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Tau-Functions and Generalized Integrable Hierarchies

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Abstract. The tau-function formalism for a class of generalized "zero-curvature" integrable hierarchies of partial differential equations is constructed. The class includes the Drinfel'd–Sokolov hierarchies. A direct relation between the variables of the zero-curvature formalism and the tau-functions is established. The formalism also clarifies the connection between the zero-curvature hierarchies and the Hirota-type hierarchies of Kac and Wakimoto.

1. Introduction

The evolution of the subject of integrable hierarchies of equations has exhibited many unexpected twists. Arguably, the first important mathematical result was the demonstration of the integrability of the Korteweg–de Vries (KdV) equation

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} + 6u\frac{\partial u}{\partial x}\,. \tag{1.1}$$

Since then much effort has been devoted to finding the underlying "causes" for integrability. Such an endeavour is intimately linked to the problem of classification, because in a general framework one can separate out the underlying important "wheat" of the problem from the example-dependent "chaff." We believe that one of the most important and seminal works in this regard was that of Drinfel'd and Sokolov [1]. These authors provided the most general classification of integrable hierarchies of equations up to that time. Their construction is based on a zero-curvature, or Lax-type, method, where integrability is manifest. The central objects in the construction are gauge fields in the loop algebra of a finite Lie algebra. Crudely speaking, they arrive at a picture where there is a *modified* KdV

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(mKdV) hierarchy for each loop algebra, and then associated KdV hierarchies for each of the nodes of the corresponding Dynkin diagram.

The Drinfel'd–Sokolov hierarchies make use of the "principal" gradation of the loop algebra in an essential way. In particular, the construction involves the *principal Heisenberg subalgebra*. On the other hand, it is well known that affine Kac–Moody algebras have many Heisenberg subalgebras [2, 3], an observation that was exploited in [4] (see also [5]) to construct a more general class of integrable hierarchies. These hierarchies share all the features of the Drinfel'd–Sokolov hierarchies: there are mKdV and KdV-type hierarchies with (bi-)Hamiltonian structures [6].

However, there are ways, other than the zero-curvature method, to investigate integrable hierarchies. One of the most remarkable developments in the subject started with the work of R. Hirota (see for example [7]), who discovered a way to construct various types of solutions to the hierarchies directly; in particular the multiple soliton solutions can easily be found. This led to the so-called "tau-function" approach pioneered by the Japanese school (see for example [8]). The idea is to find a new set of variables, called the tau-functions, which then sastisfy a new type of bi-linear equation known as the Hirota equation. For instance, the tau-function of the KdV equation – in standard conventions – is related to the original variable by the celebrated formula

$$u = 2\frac{\partial^2}{\partial x^2}\log\tau\ . \tag{1.2}$$

Correspondingly, for the *modified* KdV hierarchy the relation is

$$v = \frac{\partial}{\partial x}\log\left(\frac{\tau_0}{\tau_1}\right), \tag{1.3}$$

there being two separate tau-functions in this case.

Far from being just a new solution-method, the tau-function approach uncovered a deep underlying structure of integrable hierarchies. The story is quite long and complicated involving unexpected connections to other branches of mathematics; for which we refer the reader to the original literature [8, 9, 10], and references therein. It is clear from this approach that affine Kac–Moody algebras (central extensions of loop algebras) again play a central role. This was so in the original work of [8], but made even clearer by Kac and Wakimoto [11]. In this latter work, the authors construct hierarchies directly in Hirota form associated to vertex operator representations of Kac–Moody algebras.

It certainly occurred to Kac and Wakimoto [11] that there should be a connection between their work and that of Drinfel'd and Sokolov: both involving, as they do, Kac–Moody algebras. This present work is an attempt to make this connection explicit. The situation for the affine algebra $A_1^{(1)}$, which leads to the KdV and mKdV hierarchies is well established [9, 10]. Some extensions to other algebras and the *homogeneous* Heisenberg subalgebra were considered in [12], and an example involving the "intermediate" Heisenberg subalgebra in $A_2^{(1)}$, which is related to the $W_3^{(2)}$-algebra, was considered in [13]. Our approach follows very closely the spirit of [12], and we shall only mention the *Grassmannian* approach in passing.

The central goal of this work is to provide an explicit relation between the tau-functions and the variables of the zero-curvature formalism. We shall not find a one-to-one correspondence between the zero-curvature hierarchies and the Kac–Wakimoto hierarchies: only a subset of both classes are related.

The paper is organized as follows. In Sect. 2 we describe the general class of zero-curvature integrable hierarchies of [4], which contain the Drinfel'd–Sokolov hierarchies as special cases. Section 3 introduces the Kac–Wakimoto hierarchies which are defined directly on the tau-functions, in terms of a vertex operator representation of a Kac–Moody algebra. Section 4 considers the "dressing transformation" which allows one to construct solutions of the zero-curvature hierarchies, and which also provides the key for establishing a connection between the formalisms of Sects. 2 and 3. The explicit connection is established in Sect. 5 and some examples are considered in Sect. 6 for the purposes of illustration. Our conventions and some properties of Kac–Moody algebras are presented in the appendix.

2. The Zero-Curvature Hierarchies

The purpose of this section is two-fold: it should provide a summary of some of the important details of refs. [4, 6] and sets up some new results that will be required in later sections. Our conventions concerning affine Kac-Moody algebras are summarized in the appendix.

In refs. [4, 6], a generalized integrable hierarchy was associated to each affine Kac–Moody algebra $\mathfrak{g}$, a particular Heisenberg subalgebra $\mathfrak{s} \subset \mathfrak{g}$ (with an associated gradation $\mathbf{s}'$) and an additional gradation $\mathbf{s}$, such that $\mathbf{s}' \succeq \mathbf{s}$, with respect to a partial ordering (see the appendix). The auxiliary gradation $\mathbf{s}$ sets the "degree of modification" of the hierarchy: the larger $\mathbf{s}$ becomes the more "modified" the hierarchy becomes. In [4], the construction was undertaken in the loop algebra (Kac–Moody algebra with zero centre), whereas, for present purposes, it will actually prove more convenient to present the construction in a representation independent way in the full Kac–Moody algebra with centre; although we should stress that the resulting hierarchy of equations is identical.

There is a flow of the hierarchy for each element of $\mathfrak{s}$ of non-negative $\mathbf{s}'$-grade, this is the set $\{b_j, j \in E \geqq 0\}$. The flows are defined in terms of the gauge connections, or "Lax operators," of the form

$$\mathscr{L}_j = \frac{\partial}{\partial t_j} - b_j - q(j) \quad j \in E \geqq 0 \,, \tag{2.1}$$

where $q(j)$ is a function of the t_j's on the intersection

$$Q(j) \equiv \mathfrak{g}_{\geqq 0}(\mathbf{s}) \cap \mathfrak{g}_{<j}(\mathbf{s}') \,. \tag{2.2}$$

In order to ensure that the flows t_j are uniquely associated to elements of the set $\{b_j, j \in E \geqq 0\}$ we will also, without loss of generality, demand that $q(j)$ has no constant terms proportional to b_i with $i < j$. The integrable hierarchy of equations is defined by the zero-curvature conditions

$$[\mathscr{L}_i, \mathscr{L}_j] = 0 \,. \tag{2.3}$$

In general, the above systems exhibit a *gauge invariance* of the form

$$\mathscr{L}_j \mapsto U \mathscr{L}_j U^{-1} \tag{2.4}$$

preserving $q(j) \in Q(j)$, where U is a function on the group generated by the finite dimensional subalgebra given by the intersection

$$P \equiv \mathfrak{g}_0(\mathbf{s}) \cap \mathfrak{g}_{<0}(\mathbf{s}') . \tag{2.5}$$

The equations of the hierarchy are to be thought of as equations on the equivalence of classes of $Q(j)$ under the gauge transformations. Notice that if $\mathbf{s} \simeq \mathbf{s}'$ then $P = \emptyset$. The only difference between the situation in [4] and the situation here is that $q(j)$ may have a component in the center of $\mathfrak{g}$, say $q_c(j)$. This also means that the system is also gauge invariant under (2.4) with U being just a function, i.e. related to the exponentiation of the center of $\mathfrak{g}$. Obviously, only $q_c(j)$ is sensitive to these particular transformations and they can, in fact, be used to set $q_c(j)$ to any arbitrary value. Consequently, we conclude that it is not a dynamical degree of freedom but a purely gauge dependent quantity. This, and the fact that $q_c(j)$ cannot contribute to the time evolution of the other components of $q(j)$, is the reason why the resulting hierarchy is identical to the one constructed in [4].

The equations of the hierarchy (2.3) can be interpreted as a system of partial differential equations on some set of functions in a number of different ways. For each *regular* element $b_k \in \mathfrak{s}$ $(k > 0)$, so that $\mathfrak{g}$ admits the decomposition

$$\mathfrak{g} = \mathfrak{s} \oplus \mathrm{Im}(\mathrm{ad}\, b_k) , \tag{2.6}$$

(we shall use the notation $\mathfrak{s}^{\perp}$ to denote the complement of $\mathfrak{s}$, meaning $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{s}^{\perp}$) we may regard (2.3) as an integrable hierarchy of partial differential equations on the functions $q(k)$, modulo the action of the gauge symmetry discussed above. (In the language of [4] these are the "type-I" hierarchies of equations.)

Below we repeat some of the analysis of [4], to show how the results of that reference are modified when the algebra has a non-trivial centre. First of all, we consider the analogue of Proposition 3.2 of [4].

Proposition 2.1. *For a given* $b_k \in \mathfrak{s}$, *for which* $\mathfrak{g}$ *has the decomposition* (2.6), *there is a unique* $y \in \mathfrak{s}^{\perp}_{<0}(\mathbf{s}')$ *and* $h(k) \in \mathfrak{s}_{<k}(\mathbf{s}')$, *which are functions of* $q(k)$ *and its* t_k *derivatives, such that*

$$q(k) = -\Phi\left(\frac{\partial}{\partial t_k} - b_k - h(k)\right)\Phi^{-1} - b_k , \tag{2.7}$$

where $\Phi = \exp y$.

Proof. The proof is exactly the same as that of Proposition 3.2 of [4], the only difference being that now $q(k)$ has a component in the centre of $\mathfrak{g}$. We equate terms in (2.7) of equal $\mathbf{s}'$-grade to get a recursion relation of the form

$$h_j(k) + [b_k, y_{j-k}] = \star . \tag{2.8}$$

In the above $h_j(k)$ and y_j are the components of $h(k)$ and y of $\mathbf{s}'$-grade equal to j, and $\star$ denotes terms which depends on $h_i(k)$, for $i > j$, and y_i, for $i > j - k$, and $q(k)$. The proof proceeds by induction. The first equation of the series states that

$$h_{k-1}(k) + [b_k, y_{-1}] = q_{k-1}(k) . \tag{2.9}$$

We now appeal to the decomposition (2.6) in order to solve uniquely for $h_{k-1}(k)$ and for y_{-1}. The same decomposition means that we can solve (2.8) iteratively for y and $h(k)$. □

Once more, the only difference between the situation in [4] and the situation here, is that both $h(k)$ and $q(k)$ have a component in the centre of $\mathfrak{g}$, say $h_c(k)$ and $q_c(k)$, respectively, and one finds $q_c(k) = h_c(k) + (\Phi b_k \Phi^{-1})_c$.

The arguments of [4] can then be applied, with some minor modifications, to find the other variables $q(j)$, for $j \neq k$, in terms of $q(k)$ and its t_k derivatives:

$$q(j) = P_{\geqq 0[\mathbf{s}]}(\Phi b_j \Phi^{-1}) - b_j + h_c(j) , \tag{2.10}$$

where $P_{\geqq 0[\mathbf{s}]}$ is the projector onto $\mathfrak{g}_{\geqq 0}(\mathbf{s})$. The variables $q(k)$ then satisfy the partial differential equations

$$\frac{\partial q(k)}{\partial t_j} = [q(j) + b_j, \mathscr{L}_k] . \tag{2.11}$$

Before proceeding, let us clear up some technical details. Proposition 3.6 of ref. [4] states that the quantities $h(k)$ proportional to elements of $\mathfrak{s}$ with $\mathbf{s}'$-grade $\geqq 0$ are constants under the flows (2.11). Hence, they contribute constant terms to $q(k)$ proportional to elements of the Heisenberg subalgebra b_j with $j < k$; an eventuality that we disallowed in the discussion following Eq. (2.2). Hence we must impose the conditions $P_{\geqq 0[\mathbf{s}']}[h(k) - h_c(k)] = 0$. In contrast, the element of $h(k)$ in the center cannot be set to zero: it has to be compatible with the zero curvature conditions (2.3)

$$\frac{\partial h_{-m}(k)}{\partial t_j} - \frac{\partial h_{-m}(j)}{\partial t_k} = 0 ,$$

$$\frac{\partial h_c(k)}{\partial t_j} - \frac{\partial h_c(j)}{\partial t_k} = c(jh_{-j}(k) - kh_{-k}(j)) \quad j, k, m \in E \geqq 0 . \tag{2.12}$$

We notice that the value of $h_c(k)$ is completely arbitrary, up to these consistency equations.

The equations of the hierarchy are to be thought of as a set of partial differential equations on a consistent gauge slice of $q(k)$, denoted $\tilde{q}(k)$, under the gauge symmetry (2.4). The hierarchy of equations, which are labelled by the data $\{\mathfrak{g}, \mathfrak{s}, \mathbf{s}; b_k\}$, are then of the form

$$\frac{\partial \tilde{q}(k)}{\partial t_j} = F_j\left(\tilde{q}(k), \frac{\partial \tilde{q}(k)}{\partial t_k}, \ldots\right) \quad j \in E \geqq 0 , \tag{2.13}$$

for some functions F_j of $\tilde{q}(k)$ and its t_k-derivatives.

Notice that there is one-to-one correspondence between the solutions of two hierarchies $\{\mathfrak{g}, \mathfrak{s}, \mathbf{s}; b_k\}$ and $\{\mathfrak{g}, \mathfrak{s}, \mathbf{s}; b_l\}$ (where b_k and b_l both admit the decomposition (2.6)). The maps are given by (2.10). In this sense, one does not distinguish between such hierarchies. However, it was further shown in [6] that the above system of equations could be written in a one-parameter family of coordinated Hamiltonian forms. In particular, one of the Poisson bracket algebras is a classical W-algebra. So there exists a classical W-algebra associated to each $\{\mathfrak{g}, \mathfrak{s}, \mathbf{s}; b_k\}$ hierarchy. Although the hierarchies corresponding to different b_k's are in a sense the same as regards their space of solutions, the associated canonical formalisms are not the same and different W-algebras are obtained. For instance the hierarchies constructed by Drinfel'd and Sokolov, with $\mathfrak{g} = sl(n)^{(1)}$, $\mathfrak{s}$ being the *principal* Heisenberg subalgebra and $\mathbf{s} = (1, 0, \ldots, 0)$ (the homogeneous gradation) lead to the $W_n^{(k)}$-algebras, where k labels the different choices for the element $b_k \in \mathfrak{s}$.

We now return to the question of gauge invariance. A convenient choice for the gauge slice is suggested by the requirement that (2.10) is also true for $j = k$. From (2.7), it follows that

$$q(k) = P_{\geqq 0[\mathbf{s}]}(\Phi b_k \Phi^{-1}) - b_k + h_c(k) - P_{0[\mathbf{s}]}\left[\Phi\left(\frac{\partial}{\partial t_k} - h(k) + h_c(k)\right)\Phi^{-1}\right]. \tag{2.14}$$

Moreover, $h(k)$ satisfies (2.12), which are integrability conditions for

$$h(j) - h_c(j) = \frac{\partial \omega}{\partial t_j}, \tag{2.15}$$

with $\omega \in \mathfrak{s}_{<0[\mathbf{s}']}$ and $j \in E \geqq 0$. Therefore, the last term in (2.14) can be written as $P_{0[\mathbf{s}]}\,[(\Phi e^{\omega})\,\partial/\partial t_k(\Phi e^{\omega})^{-1}]$, and our gauge choice is given by the following proposition.

Proposition 2.2. *There exists a consistent gauge slice where*

$$P_{0[\mathbf{s}]}(v) = 0\,, \tag{2.16}$$

where $\Phi e^{w} \equiv \Theta = e^{v}$, and $v \in \mathfrak{g}_{<0[\mathbf{s}']}$ is considered as a function of $q(k)$ and its t_k derivatives.

Proof. Gauge transformations act on Φ as $\Phi' = U\Phi$, with $U = \exp u$ where u is a function on the algebra P in (2.5). In contrast, $h(k)$, and hence w, are gauge invariant; consequently, Θ transforms as $\Theta' = U\Theta$. Denoting the components of v', v and u with zero $\mathbf{s}$-grade as v'_0, v_0 and u_0, respectively, and projecting onto zero $\mathbf{s}$-grade we have

$$\exp v'_0 = \exp u_0 \exp v_0\,. \tag{2.17}$$

Therefore, by choosing $u_0 = -v_0$ we can gauge away the component $P_{0[\mathbf{s}]}(v)$. Notice that this is consistent because $v \in \mathfrak{g}_{<0[\mathbf{s}']}$. □

The result of this proposition is that there exists a unique gauge slice $\tilde{q}$ for which $v(=\tilde{v})$ is a function on $\mathfrak{g}_{<0[\mathbf{s}]}$ and $\Theta = \exp\tilde{v} \in U_-(\mathbf{s})$; from now on we will assume that this gauge slice has been chosen and we shall denote $\Theta \equiv \tilde{\Phi}\exp\omega \equiv \exp\tilde{v}$. As we have noted, if $\mathbf{s} \simeq \mathbf{s}'$, then there is no gauge symmetry in the hierarchy leading to a *modified* hierarchy (mKdV hierarchy) in the language of [1]. For $\mathbf{s} \prec \mathbf{s}'$ the hierarchies are *partially modified* (pmKdV hierarchies); these include the KdV hierarchies for which $\mathbf{s}$ is a "minimal" gradation, i.e. one for which all the s_j are equal to zero except for say $s_k = 1$.

For a given choice of Heisenberg subalgebra $\mathfrak{s}$, the pmKdV hierarchies are related to the mKdV hierarchy by a *Miura map* which takes solutions of the mKdV hierarchy into solutions of the pmKdV hierarchy. The Miura maps have been discussed in detail in [6], however in the present context they can be discussed in a slightly different way. Given two hierarchies $\{\mathfrak{g}, \mathfrak{s}, \mathbf{s}_1; b_k\}$ and $\{\mathfrak{g}, \mathfrak{s}, \mathbf{s}_2; b_k\}$, with $\mathbf{s}_2 \succeq \mathbf{s}_1$, it follows that a solution of the second hierarchy gives a solution to the first hierarchy since $Q(k; \mathbf{s}_2) \subset Q(k; \mathbf{s}_1)$ (where we have made the dependence of the space $Q(k)$ on the "degree of modification" explicit). With the choice of gauge in Proposition 2.2 we can make the Miura map more explicit.

Proposition 2.3. *The Miura map, which takes solutions of a hierarchy $\{\mathfrak{g}, \mathfrak{s}, \mathbf{s}_2; b_k\}$ to solutions of the hierarchy $\{\mathfrak{g}, \mathfrak{s}, \mathbf{s}_1; b_k\}$ with $\mathbf{s}_2 \succeq \mathbf{s}_1$, with the choice of gauge in*

Proposition 2.2, *is the projection*

$$\tilde{v}_1 = P_{<0[\mathbf{s}_1]}(\tilde{v}_2)\,, \tag{2.18}$$

where $\tilde{v}_2$ *is considered as a function of* $\tilde{q}_2(k)$ *and* $\tilde{q}_1(k)$ *is then given in terms of* $\tilde{q}_2(k)$ *via* (2.18) *and* (2.7).

Proof. The existence of the Miura map follows from the fact that if $\mathbf{s}_2 \succeq \mathbf{s}_1$ then $Q(k;\mathbf{s}_2) \subset Q(k;\mathbf{s}_1)$, hence $\tilde{q}_2(k) \in Q(k;\mathbf{s}_1)$. In order to ensure that the image under the Miura map is in the gauge of Proposition 2.2 one has to "gauge away" $P_{0[\mathbf{s}_1]}(\tilde{v}_2)$, and hence the Miura map can be thought of as the projection in (2.18). □

Along with each integrable hierarchy there is an associated linear problem:

$$\left(\frac{\partial}{\partial t_j} - b_j - q(j)\right)\Psi = 0 \quad j \in E \geqq 0\,, \tag{2.19}$$

where Ψ is a function of the t_j's on the group G formed by *exponentiating* $\mathfrak{g}$.

We now prove a central theorem.

Theorem 2.4. *There is a one-to-one map from solutions of the (gauge fixed) associated linear problem of the form*

$$\tilde{\Psi} = \Theta\Gamma\,, \tag{2.20}$$

where

$$\Gamma = \exp\left[\sum_{j\in E\geqq 0} t_j b_j\right], \tag{2.21}$$

and Θ *being a function on the subgroup* $U_-(\mathbf{s})$, *to solutions of the hierarchy* (2.13) *with the arbitrary functions* $h_c(k)$ *fixed by the conditions* $\partial h_c(k)/\partial t_j + ckh_{-k}(j) = 0$ *for any* $j, k \in E \geqq 0$.

Proof. First of all, using (2.7) and the equations for $h_c(k)$, the gauge fixed Lax operators can be written in the form

$$\mathscr{L}_j = \frac{\partial}{\partial t_j} + (\tilde{\Phi}\mathbf{e}^{\omega})\left(\frac{\partial}{\partial t_j} - b_j\right)(\tilde{\Phi}\mathbf{e}^{\omega})^{-1}\,, \tag{2.22}$$

where $\omega \in \mathfrak{s}_{<0[\mathbf{s}']}$ has been defined in (2.15). Therefore, we can build a solution of the linear problem of the form (2.20) with

$$\Theta = \tilde{\Phi}\mathbf{e}^{\omega}\,. \tag{2.23}$$

On the other hand, if we are given a solution of the linear problem of the form (2.20) then it is straightforward to see that

$$\tilde{q}(k) = -\Theta\left(\frac{\partial}{\partial t_k} - b_k\right)\Theta^{-1} - b_k\,, \tag{2.24}$$

however, what is not so clear is that the quantity $P_{\geqq 0[\mathbf{s}']}[h(k) - h_c(k)]$ equals zero for $\tilde{q}(k)$ given by (2.24). To see this one first finds $\tilde{\Phi}$ as a function of $\tilde{q}(k)$ and its t_k derivatives, via Proposition 2.1, and therefore ultimately as a function of Θ via (2.24). The result is that $\tilde{\Phi}^{-1}\Theta = \mathbf{e}^{u}$ with $u = \sum_{j<0} u_j b_j \in \mathfrak{s}_{<0[\mathbf{s}']}$, which gives

$$\begin{aligned} h(k) &= -(\Theta^{-1}\tilde{\Phi})^{-1}\frac{\partial\Theta^{-1}\tilde{\Phi}}{\partial t_k} - b_k + [(\Theta^{-1}\tilde{\Phi})^{-1} b_k \Theta^{-1}\tilde{\Phi}]_c \\ &= \sum_{j<0}\frac{\partial u_j}{\partial t_k} b_j - cku_{-k}\,. \end{aligned} \tag{2.25}$$

Therefore, it satisfies the two conditions $P_{\geqq 0[\mathbf{s}']}[h(k) - h_c(k)] = 0$, and $\partial h_c(k)/\partial t_j + ckh_{-k}(j) = 0$. $\square$

The result (2.24) admits a very useful simplification. The solution $\tilde{q}(k)$ has s-grade $\geqq 0$, hence using the fact that $\Theta \in U_-(\mathbf{s})$,

$$\tilde{q}(k) = P_{\geqq 0[\mathbf{s}]}(\Theta b_k \Theta^{-1}) - b_k . \tag{2.26}$$

The importance of this result is that it only needs the finite number of terms of Θ with **s**-grade greater than $-k-1$ to be applied.

3. The Kac–Wakimoto Hierarchies and Tau-Functions

In this section we provide a short review of the construction of integrable hierarchies by Kac and Wakimoto [11]. This will lead us to introduce the tau-functions.

The construction of Kac and Wakimoto leads directly to the equations of the hierarchy in *Hirota form* [7]. The idea is the following: the tau-function $\tau_{\mathbf{s}}$ associated to an integrable highest weight representation $L(\mathbf{s})$ of an affine Kac–Moody algebra $\mathfrak{g}$ is characterized by saying that it lies in the G-orbit of the highest weight vector $v_{\mathbf{s}}$. Here G is the group associated to $\mathfrak{g}$.

Let $\{u_i\}$ and $\{u^j\}$ be dual bases of the larger algebra $\mathfrak{g} \oplus \mathbf{C}d$, with respect to the non-degenerate bi-linear inner product $(\cdot|\cdot)$. It can be shown [11, 14] that $\tau_{\mathbf{s}}$ lies in the G-orbit of $v_{\mathbf{s}}$ if and only if

$$\sum u_j \otimes u^j(\tau_{\mathbf{s}} \otimes \tau_{\mathbf{s}}) = (\Lambda_{\mathbf{s}}|\Lambda_{\mathbf{s}})\tau_{\mathbf{s}} \otimes \tau_{\mathbf{s}} , \tag{3.1}$$

where $\Lambda_{\mathbf{s}}$ is the eigenvalue of $\mathfrak{g}_0(\mathbf{s})$ on $v_{\mathbf{s}}$. We can think of $\mathscr{C} = \sum u^j \otimes u_j$ as a generalized Casimir operator. Furthermore, the condition (3.1) is also equivalent to the statement that

$$\tau_{\mathbf{s}} \otimes \tau_{\mathbf{s}} \in L(2\mathbf{s}) . \tag{3.2}$$

It follows from the definition of the action of a group on a tensor product that, for the representation $L(s)$,

$$\tau_{\mathbf{s}} = \bigotimes_{i=0}^{r} \{\tau_i^{\otimes s_i}\} , \tag{3.3}$$

where τ_i is the tau-function corresponding to the fundamental representation with $s_j = \delta_{ij}$.

At the moment, the conditions (3.1) are completely "group theoretic," with no apparent connection to integrable hierarchies of equations. However, for cases where the representations are of "vertex type," so they are carried by Fock spaces, then (3.1) can be interpreted as differential equations on the tau-functions. In fact, they are precisely the Hirota equations of an integrable hierarchy. In order to explain this, we restrict ourselves to cases where $\mathfrak{g} = g^{(1)}$ is the untwisted affinization of a finite simply-laced algebra g. In that case, level one representations (or *basic* representations, those for which $s_j = \delta_{ji}$ for some i such that $k_i^\vee = k_i = 1$) are isomorphic to the Fock space of any one of the Heisenberg subalgebras of $\mathfrak{g}$. It is known that inequivalent Heisenberg subalgebras are classified by the conjugacy classes of the Weyl group of g [2.3]. The connection between the Weyl group element, say w (up to conjugacy), and the associated Heisenberg subalgebra $\mathfrak{s}_w$, is that there is a lift of w, denoted $\hat{w}$, onto $\mathfrak{g}$, which acts on the Heisenberg subalgebra as

$$\hat{w}(b_j) = \exp\left(\frac{2\pi i j}{N}\right) b_j . \tag{3.4}$$

The Heisenberg subalgebra $\mathfrak{s}_w$ is realized on the Fock space $\mathbf{C}[x_j, j \in E > 0]$ in the standard way: $c = 1$ and

$$b_j = \begin{cases} \dfrac{\partial}{\partial x_j} & j > 0 \\ -jx_{-j} & j < 0 \,. \end{cases} \tag{3.5}$$

A rather different treatment is required for any zero-graded generators of $\mathfrak{s}_w$ which correspond to the invariant subspace of w. These zero-modes are represented on the space

$$\mathbf{C}(Q) = \{\exp \beta \cdot x_0, \beta \in Q\} , \tag{3.6}$$

where Q is the root lattice of g projected onto the invariant subspace of w. b_0 acts as $\partial/\partial x_0$ so

$$b_0 e^{\beta \cdot x_0} = \beta e^{\beta \cdot x_0}, \quad \beta \in Q \,. \tag{3.7}$$

The level-one representation is isomorphic to $\mathbf{C}[x_j] \otimes \mathscr{V}$, where $\mathscr{V} = \mathbf{C}(Q) \otimes V$ is the *zero-mode space.* Here, V is an additional finite-dimensional vector space [3, 15]. The elements of $\mathfrak{g}$ not in $\mathfrak{s}_w$ are the modes of vertex operators, the centre is the identity ($c = 1$), and the derivation $d_{s'}$ is the zero-mode of the Sugawara current, up to a constant. Notice that the construction does not distinguish between the different level-one representations of $g^{(1)}$, this is a reflection of the fact that all such representations are isomorphic due to symmetries of the extended Dynkin diagram.

Equations (3.1) after expressing the generators of $\mathfrak{g}$ in terms of operators on the Fock-space, are then bi-linear Hirota equations for the functions $\tau_i^{(\beta)}(x_j)$, which are projections onto a basis for $\mathbf{C}(Q)$:

$$\tau_i(x_0; x_j) = \sum_{\beta \in Q} \tau_i^{(\beta)}(x_j) e^{\beta \cdot x_0} \,. \tag{3.8}$$

We wish to emphasize that there is a different realization of each level-one representation for each inequivalent Heisenberg subalgebra of $\mathfrak{g}$, and moreover, although these realizations are isomorphic as representations they lead to different Hirota equations for the corresponding tau-functions.

In the following we shall often deal with the vertex representation of $L(s)$ realized on the tensor product of fundamental representations, where s_i gives the multiplicity of the i^{th} fundamental representation in the product (so any non-zero s_i corresponds to $k_i^\vee = k_i = 1$). They will be carried by a tensor product of the Fock spaces:

$$\bigotimes_{i=1}^{N} \{\mathbf{C}[x_j, j \in E > 0] \otimes \mathscr{V}\} , \tag{3.9}$$

where $N = \sum_{i=0}^{r} s_i$. We shall use the notation $x_j^{(i)}$ to indicate the Fock space variables of the i^{th} space in the tensor product, and $x_j \equiv \sum_{i=1}^{N} x_j^{(i)}$.

4. Dressing Transformations

In this section we define a set of transformations on the zero-curvature hierarchies, which take known solutions of a hierarchy to new solutions. These "dressing transformations" will be crucial for establishing the link to the tau-function formalism of the next section.

Consider the gauge-fixed linear problem associated to the hierarchy:

$$\left(\frac{\partial}{\partial t_j} - b_j - \tilde{q}(j)\right)\tilde{\Psi} = 0 \,. \tag{4.1}$$

We have already established in Theorem 2.4 a map between solutions of the gauge-fixed linear problem of the form

$$\tilde{\Psi} = \Theta\Gamma \qquad \Theta \in U_-(\mathrm{s}) \,, \tag{4.2}$$

where Γ is defined in (2.19), and solutions of the gauge-fixed hierarchy.

The *dressing transformation* is a map between solutions of (4.1), which preserves the form (4.2). Many of the technical aspects of these transformations are considered in [10] for the case $\mathfrak{g} = A_1^{(1)}$ and the principal Heisenberg subalgebra; however, our approach is closer to that of ref. [12]. One difference with these other works is that we will deal with the KdV hierarchies directly rather than considering the mKdV hierarchies and then using the Miura map to find the resulting transformations for the KdV hierarchies.

Theorem 4.1. *Given a solution* $\tilde{\Psi}_1$ *of the linear problem* (4.1), *of the form* (4.2), *and* $g \in G$, *with* $\tilde{\Psi}_1 \cdot g = (\tilde{\Psi}_1 \cdot g)_-(\tilde{\Psi}_1 \cdot g)_0(\tilde{\Psi}_1 \cdot g)_+$ *lying in the "big cell"* $U_-(\mathrm{s})H(\mathrm{s})U_+(\mathrm{s})$ *then*

$$\tilde{\Psi}_2 = \mathscr{D}_g(\tilde{\Psi}_1) \equiv [(\tilde{\Psi}_1 \cdot g)_-]^{-1}\tilde{\Psi}_1 = (\tilde{\Psi}_1 \cdot g)_0(\tilde{\Psi}_1 \cdot g)_+ g^{-1} \,, \tag{4.3}$$

is also a solution of the linear problem with the form (4.2).

Proof. We have to show that $\tilde{q}(j)$ defined by

$$\frac{\partial}{\partial t_j}\tilde{\Psi}_2 = (\tilde{q}(j) + b_j)\tilde{\Psi}_2 \,, \tag{4.4}$$

lies in the subspace $\tilde{Q}(j)$ defined in (2.2). The proof follows from the two expressions for $\tilde{\Psi}_2$ in (4.3). From the first expression we find $\tilde{q}(j) + b_j = b_j +$ terms with s'-grade $< j$ and from the second expression we find $\tilde{q}(j) + b_j =$ terms with s-grade $\geqq 0$. Hence $\tilde{q}(j)$ lies in $\tilde{Q}(j)$ as required. Moreover, it is clear from the first expression in (4.3) that $\tilde{\Psi}_2$ has the form of (4.2). □

Corollary. *The expressions*

$$\tilde{\Psi} = \mathscr{D}_g(\Gamma) \,, \tag{4.5}$$

where Γ *is defined in* (2.19), *are solutions of the linear problem of the form* (4.2) *and consequently using Theorem* 2.4 *solutions of the hierarchy.*

Proof. The proof is elementary. One just has to notice that

$$\Gamma = \exp\left[\sum_{j \in E \geqq 0} b_j t_j\right], \tag{4.6}$$

satisfies the linear problem with $\tilde{q}(j) = 0$, and obviously has the form (4.2). □

5. The Zero-Curvature Hierarchies and the Tau-Functions

In Sect. 2 we introduced a series of integrable hierarchies constructed via a zero-curvature method, whilst in Sect. 3 we described a series of hierarchies in the form of a set of Hirota equations. It is now time to connect these two formalisms, the bridge being provided by the dressing transformation.

Let us review again the situation in the two formalisms. In the zero curvature formalism a hierarchy was defined in terms of the following data $\{\mathfrak{g}, \mathfrak{s}, \mathbf{s}; b_k\}$:

(i) An affine Kac–Moody algebra $\mathfrak{g}$.
(ii) A Heisenberg subalgebra $\mathfrak{s} \subset \mathfrak{g}$ (with an associated gradation $\mathbf{s}'$).
(iii) A gradation $\mathbf{s}$ ("the degree of modification") such that $\mathbf{s} \preceq \mathbf{s}'$.
(iv) An element $b_k \in \mathfrak{s}$ with positive grade such that $\mathfrak{g}$ admits the decomposition (2.6), i.e. $\mathrm{Im}(\mathrm{ad}\, b_k) = \mathfrak{s}^{\perp}$.

In the tau-function formalism, a hierarchy of Hirota equations was associated to the following data $\{g, w, \mathbf{s}\}$:

(i) A simply-laced finite Lie algebra g.
(ii) A vertex operator realization of $L(\mathbf{s})$, associated to the untwisted affinization of g, corresponding to some conjugacy class of the Weyl group of g, containing w. This requires that $s_i = 0$ if $k_i > 1$.

To connect the zero-curvature and tau-function formalisms we notice that the dressing transformations introduced in the last section involve a group element $g \in G$. The idea is to use the group element which appears in the characterization of the tau-function to "dress" the "vacuum" solution to the linear system [12], in the manner of the corollary of Theorem 4.1. We are led to the following key theorem.

Theorem 5.1. *There exists a map from solutions of the Kac–Wakimoto hierarchy associated to the data* $\{g, w, \mathbf{s}\}$ *(with the gradation associated to the Heisenberg subalgebra* $\mathfrak{s}_w$ *satisfying* $\mathbf{s}_w \succeq \mathbf{s}$ *(and also* $s_i > 0$ *only if* $k_i = 1$*) and a zero-curvature hierarchy associated to the data* $\{g^{(1)}, \mathfrak{s}_w, \mathbf{s}; b_k\}$*, given by*

$$\Theta^{-1} \cdot v_{\mathbf{s}} = \frac{\tau_{\mathbf{s}}(x_j + t_j)}{\tau_{\mathbf{s}}^{(0)}(t_j)}, \tag{5.1}$$

where $\Theta \in U_-(\mathbf{s})$ *gives* $q(k)$ *via* (2.25).

Proof. Consider solutions of the linear problem which follow from the corollary of Theorem 4.1, from which we deduce that in representation $L(\mathbf{s})$

$$\Gamma \cdot \tilde{\Psi}^{-1} \cdot v_{\mathbf{s}} = (\Gamma \cdot g)_- \cdot v_{\mathbf{s}}\,. \tag{5.2}$$

Now we have

$$\tau_{\mathbf{s}}(x_j + t_j) = \Gamma \cdot g \cdot v_{\mathbf{s}} = (\Gamma \cdot g)_-(\Gamma \cdot g)_0 \cdot v_{\mathbf{s}} = f(t_j)(\Gamma \cdot g)_- \cdot v_{\mathbf{s}}\,, \tag{5.3}$$

where $f(t_j)$ is the eigenvalue of $(\Gamma \cdot g)_0$ on $v_{\mathbf{s}}$. (It should be emphasized that the representation is defined by the Fock space of the variables x_j; the variables t_j are to be thought of as auxiliary variables.) To compute the eigenvalue we notice that

$$\tau_{\mathbf{s}}(x_j + t_j) = \Gamma \cdot g \cdot v_{\mathbf{s}} = (\Gamma \cdot g)_0 \cdot v_{\mathbf{s}} + \cdots, \tag{5.4}$$

where the ellipsis represents states in the representation with lower $\mathbf{s}$-grade; in other words $f(t_j)$ is given by the coefficient of the projection of $\tau_{\mathbf{s}}(x_j + t_j)$ onto $v_{\mathbf{s}}$; but this is

$$f(t_j) = \tau_{\mathbf{s}}^{(0)}(t_j)\,, \tag{5.5}$$

where $\tau_{\mathbf{s}}^{(0)}$ is the component of the tau-function with zero "momentum" in (3.8)[1]. Hence

$$\Theta^{-1}\cdot v_{\mathbf{s}} = \Gamma\cdot\tilde{\Psi}^{-1}\cdot v_{\mathbf{s}} = \frac{\tau_{\mathbf{s}}(x_j+t_j)}{\tau_{\mathbf{s}}^{(0)}(t_j)}\,, \tag{5.6}$$

where $\tau_{\mathbf{s}} = g\cdot v_{\mathbf{s}}$ is the tau-function for the representation $L(\mathbf{s})$. Now given $\tau_{\mathbf{s}}$ then (5.6) uniquely determines $\Theta \in U_-(\mathbf{s})$ – since $U_-(\mathbf{s})$ is faithful on $v_{\mathbf{s}}$ – and hence via Theorem 2.4 a solution of the zero-curvature hierarchy $\{g^{(1)}, \mathfrak{s}_w, \mathbf{s}; b_k\}$, for any b_k admitting the decomposition (2.6). □

The theorem allows one to find the direct relation between the tau-functions and the variables of the zero-curvature hierarchies. Notice that not all zero-curvature hierarchies can be related to tau-functions (s must correspond to products of level one representations), and conversely not all Kac–Wakimoto hierarchies can be related to zero-curvature hierarchies (due to the condition $\mathbf{s}_w \succeq \mathbf{s}$). Notice that in our formalism the KdV (and pmKdV) hierarchies can be dealt with directly without recourse to the Miura map, indeed, one of the pleasant results of the above formalism is that one can see immediately, from the "degree of modification" s, which tau-functions are required. The Miura map at the level of the tau-functions is the trivial statement, which follows from (3.3), that if $\mathbf{s}_2 \succeq \mathbf{s}_1$ then $\tau_{\mathbf{s}_1} \subset \tau_{\mathbf{s}_2}$.

6. Examples

In this section we consider in some detail some examples of the preceding formalism, in order to illustrate some of the issues involved. The main idea is to use Theorem 5.1 to find an expression for the variables of the zero-curvature hierarchies in terms of the tau-functions, generalizing the maps in (1.2) and (1.3). In order to follow the calculations in this section some knowledge of the vertex operator calculus is required, for which we refer the reader to the original literature [11, 3, 15].

The Drinfel'd–Sokolov hierarchies [1] are recovered in our formalism by choosing $\mathfrak{s}$ to be the principal Heisenberg subalgebra. For example, consider the case when $g = sl(N)^{(1)}$. The principal Heisenberg subalgebra has generators with grades in $E = \{1, 2, \ldots, N-1, \mathrm{mod}\, N\}$, and the associated gradation is $\mathbf{s}' = (1, 1, \ldots, 1)$. The basic representations of $g = sl(N)^{(1)}$ are then represented in terms of the principal Heisenberg subalgebra on the Fock space $\mathbf{C}[x_j, j\in E > 0]$; there are no zero-modes in this case and so $\mathscr{V} \simeq 1$.

The Drinfel'd–Sokolov hierarchies were originally defined in ref. [1] in terms of the loop algebra $sl(N)^{(1)}$; taken to be Laurent polynomials in a variable z with coefficients in the N-dimensional representation of $sl(N)$. The elements of the principal Heisenberg subalgebra are

$$b_j = \Lambda^j \quad j \neq N\mathbf{Z}\,, \tag{6.1}$$

[1] We remark at this point that $\tau_{\mathbf{s}}^{(0)}(t_j)$ is a scalar quantity, because if $\mathbf{s}_w \succeq \mathbf{s}$ then the auxiliary vector space V is trivial ($\dim V = 1$), which follows from the corollary to Lemma A.1

where

$$\Lambda = \begin{pmatrix} & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ z & & & & \end{pmatrix}, \tag{6.2}$$

with zeros elsewhere.

The $sl(N)^{(1)}$ modified KdV hierarchy is generated from the Lax operator

$$\mathscr{L}_1 = \frac{\partial}{\partial x} - \begin{pmatrix} v_0 & & & \\ & v_1 & & \\ & & \ddots & \\ & & & v_{N-1} \end{pmatrix} - \Lambda\,, \tag{6.3}$$

where $x \equiv t_1$ and $\sum_{i=1}^{N} v_{i-1} = 0$. The form of this operator follows from the systematic construction in Sect. 1. We now relate the variables $q(1) = \{v_i, i = 0, 1, \ldots, N-1\}$ to the tau-functions τ_i, $i = 0, 1, \ldots, N-1$. From (2.25) we have

$$q(1) = [\Theta_{-1}, \Lambda]\,, \tag{6.4}$$

where Θ_{-1} is the component of Θ of $\mathbf{s}(=\mathbf{s}')$-grade -1. Now $\Theta^{-1} \in U_-(\mathbf{s})$, so we may write for some functions a_i

$$\Theta^{-1} = \exp\left[\sum_{i=0}^{N-1} a_i e_i^- + \cdots\right], \tag{6.5}$$

where the ellipsis represents terms with lower $\mathbf{s}$-grade. Acting on the highest weight vector in the representation $L(\mathbf{s})$ we have

$$\Theta^{-1} \cdot v_{\mathbf{s}} = \left[1 + \sum_{i=0}^{N-1} a_i e_i^- + \cdots\right] \cdot v_{\mathbf{s}}\,. \tag{6.6}$$

Using properties of the reducible representation $L(\mathbf{s})$ with $\mathbf{s} = (1, 1, \ldots, 1)$ in terms of vertex operators on the Fock space $\bigotimes_{i=0}^{N-1} \mathbf{C}[x_j^{(i)}, j \in E > 0]$ with highest weight vector $v_{\mathbf{s}} = \bigotimes_{i=0}^{N-1} v_i$, one finds

$$e_i^- \cdot v_j = (x_1^{(j)} \cdot v_j)\,\delta_{ij}\,. \tag{6.7}$$

From (6.6) and (6.7) and using Theorem 5.1 one deduces

$$a_i = \frac{\partial}{\partial x} \log \tau_i\,, \tag{6.8}$$

and therefore

$$\Theta_{-1} = -\sum_{i=0}^{N-1} e_i^- \frac{\partial}{\partial x} \log \tau_i\,. \tag{6.9}$$

Now one can find $q(1)$ from (6.4), however, to make the connection with the formalism of Drinfel'd and Sokolov we must move to the loop algebra. This is achieved by noting that in the loop algebra

$$e_i^- = \begin{cases} \mathbf{e}_{i,i+1} & i = 1, 2, \ldots, N-1 , \\ z^{-1}\mathbf{e}_{N,1} & i = 0 , \end{cases} \tag{6.10}$$

where $\mathbf{e}_{i,j}$ is the matrix with a 1 in the $(i, j)^{\text{th}}$ position and zero elsewhere. Applying (6.4) one finds the well-known relation

$$v_i = \frac{\partial}{\partial x} \log \left[\frac{\tau_i}{\tau_{i+1}} \right] , \tag{6.11}$$

with $\tau_{i+N} \equiv \tau_i$. The expression (1.3) for the original mKdV hierarchy is a particular example of this.

There is no obstacle in extending the analysis to the Drinfel'd–Sokolov KdV hierarchies, however, in the general case the formulas are complicated and not very illuminating. Rather than treating the general case we shall be satisfied with re-deriving the famous relation (1.2) of the original KdV hierarchy – which arises from choosing $\mathfrak{g} = sl(2)^{(1)}$ and $\mathfrak{s}$ to be the principal Heisenberg subalgebra, as above. In this case $\mathbf{s}' = (1, 1)$ (the principal gradation) and the "degree of modification" $\mathbf{s} = (1, 0)$ (the homogeneous gradation).

The KdV hierarchy is defined via the Lax operator

$$\mathscr{L}_1 = \frac{\partial}{\partial x} - \begin{pmatrix} w & 0 \\ v & -w \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ z & 0 \end{pmatrix} , \tag{6.12}$$

where as before $x \equiv t_1$. The gauge symmetry acts as

$$\mathscr{L}_1 \mapsto \begin{pmatrix} 1 & 0 \\ g & 1 \end{pmatrix} \mathscr{L}_1 \begin{pmatrix} 1 & 0 \\ -g & 1 \end{pmatrix} . \tag{6.13}$$

The choice of gauge made by Drinfel'd and Sokolov is

$$\tilde{q}(1) = \begin{pmatrix} 0 & 0 \\ -u & 0 \end{pmatrix} , \tag{6.14}$$

for which u is then the conventional variable of the KdV hierarchy.

We now follow the same steps as for the mKdV hierarchies, but now with $\mathbf{s} = (1, 0)$. Putting

$$\Theta^{-1} = \exp(a e_0^- + b[e_1^-, e_0^-] + \cdots) , \tag{6.15}$$

for some functions a and b, where the ellipsis represents terms with lower $\mathbf{s}$-grade which will not be required. Acting on the highest weight state one finds

$$\Theta^{-1} \cdot v_0 = (1 + a e_0^- + b e_1^- e_0^- + \cdots) \cdot v_0 . \tag{6.16}$$

Using properties of the vertex operator representation and Theorem 5.1 one finds

$$a = \frac{1}{\tau} \frac{\partial \tau}{\partial x} , \quad b = \frac{1}{2\tau} \frac{\partial^2 \tau}{\partial x^2} , \tag{6.17}$$

where $\tau \equiv \tau_0$ and $x \equiv t_1$. Moving to the loop algebra using

$$e_0^- = \begin{pmatrix} 0 & z^{-1} \\ 0 & 0 \end{pmatrix} , \quad e_1^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} , \tag{6.18}$$

and evaluating (2.25) we find

$$\tilde{q}(1) = \begin{pmatrix} -\tau'/\tau & 0 \\ -\tau''/\tau & \tau'/\tau \end{pmatrix}, \tag{6.19}$$

where $' \equiv \partial/\partial x$. The result is not in the Drinfel'd and Sokolov gauge (6.14), however, it is straightforward to find the gauge transformation connecting the gauge choices. Making the required gauge transformation one re-derives the classic result (1.2),

$$u = 2\frac{\partial^2}{\partial x^2}\log\tau\,. \tag{6.20}$$

As the last example we consider the "homogeneous hierarchies" which are obtained by taking $\mathbf{s} = \mathbf{s}' = (1, 0, 0, \ldots, 0)$, the homogeneous gradation. This includes the non-linear Schrödinger hierarchy when $\mathfrak{g} = sl(2)^{(1)}$. The homogeneous Heisenberg subalgebra is spanned, in the loop algebra, by $H \otimes z^n$, where H is the Cartan subalgebra of the finite horizontal subalgebra $g \subset \mathfrak{g}$. So at each level there is as vector space of flows corresponding to the Cartan subalgebra of g. The hierarchies are defined by the Lax operator

$$\mathscr{L}_1 = \frac{\partial}{\partial x} - \sum_{\alpha\in\Phi_g} q^\alpha E_\alpha - zH\,, \tag{6.21}$$

where $x \equiv t_1$ and the variables q^α are Cartan subalgebra-valued. In the above we have introduced the Cartan Weyl basis for g (see ref. [4] for a more thorough discussion), and Φ_g is the root system of g.

Repeating the arguments above, we have for some arbitrary functions a^α and b

$$\begin{aligned}\Theta^{-1}\cdot v_s &= \exp\left[\sum_{\alpha\in\Phi_g} a^\alpha (E_\alpha)_{-1} + b\cdot H_{-1} + \cdots\right]\cdot v_0 \\ &= \left[1 + \sum_{\alpha\in\Phi_g} a^\alpha (E_\alpha)_{-1} + b\cdot H_{-1} + \cdots\right]\cdot v_0\,,\end{aligned} \tag{6.22}$$

where the ellipsis represents terms of lower homogeneous grade, which wil not be required. Now using properties of the vertex operator representation one finds

$$a^\alpha = \frac{\tau^{(\alpha)}}{\tau^{(0)}} e^{\alpha\cdot t_0} \quad \forall\alpha\in\Phi_g, \quad b = \frac{\partial}{\partial x}\log\tau^{(0)}\,. \tag{6.23}$$

Moving to the loop algebra $(E_\alpha)_{-1} = z^{-1}E_\alpha$ and $H_{-1} = z^{-1}H$ and applying (6.4) one finds

$$q^\alpha = \alpha\frac{\tau^{(\alpha)}}{\tau^{(0)}} e^{\alpha\cdot t_0}\,. \tag{6.24}$$

This agrees with that found by Kac and Wakimoto [11], and by Imbens [12], for the particular case when $\mathfrak{g} = sl(2)^{(1)}$ (which is the non-linear Schrödinger hierarchy) – although in both these references the trivial t_0 evolution is not considered.

An interesting example which involves the "intermediate" Heisenberg subalgebra of $A_2^{(1)}$, and hence goes beyond the principal and homogeneous constructions, has been considered in [13].

7. Discussion

In this paper we have established the connection between some of the zero-curvature hierarchies and some of the Kac–Wakimoto hierarchies. We provided an algorithm for finding the variables of the zero-curvature hierarchies in terms of the tau-functions.

Given the relation between the variables of the zero-curvature hierarchies and the tau-functions it is straightforward to write down the expression for the multiple soliton solutions of the hierarchies. If we denote by $V(\alpha, z)$ the vertex operator associated to the orbit of the root α of the finite Lie algebra, under the cyclic subgroup of the Weyl group of g generated by w – the Weyl group element that defines the vertex operator representation – then the N soliton solution is given by the tau-function

$$\tau_s(x_j) = (1 + a_1 V(\alpha_1, z_1))(1 + a_2 V(\alpha_2, x_2)) \cdots (1 + a_N V(\alpha_N, z_N)) \cdot v_s \, , \quad (7.1)$$

where the α_p's are roots of g, indicating that each soliton carries a "flavour" – labelled by an orbit of a root under the cyclic group generated by w – the a_p's are constants and the z_p's are "velocity parameters" of the solitons. Given some familiarity with vertex operator representations, it is not difficult to find the explicit form for the soliton solutions.

In our exposition we did not mention the *Grassmannian* approach of refs. [9, 10], and so a few comments are in order. Given that $\tau_s = g \cdot v_s$ for g in the Group G associated to the Kac–Moody algebra $\mathfrak{g}$, and the fact that the highest weight vector v_s is annihilated by the subalgebra generated by the e_i^-, for each i such that $s_i = 0$, and the e_i^+, for $i = 0, 1, \ldots, r$, two group element yield the same tau-function if they correspond to the same class in the quotient G/P_s, where P_s is the *parabolic* subgroup generated by this subalgebra. Now consider the case $\mathfrak{g} = sl(2)^{(1)}$. In this case solutions of the KdV hierarchy are associated to G/P, where $P = P_{(1,0)}$, and solutions of the mKdV hierarchy are associated to G/B, where $B = P_{(1,1)}$. In refs. [9, 10] it is shown that G/P is the Grassmannian and G/B is a *flag manifold*, and there exists a natural projection $G/B \to G/P$, which is nothing else than the Miura map taking solutions of the mKdV hierarchy to solutions of the KdV hierarchy. In our more general setting the Grassmannian is replaced by G/P_s, and there is a natural projection $G/P_{s_1} \to G/P_{s_2}$, if $\mathbf{s}_1 \succeq \mathbf{s}_2$, which is again a geometrical statement of the Miura map between the solutions of the two hierarchies.

One of the motivations which lies behind this work, comes from recent developments regarding quantum gravity theories in two-dimensions. In this context the partition function of pure gravity is the tau-function of the KdV hierarchy, where the flows t_i are the coupling constants for all the operators in the theory (the gravitational descendents of the puncture operator which couples to $t_1 \equiv x$), supplemented by an additional condition which is called the "string equation." Remarkably, the string equation and the condition that the partition function is a tau-function of the KdV hierarchy are equivalent to the Virasoro constraints [16]:

$$L_n \tau = 0 \quad n \geqq -1 \, , \quad (7.2)$$

where the Virasoro generators are constructed from the Heisenberg subalgebra. We shall show that there is a very natural generalization of this structure to the other hierarchies that we have considered in this paper; in the general case one obtains W-algebra constraints, generalizing the situation for the $sl(N)$ Drinfel'd–Sokolov KdV hierarchies [16, 17], where the W-currents are constructed from the appropriate Heisenberg subalgebra [18].

Appendix A

In this appendix we review some of the details of affine Kac–Moody algebras which will be important for the following constructions. A complete treatment of such algebras may be found in [2], and references therein.

An affine Kac–Moody algebra $\mathfrak{g}$ is defined by a generalized Cartan matrix a, of dimension $r+1$ and rank r, and is generated by $\{h_i, e_i^+, e_i^-, i = 0, 1, \ldots, r\}$ subject to the relations

$$[h_i, h_j] = 0, \quad [h_i, e_j^\pm] = \pm a_{ij} e_j^\pm ,$$

$$[e_i^+, e_j^-] = \delta_{ij} h_i, \quad (\operatorname{ad} e_i^\pm)^{1-a_{ij}}(e_j^\pm) = 0 . \tag{A.1}$$

The algebra $\mathfrak{g}$ has a centre $\mathbf{C}c$ where

$$c = \sum_{i=0}^{r} k_i^\vee h_i , \tag{A.2}$$

where $k_i^\vee$ (the *dual Kac labels*) are the components of the left null eigenvector of a:

$$\sum_{i=0}^{r} k_i^\vee a_{ij} = 0 . \tag{A.3}$$

A derivation $d_{\mathbf{s}}$, with $\mathbf{s} = (s_0, s_1, \ldots, s_r)$ being a set of $r+1$ non-negative integers [2], induces a $\mathbf{Z}$ grading on $\mathfrak{g}$ which we label $\mathbf{s}$:

$$[d_{\mathbf{s}}, e_i^\pm] = \pm s_i e_i^\pm, \quad [d_{\mathbf{s}}, h_i] = 0 . \tag{A.4}$$

Under $\mathbf{s}$, $\mathfrak{g}$ has the eigenspace decomposition

$$\mathfrak{g} = \bigoplus_{j \in \mathbf{Z}} \mathfrak{g}_j(\mathbf{s}) . \tag{A.5}$$

We shall often use the notation like $\mathfrak{g}_{>k}(\mathbf{s}) = \bigoplus_{i>k} \mathfrak{g}_i(\mathbf{s})$. There exists a partial ordering on the set of gradations, such that $\mathbf{s} \succeq \mathbf{s}'$ if $s_i \neq 0$ whenever $s_i' \neq 0$.

We shall sometimes deal with the larger algebra $\mathfrak{g} \oplus \mathbf{C}d$, formed by adjoining a derivation with $[d, d] = 0$[2]. The important difference between $\mathfrak{g} \oplus \mathbf{C}d$ and $\mathfrak{g}$, is that the former has an invariant symmetric *non-degenerate* bi-linear form $(\cdot|\cdot)$, whereas for the latter the analogous inner-product is degenerate.

In the following we shall be interested in the *Heisenberg subalgebras* of $\mathfrak{g}$ [2, 3], $\mathfrak{s} = \mathbf{C}c + \sum_{j\in E} \mathbf{C}b_j$, where $E = I + \mathbf{Z}N$, where I is a set of r integers $\geqq 0$ and $< N$, for an integer N, the algebra being

$$[b_j, b_k] = j\delta_{j,-k} c . \tag{A.6}$$

[2] The algebras formed by adjoining different derivations are equivalent because $d_{\mathbf{s}} - d_{\mathbf{s}'} \in \mathfrak{g}$

For each Heisenberg subalgebra there is an associated gradation $\mathbf{s}'$ and derivation $d_{\mathbf{s}'}$ such that

$$[d_{\mathbf{s}'}, b_j] = j b_j \, . \tag{A.7}$$

The integer N is given by $N = \sum_{i=0}^{r} k_i s_i'$, where k_i are the *Kac labels* ($\sum_{i=0}^{r} a_{ij} k_j = 0$).

Integrable highest weight modules of $\mathfrak{g}$ are defined in terms of a highest weight vector $v_{\mathbf{s}}$, labelled by a gradation $\mathbf{s}$ of $\mathfrak{g}$. The highest weight vector is annihilated by $\mathfrak{g}_{>0}(\mathbf{s})$, so

$$e_i^+ \cdot v_{\mathbf{s}} = 0 \quad \forall i \, , \tag{A.8}$$

and is an eigenvector of $\mathfrak{g}_0(\mathbf{s})$ with eigenvalues

$$\begin{aligned} h_i \cdot v_{\mathbf{s}} &= s_i v_{\mathbf{s}} \, , \\ e_i^- \cdot v_{\mathbf{s}} &= 0 \quad \forall i \text{ with } s_i = 0 \, , \\ d_{\mathbf{s}} \cdot v_{\mathbf{s}} &= 0 \, . \end{aligned} \tag{A.9}$$

The eigenvalue of the centre c on the representation $L(\mathbf{s})$ is known as the level k:

$$c \cdot v_{\mathbf{s}} = \sum_{i=0}^{r} k_i^\vee h_i \cdot v_{\mathbf{s}} = \left(\sum_{i=0}^{r} k_i^\vee s_i \right) v_{\mathbf{s}} \, , \tag{A.10}$$

hence $k = \sum_{i=0}^{r} k_i^\vee s_i$. In particular, the integrable highest weight representations with $k = 1$ are known as the *basic* representations.

We shall use the notation $v_i = v_{\mathbf{s}}$, where $s_j = \delta_{ij}$, for the highest weight vectors of the fundamental representations.

Below we prove a lemma.

Lemma A.1. *Given two gradations* $\mathbf{s}$ *and* $\mathbf{s}'$, *such that* $\mathbf{s}' \succeq \mathbf{s}$, *the highest weight vector* $v_{\mathbf{s}}$ *is an eigenvector of the subalgebra* $\mathfrak{g}_0(\mathbf{s}')$.

Proof. The proof follows from the fact that if $\mathbf{s}' \succeq \mathbf{s}$ then $\mathfrak{g}_0(\mathbf{s}') \subset \mathfrak{g}_0(\mathbf{s})$. But $v_{\mathbf{s}}$ is an eigenvector of $\mathfrak{g}_0(\mathbf{s})$ and hence also of $\mathfrak{g}_0(\mathbf{s}')$. □.

Corollary. *If* $\mathbf{s}' \succeq \mathbf{s}$ *then* $v_{\mathbf{s}}$ *is the unique vector in* $L(\mathbf{s})$ *with lowest* $\mathbf{s}'$*-grade.*

Proof. Suppose the converse was true, so there exists another vector $\phi \neq v_{\mathbf{s}}$ with the same $\mathbf{s}'$-grade as $v_{\mathbf{s}}$. This would require $\phi = a \cdot v_{\mathbf{s}}$, with $a \in \mathcal{U}(\mathfrak{g}_0(\mathbf{s}'))$ (the universal enveloping algebra of $\mathfrak{g}_0(\mathbf{s}')$). But by Lemma A.1, $v_{\mathbf{s}}$ is an eigenstate of $\mathfrak{g}_0(\mathbf{s}')$, hence $\phi \propto v_{\mathbf{s}}$ contrary to the hypothesis. □

Associated to each Kac–Moody algebra $\mathfrak{g}$, there is a group G formed by exponentiating the action of $\mathfrak{g}$ (see [14] for details). We denote by $U_{\pm}(\mathbf{s})$ and $H(\mathbf{s})$ the subgroups formed by exponentiating the subalgebras $\mathfrak{g}_{>0}(\mathbf{s})$, $\mathfrak{g}_{<0}(\mathbf{s})$ and $\mathfrak{g}_0(\mathbf{s})$, respectively. The group acts projectively on the representations.

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References

1. Drinfel'd, V.G., Sokolov, V.V.: Lie algebras and equations of the Korteweg–de Vries type. J. Sov. Math. **30**, 1975 (1985)
2. Kac, V.G.: Infinite dimensional Lie algebras (3rd ed.) Cambridge: Cambridge University Press 1990
3. Kac, V.G., Peterson, D.H.: 112 constructions of the basic representation of the loop group of E_8. In: Symp. on Anomalies, Geometry and Topology. (Bardeen, W.A., White, A.R. (eds.) Singapore: World Scientific 1985
4. de Groot, M.F., Hollowood, T.J., Miramontes, J.L.: Generalized Drinfel'd–Sokolov Hierarchies. Commun. Math. Phys. **145**, 57 (1992)
5. Burroughs, N.J., de Groot, M.F., Hollowood, T.J., Miramontes, J.L.: Generalized W-algebras and integrable hierarchies. Phys. Lett. **B277**, 89 (1992)
6. Burroughs, N.J., de Groot, M.F., Hollowood, T.J., Miramontes, J.L.: Generalized Drinfel'd–Sokolov hierarchies II: The Hamiltonian structures. Commun. Math. Phys. **153**, 187 (1993)
7. Hirota, R.: Direct methods in soliton theory. In: Soliton, p. 157: Bullough, R.K., Caudrey, P.S. (eds.) 1980
8. Jimbo, M., Miwa, T.: Solitons and infinite Lie algebras. Publ. RIMS, Kyoto Univ. **19**, 943 (1983); Date, E., Jimbo, M., Kashiwara, M., Miwa, T.: Transformation groups for soliton equations. Euclidean Lie algebras and reduction of the KP hierarchy. Publ. RIMS, Kyoto Univ. **18**, 1077 (1982)
9. Segal, G., Wilson, G.: Loop groups and equations of Korteweg–de Vries type. Inst. Hautes Études Sci. Publ. Math. **63**, 1 (1985)
10. Wilson, G.: Infinite-dimensional Lie groups and algebraic geometry in soliton theory. Phil. Trans. R. Soc. Lond. **A315**, 383 (1985); Habillage et fonctions τ, C.r. hebd. Séanc. Acad. Paris **299** (I) 587 (1984)
11. Kac, V.G., Wakimoto, M.: Exceptional hierarchies of soliton equations. Proceedings of Symposia in Pure Mathematics. Vol **49**, 191 (1989)
12. Imbens, H.-J.: Drinfel'd–Sokolov hierarchies and τ-functions. In: Infinite dimensional Lie algebras and groups. Singapore: World Scientific Adv. Ser. in Math. Phys. **7**, 352 (1989)
13. van Driel, P.: On a new hierarchy associated with the $W_3^{(2)}$ hierarchy. Phys. Lett. **B274**, 179 (1991)
14. Peterson, D.H., Kac, V.G.: Infinite flag varieties and conjugacy theorems. Proc. Nat. Acad. Sci. U.S.A. **80**, 1778 (1983)
15. Frenkel, I.B., Kac, V.G.: Basic representations of affine algebras and dual resonance models. Invent. Math. **62**, 28 (1980); Lepowsky, J., Wilson, R.L.: Construction of the affine Lie algebra. Commun. Math. Phys. **62**, 43 (1978); Kac, V.G., Kazhdan, D.A., Lepowsky, J., Wilson, R.L.: Realization of the basic representation of the Euclidean Lie algebras. Adv. in Math. **42**, 83 (1981); Lepowsky, J.: Calculus of twisted vertex operators. Proc. Natl. Acad. Sci. USA **82**, 8295 (1985)
16. Dijkgraaf, R., Verlinde, E., Verlinde, H.: Loop equations and Virasoro constraints in non-perturbative 2D quantum gravity. Nucl Phys. **B348**, 435 (1991); Fukuma, M., Kawai, H., Nakayama, R.: Continuum Schwinger–Dyson equations and universal structures in two-dimensional quantum gravity. Int. J. Mod. Phys. **A6**, 507 (1991); Kac, V.G., Schwarz, A.: Geometrical interpretation of the partition function of 2D gravity. Phys. Lett. **B257**, 329 (1991)
17. Fukuma, M., Kawai, H., Nakayama, R.: Infinite dimensional Grassmannian structure of two-dimensional gravity. Commun. Math. Phys. **143**, 403 (1992); Goeree, J.: W-constraints in 2D quantum gravity. Nucl. Phys. **B358**, 737 (1991)
18. Hollowood, T.J., Miramontes, J.L., Sánchez Guillén, J.: in preparation

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