phi}_{\mu_1}^{\mu_2 V^{(l_2)}}(z_2) \circ \tilde{\Phi}_{\mu_0}^{\mu_1 V^{(l_1)}}(z_1) = \sum_{m,n,j,k} \tilde{\Phi}_{jm} \circ \tilde{\Phi}_{kn} \otimes v_j^{(l_2)} z_2^{-m} \otimes v_k^{(l_1)} z_1^{-n} .$$

The composition of N q-vertex operators are defined in a similar fashion.

3.2. Elementary q-Vertex Operators. In [6] an integral formula for correlation functions of the local operators of the XXZ model is obtained by utilizing bosonization of the $U_q(\widehat{\mathfrak{sl}}_2)$ of level one [7] and the bosonized q-vertex operators. In the same spirit we want to derive the formulae for the q-vertex operators for arbitrary level k in terms of bosonic fields a, b and c.

Since the Drinfeld generators are successfully bosonized, we intend to know how the intertwining properties of q-VOs are expressed in those terms [17].

Proposition 3.2. *For $k \in \mathbf{Z}_{\geqq 0}$ and $l \in \mathbf{Z}_{>0}$ we have*

$$\left.\begin{aligned} \Delta(J_k^+) &= J_k^+ \otimes \gamma^k + \gamma^{2k} K \otimes J_k^+ + \sum_{i=0}^{k-1} \gamma^{(k+3i)/2} \psi_{k-i} \otimes \gamma^{k-i} J_i^+ \\ \Delta(J_{-l}^+) &= J_{-l}^+ \otimes \gamma^{-l} + K^{-1} \otimes J_{-l}^+ + \sum_{i=1}^{l-1} \gamma^{(l-i)/2} \varphi_{-l+i} \otimes \gamma^{-l+i} J_{-i}^+ \end{aligned}\right\} \bmod N_- \otimes N_+^2 ,$$

$$\left.\begin{aligned} \Delta(J_l^-) &= J_l^- \otimes K + \gamma^l \otimes J_l^- + \sum_{i=1}^{l-1} \gamma^{l-i} J_i^- \otimes \gamma^{(i-l)/2} \psi_{l-i} \\ \Delta(J_{-k}^-) &= J_{-k}^- \otimes \gamma^{-2k} K^{-1} + \gamma^{-k} K \otimes J_{-k}^- \\ &\quad + \sum_{i=0}^{k-1} \gamma^{i-k} J_{-i}^- \otimes \gamma^{-(k+3i)/2} \varphi_{i-k} \end{aligned}\right\} \bmod N_-^2 \otimes N_+ ,$$

$$\left.\begin{aligned} \Delta(J_l^3) &= J_l^3 \otimes \gamma^{l/2} + \gamma^{3l/2} \otimes J_l^3 \\ \Delta(J_{-l}^3) &= J_{-l}^3 \otimes \gamma^{-3l/2} + \gamma^{-l/2} \otimes J_{-l}^3 \end{aligned}\right\} \bmod N_- \otimes N_+ .$$

Here $N_\pm$ and $N_\pm^2$ are left $\mathbf{Q}(q)[\gamma^\pm, \psi_r, \varphi_s | r, -s \in \mathbf{Z}_{\geqq 0}]$-modules generated by $\{J_m^\pm | m \in \mathbf{Z}\}$ and $\{J_m^\pm J_n^\pm | m, n \in \mathbf{Z}\}$ respectively.

By using Propositions 2.2, 3.2 and noting that $N_+ v_0^{(l)} = N_- v_l^{(l)} = 0$, $N_\pm v_m^{(l)} \subset \mathbf{Q}(q)[z, z^{-1}] v_{m\mp 1}^{(l)}$, we get the exact relations

$$
\begin{aligned}
&[J_n^3, \tilde{\Phi}_{\lambda l}^{\mu V^{(l)}}(z)] = q^{2n} z^n \frac{[nl]}{n} \cdot q^{k(n+|n|/2)} \tilde{\Phi}_{\lambda l}^{\mu V^{(l)}}(z) \quad n \neq 0 , \\
&[\tilde{\Phi}_{\lambda l}^{\mu V^{(l)}}(z), J^+(w)] = 0 , \\
&\quad K \tilde{\Phi}_{\lambda l}^{\mu V^{(l)}}(z) K^{-1} = q^l \tilde{\Phi}_{\lambda l}^{\mu V^{(l)}}(z) ,
\end{aligned} \tag{3.5}
$$

which follows from $V(\mu) \otimes v_l^{(l)}$ components of intertwining relation.

How can we construct $\tilde{\Phi}_{\lambda l}^{\mu V^{(l)}}(z)$ in terms of free bosons? Conditions (3.5) put stringent constraints on the possible bosonized form $\phi_l^{(l)}(z)$ of vertex operators $\tilde{\Phi}_{\lambda l}^{\mu V^{(l)}}(z)$. By the explicit calculation, we can check that if

$$
\phi_l^{(l)}(z) = : \exp \left\{ a\left(l; 2, k+2 \,\middle|\, q^k z; \frac{k+2}{2} \right) \right\} : \tag{3.6}
$$

is substituted for $\tilde{\Phi}_{\lambda l}^{\mu V^{(l)}}(z)$, then all the commutation relations (3.5) hold. Proposition 3.1 suggests that the other components of the vertex operator should be defined by the following multiple contour integral:

$$
\begin{aligned}
\phi_m^{(l)}(z) &= \frac{1}{[1][2] \cdots [l-m]} \oint \frac{dw_1}{2\pi\sqrt{-1}} \oint \frac{dw_2}{2\pi\sqrt{-1}} \cdots \oint \frac{dw_{l-m}}{2\pi\sqrt{-1}} \\
&\quad \times [\cdots [[\phi_l^{(l)}(z), J^-(w_1)]_{q^l}, J^-(w_2)]_{q^{l-2}} \cdots J^-(w_{l-m})]_{q^{-l+2m+2}} ,
\end{aligned} \tag{3.7}
$$

where $[A, B]_q := AB - qBA$.

We will call these operators (3.6), (3.7) "elementary vertex operators." A salient feature of these operators is that they are determined solely from the commutation relation with bosonized $U_q(\widehat{\mathfrak{sl}}_2)$ currents; this is completely independent of which infinite dimensional modules they intertwine[2].

Before discussing the relation between elementary q-vertex operators and bona-fide vertex operators, we need to clarify on which space these bosonized operators are acting.

4. Fock Module, Screening Charge and Correlation Function

In this section, we define the Fock module of bosons on which the $U_q(\widehat{\mathfrak{sl}}_2)$ currents $J^3(z)$, $J^\pm(z)$, and the elementary q-VOs $\phi_m^{(l)}(z)$ act. All the integrable highest weight modules are constructed in this Fock module. Further the q-VOs as the intertwiner among these modules are obtained.

4.1. Fock module and Highest Weight Module. From the observation that

$$
[J^3(z), \tilde{b}_0 + \tilde{c}_0] = 0, \quad [J^\pm(z), \tilde{b}_0 + \tilde{c}_0] = 0, \quad [\phi_m^{(l)}(z), \tilde{b}_0 + \tilde{c}_0] = 0 , \tag{4.1}
$$

[2] These elementary q-vertex operators are determined from a part of the intertwining properties, but it is very likely that they enjoy all of these properties

we can restrict the full Fock module of the boson a, b, and c to the sector such that the eigenvalue of the operator $\tilde{b}_0 + \tilde{c}_0$ is equal to 0. This requirement does not conflict with any other conditions we shall impose.[3]

Let us introduce a vacuum vector $|0\rangle$ which has the following properties:

$$a_n|0\rangle = 0, \quad b_n|0\rangle = 0, \quad c_n|0\rangle = 0, \quad n \geqq 0 .$$

Define the vectors $|r, s\rangle$ by

$$|r, s\rangle := \exp\left\{r \frac{Q_a}{k+2} + s \frac{Q_b + Q_c}{2}\right\}|0\rangle , \tag{4.2}$$

where $r \in \frac{1}{2}\mathbf{Z}$, $s \in \mathbf{Z}$.

Let F be a free $\mathbf{Q}(q)$ module generated by $\{a_{-1}, a_{-2}, \ldots, b_{-1}, b_{-2}, \ldots, c_{-1}, c_{-2}, \ldots\}$. Now we define the Fock modules $F_{r,s}$ as

$$F_{r,s} := F|r, s\rangle .$$

We can regard the currents $J^3(z)$, $J^{\pm}(z)$, $J^S(z)$, and q-VOs $\phi_m^{(l)}(z)$ as the following maps:

$$\begin{aligned} J^3(z)&\colon\ F_{r,s} \to F_{r,s} , \\ J^{\pm}(z)&\colon\ F_{r,s} \to F_{r,s\mp 1} , \\ \phi_m^{(l)}(z)&\colon\ F_{r,s} \to F_{r+l/2,s+l-m} . \end{aligned} \tag{4.3}$$

We can check that $|i/2, 0\rangle$ satisfies the highest weight condition,

$$t_1|i/2, 0\rangle = q^i|i/2, 0\rangle, \quad t_0|i/2, 0\rangle = q^{k-i}|i/2, 0\rangle, \quad e_0|i/2, 0\rangle = 0, \quad e_1|i/2, 0\rangle = 0 .$$

Thus we can identify

$$|\lambda_i\rangle = |i/2, 0\rangle ,$$

where $\lambda_i = (k-i)\Lambda_0 + i\Lambda_1$.

We construct the left highest weight representations $V(\lambda_i)$ of $U_q(\widehat{\mathfrak{sl}}_2)$ as follows[4]

$$V(\lambda_i) := U_q(\widehat{\mathfrak{sl}}_2)|\lambda_i\rangle .$$

Proposition 4.1. *Using this highest weight vector, we can embed the left highest weight module $V(\lambda_i)$ in the Fock modules as follows:*

$$V(\lambda_i) \hookrightarrow \bigoplus_{s \in \mathbf{Z}} F_{i/2,s} \tag{4.4}$$

We can not simply use the vector $|r, s\rangle$, $s \neq 0$, as the highest weight vector since $e_1|r, s\rangle$ does not vanish.

---

3 This kind of decoupling is well known in CFT when we bosonize fermionic ghosts [18]

4 As is well known in CFT, $f_1^{2i+1}|i/2, 0\rangle = 0$ but $f_0^{k-2i+1}|i/2, 0\rangle \neq 0$. So our module is reducible. In order to obtain irreducible modules, the resolution à la Bernard–Felder [19] should be discussed

4.2. Screening Charge. We see that owing to nontrivial charge assignment, naive composition of elementary vertex operators does not define a map between highest weight modules introduced in the previous section. This conundrum is solved by introducing the screening charge.

Let us define the screening operator $J^S(z)$ as follows [8]:

$$J^S(z) = -:[{}_1\partial_z \exp\{-c(2|q^{-k-2}z;0)\}] \\ \times \exp\left\{-b(2|q^{-k-2}z;-1) - a\left(k+2|q^{-2}z;-\frac{k+2}{2}\right)\right\}: . \quad (4.5)$$

Then we get the following:

$$[J_n^3, J^S(z)] = 0 ,$$
$$[J_n^+, J^S(z)] = 0 ,$$
$$[J_n^-, J^S(z)] = {}_{k+2}\partial_z\left[z^n:\exp\left\{-a\left(k+2\left|q^{-2}z;\frac{k+2}{2}\right.\right)\right\}:\right], \quad (4.6)$$

for all $n \in \mathbf{Z}$.

For $p \in \mathbf{C}$, $|p| < 1$, and $s \in \mathbf{C}^\times$, the Jackson integral is defined as

$$\int_0^{s\infty} d_p t f(t) = s(1-p) \sum_{m=-\infty}^{\infty} f(sp^m)p^m ,$$

whenever the RHS converges [20].

Note that the RHS of (4.6) is a total $p = q^{2(k+2)}$ difference. Therefore, the following Jackson integral of the screening operator (screening charge)

$$\int_0^{s\infty} d_p t\, J^S(t) \quad (4.7)$$

commutes with all the generators of $U_q(\widehat{\mathfrak{sl}}_2)$ exactly.

The screening operator enjoys the same relations

$$[J^S(z), \tilde{b}_0 + \tilde{c}_0] = 0 , \quad (4.8)$$

as (4.1), and is a map among Fock modules as follows:

$$J^S(z): F_{r,s} \to F_{r-1,s-1} . \quad (4.9)$$

We want to construct a $U_q'(\widehat{\mathfrak{sl}}_2)$-homomorphism $V(\lambda_i) \to V(\lambda_j) \otimes V_z^{(l)}$, where $\lambda_i = (k-i)\Lambda_0 + i\Lambda_1$. Let us consider the following combination of operators:

$$J^S(t_1)J^S(t_2)\ldots J^S(t_{f(\lambda_j,l,\lambda_i)})\phi_m^{(l)}(z): F_{i/2,s} \to F_{j/2,s-m-(i-j-l)/2} ,$$

where, we denote $f(\lambda_j, l, \lambda_i) := (i-j+l)/2$. By performing the Jackson integral of this operator we obtain a $U_q'(\widehat{\mathfrak{sl}}_2)$-linear map,

$$\sum_m \int_0^{s_1\infty} d_p t_1 J^S(t_1)\ldots \int_0^{s_{f(\lambda_j,l,\lambda_i)}\infty} d_p t_{f(\lambda_j,l,\lambda_i)} J^S(t_{f(\lambda_j,l,\lambda_i)})\phi_m^{(l)}(z) \otimes v_m^{(l)}: \\ V(\lambda_i) \to V(\lambda_j) \otimes V_z^{(l)} ,$$

for arbitrary $\lambda_i, \lambda_j \in P_k$. Since we fixed the normalization of q-vertex operator in (3.13), we have to choose an appropriate normalization factor for each $\lambda_i, \lambda_j \in P_k$ and $V^{(l)}$.

Now we are in a position to state our main conjecture:

Conjecture 4.2. *The q-vertex operator is bosonized as*

$$\tilde{\Phi}_{\lambda}^{\mu V^{(l)}}(z) = \sum_{m=0}^{l} \tilde{\Phi}_{\lambda m}^{\mu V^{(l)}}(z) \otimes v_m^{(l)} : V(\lambda) \to V(\mu) \otimes V_z^{(l)} ,$$

for $\lambda, \mu \in P_k$. Here,

$$\tilde{\Phi}_{\lambda m}^{\mu V^{(l)}}(z) = g_{\lambda}^{\mu V^{(l)}}(z) \int_0^{s_1 \infty} d_p t_1 J^S(t_1) \dots \int_0^{s_{f(\mu,l,\lambda)} \infty} d_p t_{f(\mu,l,\lambda)} J^S(t_{f(\mu,l,\lambda)}) \phi_m^{(l)}(z) ,$$

and $g_{\lambda}^{\mu V^{(l)}}(z)$ is the normalization factor mentioned above.

The N-point function of the q-vertex operators is by definition the expectation value of the composition

$$\Phi_{\mu_{N-1}}^{\mu_N V_N}(z_N) \circ \cdots \circ \Phi_{\mu_0}^{\mu_1 V_1}(z_1) : V(\mu_0) \to V(\mu_N) \otimes V_{z_N}^{(l_N)} \otimes \cdots \otimes V_{z_1}^{(l_1)} .$$

The above construction naturally leads us to the next:

Conjecture 4.3. *If we expand the N-point function of q-vertex operators as*

$$\langle \mu_N | \Phi_{\mu_{N-1}}^{\mu_N V_N}(z_N) \circ \cdots \circ \Phi_{\mu_0}^{\mu_1 V_1}(z_1) | \mu_0 \rangle$$

$$= \sum_{m_1, \dots, m_N} f_{m_1, \dots, m_N}(z_1, \dots, z_N) v_{m_N}^{(l_N)} \otimes \cdots \otimes v_{m_1}^{(l_1)} \in V^{(l_N)} \otimes \cdots \otimes V^{(l_1)} ,$$

where $\mu_0, \dots, \mu_N \in P_k$, then each component has the following integral form:

$$f_{m_1, \dots, m_N}(z_1, \dots, z_N)$$

$$= \prod_{i=1}^{N} z_i^{\Delta_{\mu_i} - \Delta_{\mu_{i-1}}} g_{\mu_{i-1}}^{\mu_i V_i}(z_i) \langle \mu_N | \int_0^{s_1^{(N)} \infty} d_p t_1^{(N)} J^S(t_1^{(N)}) \dots \int_0^{s_{h_N}^{(N)} \infty} d_p t_{h_N}^{(N)} J^S(t_{h_N}^{(N)}) \phi_{m_N}^{(l_N)}(z_N)$$

$$\times \int_0^{s_1^{(1)} \infty} d_p t_1^{(1)} J^S(t_1^{(1)}) \dots \int_0^{s_{h_1}^{(1)} \infty} d_p t_{h_1}^{(1)} J^S(t_{h_1}^{(1)}) \phi_{m_1}^{(l_1)}(z_1) | \mu_0 \rangle ,$$

where $h_i = f(\mu_i, l_i, \mu_{i-1})$.

5. Calculation of Two-Point Function

In what follows we denote $V := V^{(1)} = \mathbf{C}v_+ \otimes \mathbf{C}v_-$, and $z := z_1/z_2$, for short. Let $\Psi(z_1, z_2) \in V \otimes V \otimes z^{3/4(k+2)} \mathbf{Q}(q)[\![z]\!]$ be the following two-point function:

$$\Psi(z_1, z_2) := \langle \lambda_0 | \Phi_{\lambda_1}^{\lambda_0 V}(z_2) \circ \Phi_{\lambda_0}^{\lambda_1 V}(z_1) | \lambda_0 \rangle , \tag{5.1}$$

where

$$\Phi_{\lambda_0}^{\lambda_1 V}(z_1) = z^{3/4(k+2)} \tilde{\Phi}_{\lambda_0}^{\lambda_1 V}(z_1), \quad \Phi_{\lambda_1}^{\lambda_0 V}(z_2) = z^{-3/4(k+2)} \tilde{\Phi}_{\lambda_1}^{\lambda_0 V}(z_2) ,$$

and $\lambda_0 = k\Lambda_0$, $\lambda_1 = (k-1)\Lambda_0 + \Lambda_1$.

In this section by evaluating this correlation function, we check Conjecture 4.3 for $N = 2$, $l_1 = l_2 = 1$, and $k \in \mathbf{Z}_{>0}$.

5.1. Jackson Integral Formula for Two-Point Function. From Conjecture 4.2 q-VOs have the following bosonization:

$$\tilde{\Phi}_{\lambda_0}^{\lambda_1 V}(z_1) = g_{\lambda_0}^{\lambda_1 V}(z_1)(\phi_+(z_1)\otimes v_+ + \phi_-(z_1)\otimes v_-)\,, \tag{5.2}$$

$$\tilde{\Phi}_{\lambda_1}^{\lambda_0 V}(z_2) = g_{\lambda_1}^{\lambda_0 V}(z_2)\int_0^{s\infty} d_p t J^S(t)(\phi_+(z_2)\otimes v_+ + \phi_-(z_2)\otimes v_-)\,, \tag{5.3}$$

where $\phi_+(z_i) = \phi_0^{(1)}(z_i)$, and $\phi_-(z_i) = \phi_1^{(1)}(z_i)$ $(i = 1, 2)$. Here $g_{\lambda_0}^{\lambda_1 V}(z_1) = 1$ and $g_{\lambda_1}^{\lambda_0 V}(z_2) =: g(z_2)$ are the normalization factors of $\tilde{\Phi}_{\lambda_0}^{\lambda_1 V}(z_1)$ and $\tilde{\Phi}_{\lambda_1}^{\lambda_0 V}(z_2)$, respectively. Explicitly, (for a detailed calculation, see Appendix B)

$$g(z_2) = -q^{-2-(k+8)/2(k+2)} z_2^{-1/2(k+2)}\left[\int_0^{s\infty} d_p t t^{-1-2/(k+2)} \frac{(p^2 q z_2/t; p)_\infty}{(pq^{-1}z_2/t; p)_\infty}\right]^{-1}, \tag{5.4}$$

where $p = q^{2(k+2)}$, and

$$(a; q)_\infty := \prod_{n=0}^{\infty}(1 - aq^n)\,.$$

Since the q-VOs preserve the weight modulo δ we have

$$\Psi(z_1, z_2) = f_1(z_1, z_2)v_+ \otimes v_- + f_2(z_1, z_2)v_- \otimes v_+\,.$$

Using the free boson representation of the q-VOs we can rewrite $f_1(z_1, z_2)$ as

$$\begin{aligned} f_1(z_1, z_2) &= z^{3/4(k+2)} g(z_2)\int_0^{s\infty} d_p t\langle\lambda_0|J^S(t)\phi_+(z_2)\phi_-(z_1)|\lambda_0\rangle \\ &= z^{3/4(k+2)} g(z_2)\int_0^{s\infty} d_p t\oint \frac{dw}{2\pi\sqrt{-1}} \\ &\quad\times\langle\lambda_0|J^S(t)[\phi_-(z_2), J^-(w)]_q\,\phi_-(z_1)|\lambda_0\rangle\,. \end{aligned} \tag{5.5}$$

Thanks to the formulae of the OPE given in Appendix A we obtain

$$\begin{aligned} f_1(z_1, z_2) &= z^{3/4(k+2)} g(z_2)\int_0^{s\infty} d_p t\left\{\oint_{q^{k+3}z_1}\frac{dw}{2\pi\sqrt{-1}}\langle\lambda_0|J^S(t)\phi_-(z_2)J^-(w)\phi_-(z_1)|\lambda_0\rangle\right. \\ &\quad\left. - q\oint_{q^{k+3}z_i,\, i=1,2}\frac{dw}{2\pi\sqrt{-1}}\langle\lambda_0|J^S(t)J^-(w)\phi_-(z_2)\phi_-(z_1)|\lambda_0\rangle\right\} \\ &= -q^{1+4/(k+2)} z^{3/4(k+2)} g(z_2) G(z_1, z_2)\int_0^{s\infty} d_p t\varphi_1(z_1, z_2, t)\,, \end{aligned} \tag{5.6}$$

where $G(z_1, z_2)$ comes from OPE of the q-VOs,

$$\begin{aligned} &\phi_-(z_2)\phi_-(z_1) = G(z_1, z_2) :\phi_-(z_2)\phi_-(z_1):\,, \\ &G(z_1, z_2) = (q^k z_2)^{1/2(k+2)}\prod_{m=1}^{\infty}\frac{(p^m z; q^4)_\infty (p^m q^4 z; q^4)_\infty}{(p^m q^2 z; q^4)_\infty^2}\,, \end{aligned} \tag{5.7}$$

while the integrand of the Jackson integral is given as follows:

$$\varphi_1(z_1, z_2, t) = qt^{-1-2/(k+2)}\frac{(qp^2 z_1/t; p)_\infty (qp^2 z_2/t; p)_\infty}{(q^{-1}p^2 z_1/t; p)_\infty (q^{-1}pz_2/t; p)_\infty}\,. \tag{5.8}$$

Let us check that $f_1(z_1, z_2)$ depends upon $z = z_1/z_2$ only and hence we may denote $f_1(z_1, z_2) = f_1(z)$. Using the freedom of redefinition $t \mapsto z_2 t$ in the Jackson integral, we can rewrite

$$g(z_2) = -q^{-2-(k+8)/2(k+2)} z_2^{3/2(k+2)} \left[\int_0^{s\infty} dt\, t^{-1-2/(k+2)} \frac{(p^2q/t;p)_\infty}{(pq^{-1}/t;p)_\infty} \right]^{-1} , \tag{5.9}$$

$$\int_0^{s\infty} d_p t\, \varphi_1(z_1, z_2, t) = q z_2^{-2/(k+2)} \int_0^{s\infty} d_p t\, t^{-1-2/(k+2)} \frac{(p^2qz/t;p)_\infty (p^2q/t;p)_\infty}{(p^2q^{-1}z/t;p)_\infty (pq^{-1}/t;p)_\infty} . \tag{5.10}$$

Therefore we can regard $f_1(z_1, z_2)$ as a function of z:

$$f_1(z) = z^{3/4(k+2)} \prod_{m=1}^{\infty} \frac{(p^m z;q^4)_\infty (p^m q^4 z;q^4)_\infty}{(p^m q^2 z;q^4)_\infty^2} \times \frac{\int_0^{s\infty} d_p t\, t^{-1-2/(k+2)} \frac{(p^2qz/t;p)_\infty (p^2q/t;p)_\infty}{(p^2q^{-1}z/t;p)_\infty (pq^{-1}/t;p)_\infty}}{\int_0^{s\infty} d_p t\, t^{-1-2/(k+2)} \frac{(p^2q/t;p)_\infty}{(pq^{-1}/t;p)_\infty}} . \tag{5.11}$$

Let us repeat the same argument with respect to $f_2(z_1, z_2)$.

$$\begin{aligned} f_2(z_1, z_2) &= z^{3/4(k+2)} g(z_2) \int_0^{s\infty} d_p t \langle \lambda_0 | J^S(t) \phi_-(z_2) \phi_+(z_1) | \lambda_0 \rangle \\ &= z^{3/4(k+2)} g(z_2) \int_0^{s\infty} d_p t \oint \frac{dw}{2\pi\sqrt{-1}} \\ &\quad \times \langle \lambda_0 | J^S(t) \phi_-(z_2) [J^-(w), \phi_-(z_1)]_q | \lambda_0 \rangle . \end{aligned} \tag{5.12}$$

Similarly we have

$$f_2(z_1, z_2) = -q^{1+4/(k+2)} z^{3/4(k+2)} g(z_2) G(z_1, z_2) \int_0^{s\infty} d_p t\, \varphi_2(z_1, z_2, t) , \tag{5.13}$$

where

$$\varphi_2(z_1, z_2, t) = t^{-1-2/(k+2)} \frac{(qp^2 z_1/t;p)_\infty (qpz_2/t;p)_\infty}{(q^{-1}pz_1/t;p)_\infty (q^{-1}pz_2/t;p)_\infty} . \tag{5.14}$$

Thus we obtain

$$f_2(z) = q^{-1} z^{3/4(k+2)} \prod_{m=1}^{\infty} \frac{(p^m z;q^4)_\infty (p^m q^4 z;q^4)_\infty}{(p^m q^2 z;q^4)_\infty^2} \times \frac{\int_0^{s\infty} d_p t\, t^{-1-2/(k+2)} \frac{(p^2qz/t;p)_\infty (pq/t;p)_\infty}{(pq^{-1}z/t;p)_\infty (pq^{-1}/t;p)_\infty}}{\int_0^{s\infty} d_p t\, t^{-1-2/(k+2)} \frac{(p^2q/t;p)_\infty}{(pq^{-1}/t;p)_\infty}} . \tag{5.15}$$

Note that $z^{-3/4(k+2)} f_i(z)$ is analytic around $z = 0$ ($i = 1, 2$).

5.2. q-KZ Equation. Now we show that the q-difference system for $f_1(z_1, z_2)$, $f_2(z_1, z_2)$ gives the q-KZ equation for the two-point function. Let us study the effect of p-shift $z_1 \mapsto pz_1, z_2 \mapsto z_2$. The change of $f_i(z_1, z_2)$ results from $z^{3/4(k+2)}$, $G(z_1, z_2)$ and $\phi_i(z_1, z_2, t)$, $(i = 1, 2)$. First $G(z_1, z_2)$ transforms as follows:

$$G(pz_1, z_2) = q^{1/2}\rho(pz)G(z_1, z_2)\,, \tag{5.16}$$

where

$$\rho(z) = q^{-1/2}\frac{(q^2z;q^4)_\infty^2}{(qz;q^4)_\infty(q^4z;q^4)_\infty}\,, \tag{5.17}$$

is precisely the same factor which appeared in the image of the universal $\mathscr{R}$ matrix of $U_q(\widehat{\mathfrak{sl}}_2)$ [2]. Next the contribution from the Jackson integral is given as follows: (See Appendix C, for details)

$$\begin{aligned}&\left(\int_0^{s\infty} d_pt\,\varphi_1(pz_1, z_2, t),\ \int_0^{s\infty} d_pt\,\varphi_2(pz_1, z_2, t)\right)\\ &\quad = \left(\int_0^{s\infty} d_pt\,\varphi_1(z_1, z_2, t),\ \int_0^{s\infty} d_pt\,\varphi_2(z_1, z_2, t)\right)\bar{R}(pz)\,,\end{aligned} \tag{5.18}$$

where

$$\bar{R}(z) = \frac{1}{1-q^2z}\begin{pmatrix} 1-z & q^{-1}-q \\ (q^{-3}-q^{-1})z & q^{-2}(1-z)\end{pmatrix}, \tag{5.19}$$

is just the zero-weight part of the R matrix of the six vertex model up to a similarity transformation.

By combining Eqs. (5.16), (5.18), 5.19), and the factor from $z^{3/4(k+2)}$ we obtain

$$\begin{pmatrix} f_1(pz) \\ f_2(pz)\end{pmatrix} = \frac{\rho(pz)}{1-pq^2z}\begin{pmatrix} q^2(1-pz) & pq^{-1}(1-q^2)z \\ q(1-q^2) & (1-zp)\end{pmatrix}\begin{pmatrix} f_1(z) \\ f_2(z)\end{pmatrix}. \tag{5.20}$$

It coincides with the q-KZ equation [2] for the two-point function.

This recursion formula implies

$$qf_1(pz) + f_2(pz) = \rho(pz)(qf_1(z) + f_2(z))\,.$$

Comparing the coefficients of $z^{3/4(k+2)}$ of both sides of the equation above, we obtain

$$qf_1(z) + f_2(z) = 0\,.$$

Therefore we have the q-difference equation of the first order

$$\frac{f_1(pz)}{f_1(z)} = q^{3/2}\frac{(pq^{-2}z;q^4)_\infty(pq^6z;q^4)_\infty}{(pqz;q^4)_\infty(pq^4z;q^4)_\infty}\,, \tag{5.21}$$

$$f_2(z) = -qf_1(z)\,. \tag{5.22}$$

In particular if we put $k = 1$, by solving the above q-difference equation we have

$$\Psi(z_1, z_2) = z^{1/4}\frac{(q^6z;q^4)_\infty}{(q^4z;q^4)_\infty}(v_+ \otimes v - qv_- \otimes v_+)\,, \tag{5.23}$$

which reproduces the known results[5].

[5] We use the opposite ordering of two V's as that of ref. [5]

6. Conclusion

In this paper we discuss a bosonization of the q-vertex operator on the basis of the Fock representation of $U_q(\widehat{\mathfrak{sl}}_2)$. We propose an integral formula for N-point functions of the q-vertex operators with the help of the screening charges. Matsuo [21] and Reshetikhin [22] have obtained integral formulae from the viewpoint of the q-KZ equation. The relations among these three integral formulae should be clarified.

After performing all the residue calculi of the two-point function, we have a Jackson integral of Jordan–Pochhammer type [23, 20]. It is intriguing that the scalar factor which arises in the image of the universal $\mathscr{R}$ matrix naturally appears in the OPE of elementary vertex operators.

We would like to check all the intertwining properties of the elementary q-vertex operators for the general case. The analogy with CFT is quite remarkable; we can deform $(\widehat{\mathfrak{sl}}_2)$ currents, the screening current, and vertex operators à la Tsuchiya–Kanie [24]. However, we have no counterpart of Virasoro algebra, and the meaning of the spectral parameters of the q-vertex operators is not yet obvious.

Recently Matsuo [25] constructed another bosonization of $U_q'(\widehat{\mathfrak{sl}}_2)$. It is interesting to investigate the connection between his bosonization and ours. After completing this work we received a preprint by Abada et al. [26].

Acknowledgement. The authors thank T. Eguchi, M. Jimbo, A. Matsuo, T. Miwa and K. Sugiyama for helpful discussions and useful comments. They also thank H. Awata, T. Inami, T. Nakatsu, A. Nakayashiki, T. Sano and Y. Yamada for encouragement and their interest in this work. J.S. is grateful for kind hospitality at YITP and RIMS.

This work was supported in part by the Grant-in-Aid for Scientific Research for the Ministry of Education, Science and Culture (No. 04245206 and No. 04-2297).

A. Operator Product Expansion Formulae

In this appendix we list the operator product expansion formulae among the $U_q(\widehat{\mathfrak{sl}}_2)$-current $J^-(z)$, the screening current $J^S(z)$ and the elementary q-vertex operator $\phi_l^{(l)}(z)$.

We split $J^-(z)$ into two parts:

$$J^-(z) \equiv \frac{1}{(q-q^{-1})z}\{J_{\mathrm{I}}^-(z) - J_{\mathrm{II}}^-(z)\}\,,$$

$$J_{\mathrm{I}}^-(z) = :\exp\left\{a\left(k+2\left|q^k z; -\frac{k+2}{2}\right.\right) - a\left(k+2\left|q^{-2}z; \frac{k+2}{2}\right.\right)\right.$$
$$\left. + b(2|z;-1) + c(2|q^{-1}z;0)\right\}:\,,$$

$$J_{\mathrm{II}}^-(z) = :\exp\left\{a\left(k+2\left|q^{-k-4}z; -\frac{k+2}{2}\right.\right) - a\left(k+2\left|q^{-2}z; \frac{k+2}{2}\right.\right)\right.$$
$$\left. + b(2|q^{-2k-4}z;-1) + c(2|q^{-2k-3}z;0)\right\}:\,.$$

Similarly, we put

$$J^S(z) \equiv -\frac{1}{(q-q^{-1})z}\{J^S_{\mathrm{I}}(z) - J^S_{\mathrm{II}}(z)\}\,,$$

$$J^S_{\mathrm{I}}(z) = :\exp\left\{-a\left(k+2\left|q^{-2}z; -\frac{k+2}{2}\right.\right)\right.$$
$$\left. - b(2|q^{-k-2}z; -1) - c(2|q^{-k-1}z; 0)\right\}:\,,$$

$$J^S_{\mathrm{II}}(z) = :\exp\left\{-a\left(k+2\left|q^{-2}z; -\frac{k+2}{2}\right.\right)\right.$$
$$\left. - b(2|q^{-k-2}z; -1) - c(2|q^{-k-3}z; 0)\right\}:\,.$$

$$\begin{aligned}
J^-_{\mathrm{I}}(z)\phi^{(l)}_l(w) &= \frac{q^l z - q^{k+2}w}{z - q^{l+k+2}w} :J^-_{\mathrm{I}}(z)\phi^{(l)}_l(w): \quad |z| > q^{k+2-l}|w|\,,\\
J^-_{\mathrm{II}}(z)\phi^{(l)}_l(w) &= q^{-l} :J^-_{\mathrm{II}}(z)\phi^{(l)}_l(w):\,,\\
\phi^{(l)}_l(w)J^-_{\mathrm{I}}(z) &= :\phi^{(l)}_l(w)J^-_{\mathrm{I}}(z):\,,\\
\phi^{(l)}_l(w)J^-_{\mathrm{II}}(z) &= \frac{w - q^{-l-k-2}z}{w - q^{l-k-2}z} :\phi^{(l)}_l(w)J^-_{\mathrm{II}}(z): \quad |w| > q^{-l-k-2}|z|\,,
\end{aligned} \tag{A.1}$$

$$\begin{aligned}
J^S_{\mathrm{I}}(w)J^-_{\mathrm{I}}(z) &= q^{-1} :J^-_{\mathrm{I}}(z)J^S_{\mathrm{I}}(w):\,,\\
J^S_{\mathrm{I}}(w)J^-_{\mathrm{II}}(z) &= \frac{q^{-1}w - q^{-k-1}z}{w - q^{-k-2}z} :J^-_{\mathrm{II}}(z)J^S_{\mathrm{I}}(w): \quad |w| > q^{-k-2}|z|\,,\\
J^S_{\mathrm{II}}(w)J^-_{\mathrm{I}}(z) &= \frac{qw - q^{k+1}z}{w - q^{k+2}z} :J^-_{\mathrm{I}}(z)J^S_{\mathrm{II}}(w): \quad |w| > q^{k}|z|\,,\\
J^S_{\mathrm{II}}(w)J^-_{\mathrm{II}}(z) &= q :J^-_{\mathrm{II}}(z)J^S_{\mathrm{II}}(w):\,,
\end{aligned} \tag{A.2}$$

$$\begin{aligned}
J^-_{\mathrm{I}}(z)J^S_{\mathrm{I}}(w) &= q^{-1} :J^-_{\mathrm{I}}(z)J^S_{\mathrm{I}}(w):\,,\\
J^-_{\mathrm{I}}(z)J^S_{\mathrm{II}}(w) &= \frac{q^{-1}z - q^{-k-1}w}{z - q^{-k-2}w} :J^-_{\mathrm{I}}(z)J^S_{\mathrm{II}}(w): \quad |z| > q^{-k-2}|w|\,,\\
J^-_{\mathrm{II}}(z)J^S_{\mathrm{I}}(w) &= \frac{qz - q^{k+1}w}{z - q^{k+2}w} :J^-_{\mathrm{II}}(z)J^S_{\mathrm{I}}(w): \quad |z| > q^{k}|w|\,,\\
J^-_{\mathrm{II}}(z)J^S_{\mathrm{II}}(w) &= q :J^-_{\mathrm{II}}(z)J^S_{\mathrm{II}}(w):\,,
\end{aligned} \tag{A.3}$$

$$\begin{aligned}
J^S_{\mathrm{I}}(z)\phi^{(l)}_l(w) &= \frac{(q^l pw/z; p)_\infty}{(q^{-l}pw/z; p)_\infty}(q^{-2}z)^{-l/2(k+2)} :J^S_{\mathrm{I}}(z)\phi^{(l)}_l(w): \quad |z| > q^{-l}p|w|\,,\\
J^S_{\mathrm{II}}(z)\phi^{(l)}_l(w) &= \frac{(q^l pw/z; p)_\infty}{(q^{-l}pw/z; p)_\infty}(q^{-2}z)^{-l/2(k+2)} :J^S_{\mathrm{II}}(z)\phi^{(l)}_l(w): \quad |z| > q^{-l}p|w|\,,\\
\phi^{(l)}_l(w)J^S_{\mathrm{I}}(z) &= \frac{(q^l z/w; p)_\infty}{(q^{-l}z/w; p)_\infty}(q^{k}w)^{-l/2(k+2)} :J^S_{\mathrm{I}}(z)\phi^{(l)}_l(w): \quad |w| > q^{-l}|z|\,,\\
\phi^{(l)}_l(w)J^S_{\mathrm{II}}(z) &= \frac{(q^l z/w; p)_\infty}{(q^{-l}z/w; p)_\infty}(q^{k}w)^{-l/2(k+2)} :J^S_{\mathrm{II}}(z)\phi^{(l)}_l(w): \quad |w| > q^{-l}|z|\,,
\end{aligned} \tag{A.4}$$

$$\phi_l^{(l)}(z)\phi_l^{(l)}(w) = (q^k z)^{l^2/2(k+2)} \prod_{m=1}^{\infty} \frac{(p^m q^{2(1-l)} w/z; q^4)_\infty (p^m q^{2(1+l)} w/z; q^4)_\infty}{(p^m q^2 w/z; q^4)_\infty^2} \times :\phi_l^{(l)}(z)\phi_l^{(l)}(w): . \tag{A.5}$$

B. Normalization of q-Vertex Operators

Here we consider the normalization of the following q-VOs:

$$\tilde{\Phi}_{\lambda_0}^{\lambda_1 V^{(1)}}(z)\colon V(\lambda_0) \to V(\lambda_1) \otimes V_z^{(1)} ,$$
$$\tilde{\Phi}_{\lambda_1}^{\lambda_0 V^{(1)}}(z)\colon V(\lambda_1) \to V(\lambda_0) \otimes V_z^{(1)} . \tag{B.1}$$

These q-VOs have the following leading terms:

$$\tilde{\Phi}_{\lambda_0}^{\lambda_1 V^{(1)}}(z)|\lambda_0\rangle = |\lambda_1\rangle \otimes v_- + \cdots ,$$
$$\tilde{\Phi}_{\lambda_1}^{\lambda_0 V^{(1)}}(z)|\lambda_1\rangle = |\lambda_0\rangle \otimes v_+ + \cdots , \tag{B.2}$$

These q-VOs are bosonized as

$$\tilde{\Phi}_{\lambda_0}^{\lambda_1 V^{(1)}}(z) = g_{\lambda_0}^{\lambda_1 V^{(1)}}(z)\,[\phi_+(z) \otimes v_+ + \phi_-(z) \otimes v_-] ,$$
$$\tilde{\Phi}_{\lambda_1}^{\lambda_0 V^{(1)}}(z) = g_{\lambda_1}^{\lambda_0 V^{(1)}}(z) \left[\int_0^{s\infty} d_p t J^S(t) \phi_+(z) \otimes v_+ + \int_0^{s\infty} d_p t J^S(t) \phi_-(z) \otimes v_- \right] . \tag{B.3}$$

We can get these normalization functions $g_{\lambda_0}^{\lambda_1 V^{(1)}}(z)$, $g_{\lambda_1}^{\lambda_0 V^{(1)}}(z)$ by calculating the leading term explicitly.

First we have

$$\phi_-(z)|0,0\rangle = :\exp\left\{a\left(1;2,k+2\middle|q^k z;\frac{k+2}{2}\right)\right\}:|0,0\rangle$$
$$= |1/2,0\rangle + \cdots , \tag{B.4}$$

then we get

$$g_{\lambda_0}^{\lambda_1 V^{(1)}}(z) = 1 . \tag{B.5}$$

Next we can see

$$\int_0^{s\infty} d_p t J^S(t) \phi_+(z)|1/2,0\rangle$$
$$= \int_0^{s\infty} d_p t J^S(t) \int \frac{dw}{2\pi\sqrt{-1}} [\phi_-(z), J^-(w)]_q \exp\left\{\frac{Q_a}{2(k+2)}\right\}|0,0\rangle$$
$$= -q^{2+(k+8)/2(k+2)} z^{1/2(k+2)} \int_0^{s\infty} d_p t\, t^{-1-2/(k+2)} \frac{(qp^2 z/t;p)_\infty}{(q^{-1}pz/t;p)_\infty}|0,0\rangle + \cdots , \tag{B.6}$$

likewise. Then in this case

$$g_{\lambda_1}^{\lambda_0 V^{(1)}}(z) = -q^{-2-(k+8)/2(k+2)} z^{-1/2(k+2)} \left[\int_0^{s\infty} d_p t\, t^{-1-2/(k+2)} \frac{(qp^2 z/t;p)_\infty}{(q^{-1}pz/t;p)_\infty} \right]^{-1} \tag{B.7}$$

holds.

C. Difference Properties of Jackson Integrals

In this appendix, we calculate the difference system satisfied by the following functions:

$$\int_0^{s\infty} d_pt\varphi_1(z_1, z_2, t), \quad \int_0^{s\infty} d_pt\varphi_2(z_1, z_2, t)\,, \tag{C.1}$$

where $\varphi_1(z_1, z_2, t)$, and $\varphi_2(z_1, z_2, t)$ are given as

$$\varphi_1(z_1, z_2, t) = q\frac{(qp^2z_1/t;p)_\infty(qp^2z_2/t;p)_\infty}{(q^{-1}p^1z_1/t;p)_\infty(q^{-1}p^2z_2/t;p)_\infty}t^{-1-2/(k+2)}\,,$$

$$\varphi_2(z_1, z_2, t) = q\frac{(qpz_1/t;p)_\infty(qp^2z_2/t;p)_\infty}{(q^{-1}pz_1/t;p)_\infty(q^{-1}pz_2/t;p)_\infty}t^{-1-2/(k+2)}\,. \tag{C.2}$$

We note that these functions are the Jackson integrals of Jordan-Pochhammer type. For the general theory of the difference system for the Jackson integrals of Jordan–Pochhammer type, we refer the reader to [23, 20].

To find the difference equation, we use the following identity:

$$\int_0^{s\infty} d_pt\varphi_i(z_1, z_2, t) = \int_0^{s\infty} d_ptp\varphi_i(z_1, z_2, pt)\,. \tag{C.3}$$

Since we have

$$p\varphi_1(pz_1, z_2, pt) = p\varphi_1(z_1, z_2, pt)\frac{1-pz}{1-pq^2z} + \varphi_2(z_1, z_2, t)\frac{(1-q^2)pq^{-3}z}{1-pq^2z}\,,$$

$$p\varphi_2(pz_1, z_2, pt) = p\varphi_1(z_1, z_2, pt)\frac{(1-q^2)q^{-1}}{1-pq^2z} + \varphi_2(z_1, z_2, t)\frac{(1-zp)q^{-2}}{1-pq^2z}\,, \tag{C.4}$$

we get the following difference equation:

$$\left(\int_0^{s\infty} d_pt\varphi_1(pz_1, z_2, t), \int_0^{s\infty} d_pt\varphi_2(pz_1, z_2, t)\right)$$
$$= \left(\int_0^{s\infty} d_pt\varphi_1(z_1, z_2, t), \int_0^{s\infty} d_pt\varphi_2(z_1, z_2, t)\right)\begin{pmatrix} \dfrac{1-pz}{1-pq^2z} & \dfrac{(1-q^2)q^{-1}}{1-pq^2z} \\ \dfrac{(1-q^2)pq^{-3}z}{1-pq^2z} & \dfrac{(1-zp)q^{-2}}{1-pq^2z} \end{pmatrix}. \tag{C.5}$$

References

1. Itzykson, C., Saleur, H., Zuber, J.-B. (eds.): Conformal Invariance and Applications to Statistical Mechanics. Singapore: World Scientific 1988
2. Frenkel, I.B., Reshetikhin, N.Yu.: Quantum Affine Algebras and Holonomic Difference Equations. Commun. Math. Phys. **146**, 1–60 (1992)
3. Andrews, G.E., Baxter, R.J., Forrester, P.J.: Eight-Vertex SOS Model and Generalized Rogers–Ramanujan-Type Identities. J. Stat. Phys. **35**, 193–266 (1984)
4. Jimbo, M., Miwa, T., Okado, M.: Solvable Lattice Models Related to the Vector Representation of Classical Simple Lie Algebras. Commun. Math. Phys. **116**, 507–525 (1988)

5. Davies, B., Foda, O., Jimbo, M., Miwa, T., Nakayashiki, A.: Diagonalization of the XXZ Hamiltonian by Vertex Operators. RIMS preprint **873** (1992)
6. Jimbo, M., Miki, K., Miwa, T., Nakayashiki, A.: Correlation Functions of the XXZ model for $\Delta < -1$. RIMS preprint **877** (1992)
7. Frenkel, I.B., Jing, N.H.: Vertex Representations of Quantum Affine Algebras. Proc. Nat'l. Acad. Sci. USA **85**, 9373–9377 (1988)
8. Shiraishi, J.: Free Boson Representation of $U_q(\widehat{\mathfrak{sl}}_2)$. Univ. Tokyo preprint **UT-617** (1992)
9. Wakimoto, M.: Fock Representations of the Affine Lie Algebra $A_1^{(1)}$. Commun. Math. Phys. **104**, 605–609 (1986)
10. Idzumi, M., Iohara, K., Jimbo, M., Miwa, T., Nakashima, T., Tokihiro, T.: Quantum Affine Symmetry in Vertex Models. RIMS preprint (1992)
11. Drinfeld, V.G.: A New Realization of Yangians and Quantum Affine Algebas. Soviet Math. Doklady **36**, 212–216 (1988)
12. Kac, V.G.: Infinite Dimensional Lie Algebras. 3rd ed., Cambridge: Cambridge Univ. Press 1990
13. Drinfeld, V.G.: Quantum Groups. Proceedings of ICM, Berkeley, 1986, pp. 798–820
14. Jimbo, M.: A q-Difference Analogue of $U(\mathfrak{g})$ and the Yang–Baxter Equation. Lett. Math. Phys. **10**, 63–69 (1985)
15 Kashiwara, M.: On Crystal Bases of the q-analog of the Universal Enveloping Algebra. Duke Math. J. **63**, 465–516 (1991)
16. Date, E., Jimbo, M., Okado, M.: Crystal Base and q-Vertex Operators. Commun. Math. Phys. **155**, 47–69 (1993)
17. Chari, V., Pressley, A.: Quantum Affine Algebras. Commun. Math. Phys. **142**, 261–283 (1991)
18. Friedan, D., Martinec, E., Shenker, S.: Conformal Invariance, Supersymmetry and String Theory. Nucl. Phys. **B271**, 93–165 (1986)
19. Bernard, D., Felder, G.: Fock Representations and BRST Cohomology in $SL(2)$ Current Algebra. Commun. Math. Phys. **127**, 145–168 (1990)
20. Matsuo, A.: Jackson Integrals of Jordan–Pochhammer Type and Knizhnik–Zamolodchikov Equations. Preprint (1992)
21. Matsuo, A.: Quantum Algebra Structure of Certain Jackson Integrals. Preprint (1992)
22. Reshetikhin, N.: Jackson-Type Integrals, Bethe Vectors, and Solutions to a Difference Analog of the Knizhnik–Zamolodchikov System. Preprint
23. Mimachi, K.: Connection Problem in Holonomic q-Difference System Associated with a Jackson Integral of Jordan–Pochhammer Type. Nagoya Math. J. **116**, 149–161 (1989)
24. Tsuchiya, A., Kanie, Y.: Vertex Operators in Conformal Field Theory on $\mathbf{P}^1$ and Monodromy Representations of Braid Group. Adv. Stud. Pure Math. **16**, 297–372 (1988)
25. Matsuo, A.: Free Field Realization of Quantum Affine Algebra $U_q(\widehat{\mathfrak{sl}}_2)$. Preprint (1992)
26. Abada, A., Bougourzi, A.H., El Gradechi, M.A.: Deformation of the Wakimoto construction. Preprint (1992)

Communicated by H. Araki