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Multivalued Fields on the Complex Plane and Conformal Field Theories

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Abstract. In this paper a class of conformal field theories with nonabelian and discrete group of symmetry is investigated. These theories are realized in terms of free scalar fields starting from the simple $b-c$ systems and scalar fields on algebraic curves. The Knizhnik-Zamolodchikov equations for the conformal blocks can be explicitly solved. Besides the fact that one obtains in this way an entire class of theories in which the operators obey nonstandard statistics, these systems are interesting in exploring the connection between statistics and curved space-times, at least in the two dimensional case.

1. Introduction

In this paper we investigate the connections between conformal field theories on the complex plane and field theories on algebraic curves. These connections were first explored in [1] in the case of hyperelliptic curves and then in [2–4] in the more general case of curves with an abelian group of monodromy. Other examples of these techniques, in which the monodromy group is abelian, are given in [5, 6].

Here we study the simplest class of curves with a nonabelian group of monodromy. They can be viewed as multivalued mappings from the complex sphere to a Riemann surface having a discrete group of automorphisms D_m. Alternatively they can be viewed as cyclic coverings of hyperelliptic curves. The case $m = 3$ was briefly treated in [7].

In general, the construction of the amplitudes of a theory with nontrivial monodromy properties requires the solution of a Riemann monodromy problem (RMP) and of the related Schlesinger equations [8, 9]. Even if we are able to solve the RMP, the problem still remains of determining what combinations of the solutions enter in the amplitudes, in such a way that the physical properties of locality, associativity and so on are preserved [10]. In the case in which the monodromy group coincides

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with the monodromy group G of a known algebraic curve, there is the possibility of simplifications, since the most general function exchanging its branches according to G can be constructed using the techniques of algebraic geometry [11].

This is the case for example in which the monodromy group G describes the class of algebraic curves with discrete group of symmetry D_m. For these curves we can in fact construct a finite set of functions (and more in geneal λ-differentials) $F_k(z)$, $k = 0, \dots, 2m-1$, characterized by all possible monodromy properties that are compatible with the monodromy group G. We show that the elements of this set are rationally independent, i.e. the ratio of two of them is not a singlevalued function and that all the other multivalued functions are linear combinations of the $F_k(z)$'s. Moreover, our set of functions satisfies partial differential equations similar to the equations of parallel transport for the conformal blocks of [12, 13]. Finally, following [8], we show that it is possible to express the multivalued functions $F_k(z)$ in terms of free fields and twist fields. Therefore, starting from the $F_k(z)'s$, we are able to construct conformal blocks, whose monodromy properties correspond to the monodromy group G.

It is difficult to associate a conformal field theory defined on the complex plane to these conformal blocks. However they are surely tightly related to the $b-c$ systems on the algebraic curve Σ_g with D_m group of symmetry, as we will see.

The method presented here is interesting because it allows the construction of conformal blocks with nontrivial monodromy properties, provided the underlying monodromy group is that of a known algebraic curve. Moreover, the twist fields turn out to be anyons, exchanged in the conformal blocks according to a non-abelian braid group statistics. Unlike the usual anyons realized starting from a nonabelian Chern-Simons field theory [14], the exchange relations between the twist fields become nonabelian due to the presence of the group of automorphisms D_m of the algebraic curve. The statistics of the twist fields has been studied in a separate publication [15]. Finally we provide a nice interpretation of the twist fields as electrostatic charges induced by the topology of the algebraic curve.

The disadvantage of our approach is that we are not able to prove that the conformal blocks satisfy a Riemann monodromy problem. However, they obey a simplified system of equations given by Plemelj, which is strictly related to the Riemann monodromy problem (see [15]). Moreover, our method can surely be extended to the other classes of curves but not, we believe, to the most general cases where, apparently, there seems to be obstructions in the construction of some of the functions $F_k(z)$ which satisfy the requirements given in Sect. 2.

The material contained in this paper is organized as follows. In Sect. 2 we find the conditions for which a λ differential on a general algebraic curve can be represented as a ratio of conformal blocks containing free fields and twist fields. The general form of the twist fields is given. Starting from Sect. 3 we restrict ourselves to the class of D_m symmetric curves. We construct a basis of λ differentials satisfying the conditions of Sect. 2. They are rationally independent and exhibit all possible monodromy behaviors at the branch points compatible with the monodromy group of the algebraic curve. Moreover, all other meromorphic λ differentials are linear combinations of them. In Sect. 4 the full n-point functions of the $b-c$ systems on the D_m symmetric curves are computed. The n-point functions turn out to be superpositions of the solutions of the conformal blocks defined in Sect. 3. In Sect. 5 it is shown that the $b-c$ systems on an algebraic curve with D_m group of symmetry contain multivalued operators with fractional ghost charges. These twist fields simulate the presence of the branch points in the amplitudes and are primary fields. The appearance of primary fields in the amplitudes of the $b - c$ systems is explained in terms of electrostatics in Sect. 6. In Sect. 7 the form of the twist fields is explicitly given in terms of free fields using bosonization and the method introduced in Sect. 2. We prove that, apart from zero modes, the two-point function of the $b - c$ systems on an algebraic curve can be seen as conformal field theories. Each conformal field theory is characterized by particular monodromy properties at the branch points of the algebraic curve. The conformal blocks satisfy differential equations of the kind of the Knizhnik-Zamolodchikov equations [16, 12, 13]. Finally the exchange relations between the twist fields are derived showing that they satisfy a nonabelian braid group statistics [17, 18].

2. Monodromy Properties and Twist Fields

Let us consider a classical field $B(z)dz^\lambda$, λ integer or half-integer, satisfying a Fermi statistics, analytic in z and taking its values on an affine algebraic curve Σ_g defined by the vanishing of a Weierstrass polynomial $F(z, y)$:

$$F(z, y) = P_n(z)y^n + \ldots + P_0(z) = 0\,. \tag{2.1}$$

Each affine algebraic curve is equivalent, apart from conformal transformations, to a closed and orientable Riemann surface. The genus g of the Riemann surface is given by the Riemann-Hurwitz formula [19], which we will not discuss here. The $P_i(z)$'s, $i = 0, \ldots, n$, are polynomials in the complex variable $z \in \mathbf{CP}_1$, $\mathbf{CP}_1$ denoting the Riemann sphere. Here it is useful to regard the sphere as a compactified complex plane $\mathbf{C} \cup \{\infty\}$, covered by the two open sets U_1 and U_2 which contain the points 0 and ∞ respectively. z is the local coordinate in U_1 and z' in U_2. At the intersections of these two sets $z' = 1/z$. Solving Eq. (2.1) in y, we get a multivalued function $y(z)$ with n branches, denoted here by $y^{(l)}(z)$, $l = 0, \ldots, n-1$. As a consequence, the complex field $B(z)dz^\lambda$ becomes multivalued when transported along a closed small path encircling the branch points of $y(z)$:

$$B^{(l)}(z)dz^\lambda \equiv B(z, y^{(l)}(z))dz^\lambda\,. \tag{2.2}$$

On an algebraic curve, dz^λ represents a true λ differential with zero and poles [11]. The degree of its divisor is $2\lambda(g-1)$. Therefore we can consider $B^{(l)}(z)$ in Eq. (2.2) as a function multiplied by the λ differential dz^λ. Let us suppose that $B^{(l)}(z)$ has zeros z_i and poles p_j, $i, j = 1, \ldots, N$ of multiplicities $\nu_i(l_i)$ and $\mu_j(l_j)$ respectively. The zeros and poles occur only for certain values of the branch l of $B^{(l)}(z)$ and therefore the multiplicities $\nu(l_i)$ and $\mu_j(l_j)$ should also depend on the branch index. Now we associate to $B^{(l)}(z)dz^\lambda$ another $1 - \lambda$ differential defined as follows:

$$C^{(l)}(z)dz^{1-\lambda} = \frac{dz^{1-\lambda}}{B^{(l)}(z)}\,. \tag{2.3}$$

At this point we investigate the conditions under which the tensor

$$G(z, w)dz^\lambda dw^{1-\lambda} = B^{(l)}(z)C^{(l')}(w)\,\frac{dz^\lambda dw^{1-\lambda}}{z - w}$$

in the two independent complex variables z and w can be written in terms of conformal blocks. To this purpose, we introduce free fields $\tilde{b}(z)dz^\lambda$ and $\tilde{c}(z)dz^{1-\lambda}$ on Σ_g, which are however singlevalued in the variable z. Since they do not depend on $y(z)$, their expansion is the usual Laurent series of the genus zero case. The fields $\tilde{b}(z)$ and $\tilde{c}(z)$ are fermions or ghosts according to the values of λ. Moreover we introduce multivalued "twist fields" $V(z_i)$ and $V(p_j)$ with the following multivalued operator product expansions (OPE):

$$\begin{aligned}
\tilde{b}(z)V^{(l_i)}(z_i) &\sim (z-z_i)^{\nu_i(l_i)} :\tilde{b}(z)V^{(l_i)}(z_i): + \dots ,\\
\tilde{b}(z)V^{(l_j)}(p_j) &\sim (z-p_j)^{-\mu_j(l_j)} :\tilde{b}(z)V^{(l_j)}(p_j): + \dots ,\\
\tilde{c}(z)V^{(l_i)}(z_i) &\sim (z-z_i)^{\nu_i(l_i)} :\tilde{c}(z)V^{(l_i)}(z_i): + \dots ,\\
\tilde{c}(z)V^{(l_j)}(p_j) &\sim (z-p_j)^{-\mu_j(l_j)} :\tilde{c}(z)V^{(l_j)}(p_j): + \dots .
\end{aligned} \tag{2.4}$$

Apart from zero modes, which we ignore for the moment, we express the tensor $G(z,w)dz^\lambda dw^{1-\lambda}$ in the form:

$$\frac{B_k^{(l)}(z)C_k^{(l')}(w)}{z-w}\,dz^\lambda\, dw^{1-\lambda} = \frac{\langle 0|\tilde{b}(z)\tilde{c}(w)\prod_{i=1}^{N} V(z_i)\prod_{j=1}^{N} V(p_j)|0\rangle}{\langle 0|\prod_{i=1}^{N} V_k(z_i)\prod_{j=1}^{N} V(p_j)|0\rangle}, \tag{2.5}$$

$\langle 0|$ being the usual vacuum at genus zero. For the twist fields $V(z_i)$ and $V(p_j)$ we can try the simple ansatz of [7]:

$$\begin{aligned}
V_k^{(l)}(z_i) &= \exp\left[i\oint_{C_{z_i}} dt\partial_t \log[C^{(l)}(t)]\varphi(t)\right],\\
V_k^{(l)}(p_j) &= \exp\left[i\oint_{C_{p_j}} dt\,\partial_t \log[B^{(l)}(t)]\varphi(t)\right],
\end{aligned} \tag{2.6}$$

after using bosonization:

$$\tilde{b}(z) \sim e^{-i\varphi(z)}, \qquad \tilde{c}(z) \sim e^{i\varphi(z)}, \tag{2.7}$$

$$\langle 0|\varphi(z)\varphi(w)|0\rangle = -\log(z-w). \tag{2.8}$$

The multivaluedness of the twist fields, caused by the fact that the zeros and poles of $B^{(l)}(z)$ and $C^{(l)}(z)$ occur only for certain values of the branches, implies that they are nonlocal operators in the most general case, as Eq. (2.6) shows. Moreover, since the OPE's with the free fields turns out to be multivalued, the right-hand side (rhs) in (2.5) is also multivalued in z and w. Consistently with the left hand side (lhs), the branches in z and w of the rhs should be l and l' respectively. We remember here another similar example in which the presence of nonabelian groups of symmetries introduce nonlocal fields in the amplitudes, namely the solitonic sectors of scalar field theories with discrete group of symmetries discussed in [17, 20]. Exploiting Eq. (2.8), we evaluate the OPE's between the twist fields and the free fields as in the genus zero case. More OPE's are not needed to evaluate Eq. (2.5). Proceeding as in [7] we can compute the rhs of (2.5) obtaining the following result:

$$B^{(l)}(z)C^{(l')}(w) = \exp\left[-\left(\sum_{i=1}^{N}\oint_{C_{z_i}} + \sum_{i=1}^{N}\oint_{C_{p_j}}\right)\partial_t \log C(t)\log\left(\frac{t-w}{t-z}\right)\right]. \tag{2.9}$$

Here we have used the fact that, by definition, $\log B(z) = -\log C(z)$. Equation (2.9) can be rewritten as follows:

$$B^{(l)}(z)C^{(l')}(w)dz^{\lambda}\,dw^{1-\lambda} = \exp\left[-\oint\limits_{C} \partial_t \log C(t) \log\left(\frac{t-w}{t-z}\right)\right], \tag{2.10}$$

C being a contour surrounding all the poles and zeros of $C(z)$. Unfortunately it is impossible to apply the theorem of residues in (2.10). The function in the integrand is in fact multivalued inside the contour C and, in general, also on the contour itself. For this reason we additionally require that all the branch points of $B^{(l)}(z)$ are included in the set of points z_i and p_j. This is a reasonable request in view of our applications, since in conformal field theories on an algebraic curve the physical zeros and poles in the amplitudes are given by the branch points of the algebraic curve (see Sect. 4). Under the above requirement the integrand in (2.10) becomes one-valued on the contour C because it surrounds all the branch points of $B^{(l)}(t)$ and $C^{(l)}(t)$. Moreover, since we are on the compact sphere $\mathbf{CP}_1$, we can deform the contour C in such a way that only the other two singularities of the integrand are included, namely the points $t = z$ and $t = w$. The integration by parts in the exponent of Eq. (2.10) is then made possible and yields:

$$\oint\limits_{C} \log C(t) \log\left(\frac{t-w}{t-z}\right) = \oint\limits_{C_w+C_z} \log C(t)\left(\frac{1}{t-w} - \frac{1}{t-w}\right). \tag{2.11}$$

$C_w + C_z$ describes a simple contour equivalent to C containing the points w and z. The integrand of the lhs of Eqs. (2.11) is now one-valued inside and on the contour $C_w + C_z$, so that we can easily compute its residue:

$$\oint\limits_{C_w+C_z} \log C(t)\left(\frac{dt}{t-w} - \frac{dt}{t-z}\right) = \log C(w) - \log C(z)\,. \tag{2.12}$$

Substituting Eq. (2.12) in the rhs of Eq. (2.10) we obtain an identity, proving that Eq. (2.5) makes sense if all the ramification points of $B^{(l)}(z)$ are included in the set z_i and p_j.

3. Conformal Blocks for the $b - c$ Systems on an Algebraic Curve

At this point we specify a class of Riemann surfaces Σ_g of genus g associated to the Weierstrass polynomial

$$y^{2m} - 2q(z)y^m + q^2(z) - p(z) = 0 \tag{3.1}$$

$q(z)$ and $p(z)$ are polynomials in the variable z. The genus g is given in Appendix A in terms of the degrees mr and $2r'$ of $q(z)$ and $p(z)$ respectively. The algebraic curve $y(z)$ has $2m$ branches denoted by $y^{(l)}(z)$, $l = 0, \ldots, 2m-1$, that are exchanged at the branch points α_i and β_j as shown in Appendix A. i and j label the number of the independent roots of the equations

$$q^2(z) - p(z) = 0\,, \qquad p(z) = 0 \tag{3.2}$$

respectively. The first equation has $N_\alpha = \max(2mr, 2r')$ solutions α_i while the second equation has $N_\beta = 2r'$ solutions β_j. The integers r and r' are fixed in such a way that the point at infinity is not a branch point. This is not an essential limitation and it is introduced only in order to keep the notations as simple as possible.

Equation (3.1) is invariant under a D_m group of symmetry, generated by the transformations:

$$(z, y) \to (z, \varepsilon y) \quad \text{and} \quad y^m - q(z) \to -y^m + q(z)\,, \tag{3.3}$$

where $\varepsilon^m = 1$. The local monodromy group contains D_m as a subgroup. It is possible to view Σ_g as a Z_m cyclic of an hyperelliptic curve H_g of genus $g' = r' + 1$ and branch points β_j. The multivaluedness at the branch points α_i is then related to the Z_m branched covering of H_g.

We start constructing a basis $B_k(z)$, $0 \le k \le 2m-1$, of $2m$ rationally independent functions on Σ_g such that all other functions are linear combinations of them, the coefficients entering the linear combination being at most singlevalued functions of z. Two functions are said to be rationally independent if their ratio is not a singlevalued function on $\mathbf{CP}_1$. A basis of that kind is for example given by $B_k(z) = [y(z)]^k$, $0 \le k \le 2m-1$. However, the elements of this basis do not satisfy in general the requirement to have all their ramification points included in their divisor. Therefore we seek for a basis $B_k^{(l)}(z)$ with the following leading order expansions at the branch points:

$$\begin{aligned} B_k^{(l)}(z) &\sim (z - \alpha_i)^{-q_{k,\alpha_i}(l)} + \dots\,, \\ B_k^{(l)}(z) &\sim (z - \beta_j)^{-q_{k,\beta_j}(l)} + \dots\,. \end{aligned} \tag{3.4}$$

Transporting the functions $B_k^{(l)}(z)$ around a branch point on a closed path, one obtains the phases $\exp(-2\pi i q_{k,\alpha_i}(l))$, $\exp(-2\pi i q_{k,\beta_j}(l))$ that depend on the initial branch l of the function and on the index k characterizing the rationally independent functions. The $q_{k,\alpha_j}(l)$ and $q_{k,\beta_j}(l)$ must be rational numbers for some values of l, otherwise there is no multivaluedness at all. In principle, in order to find the $B_k^{(l)}(z)$, one needs to solve a Riemann monodromy problem and the related Schlesinger equations [8–10]. However, this is not so simple and the boundary conditions of the Schlesinger equations are not known. Fortunately we can rely on a theorem of algebraic geometry stating that a general function on an algebraic curve, therefore also a function satisfying Eqs. (3.4), should be a rational function in $y(z)$ and z. The construction of a function with a nontrivial behavior at the branch points of the kind (3.4) can be done using techniques of algebraic geometry. The parameters $q_{k,\alpha_i}(l)$ and $q_{k,\beta_j}(l)$, however, are still defined only up to integers. For example one can multiply $B_k^{(l)}(z)$ with singlevalued functions whose zeros lie at the branch points. This freedom is fixed by the physical properties that the correlation functions of the conformal field theories should satisfy, for example associativity, locality and statistics of the fields.

In this paper we choose a particularly simple conformal field theory, the $b - c$ systems [21] with spin λ and action:

$$S = \int_{\Sigma_g} d^2 z b \bar{\partial} c + \text{c.c.}\,; \tag{3.5}$$

$b(z)dz^\lambda$ and $c(z)dz^{1-z}$ are now fields on Σ_g and consequently they are multivalued fields in z in the sense of Eq. (2.2). For each value of λ, the physical requirements mentioned above are dictated by the fermionic statistics of the $b - c$ systems. In other words, their correlation functions should have simple poles whenever the coordinates of one field b and one field c coincide and simple zeros in the case in which the coordinates of two fields b or two fields c coincide [22, 23]. It is easy to check that, as a consequence, the parameters $q_{k,\alpha_i}(l)$ and $q_{k,\beta_j}(l)$ must depend also on λ. Therefore it is convenient to introduce two different basis $B_k^{(l)}(z)$ and $C_k^{(l)}(z)$ for the fields b and c respectively. Finally, the freedom of multiplying the basis with a singlevalued function with zeros and poles at the branch points will be exploited in such a way that the correlation functions of the $b-c$ systems on Σ_g can be expanded in the simplest way in the elements of the basis.

First of all we consider the case $\lambda = 0$. The following $2m$ functions $F_k(z)$ are an example of a basis satisfying the above requirements and those of Sect. 2:

$$\begin{aligned} F_k(z) &= y^k(z) && 0 \leq k \leq m-1\,, \\ &= y^{2m-1-k}(z)\sqrt{p(z)} && m \leq k \leq 2m-1\,. \end{aligned} \tag{3.6}$$

It is easy to check that the functions $F_k(z)$ are rationally independent and that they have the behavior (3.4) at the branch points with nontrivial rational values of $q_{k,\alpha_i}(l)$ and $q_{k,\beta_i}(l)$.

Now we will prove that any rational function $R(z, y(z))$ of z and $y(z)$ is a linear combination of the functions $F_k(z)$ of the kind:

$$R(z, y^{(l)}(z)) = \sum_k c_k(z) F_k^{(l)}(z)\,, \tag{3.7}$$

where the coefficients $c_k(z)$ are singlevalued in z. Equation (3.7) is certainly true if $R(z, y(z))$ is a sum of monomials of z and $y(z)$. In fact, from Eq. (3.1) we have $y^m(z) = q(z) \pm \sqrt{p(z)}$. Therefore monomials containing powers in $y(z)$ greater than $m-1$ are still expressible in terms of the basis (3.7). At this point we have only to consider the rational functions of the kind

$$R(z, y(z)) = \frac{1}{\sum\limits_k c_k(z) F_k(z)}\,.$$

A simple consequence of Eq. (3.1) is the following equation:

$$\prod_{l=0}^{m-1} \left(\sum_k c_k(z) \varepsilon^{kl} F_k(z) \right) [R(z, y(z))]^{-1} = Q(z)\sqrt{p(z)} + P(z)\,, \tag{3.8}$$

$Q(z)$ and $P(z)$ being singlevalued in z. Therefore

$$R(z, y(z)) = \frac{(Q(z)\sqrt{p(z)} - P(z)) \prod\limits_{l=0}^{m-1} \left(\sum\limits_k c_k(z) \varepsilon^{kl} F_k(z) \right)}{Q^2(z) p(z) - P^2(z)} \tag{3.9}$$

which is again of the kind (3.7). Thus we have shown that the functions $F_k(z)$ are $2m$ multivalued, rationally independent functions and that all other functions, the solutions of the RMP included, are linear superpositions of them.

The case of general λ is solved as follows. As can be seen from the divisors written in Appendix A, the λ-differential

$$B_0(z) dz^\lambda = \frac{dz^\lambda}{[y(z)]^{\lambda(m-1)} [p(z)]^{\frac{-\lambda}{2}}} \tag{3.10}$$

has neither poles nor zeros at the branch points. Therefore, multiplying $B_0(z)dz^\lambda$ with the functions $F_k(z)$ of Eq. (3.6), we get $2m$ λ-differentials $B_k(z)dz^\lambda$ with all the possible independent behaviors at the branch points. The final result is:

$$\begin{aligned} B_k^{(l)}(z)dz^\lambda &= (y^{(l)}(z))^{mq_{k,\alpha_i}}(p(z))^{q_{k,\beta_j}}\, dz^\lambda\,, \\ C_k^{(l)}(z)dz^{1-\lambda} &= (y^{(l)}(z))^{-mq_{k,\alpha_i}}(p(z))^{-q_{k,\beta_j}}\, dz^{1-\lambda}\,, \end{aligned} \tag{3.11}$$

where the "charges" q_{k,α_i} and q_{k,β_j} are defined by:

$$q_{k,\alpha_i} = \frac{[k]_m + \lambda(1-m)}{m}\,, \qquad [k_m] = [k+m]_m = k\,, \tag{3.12}$$

and

$$\begin{aligned} q_{k,\beta_j} &= -\frac{\lambda}{2}\,, \qquad k = 0,\dots,m-1\,, \\ &= \frac{1-\lambda}{2}\,, \quad k = m,\dots,2m-1\,. \end{aligned} \tag{3.13}$$

The significance of charges of the parameters q_{k,α_i} and q_{k,β_j} will be clarified below (see also [7, 24]). It is easy to show that the elements in the basis (3.11) are rationally independent and that the functions $B_k(z)$, $C_k(z)$ are linear combinations with rational coefficients of the $F_k(z)$'s. The leading order behavior of $B_k^{(l)}(z)$ and $C_k^{(l)}$ at the branch points is again of the form given in Eq. (3.4). The parameters $q_{k,\alpha_i}(l)$ and $q_{k,\beta_j}(l)$ are given by:

$$\begin{aligned} q_{k,\alpha_i}(l) &= 0\,, \qquad 0 \le l \le m-1\,, \\ &= q_{k,\alpha_i}\,, \quad m \le l \le 2m-1\,, \end{aligned} \tag{3.14}$$

and

$$q_{k,\beta_j}(l) = q_{k,\beta_j}\,, \qquad 0 \le l \le 2m-1\,. \tag{3.15}$$

4. The n-Point Functions of Free Field Theories on a D_m Symmetric Algebraic Curve

In this section we derive the correlation functions of the $b-c$ systems showing that they are superpositions of the basis given in Eq. (3.11). The $N_b = (2\lambda-1)(g-1)$ zero modes $\Omega_{1,\lambda}(z)dz^\lambda, \dots, \Omega_{N_b,\lambda}(z)dz^\lambda$ are computed in Appendix A in terms of the basis (3.11). In the Appendix we have however exploited a different notation to number the zero modes introducing a double index i_k, k. The index k labels the sector of zero modes having the same behavior at the branch points of the λ-differential $B_k(z)dz^\lambda$, while i_k labels the zero modes inside a given sector. This notation stresses the fact that the zero modes are constructed in terms of the basis (3.11). Here, however, it complicates the expressions of the correlation functions and therefore will not be used.

When $\lambda > 1$, the following meromorphic tensor with a single pole in $z = w$ will be necessary:

$$K_\lambda^{(ll')}(z,w)dz^\lambda\, dw^{1-\lambda} = \frac{1}{2m}\,\frac{dz^\lambda\, dw^{1-\lambda}}{z-w}\sum_{k=0}^{2m-1} B_k^{(l)}(z)C_k^{(l')}(w)\,. \tag{4.1}$$

If $\lambda = 1$ we need instead a differential of the third kind $\omega_{ab}(z)dz$ with two simple poles in $z = a$ and $z = b$ and with residue $+1$ and -1 respectively:

$$\omega^{(l)}_{a_{(l')}b_{(l'')}}(z)\,dz = K^{(ll')}_{\lambda=1}(z,a)\,dz - K^{(ll'')}_{\lambda=1}(z,b)\,dz\,. \tag{4.2}$$

The pole in $z = a$ is active only if $l = l'$. Analogously there is a divergence in $z = b$ only if $l = l''$. The zero modes, the tensors (4.1) and the differentials of the third kind (4.2) are derived using the formalism developed in [11]. At this point we are ready to compute the n-point functions exploiting the method of fermionic construction [25,26]. For $\lambda > 1$ the n-point functions are ratios of the following correlators [26]:

$$\left\langle \prod_{s=1}^{M} b^{(l_s)}(z_p) \prod_{t=1}^{N} c^{(l'_t)}(w_t) \right\rangle = \det \begin{vmatrix} \Omega^{(l_1)}_{1,\lambda}(z_1) & \dots & \Omega^{(l_1)}_{N_b,\lambda}(z_1) & K^{(l_1 l'_1)}_{\lambda}(z_1,w_1) & \dots & K^{(l_1 l'_N)}_{\lambda}(z_1,w_N) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \Omega^{(l_M)}_{1,\lambda}(z_M) & \dots & \Omega^{(l_M)}_{N_b,\lambda}(z_M) & K^{(l_M l'_1)}_{\lambda}(z_M,w_1) & \dots & K^{(l_M l'_N)}_{\lambda}(z_M,w_N) \end{vmatrix}\,. \tag{4.3}$$

where $M - N = (2\lambda - 1)(g-1) = N_b$. The tensor $K^{(ll')}_{\lambda}(z,w)$ has spurious poles in the limit $w \to \infty$. However one can show as in [11] and [26] that these poles do not contribute to the determinant (4.3). For $\lambda = 1$ we have an analogous equation:

$$\left\langle \prod_{i=1}^{N} b^{(l_i)}(z_i) \prod_{j=1}^{M} c^{(l'_j)}(w_j) \right\rangle = \det \begin{vmatrix} \omega^{(l_1)}(z_1)_{w_2 w_1} & \dots & \omega^{(l_1)}(z_1)_{w_M w_1} & \Omega^{(l_1)}_{1,1}(z_1) & \dots & \Omega^{(l_1)}_{g,1}(z_1) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \omega^{(l_N)}(z_N)_{w_2 w_1} & \dots & \omega^{(l_N)}(z_N)_{w_M w_1} & \Omega^{(l_N)}_{1,1}(z_N) & \dots & \Omega^{(l_N)}_{g,1}(z_N) \end{vmatrix}\,, \tag{4.4}$$

where $N - M = g - 1$. In order to simplify the notations we have omitted in the rhs of Eq. (4.4) the indices of the branches for the variables $w_1, \dots, w_M$. In the next section we will mainly use the two point functions of the $b - c$ systems:

$$G^{(ll')}_{\lambda}(z,w) = \frac{\left\langle b^{(l)}(z) c^{(l')}(w) \prod_{s=1}^{N_b} b^{(l_s)}(z_s) \prod_{r=1}^{N_c} c(w_r) \right\rangle}{\det|\Omega_{\bar{s},\lambda}(z_t)|}\,. \tag{4.5}$$

From Eq. (4.3) it turns out that the above propagator has the following form:

$$G_{\lambda}(z,w)\,dz_{\lambda}\,dw^{1-\lambda} = K^{(ll')}_{\lambda}(z,w)\,dz^{\lambda}\,dw^{1-\lambda} + \sum_{s=1}^{N_b}(-1)^s K^{(ll')}_{\lambda}(z_s,w) \times \frac{\langle b^{(l_1)}(z_1)\dots b^{(l_{s-1})}(z_{s-1}) b^{(l)}(z) b^{(l_{s+1})}(z_{s+1}) \dots b^{(l_{N_b})}(z_{N_b})\rangle}{\langle b^{(l_1)}(z_1)\dots b^{(l_{N_b})}(z_{N_b})\rangle}\,, \tag{4.6}$$

where

$$\langle b^{(l_1)}(z_1)\dots b^{(l_{N_b})}(z_{N_b})\rangle = \det|\Omega^{(l_j)}_{\bar{i},\lambda}(z_j)|\,. \tag{4.7}$$

Equation (4.6) and the form of $K_\lambda^{(ll')}(z,w)$ given in Eq. (4.1) show that the propagator (4.5) is just a superposition of the elements of the basis (3.11). An analogous equation can be written when $\lambda = 1$.

Before concluding this section, we prove that also the correlation functions of the scalar fields

$$S[X] = \int_{\Sigma_g} d^2z \partial_z X \partial_{\bar{z}} X \tag{4.8}$$

can be expressed as linear combinations of the elements of the basis (3.11). The correlator $\langle \partial X X \rangle$ is a differential of the third kind that coincides with the propagator of the $b-c$ systems with $\lambda = 1$ up to zero modes in z:

$$\begin{aligned} &\langle \partial_z X(z,\bar{z}) \left[X(w,\bar{w}) - X(w',\bar{w}') \right] \rangle \\ &= \Re \left[\langle b(z)c(w) \prod_{i=1}^{g} b(z_i)c(w') \rangle + \text{zero modes} \right] + (w \to w')\,, \end{aligned} \tag{4.9}$$

where the symbol $\Re[T]$ means that the real part of the tensor T is taken. The correlation function $\langle \partial X \partial X \rangle$ can be obtained deriving Eq. (4.9) in w and $\bar{w}$. The derivation in w of the correlation function of the $b-c$-systems (4.6) is quite simple. The variable w appears only in the tensor $K_{\lambda=1}^{(l_s l')}(z_s, w)$ and $K_{\lambda=1}^{(ll')}(z,w)$. The latter tensors are linear comnbinations of the basis $C_k(w)\, dw^{1-\lambda}$ and can be easily differentiated using eqs. (B.3), (B.4) of Appendix B.

5. Conformal Field Theories with D_m Group of Symmetry

In this section we prove that the $b-c$ systems on an algebraic curve are, apart from zero modes, a conformal field theory, in the sense that they contain primary fields concentrated at the branch points. To this purpose we study the vacuum expectation values (vev's) of the ghost current $J(z) = \,:b(z)c(z):$ and of the energy momentum tensor at the branch points. These vev's can be computed starting from the two point functions (4.6). We start considering the vev of the ghost current, which is given by:

$$\langle J^{(l)}(z) \rangle \lim_{\substack{z\to w \\ l=l'}} \left[G_\lambda^{(ll')}(z,w)\, dz_\lambda\, dw^{1-\lambda} - \frac{dz^\lambda\, dw^{1-\lambda}}{z-w} \right]. \tag{5.1}$$

From Eq. (4.6) it is clear that the divergences at the branch points are generated only by the term $K_\lambda^{(ll')}(z,w)\, dz^\lambda\, dw^{1-\lambda}$. The other terms forming the propagator are in fact zero modes in z and the poles in the variable w occur only at the locations of the zero modes z_s or in $z = \infty$. Therefore, inserting Eq. (4.1) in Eq. (5.1) we get:

$$\langle J_z^{(l)}(z) \rangle dz = \frac{1}{2m} \sum_{k=0}^{2m-1} \partial_z \log C_k^{(l)}(z)\, dz\,. \tag{5.2}$$

It is possible to regard the differential

$$J_k^{(l)}(z)\, dz = \partial_z \log[C_k^{(l)}(z)]\, dz \tag{5.3}$$

as the vev of the current associated to the ghost number conservation in a given sector k, i.e. in the sector in which the fields have the same monodromy properties of $B_k(z)\,dz^{\lambda}$ and $C_k(z)\,dz^{1-\lambda}$. The leading order of Eq. (5.2) at the branch points confirms Eqs. (3.14)–(3.15):

$$\begin{aligned} \langle J^{(l)}(z)\rangle &\sim \text{reg. terms}\,, && 0\le l\le m-1\,,\\ &\sim \frac{1}{2m}\sum_{k=0}^{2m-1}\sum_{i=1}^{N_\alpha}\frac{q_{k,\alpha_i}}{z-\alpha_i}\,, && m\le l\le 2m-1\,, \end{aligned} \tag{5.4}$$

and

$$\langle J^{(l)}(z)\rangle \sim \frac{1}{2m}\sum_{k=0}^{2m-1}\sum_{j=1}^{N_\beta}\frac{q_{k,\beta_j}}{z-\beta_j}\,. \tag{5.5}$$

Now we compute also the vev of $\langle T(z)\rangle$ at the branch points in the first order approximation. As before, the only contribution comes from the tensor $K_\lambda^{(ll')}(z,w)\,dz^{\lambda}\,dw^{1-\lambda}$ as we will show immediately. The proof is a slight generalization of a simple argument given in [2] in the case of the Z_m symmetric curves.

On the algebraic curve Σ_g the fields b and c are singlevalued. Therefore, in the proper system of coordinates, the energy momentum tensor must be regular. The proper coordinate near a branch point α_i is defined as follows:

$$\begin{aligned} t &= z\,, && 0\le l\le m-1\,,\\ t^m &= z-\alpha_i\,, && m\le l\le 2m-1\,. \end{aligned} \tag{5.6}$$

At the points β_j the local uniformizer becomes instead:

$$t^2 = z-\beta_j\,,\qquad 0\le l\le 2m-1\,. \tag{5.7}$$

Since the vev of the energy momentum tensor is not a tensor, a change of coordinates like that given in Eqs. (5.6) and (5.7) yields an extra term which is nothing but a partial derivative:

$$\langle T(t)\rangle = \left(\frac{dz}{dt}\right)^2\langle T(z)\rangle - \frac{c_\lambda}{6}\left[\frac{d^3z/dt^3}{dz/dt}-\frac{3}{2}\left(\frac{d^2z/dt^2}{dz/dt}\right)^2\right], \tag{5.8}$$

where $c_\lambda = (6\lambda^2-6\lambda-1)$. The Schwarzian derivative appearing in the rhs of Eq. (5.8) gives poles of the second order at the branch points. In order to eliminate these singularities from $\langle T(t)\rangle$, the correlator $\langle T(z)\rangle$ in Eq. (5.8) should have the same singularities but with opposite signs, i.e.:

$$\begin{aligned} \langle T(z)\rangle &= \text{reg. terms} && 0\le l\le m-1\,,\\ &= \frac{1}{(z-\alpha_i)^2}\frac{c_\lambda}{12}\left(\frac{1}{m^2}-1\right) && 0\le l\le 2m-1\,. \end{aligned} \tag{5.9}$$

and at the branch points β_j:

$$\langle T(z)\rangle = \frac{1}{(z-\beta_j)^2}\frac{c_\lambda}{16}\,,\qquad 0\le l\le 2m-1\,. \tag{5.10}$$

Now we compute the correlator $\langle T(z)\rangle$ explicitly. We apply the following formula given in [25]:

$$\langle T^{(l)}(z)\rangle = \lim_{\substack{z\to w\\ l=l'}}\left[-\lambda\partial_w G_\lambda^{(ll')}(z,w)+(1-\lambda)\partial_z G_\lambda^{(ll')}(z,w)-\frac{1}{(z-w)^2}\right]. \tag{5.11}$$

After inserting in Eq. (5.11) the tensor $K_\lambda^{(ll')}(z,w)$ instead of the entire propagator the result is:

$$\langle T^{(l)}(z)\rangle = \frac{1}{2m}\sum_{k=0}^{2m-1}\left[\left(\lambda-\frac{1}{2}\right)\left(\frac{d^2C_k^{(l)}(z)/dz^2}{C_k^{(l)}(z)}\right) - (\lambda-1)\left(\frac{dC_k^{(l)}(z)/dz}{C_k^{(l)}(z)}\right)^2\right]. \tag{5.12}$$

In the first order approximation at the branch points Eq. (5.12) becomes:

$$\begin{aligned}\langle T^{(l)}(z)\rangle &\sim \text{reg. terms} && 0\le l\le m-1\,,\\ &\sim \frac{1}{2m}\sum_{k=0}^{2m-1}\left[\frac{1}{2}q_{k,\alpha_i}^2+\left(\lambda-\frac{1}{2}\right)q_{k,\alpha_i}\right],\frac{1}{(z-\alpha_i)^2} && m\le l\le 2m-1\,, \end{aligned}\tag{5.13}$$

$$\langle T^{(l)}(z)\rangle \sim \frac{1}{2m}\sum_{k=0}^{2m-1}\left[\frac{1}{2}q_{k,\beta_j}^2+\left(\lambda-\frac{1}{2}\right)q_{k,\beta_j}\right]\frac{1}{(z-\beta_j)^2}\,, \quad 0\le l\le 2m-1\,. \tag{5.14}$$

Summing over k in Eqs. (5.13) and (5.14) we obtain exactly Eqs. (5.9) and (5.10). Concluding, we have shown that the amplitudes of the $b-c$ systems on an algebraic curve with D_m group of symmetry contain primary fields with charges and conformal dimensions given by Eqs. (5.4)–(5.5) and (5.11)–(5.12) respectively. In Sect. 6 we interpret these primary fields as twist fields simulating the presence of the branch points in the correlation functions.

6. On the Geometrical Meaning of the Twist Fields and Their Electrostatic Interpretation

In this section we notice an important point coming from the previous analysis. The $b-c$ systems are singlevalued on Σ_g, so that the energy momentum tensor has no singularities at the branch points in the proper coordinates (5.6) and (5.7). Instead, the poles of the ghost current remain[1]. They can be explained as a topological effect induced by the fact that we are considering a theory on a curved space-time. Already Wheeler pointed out that topology is equivalent to charge on some manifolds. For example in Fig. 1 the total effect of the potential lines is to simulate a positive charge inside the left hole and a negative charge inside the right one. In our case, we have a more complicated surface, similar to that of Fig. 1 but with many handles. Therefore it is natural to interpret the poles of the ghost current at the branch points as virtual (and factional) ghost charges generated by the nontrivial topology of the world-sheet. On that point see also [7].

Now we explain this phenomenon in a somewhat heuristic way using electrostatic considerations. We consider the $b-c$ theory on Σ_g as a multivalued field theory on $\mathbf{CP}_1$:

$$S^{(l)}(b,c) = \int_{\mathbf{CP}_1} d^2z\, b^{(l)}(z)\bar\partial c^{(l)}(z) + \text{c.c.}\,. \tag{6.1}$$

[1] We thank J. Sobczyk for having pointed out this fact

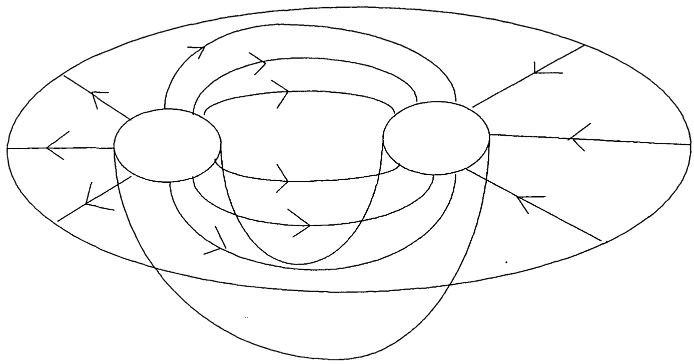

Fig. 1. Lines of force on a wormhole in absence of charge. The total effect of the lines is the appearance of virtual charges in the two "mouths" of the wormhole

As we showed in Sect. 3, the fields $b^{(l)}(z)$ and $c^{(l)}(z)$ can be expanded in the basis (3.11):

$$\begin{aligned} b^{(l)}(z) &= \sum_{k=0}^{2m-1} B_k^{(l)}(z)\tilde{b}_k(z)\,, \\ c^{(l)}(z) &= \sum_{k=0}^{2m-1} C_k^{(l)}(z)\tilde{c}_k(z)\,, \end{aligned} \tag{6.2}$$

where $\tilde{b}_k(z)$ and $\tilde{c}_k(z)$ are singlevalued fields on $\mathbf{CP}_1$, interacting only if $k = k'$ as the usual $b - c$ systems on the sphere. This expansion is valid only locally, i.e. away from the branch points and from the point at infinity. At these points one should use the local uniformizer and the coordinate $z' = 1/z$ respectively. Equation (6.2) defines an operator formalism on Σ_g and, substituting Eq. (6.2) in Eq. (6.1), we get:

$$S^{(l)}(b,c) = \int_{\mathbf{CP}^1} d^2z \sum_{k=0}^{2m-1} \tilde{b}_k(z)\bar{\partial}\tilde{c}_k(z) + \int_{\mathbf{CP}_1} d^2z \sum_{k=0}^{2m-1} \bar{\partial}\log(C^{(l)}(z))\tilde{b}_k(z)\tilde{c}_k(z)\,. \tag{6.3}$$

Now the fields are considered as operators, so that everywhere a normal ordering should be understood. In Eq. (6.3) the multivaluedness of the action is in the second term of the rhs. At this point we can bosonize the action (6.3) using the formulas (2.7) and (2.8) for each field $\tilde{b}_k(z)$ and $\tilde{c}_k(z)$. As a consequence, after an integration by part, Eq. (6.3) becomes:

$$\begin{aligned} S^{(l)}(\varphi_k(z)) = &\int_{\mathbf{CP}_1} d^2z \sum_{k=0}^{2m-1} [\partial\varphi_k\bar{\partial}\varphi_k + R_{z\bar{z}}(z,\bar{z})\varphi_k] \\ &+ \int_{\mathbf{CP}_1} d^2z \sum_{k=0}^{2m-1} \partial\bar{\partial}\log|C_k^{(l)}(z)|^2\varphi_k\,. \end{aligned} \tag{6.4}$$

The term in $R_{z\bar{z}}(z,\bar{z})$ comes from the usual bosonization of the $b - c$ systems on the sphere and it is given by the distribution:

$$R_{z\bar{z}}(z,\bar{z}) = (1 - 2\lambda)\delta^{(2)}(z,\infty)\,. \tag{6.5}$$

The third term in the rhs of Eq. (6.4) represents instead the additional curvature requested by the fact that we are treating a $b-c$ system on a Riemann surface. This curvature corresponds to a distribution consisting in a sum of δ-functions concentrated at the branch points. The classical equations of motions of the propagators $G(z,w)$ of the fields φ_k coming from the action (6.4) are:

$$\partial\bar{\partial}(G(z;\alpha_i,\beta_j) = (1-2\lambda)\delta^{(2)}(z,\infty) + \sum_{i=1}^{N_\alpha} q_{k,\alpha_i}\delta^{(2)\,(l)}(z,\alpha_i) + \sum_{j=1}^{N_\beta} q_{k,\beta_j}\delta^{(2)}(z,\beta_j)\,. \tag{6.6}$$

$G(z;\alpha_i,\beta_j)$ turns out to be the Green function of electrostatics in the presence of a charge $1-2\lambda$ at $z=\infty$ and fractional charges q_{k,α_i} and q_{k,β_j} at the branch points. Only the δ-function at the branch points α_i is multivalued. The total charge of the system is zero as it should be due to the presence of the zero modes, as we will see in the next section.

7. Multivalued Complex Fields on the Punctured Complex Plane

In this section we construct the Green function $K_\lambda(z,w)\,dz^\lambda\,dw^{-\lambda}$ of Eq. (4.1) in terms of free fields using the techniques of Sect. 2. This tensor represents the two point function of the $b-c$ systems apart from zero mode contributions and gives the vev's of the ghost currents and of the energy momentum tensor as we have already seen. We show in this way that $K_\lambda(z,w)\,dz^\lambda\,dw^{1-\lambda}$ is a superposition of ratios of conformal blocks of the kind (2.5). Each conformal block corresponds to a conformal field theory whose monodromy properties are characterized by Eqs. (3.4) and (3.14)–(3.15).

First of all we consider the tensors $G_{\lambda,k}(z,w)$ that play the role of $G(z,w)$ in (2.5). For each fixed value of k, they can be interpreted as the propagators of the sector of the $b-c$ fields having the same boundary conditions at the branch points of $B_k(z)\,dz^\lambda$ and $C_k(z)\,dz^{1-\lambda}$:

$$G^{(ll')}_{\lambda,k}(z,w)\,dz^\lambda\,dw^{1-\lambda} = \frac{B^{(l)}_k(z)C^{(l')}_k(w)}{z-w}\,dz^\lambda\,dw^{1-\lambda}\,. \tag{7.1}$$

Indeed, summing the partial propagators $G_{\lambda,k}(z,w)$ of Eq. (7.1) over k, we get exactly the tensor $K_\lambda(z,w)$ which is, as we have previously seen, the total propagator of the $b-c$ systems up to zero modes.

As in Sect. 2 we express $G^{(ll')}_{\lambda,k}(z,w)$ in terms of free $b-c$ systems $\tilde{b}_k(z)\,dz^\lambda$ and $\tilde{c}_k(z)\,dz^{1-\lambda}$ defined on the complex plane, $0\le k\le 2m-1$. The effect of the branch points is simulated by the twist fields $V_k(\alpha_i)$ and $V_k(\beta_j)$:

$$G^{(ll')}_{\lambda,k}(z,w)\,dz^\lambda\,dw^{1-\lambda} = \frac{{}_k\langle 0|\tilde{b}_k(z)\tilde{c}_k(w)\,(\text{z.m.})_k \prod_{i=1}^{N_\alpha} V_k(\alpha_i)\prod_{j=1}^{N_\beta} V_k(\beta_j)\,|0\rangle_k}{{}_k\langle 0|\,(\text{z.m.})_k \prod_{i=1}^{N_\alpha} V_k(\alpha_i)\prod_{j=1}^{N_\beta} V_k(\beta_j)\,|0\rangle_k}\,. \tag{7.2}$$

$|0\rangle_k$ is the usual $SL(2,\mathbf{C})$ invariant vacuum of the flat case and $(\text{z.m.})_k$ represents an insertion of zero modes in Eq. (7.2). This will be necessary in order to set the total ghost charge to zero in the correlators appearing in Eq. (7.2). To compute the number of zero modes we need to insert, we look at the residues of the "current" $J_k(z) = \partial_z \log C_k^{(l)}(z)$. From Eqs. (5.4)–(5.5) we know already the total ghost charge introduced by the presence of the branch points in each sector k with independent monodromy properties. The computation of the total charge $q_{k,\infty}$ at infinity is easy to find and yields $q_{k,\infty} = 1 - 2\lambda$. All the k-sectors have the same charge at infinity and moreover $q_{k,\infty}$ does not depend on l confirming Eq. (6.5). Summing all the charges at the branch points and at infinity we get

$$\sum_{l=0}^{2m-1}\left(\sum_{i=1}^{N_\alpha}\oint_{C_{\alpha_i}} + \sum_{j=1}^{N_\beta}\oint_{C_{\beta_j}} + \oint_{C_\infty}\right) dz\,\partial_z \log C_k^{(l)}(z) = N_{b_k} - N_{c_k}\,, \tag{7.3}$$

where C_{α_i}, C_{β_j}, and C_∞ are closed infinitesimal paths on the complex plane surrounding the points α_i, β_j and ∞ respectively. In Eq. (7.3) N_{b_k} and N_{c_k} are exactly the numbers of the zero modes $\Omega_{i_k,k}(z)\,dz^\lambda$ computed in the Appendix and having the same behavior at the branch points of $B_k(z)\,dz^\lambda$ and $C_k(z)\,dz^{1-\lambda}$. In order to get nonvanishing amplitudes in Eq. (7.2), we therefore have to add the following insertion of zero modes:

$$(\text{z.m.})_k = \prod_{s=1}^{N_{b_k}} \tilde{b}_k(z_s) \prod_{t=1}^{N_{c_k}} \tilde{c}_k(z_t)\,. \tag{7.4}$$

Still we need the explicit expression of the twist fields. These fields are derived in [7] in the case of D_3 symmetric curves using bosonization. In the general case we just apply Eq. (2.6). Let us introduce a set of free scalar fields $\varphi_k(z)$ with propagator

$$\langle \varphi_k(z)\varphi_{k'}(w)\rangle = -\delta_{kk'} \log(z-w)\,.$$

Then the final form of the twist fields reads:

$$V_k^{(l)}(\alpha_i) = \exp\left[i \oint_{C_{\alpha_i}} dt\,\partial_t \log[C_k^{(l)}(t)]\varphi_k(t)\right], \tag{7.5}$$

$$V_k^{(l)}(\beta_j) = \exp\left[i \oint_{C_{\beta_j}} dt\,\partial_t \log[C_k^{(l)}(t)]\varphi_k(t)\right]. \tag{7.6}$$

Equation (7.6) can be further simplified and becomes:

$$V_k^{(l)}(\beta_j) = e^{-iq_{k,\beta_j}\varphi_k(\beta_j)}\,. \tag{7.7}$$

The asymptotic form of the twist fields $V_k(\alpha_i)$ at the branch points is in agreement with Eqs. (5.4) and (3.4). Using the formulas given in Appendix B to compute the residues at α_i and β_j in Eq. (7.5) we get in fact:

$$V_k^{(l)}(\alpha_i) \sim e^{iq_{k,\alpha_i}(l)\varphi_k(\alpha_i)}\,, \qquad m \le l \le 2m-1\,. \tag{7.8}$$

Let us now show that the rhs of Eq. (7.2) is the desired subcorrelator (7.1) following the formalism of Sect. 2. Inserting in Eq. (7.2) the multivalued OPE's:

$$V_k(\gamma) e^{-i\varphi_k(z)} e^{i\varphi_k(w)} = \exp \oint_{C_\gamma} dt\, \partial_t \log[C_k^{(t)}(t)] \times \log\left(\frac{z-t}{w-t}\right) : V_k(\gamma) e^{-i\varphi_k(z)} e^{i\varphi_k(w)} : \tag{7.9}$$

with $\gamma = \alpha_i, \beta_j$ and remembering the contribution of the charge at infinity, we have:

$$G^{(ll')}_{\lambda,k}(z,w)\, dz^\lambda\, dw^{1-\lambda} = \frac{dz^\lambda\, dw^{1-\lambda}}{z-w} \prod_{s=1}^{N_{b_k}} \left(\frac{z-z_s}{w-z_s}\right) \prod_{t=1}^{N_{c_k}} \left(\frac{w-w_s}{z-w_t}\right)$$
$$\times \exp\left[\left(\sum_{i=1}^{N_\alpha} \oint_{C_{\alpha_i}} dt + \sum_{j=1}^{N_\beta} \oint_{C_{\beta_j}} dt + \oint_{C_\infty} dt\right) \partial_t \log[C_k^{(l)}(t)] \log\left(\frac{z-t}{w-t}\right)\right]. \tag{7.10}$$

The contour $C = \sum_i C_{\alpha_i} + \sum_j C_{\beta_j} + C_\infty$ contains all the branch points as required in Sect. 2 and therefore we get: the final result:

$$G^{(ll')}_{\lambda,k}(z,w)\, dz^\lambda\, dw^{1-\lambda} = \frac{dz^\lambda\, dw^{1-\lambda}}{z-w} \frac{C_k^{(l')}(w)}{C_k^{(l)}(z)} \times \prod_{i=1}^{N_{b_k}} \left(\frac{z-z_s}{w-z_s}\right) \prod_{t=1}^{N_{c_k}} \left(\frac{w-w_t}{z-w_t}\right). \tag{7.11}$$

Remembering that $dz^\lambda / C_k^{(l)}(z) = B_k^{(l)}(z)\, dz^\lambda$ from Eq. (3.11), we conclude that Eq. (7.11) is the wanted solution of the Riemann monodromy problem. The only difference from Eq. (7.1) consists in the products involving the cordinates of the zero modes. This is not a problem, since these terms coming from the zero modes are rational functions of z and do not modify the monodromy of the tensor (7.11).

Now we investigate the possibility of writing first order differential equations for the Green function defined in Eq. (7.2). In doing this we regard the correlators (7.2) as the correlators of a conformal field theory with multivalued primary fields $V_k(\alpha_i)$.

Since Eq. (7.2) is equivalent to Eq. (7.11), we need only to study the differential equations satisfied by the $B_k(z)$'s for any value of $\lambda \in \mathbf{Z}$. It turns out that the functions $B_k(z)$ satisfy a differential equation of the following kind:

$$\frac{dB_k^{(l)}(z)}{dz} = \sum_{k'} A_{kk'}(z; \alpha_i, \beta_j) B_{k'}^{(l)}(z). \tag{7.12}$$

The elements of the matrix $A_{kk'}(z; \alpha_i, \beta_j)$ are one forms in Σ_g. They are computed in Appendix B and we simply report that result:

$$A_{kk'}(z; \alpha_i, \beta_j)\, dz = \left(m q_{k,\alpha_i} y^{-1} \frac{dy}{dz} + q_{k,\beta_j} \sum_j \frac{1}{z-\beta_j}\right) \delta_{kk'}\, dz. \tag{7.13}$$

The explicit dependence of $y^{-1}\frac{dy}{dz}$ on α_i and β_j can be found in Eqs. (B.1-2). Equation (7.13) represents a one form with simple poles at the branch points. Deriving the $B_k(z)$'s with respect to α_i we get:

$$\partial_{\alpha_i} B_k^{(l)}(z) = \tilde{A}_{kk'}(z;\alpha_i,\beta_j) B_k^{(l)}(z)\,. \tag{7.14}$$

One can check that also $\tilde{A}_{kk'}(z;\alpha_i,\beta_j)$ is a one form in the variable α_i with simple poles in $\alpha_i = z$ and $\alpha_i = \alpha_j$, $i \neq j$. The residue at these points are exactly opposite to those of the matrix $A_{kk'}(z;\alpha_i,\beta_j)$ and the two matrices differ only by zero modes. Analogous conclusions can be drawn deriving $B_k^{(l)}(z)$ with respect to the branch points β_j. If we interpret Eqs. (7.12) as a parallel transport [17, 13, and 12], then Eqs. (7.13) and (7.14) provide the connection in the variables z and α_i respectively. Using the above equations and the decomposition (4.6) one can find differential equations for all n-point functions of the $b-c$ systems on Σ_g.

The matrices $A_{kk'}(z;\alpha_i,\beta_j)$ and $\tilde{A}_{kk'}(z;\alpha_i,\beta_j)$ are not so simple as the usual Knizhnik-Zamolodchikov equations. In fact on a Riemann surface the two dimensional Poincaré group of world-sheet symmetries is explicitly broken and the connections $A_{kk'}(z;\alpha_i,\beta_j)$ and $\tilde{A}_{kk'}(z;\alpha_i,\beta_j)$ cannot be translational invariant as it happens in the flat case. Therefore they also have a multivalued dependence on the variable z. Eventually this is a consequence of Eqs. (3.4).

The twist fields (7.5) and (7.6) represent particles with nonabelian braid group statistics inside the amplitudes (7.2). To conclude this section, we derive the exchange algebras of these operators. The most difficult case occurs when two twist fields $V(\alpha_i)$ and $V(\alpha_{i'})$ are considered:

$$\begin{aligned} V_k^{(l_i)}(\alpha_i)V_k^{(l_{i'})}(\alpha_{i'}) = \exp\Bigg[&- q_{k,\alpha_i} q_{k,\alpha_{i'}} \oint_{C_{\alpha_i}} ds\, J_k^{(l_i)}(s) \\ &\times \oint_{C_{\alpha_{i'}}} ds'\, J_k^{(l_{i'})}(s') \log(s-s') \Bigg] \\ &\times V_k^{(l_{i'})}(\alpha_{i'})V_k^{(l_i)}(\alpha_i)\,. \end{aligned} \tag{7.15}$$

To see how the twist fields are locally exchanged when α_i is very near to $\alpha_{i'}$, we have to compute two residues at the branch points α_i and $\alpha_{i'}$ in Eq. (7.15). To do this it is sufficient to insert in the definition of $J_k^{(l)}(z)$ given by Eq. (5.3) the form of $p(z)$ and $q(z)$ in terms of the branch points provided by Eqs. (B.1) and (B.2). The remaining task is a simple calculation of residues and the final result is:

$$V_k^{(l_i)}(\alpha_i)V_k^{(l_{i'})}(\alpha_{i'}) = e^{i\pi q_{k,\alpha_i}(l_i) q_{k,\alpha_{i'}}(l_{i'})} V_k^{(l_{i'})}(\alpha_{i'})V_k^{(l_i)}(\alpha_i)\,, \tag{7.16}$$

$$V_k^{(l_i)}(\alpha_i)V_k^{(l_j)}(\beta_j) = e^{i\pi q_{k,\alpha_i}(l_i) q_{k,\beta_j}(l_j)} V_k^{(l_j)}(\beta_j)V_k^{(l_i)}(\alpha_i)\,, \tag{7.17}$$

$$V_k^{(l_j)}(\beta_j)V_k^{(l_{j'})}(\beta_{j'}) = e^{i\pi q_{k,\beta_j}(l_j) q_{k,\beta_{j'}}(l_{j'})} V_k^{(l_{j'})}(\beta_{j'})V_k^{(l_j)}(\beta_j)\,, \tag{7.18}$$

where $q_{k,\alpha_i}(l_i)$ and $q_{k,\beta_j}(l_j)$ are defined in Eqs. (3.12) and (3.13).

8. Conclusions

One of the results obtained here is that the $b-c$ fields on an algebraic curve with D_m group of symmetry can be decomposed into $2m$ sectors propagating with different boundary conditions at the branch points. Each k-sector, $0 \leq k \leq 2m-1$, has a well defined propagator, given by Eq. (7.2) and containing the multivalued twist fields $V_k(\alpha_i)$. The multivalued twist fields are primary fields and therefore to each k-sector corresponds a multivalued conformal block. We hope that with the formalism developed here one can treat also more physical theories on algebraic curves than the $b-c$ systems. However, the basic requirements are that the theory should be conformal and have a lagrangian. This is not for example the case of theories based on free scalar fields. The scalar fields, in fact, are not entirely conformal as their propagator, with a logarithmic singularity, shows. As a consequence an attempt to write an expansion of the kind (6.2) for the scalar fields is difficult, since they depend also on the complex conjugate variable $\bar{z}$. The free fermions, instead, are very interesting for superstring theory, but unfortunately it is not so easy to treat the spin structures on algebraic curves and therefore to construct analogues of the basis (3.11).

Another result obtained is that we have shown the presence of particles with non-standard statistics in the amplitudes of the $b-c$ systems and therefore of string theory. Following the procedure of Sect. 5 and using Eq. (4.9), it is possible to show that also the amplitudes of the free scalar fields contain multivalued twist fields. The only problem is that there is no way to obtain an explicit expression of these twist fields because bosonization does not work in the case of the scalar fields.

It is natural to ask at this point if the twist fields have some observable effect or if they are just an artifact of our way of representing the Riemann surfaces as algebraic curves. First of all we remember that also in the case of the conformal field theories there is a multivaluedness in the conformal blocks that disappears in the physical correlation functions. Despite this fact, this multivaluedness is crucial in showing the quantum group structure of conformal field theories. In our case the multivaluedness on $\mathbf{CP}_1$ of the amplitudes is allowed and therefore also the presence of the twist fields. The problem is however complicated by the fact that the space-time geometry is not flat. Surely a local observer, located in a system of reference in which the metric on the Riemann surface is induced by the mapping $y(z){:}\,\mathbf{CP}_1 \rightarrow \Sigma_g$, experiences the presence of the twist fields. The existence of these operators is in fact proved in Sect. 5 using Eqs. (4.3) and (4.4), that represent the two point functions obtained from the method of the fermionic construction of [25]. Probably an observer in another system of reference would not confirm the existence of the twist fields. Unfortunately the calculations of the n-point functions in the case of an arbitrary metric make use of the formalism of the theta functions together with bosonization and the final results are not very explicit. Therefore it is not easy to do a comparison of the observations performed in the two different frames.

The method presented here shows that a curved background can influence the statistics inside the correlation functions of free field theories. In Sect. 6 we have explained it in terms of electrostratics. Finally we have realized in Sect. 7 examples of theories with nontrivial braid group statistics [17, 27]. The problem remains to classify these theories. To this purpose we only note that the twist fields have nontrivial exchange relations but obviously they form an associative algebra when more than two branch points are permuted in the correlators of Eq. (7.2). Therefore we can construct the Yang-Baxter matrices corresponding to the exchange algebra (7.16)–

(7.18) and look if there are other integrable models yielding the same solutions of the Yang-Baxter equations. This has been done in [15].

Appendix A

The $2m$ branches of the solution of Eq. (3.1) can be written as follows:

$$\begin{cases} y^{(l)}(z) = e^{\frac{2\pi i l}{m}} \sqrt[m]{q(z) + \sqrt{p(z)}}\,, & 0 \le l \le m-1 \\ y^{(l)}(z) = e^{\frac{2\pi i l}{m}} \sqrt[m]{q(z) - \sqrt{p(z)}}\,, & m \le l \le 2m-5 \end{cases} . \tag{A.1}$$

The branches are exchanged in the following way:

$$\begin{aligned} &y^{(l)}(z) \to y^{(l+m)}(z),\ 0 \le l \le 2m-1 && \text{in } \beta_1, \dots, \beta_{N_\beta}\,, \\ &(y^{(m)}(z), \dots, y^{(2m-1)}(z) \\ &\quad \to (y^{(2m-1)}(z), y^{(m)}(z), \dots, y^{(2m-2)}(z)) && \text{in } \alpha_1, \dots, \alpha_{N_\alpha}\,. \end{aligned} \tag{A.2}$$

We can rewrite Eq. (A.2) in a matrix form:

$$y^{(m)}(z) = (M_\gamma)_{m,l} y^{(l)}(z)\,, \qquad \gamma = \alpha_i, \beta_j\,. \tag{A.3}$$

The only nonvanishing elements of the monodromy matrices M_{β_j} are $(M_{\beta_j})_{l+m,l}$. M_{α_i} has instead the following block form: $M_{\alpha_i} = \mathrm{diag}(I_m, S_m)$, where I_m is a $m \cdot m$ unit matrix and S_m generates the Z_m group of permutations. The monodromy matrices M_{α_i} and M_{β_j} provide a representation of the group D_m.

The genus of the curve Σ_g is given by the Riemann-Hurwitz formula:

$$2g-2 = 2m((m-1)r-2) + 2mr'\,, \quad mr \ge r'\,, \tag{A.4}$$
$$2g-2 = 2r'(m-1) + 2mr' - 4m\,, \quad mr \le r'\,, \tag{A.5}$$

The behavior of a multivalued tensor near the branch points α_i and β_j is studied performing the following change of variables:

$$t^m = z - \alpha_i\,, \qquad t'^2 = z - \beta_j\,. \tag{A.6}$$

t and t' are the so-called local uniformizers in α_i, β_j respectively. For example the behaviors of $y(z)$ and $p(z)$ at the branch points is:

$$y^{(l)}(z) \sim \begin{cases} \text{const} & 0 \le l \le m-1 \\ (z-\alpha_i)^{\frac{1}{m}} & m \le l \le 2m-1 \end{cases},$$
$$\sqrt{p(z)} \sim (z-\beta_j)^{1/2},\ 0 \le l \le 2m-1\,. \tag{A.7}$$

Using Eq. (A.7) we are able to write the behaviors at the branch points of the λ-differentials $B_k(z)\,dz^\lambda$.

It is possible to choose a basis of zero modes $\Omega_{i_k,k}(z)\,dz_\lambda$ in such a way that each element of the basis has the same monodromy properties of $B_k(z)\,dz^\lambda$ given in Eq. (3.11):

$$\Omega_{i_k,k}\,dz^\lambda = z^{i-1} B_k(z)\,dz^\lambda\,, \qquad 1 \le i_k \le N_{b_k}\,. \tag{A.8}$$

In order to determine the number of zero modes N_{b_k} we can use the following divisors. We denote with $a^{\nu}_{(l)}$ a zero of multiplicity ν and with $-a^{\nu}_{(l)}$ a pole of order ν occurring in the branch l of a meromorphic λ-differential. When $mr \leq r'$ we have:

$$\operatorname{div}[dz] = \sum_{i=1}^{2mr} \alpha_i + \sum_{l=0}^{m-1} \sum_{j=1}^{2r'} (\beta_j)_{(l)} - \sum_{i=0}^{2m-1} \infty^2_{(i)}\,, \tag{A.9a}$$

$$\operatorname{div}[y(z)] = \sum_{i=1}^{2mr} \alpha_i - \sum_{i=0}^{2m-1} \infty^r_{(i)}\,. \tag{A9b}$$

When $mr < r'$, Eqs. (A.9a) and (A.9b) become:

$$\operatorname{div}[dz] = \sum_{i=1}^{2r'} \alpha_i + \sum_{l=0}^{m-1} \sum_{j=1}^{2r'} (\beta_j)_{(l)} - \sum_{i=0}^{2m-1} \infty^2_{(i)}\,, \tag{A.10a}$$

$$\operatorname{div}[y(z)] = \sum_{i=1}^{2r'} \alpha_i - \sum_{i=0}^{2m-1} \infty^{\frac{r}{m}}_{(i)}\,. \tag{A.10.b}$$

In order to eliminate possible branches at infinity, r' should be a multiple of m. Studying the divisor of $\Omega_{i,k}(z)\,dz_\lambda$ it is also possible to prove that the total number of zero modes N_{b_k} with the same behavior at the branch points of $B_k(z)\,dz^\lambda$ is given by (here we suppose $\lambda > 0$):

$$N_{b_k} = 1 - 2\lambda - \sum_{l=0}^{2m-1} \sum_{i=1}^{N_\alpha} \frac{1}{2m} q_{k,\alpha_i}(l) - \sum_{j=1}^{N_\beta} q_{k,\beta_j}\,, \tag{A.11}$$

where $q_{k,\alpha_i}(l)$ is defined in Eqs. (3.14) and (3.15). Summing over k in Eq. (A.11) and using Eqs. (A.4)–(A.5) we get the total number of zero modes $N_b = (2\lambda - 1)(g-1)$. The $1-\lambda$ differentials have just a zero mode occurring when $\lambda = 1$:

$$1 = C_0(z)\,dz^0\,. \tag{A.12}$$

Therefore

$$N_{c_k} = \delta_{1\lambda}\,. \tag{A.13}$$

Appendix B

In this appendix we prove that the basis (3.11) satisfies Eq. (7.12). We consider only the tensors $B_k(z)\,dz^\lambda$ because the proof for the $1-\lambda$ differentials can be performed in a completely analogous way. First of all from Eq. (3.2) we have:

$$q^2(z) - p(z) = \prod_{i=1}^{N_\alpha} (z - \alpha_i)\,, \qquad p(z) = \prod_{j=1}^{N_\beta} (z - \beta_j)\,. \tag{B.1}$$

Solving Eq. (B.1) we get:

$$q(z) = \left(\prod_{i=1}^{N_\alpha} (z - \alpha_i) + \prod_{j=1}^{N_{\beta_j}} (z - \beta_j) \right)^{1/2}\,. \tag{B.2}$$

Equation (B.2) is useful because we can express in this way the polynomials $q(z)$ and $p(z)$ appearing in Eq. (3.1) in terms of the branch points. Of course not all the branch points α_i can be independent. From Eq. (B.1) they turn out to be functions of the other branch points β_j and of the zeros of $q(z)$. This dependence of the α_i on the other branch points is necessary because otherwise the polynomial $\prod_{i=1}^{N_\alpha}(z-\alpha_i)+\prod_{j=1}^{N_{\beta_j}}(z-\beta_j)$ has not quadratic zeros. This would be in contradiction with the fact that, from Eq. (B.2), this polynomial should be equal to $q^2(z)$. The fact that on the curves with nonabelian monodromy group the branch points are not completely independent, makes it difficult to study the properties of the twist fields and their OPE's under modular transformations [26]. For the same reason it is not possible to compute explicitly the matrix $\tilde{A}_{kk'}(z;\alpha_i,\beta_i)$ of Eq. (7.14) apart from its pole structure.

Now we consider the analytic tensor $B_k^{(l)}(z)\,dz^\lambda$ of Eq. (2.9) as a function $B_k^{(l)}(z)$ multiplied by the λ differential dz^λ. For the functions $B_k^{(l)}(z)$ we compute the ratio $(B_k^{(l)}(z))^{-1}(dB_k^{(l)}(z)/dz)$. The result is:

$$[B_k^{(l)}(z)]^{-1}\,\frac{dB_k^{(l)}(z)}{dz} = mq_{k,\alpha_i}y^{-1}\,\frac{dy}{dz} + q_{k,\beta_j}\sum_j \frac{1}{z-\beta_j}\,. \tag{B.3}$$

More explicitly, from Eq. (3.1) we have $y(z) = (q+\sqrt{p})^{\frac{1}{m}}$, and therefore

$$y^{-1}\,\frac{dy}{dz}\,dz = \frac{1}{m}\,\frac{dz}{q+\sqrt{p}}\left(q' + \frac{1}{2}\,\frac{p'}{\sqrt{p}}\right). \tag{B.4}$$

The above differential has a simple pole in $z=\infty$ which does not depend on the branches of $y(z)$. Moreover, with the aid of the divisors of Appendix A, it is clear that there are no poles when $z\to\beta_j$ in Eq. (B.4) despite of the fact that $\frac{1}{2}\frac{p'}{\sqrt{p}}$ diverges in $z=\beta_j$. However, in order to show that, we have to perform a change of coordinates in Eq. (B.4) switching to the local uniformizer (5.7). The reason is that the differential dz has exactly a zero in β_j which cancels this singularity. Unfortunately the differential equation (7.12) is not covariant under transformations of coordinate when λ-differentials are involved. Motivated by these difficulties in the approach of [12, 18] we have introduced an alternative procedure as explained in Sect. 2. Finally, exploiting Eqs. (B.1) and (B.2), it is possible to see that when $z\to\alpha_i$, the rhs of Eq. (7.13) has only a simple pole provided the branch l of $y(z)$ is in the interval $m\le l\le 2m-1$. No other singularities are possible. Therefore Eq. (B.3) describes a system of linear partial differential equations in which the 1-form matrix is

$$A_{kk'}(z) = mq_{k,\alpha_i}y^{-1}\,\frac{dy}{dz} + q_{k,\beta_j}\sum_j \frac{1}{z-\beta_j}\,. \tag{7.13}$$

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