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Subalgebras of Infinite C*-Algebras with Finite Watatani Indices
I. Cuntz Algebras

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Abstract. Using fusion rules of sectors as a working hypothesis, we construct endomorphisms of the Cuntz algebra $\mathcal{O}_n$ whose images have finite Watatani indices. Quasi-free KMS states on $\mathcal{O}_n$ appear in a natural way associated with the endomorphisms, and we determine the Murray–von Neumann–Connes types of their GNS representations.

1. Introduction

Index theory of operator algebras was initiated by V. Jones for II_1 factors, and extended by H. Kosaki for general factors [J, K]. It has many relations to other fields of mathematics and mathematical physics, and especially the relation to the theory of superselection sectors is striking [DHR, FRS, L1, L2]. In analogy with the case of quantum field theory, the notion of sectors of infinite factors was introduced by R. Longo [L2], and it turned out to be intrinsically significant in index theory [I1, I2, CK].

An attempt to extend index theory to C*-algebras was done by Y. Watatani [W]. He defined indices of conditional expectations in terms of quasi-basis, which is a generalization of the Pimsner–Popa basis [PP], and proved many analogous facts to the case of factors, such as the restriction of values of indices. Among other things, one of the most successful results of his theory is the existence of a close relation between K-theory and values of indices, in the case that an expectation preserves a trace. But for infinite C*-algebras such as the Cuntz algebras and the Cuntz–Krieger algebras, his theory gives little information. Up to now, known non-trivial examples of subalgebras with finite indices are separated into two groups. One consists of those with integer indices, which can be easily obtained by means of group actions. The other consists of those of AF algebras, which come from commuting squares.

One of the aims of this paper is to construct subalgebras of the Cuntz algebra $\mathcal{O}_n$ with finite indices, by using fusion rules of sectors [I1]. Many of our examples have non-integer indices, for example we shall construct a subalgebra of $\mathcal{O}_2$ with index $4\cos^2\frac{\pi}{5}$ (Example 3.1) and that of $\mathcal{O}_4$ with index $4\cos^2\frac{\pi}{12}$ (Subsect. 6.2). Associated with our construction, quasi-free KMS states of $\mathcal{O}_n$ appear, and using their GNS representations we shall construct pairs of type III$_\lambda$ ($< \lambda < 1$) factors.

The contents of this paper are as follows. In Sect. 2, we collect basic facts on the Cuntz algebras and Watatani index. Proposition 2.5 becomes a basic tool for our construction. In Sect. 3, we shall construct examples of endomorphisms of $\mathcal{O}_n$ whose images have finite indices. First we assume the existence of certain kinds of fusion rules of sectors, and from them we deduce the information of endomorphisms of $\mathcal{O}_n$. In Sect. 4, we shall investigate Murray–von Neumann–Connes types of quasi-free states of $\mathcal{O}_n$, and construct "good" representations for our examples. In Sect. 5, we shall argue the relation between our examples and A. Ocneanu theory. Our examples contain AF parts where our endomorphisms come from Ocneanu's connections. In Sect. 6, we shall compute principal graphs in a few examples.

Basic facts on index theory can be found in [GHJ, K], and we shall freely use the contents of them.

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2. Preliminaries

In this section, we shall collect basics of the Cuntz algebras and Watatani index to fix the notations.

2.1. The Cuntz Algebras. Let $\mathcal{O}_n$ be the Cuntz algebra generated by n ($n = 2, 3, \ldots < \infty$) isometries $S_1, S_2, \ldots, S_n$ [C1]. For a given k-tuple $\alpha = (j_1, j_2, \ldots, j_k)$, $j_i \in \{1, 2, \ldots, n\}$, we denote by $l(\alpha) = k$ the length of α and $f(\alpha) = j_k$ the last element of α. We define the isometry S_α by $S_\alpha = S_{j_1} S_{j_2} \ldots S_{j_k}$. Let ${\lambda^1}_t \in \mathrm{Aut}(\mathcal{O}_n)$, $t \in \mathbf{R}$ be the usual gauge action on $\mathcal{O}_n$ defined by ${\lambda^1}_t(S_j) = e^{\sqrt{-1}t} S_j$ ($j = 1, 2, \ldots, n$). Then the fixed point algebra of $\mathcal{O}_n$ under λ^1 is isomorphic to the UHF algebra of type n^∞. We denote it by $\mathcal{F}^n$ and define a conditional expectation $F : \mathcal{O}_n \to \mathcal{F}^n$ by

$$F(x) = \frac{1}{2\pi} \int_0^{2\pi} {\lambda^1}_t(x)\, dt \quad x \in \mathcal{O}_n .$$

We recall Evans' work on KMS states on $\mathcal{O}_n$ [E]. Let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ be a n-tuple of positive numbers and β the positive number determined by $\sum_{j=1}^n e^{-\beta\omega_j} = 1$. We define an $\mathbf{R}$ action λ^ω and a state φ^ω by $\lambda_t^\omega(S_j) = e^{\sqrt{-1}\omega_j t} S_j$ $t \in \mathbf{R}$ ($j = 1, 2, \ldots, n$), $\varphi^\omega(x) = \psi^\omega \cdot F(x)$, where ψ^ω is the product state on $\mathcal{F}^n$ with the uniform density

$$\begin{pmatrix} e^{-\beta\omega_1} & 0 & \cdots & 0 \\ 0 & e^{-\beta\omega_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{-\beta\omega_n} \end{pmatrix}.$$

Then the relation between λ^ω and φ^ω is as follows.

Proposition 2.1 ([E, Proposition 2.2.]). *φ^ω is the unique* KMS *state for λ^ω and the corresponding inverse temperature is β.*

In Sect. 5 we shall investigate the fixed point algebra $\mathcal{O}_n{}^{\lambda^\omega}$ of $\mathcal{O}_n$ under λ^ω in some special cases. For this purpose the following weighted length of k-tuple $\alpha = (j_1, j_2, \ldots, j_k)$ is useful:

$$l^\omega(\alpha) \equiv \sum_{l=1}^{k} \omega_{j_l} .$$

Let H be the linear span of $\{S_1, S_2, \ldots, S_n\}$, that is a Hilbert space with the inner product $(S, T)1 = T^*S$, $S, T \in H$. We denote by (H^s, H^t), $s, t \in \mathbf{Z}_+$ the linear span of $H^t H^{*s}$ and define the following as in [DR]:

$$^0\mathcal{O}_n^k \equiv \bigcup_{r, k+r \geqq 0} (H^r, H^{r+k}) ,$$

$$^0\mathcal{O}_n \equiv \text{linear span of } \bigcup_{k \in \mathbf{Z}} {}^0\mathcal{O}_n^k .$$

$^0\mathcal{O}_n$ is the $*$-algebra generated by $\{S_1, S_2, \ldots, S_n\}$ and hence norm dense. We define the permutation operator $\theta(r, 1)$ by

$$\theta \equiv \sum_{i,j} S_i S_j S_i^* S_j^*, \quad \theta(r, 1) \equiv \theta\sigma(\theta)\sigma^2(\theta) \ldots \sigma^{r-1}(\theta) ,$$

where σ is the canonical endomorphism of $\mathcal{O}_n$ defined by $\sigma(x) = \sum_i S_i x S_i^*$, $x \in \mathcal{O}_n$. Then we have the following.

Proposition 2.2 ([DR, Sect. 2]).

(1) $\theta(r, 1)^* S_i = \sigma^r(S_i)$.
(2) $\theta(r, 1)R = \sigma(R)\theta(s, 1)$, $R \in (H^s, H^r)$.
(3) *If* $R \in \mathcal{O}_n$ *satisfies* $\lambda_t^1(R) = e^{\sqrt{-1}kt}R$, $\sigma(R) = \lim_{r\to\infty} \theta(r + k, 1)R\theta(r, 1)^*$.

2.2. Watatani Index. Extension of Jones index to C*-algebras was argued by Y. Watatani in terms of quasi-basis [W]. In this paper we adopt his definition of index.

Definition 2.3. *Let $A \supset B$ be a pair of C*-algebras and $E: A \to B$ a conditional expectation. A finite family $\{(u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n)\}$ is called quasi-basis if the following equations hold:*

$$x = \sum_i u_i E(v_i x) = \sum_i E(x u_i) v_i, \quad x \in A .$$

We say that a conditional expectation $E: A \to B$ is of index-finite type if there exists a quasi-basis for E. In this case we define the index of E by

$$\text{Index}\, E \equiv \sum_i u_i v_i .$$

Remark 2.4. Index E belongs to the centre of A and does not depend on the choice of quasi-basis.

For two endomorphisms ρ_1, ρ_2 of a C* or W*-algebra A, we denote by (ρ_1, ρ_2) the set of intertwiners between ρ_1 and ρ_2, i.e.

$$(\rho_1, \rho_2) = \{R \in A; R\rho_1(x) = \rho_2(x)R,\ x \in A\}\ .$$

The following proposition is a key to our construction in Sect. 3. Our idea is taken from Longo's work [L2].

Proposition 2.5. *Let A, B be C*-algebras and $\rho: B \to A$, $\bar{\rho}: A \to B$ (not necessarily onto) unital isomorphisms. We assume that there exist isometries $V \in (\mathrm{id}_A, \rho\bar{\rho}) \subset A$, $W \in (\mathrm{id}_B, \bar{\rho}\rho) \subset B$ satisfying $V^*\rho(W) = \frac{1}{d}$, $W^*\bar{\rho}(V) = \frac{1}{d}$, $d > 0$. Let $E_\rho: A \to \rho(B)$, $E_{\bar{\rho}}: B \to \bar{\rho}(A)$ the positive maps defined by*

$$E_\rho(x) = \rho(W^*\bar{\rho}(x)W) \quad x \in A, \quad E_{\bar{\rho}}(y) = \bar{\rho}(V^*\rho(y)V) \quad y \in B\ .$$

Then,

(1) *E_ρ and $E_{\bar{\rho}}$ are conditional expectations.*
(2) *$\{(d \cdot V^*, d \cdot V)\}$ and $\{(d \cdot W^*, d \cdot W)\}$ are quasi-basis for E_ρ and $E_{\bar{\rho}}$ with indices d^2.*

Proof. By direct computation.

In "self-conjugate" case, i.e. assuming $A = B$, $\rho = \bar{\rho}$ and $V^*\rho(V) = c \in \mathbf{C}\backslash\{0\}$, we have the following by using $Vx = \rho^2(x)V$:

$$cV = \rho(V^*\rho(V))V = \rho(V^*)VV = \bar{c}V\ .$$

So we obtain $c \in \mathbf{R}\backslash\{0\}$ and we can take W such that $W = \pm V$. According to R. Longo [L2], we call ρ a real sector if $W = V$, i.e. $V^*\rho(V) = \frac{1}{d}$ and a pseudo-real sector if $W = -V$ i.e. $V^*\rho(V) = -\frac{1}{d}$. Every example we shall construct in Sect. 3 is a real sector.

Before closing this section, we shall prove the following technical lemma, which is helpful for checking the assumption of Proposition 2.5 in concrete examples.

Lemma 2.6. *Let ν be a unital endomorphism of $\mathcal{O}_n$. We fix $i \in \{1, 2, \ldots, n\}$ and put $T_j \equiv S_i^*\nu(S_j)S_i, j \in \{1, 2, \ldots, n\}$. If $\{T_j\}_{1 \leq j \leq n}$ satisfy the Cuntz algebra relations, i.e. $T_j^*T_k = \delta_{j,k}$, $\sum_j T_jT_j^* = 1$, then $S_i^*\nu(x)S_k = 0$, $k \neq i$, $x \in \mathcal{O}_n$ holds. In consequence, $\tilde{\nu}(x) \equiv S_i^*\nu(x)S_i$ is an endomorphism, and $S_i \in (\tilde{\nu}, \nu)$.*

Proof. By assumption we obtain

$$\sum_j T_jT_j^* = \sum_j S_i^*\nu(S_j)S_iS_i^*\nu(S_j^*)S_i = 1, \quad T_j^*T_j = S_i^*\nu(S_j^*)S_iS_i^*\nu(S_j)S_i = 1\ .$$

On the other hand, in general we have the following.

$$\sum_{j,k} S_i^*\nu(S_j)S_kS_k^*\nu(S_j^*)S_i = 1, \quad \sum_k S_i^*\nu(S_j^*)S_kS_k^*\nu(S_j)S_i = 1\ .$$

So we get $S_i^* v(S_j)S_k = S_k^* v(S_j)S_i = 0$, $k \neq i$. By using this and induction of word length, we obtain the first statement and $\tilde{v} \in \mathrm{End}(\mathcal{O}_n)$. The second statement holds as follows:

$$S_i\tilde{v}(x) = S_iS_i^* v(x)S_i = \sum_j S_jS_j^* v(x)S_i = v(x)S_i \,.$$

Q.E.D.

3. Construction of Examples

In this section we shall construct examples of endomorphisms of $\mathcal{O}_n$ satisfying the assumption of Proposition 2.5. We start with inclusions of factors and corresponding fusion rules of sectors, which we regard as a working hypothesis. In what follows, we shall use the notations in [L2, I1] for sector theory.

Example 3.1. Let $M \supset N$ be a pair of properly infinite factors with finite index and the principal graph A_4. Then there exists ρ an endomorphism of M satisfying the following. (See [I1, Proposition 3.2] [I2, Proposition 2.4].)

$$\rho(M) = N, \quad [\rho^2] = [\mathrm{id}] \oplus [\rho] \,.$$

The second equation means that there exist isometries S_1, S_2 which generate $\mathcal{O}_2$, and satisfy

$$S_1 x = \rho^2(x)S_1 \quad x \in M \,, \tag{3.1.1}$$

$$S_2\rho(x) = \rho^2(x)S_2 \quad x \in M \,, \tag{3.1.2}$$

i.e. $S_1 \in (\mathrm{id}, \rho^2)$, $S_2 \in (\rho, \rho^2)$. Note that $\dim(\mathrm{id}, \rho^2) = \dim(\rho, \rho^2) = 1$. Since ρ is self-conjugate we have $S_1^*\rho(S_1) = \pm\frac{1}{d}$, $d = 2\cos\frac{\pi}{5}$ due to [L2, Sect. 5]. From (3.1.1), (3.1.2) we obtain

$$S_2^*\rho(S_1 x) = S_2^*\rho(\rho^2(x)S_1) = \rho^2(x)S_2^*\rho(S_1) \,.$$

So we have $S_2^*\rho(S_1) \in (\rho, \rho^2)$ and hence $S_2^*\rho(S_1) = cS_2$, $c \in \mathbf{C}$. Therefore we get the following:

$$\rho(S_1) = (S_1S_1^* + S_2S_2^*)\rho(S_1) = \pm\frac{1}{d}S_1 + cS_2S_2 \,.$$

Changing the relative phase between S_1 and S_2 if necessary, we may assume that c is non-negative. So we obtain the following by using $\frac{1}{d^2} + \frac{1}{d} = 1$,

$$\rho(S_1) = \pm\frac{1}{d}S_1 + \frac{1}{\sqrt{d}}S_2S_2 \,.$$

In the same way, we have $S_1^*\rho(S_2) \in (\rho^2, \rho)$, $S_2^*\rho(S_2) \in (\rho^2, \rho^2)$. Due to $(\rho^2, \rho) = (\rho, \rho^2)^*$, $(\rho^2, \rho^2) = \mathbf{C}S_1S_1^* + \mathbf{C}S_2S_2^*$ and the Cuntz algebra relations of $\rho(S_1)$, $\rho(S_2)$, we obtain

$$\rho(S_2) = p\left(\frac{1}{\sqrt{d}}S_1 - \frac{1}{d}S_2S_2\right)S_2^* + qS_2S_1S_1^*, \quad p, q \in \mathbf{T} \,.$$

Using (3.1.1), (3.1.2) and computing $\rho^2(S_1)$, $\rho^2(S_2)$, we conclude

$$\rho(S_1) = \frac{1}{d}S_1 + \frac{1}{\sqrt{d}}S_2S_2 , \tag{3.1.3}$$

$$\rho(S_2) = \left(\frac{1}{\sqrt{d}}S_1 - \frac{1}{d}S_2S_2\right)S_2^* + S_2S_1S_1^* . \tag{3.1.4}$$

Now let us forget about the inclusion of factors and consider (3.1.3), (3.1.4) to be the *definition* of $\rho \in \mathrm{End}(\mathcal{O}_2)$. Then ρ *satisfies* (3.1.1), (3.1.2) for $x \in \mathcal{O}_2$, and consequently the assumption of Proposition 2.5 holds with $V = S_1$. So we have $\mathrm{Index}\, E_\rho = 4\cos^2\frac{\pi}{5}$ for $E_\rho(x) \equiv \rho(S_1^*\rho(x)S_1)$. Of course ρ does not commute with λ^1 but commutes with λ^ω, $\omega = (2, 1)$. Thanks to (3.1.1), (3.1.2), the following holds:

$$\rho^k(S_1S_1^*) = S_1\rho^{k-2}(S_1S_1^*)S_1^* + S_2\rho^{k-1}(S_1S_1^*)S_2^*, \quad k \geqq 2 . \tag{3.1.5}$$

We shall use this in Sect. 5.

Example 3.2. We start with a pair of factors whose principal graph is $D_5^{(1)}$ [IK, Sect. 4]. In a similar way as in the case of A_4, we obtain the following fusion rules of sectors:

$$[\rho^2] = [\mathrm{id}] \oplus [\alpha] \oplus [\rho], \quad [\alpha][\rho] = [\rho][\alpha] = [\rho], \quad [\alpha^2] = [\mathrm{id}] . \tag{3.2.1}$$

In the same way as in the proof of [I1, Proposition 3.3], we can lift ρ, and α such that

$$\alpha \cdot \rho = \rho, \quad \rho \cdot \alpha = \mathrm{Ad}(U) \cdot \rho , \tag{3.2.2}$$

where U is a unitary in (ρ^2, ρ^2) with order 2. Equation (3.2.1) shows that there exist isometries S_1, S_2, S_3 in the factor which generate $\mathcal{O}_3$ and satisfy

$$S_1 \in (\mathrm{id}, \rho^2), \quad S_2 \in (\alpha, \rho^2), \quad S_3 \in (\rho, \rho^2) .$$

From (3.2.2) we obtain

$$\alpha((\mathrm{id}, \rho^2)) = (\alpha, \rho^2), \quad \alpha((\alpha, \rho^2)) = (\mathrm{id}, \rho^2), \quad \alpha((\rho, \rho^2)) = (\rho, \rho^2) .$$

So we may assume the following by changing the relative phase between S_1 and S_2 if necessary.

$$\alpha(S_1) = S_2, \quad \alpha(S_2) = S_1, \quad \alpha(S_3) = \varepsilon_1 S_3, \quad \varepsilon_1 \in \{1, -1\} .$$

In the same way as in Example 3.1, we may assume the following due to $\alpha \cdot \rho = \rho$.

$$\rho(S_1) = \pm\frac{S_1 + S_2}{2} + \frac{S_3S_3}{\sqrt{2}} .$$

From $U \in (\rho^2, \rho^2) = \mathbf{C}S_1S_1^* + \mathbf{C}S_2S_2^* + \mathbf{C}S_3S_3^*$ and $U^2 = 1$, we may assume

$$U = S_1S_1^* + \varepsilon_2S_2S_2^* + \varepsilon_3S_3S_3^*, \quad \varepsilon_2, \varepsilon_3 \in \{1, -1\} .$$

So we have

$$\rho(S_2) = \rho\cdot\alpha(S_1) = U\rho(S_1)U = \left(\pm\frac{S_1 + \varepsilon_2 S_2}{2} + \varepsilon_3\frac{S_3S_3}{\sqrt{2}}\right)U .$$

By the orthogonality of $\rho(S_1)$ and $\rho(S_2)$, we obtain $\varepsilon_2 = 1$, $\varepsilon_3 = -1$. Using

$$S_1\rho(S_3),\ S_2^*\rho(S_3)\in(\rho^2, \rho),\quad S_3^*\rho(S_3)\in(\rho^2, \rho^2) ,$$

$\alpha\cdot\rho = \rho$, and the Cuntz algebra relations of $\rho(S_1)$, $\rho(S_2)$, $\rho(S_3)$, we obtain $\varepsilon_1 = -1$ and

$$\rho(S_3) = \eta_1\frac{S_1 - S_2}{\sqrt{2}}S_3^* + \eta_2 S_3(S_1S_1^* - S_2S_2^*) \quad \eta_1, \eta_2\in\mathbf{T} .$$

Computing $\rho^2(S_1)$, $\rho^2(S_2)$, $\rho^2(S_3)$ we have the following three solutions:

$$\rho_a(S_1) = \frac{S_1 + S_2}{2} + \frac{S_3S_3}{\sqrt{2}} , \tag{3.2.3}$$

$$\rho_a(S_2) = \left(\frac{S_1 + S_2}{2} - \frac{S_3S_3}{\sqrt{2}}\right)U , \tag{3.24}$$

$$\rho_a(S_3) = \bar{a}\frac{S_1 - S_2}{\sqrt{2}}S_3^* + aS_3(S_1S_1^* - S_2S_2^*) , \tag{3.2.5}$$

$$\alpha(S_1) = S_2,\quad \alpha(S_2) = S_1,\quad \alpha(S_3) = -S_3 , \tag{3.2.6}$$

where $U = S_1S_1^* + S_2S_2^* - S_3S_3^*$ and $a\in\mathbf{T}$ with $a^3 = 1$. Note that the above ρ_a makes sense for any $a\in\mathbf{T}$ as an endomorphism of $\mathcal{O}_3$. So we forget about the inclusion of factors again, and *define* $\rho_a\in\mathrm{End}(\mathcal{O}_3)$, $a\in\mathbf{T}$ by (3.2.3)–(3.2.5). It is easy to show (3.2.2). By direct computation using Lemma 2.6, we can show the following:

$$S_1\in(\mathrm{id}, \rho_a^2),\quad S_2\in(\alpha, \rho_a^2),\quad S_3\in(\rho_{\bar{a}^2}, \rho_a^2) . \tag{3.2.7}$$

ρ_a satisfies the assumption of Proposition 2.5 with $V = S_1$, and we obtain $\mathrm{Index}\, E_{\rho_a} = 4$ for $E_{\rho_a}(x) = \rho_a(S_1^*\rho_a(x)S_1)$. Let $\omega = (2, 2, 1)$. Then ρ_a commutes with λ^ω. By induction one can show

$$\rho_a^k(S_1S_1^*) = S_1\rho_a^{k-2}(S_1S_1^*)S_1^* + S_2\alpha\cdot\rho_a^{k-2}(S_1S_1^*)S_2^* + S_3\rho_a^{k-1}(S_1S_1^*)S_3^*,\quad k\geqq 2 . \tag{3.2.8}$$

We can generalize Example 3.2 as follows.

Example 3.3. Let G be a finite abelian group with order N, and $\hat{G}$ the dual group of G. We put $n = 2N - 1$ and write $\langle g, \sigma\rangle = \sigma(g)$, $g\in G$, $\sigma\in\hat{G}$. Let us consider $\mathcal{O}_n$ whose generators are $\{S_g, T_\sigma\}_{g\in G, \sigma\in\hat{G}\setminus\{e\}}$. We define $\rho_a\in\mathrm{End}(\mathcal{O}_n)$ $(a\in\mathbf{T})$, an action of $G\alpha$, and unitary representations of G in $\mathcal{O}_n U, U_\sigma$ ($\sigma \in \hat{G} \setminus \{e\}$) as follows:

$$\alpha_g(S_h) = S_{g+h}, \quad \alpha_g(T_\sigma) = \langle g, \sigma \rangle T_\sigma , \tag{3.3.1}$$

$$U(g) = \sum_{h \in G} S_h S_h^* + \sum_{\tau \in \hat{G}}' \overline{\langle g, \tau \rangle} T_\tau T_\tau^* \left(\sum_{\tau \in \hat{G}}' \equiv \sum_{\tau \in \hat{G} \setminus \{e\}} \right), \tag{3.3.2}$$

$$U_\sigma(g) = \langle g, \sigma \rangle \sum_{h \in G} S_h S_h^* + T_{-\sigma} T_{-\sigma}^* + \sum_{\tau \neq -\sigma}' \overline{\langle g, \tau \rangle} T_\tau T_\tau^* , \tag{3.3.4}$$

$$\rho_a(S_g) = \left(\frac{1}{N} \sum_h S_h + \frac{1}{\sqrt{N}} \sum_\tau' \overline{\langle g, \tau \rangle} T_\tau T_{-\tau} \right) U(g)^* , \tag{3.3.5}$$

$$\rho_a(T_\sigma) = \frac{\bar{a}}{\sqrt{N}} \left(\sum_g \overline{\langle g, \sigma \rangle} S_g \right) T_\sigma^* + a T_{-\sigma} \left(\sum_g \overline{\langle g, \sigma \rangle} S_g S_g^* \right)$$

$$+ \sum_{\tau \neq -\sigma}' T_\tau T_\sigma T_{\tau+\sigma}^* . \tag{3.3.6}$$

It is easy to see

$$\alpha_g \cdot \rho_a = \rho_a, \quad \rho_a \cdot \alpha_g = \mathrm{Ad}(U(g)) \rho_a . \tag{3.3.7}$$

Direct computation shows

$$S_e^* \rho_a^2(S_e) S_e = S_e, \quad S_e^* \rho_a^2(T_\sigma) S_e = T_\sigma ,$$

$$\rho_a(U(g)) = \sum_h S_h S_{h+g}^* + \sum_\tau' T_\tau U_\tau(g) T_\tau^* .$$

So we obtain

$$S_e^* \rho_a^2(S_g) S_e = S_e^* \rho_a(U(g)) \rho_a^2(S_e) \rho_a(U(g)^*) S_e = S_g^* \rho_a^2(S_e) S_g$$

$$= \alpha_g(S_e^* \rho_a^2(S_e) S_e) = \alpha_g(S_e) = S_g .$$

Thanks to Lemma 2.6 and (3.3.7), we get

$$S_g \in (\alpha_g, \rho_a^2) . \tag{3.3.8}$$

Therefore ρ_a and S_e satisfy the assumption of Proposition 2.5, and we have Index $E_{\rho_a} = N^2$ for $E_{\rho_a}(x) = \rho_a(S_e^* \rho_a(x) S_e)$. Let $\omega = (\overbrace{2, 2, \ldots, 2}^{N \text{ times}}, \overbrace{1, \ldots, 1}^{N-1 \text{ times}})$. Then ρ_a commutes with λ^ω. By induction, one can show the following:

$$\rho_a^k(S_e S_e^*) = \sum_g S_g \alpha_g \cdot \rho^{k-2}(S_e S_e^*) S_g^* + \sum_\tau' T_\tau \rho_a^{k-1}(S_e S_e^*) T_\tau^*, \quad k \geqq 2 . \tag{3.3.9}$$

Example 3.4. Let us start with the following fusion rules, which appeared in [I1, (3.3.4)],

$$[\rho^2] = [\mathrm{id}] \oplus [\alpha] \oplus 2[\rho], \quad [\alpha \cdot \rho] = [\rho \cdot \alpha] = [\rho], \quad [\alpha^2] = [\mathrm{id}] .$$

Then, in a similar way as above, we can obtain the following endomorphisms of $\mathcal{O}_4$,

$$\alpha(S_1) = S_2, \quad \alpha(S_2) = S_1, \quad \alpha(S_3) = S_3, \quad \alpha(S_4) = -S_4\,, \tag{3.4.1}$$

$$U = S_1S_1^* - S_2S_2^* + S_3S_4^* + S_4S_3^*\,, \tag{3.4.2}$$

$$\rho_\pm(S_1) = \frac{S_1 + S_2}{d} + \frac{e^{\pm\frac{\pi}{4}\sqrt{-1}}S_3^2 + e^{\mp\frac{\pi}{4}\sqrt{-1}}S_4^2}{\sqrt{d}}\,, \tag{3.4.3}$$

$$\rho_\pm(S_2) = \left[\frac{S_1 - S_2}{d} + \frac{e^{\pm\frac{\pi}{4}\sqrt{-1}}S_4S_3 + e^{\mp\frac{\pi}{4}\sqrt{-1}}S_3S_4}{\sqrt{d}}\right]U\,, \tag{3.4.4}$$

$$\begin{aligned}\rho_\pm(S_3) = c_1&\left[\frac{S_1 + S_2}{\sqrt{2}}S_3^* + \frac{S_1 - S_2}{\sqrt{2}}S_4^*\right]\\ &+ c_2[S_3(S_1S_1^* + S_2S_2^*) + S_4(S_1S_1^* - S_2S_2^*)]\\ &+ c_3[S_3S_3S_3^* + S_4S_3S_4^*] + c_4[S_3S_4S_4^* + S_4S_4S_3^*]\,,\end{aligned} \tag{3.4.5}$$

$$\begin{aligned}\rho_\pm(S_4) = c_1&\left[\frac{S_1 + S_2}{\sqrt{2}}S_3^* - \frac{S_1 - S_2}{\sqrt{2}}S_4^*\right]\\ &\pm\sqrt{-1}c_2[S_3(S_1S_1^* + S_2S_2^*) - S_4(S_1S_1^* - S_2S_2^*)]\\ &+ \pm\sqrt{-1}c_4[S_3S_3S_3^* - S_4S_3S_4^*] + \pm\sqrt{-1}c_3[S_3S_4S_4^* - S_4S_4S_3^*]\,,\end{aligned} \tag{3.4.6}$$

where

$$d = 1 + \sqrt{3}, \quad c_1 = \frac{e^{\mp\frac{5}{6}\pi\sqrt{-1}}}{\sqrt{d}}, \quad c_2 = \frac{e^{\pm\frac{7\pi}{12}\sqrt{-1}}}{\sqrt{2}}, \quad c_3 = -\frac{1}{d}, \quad c_4 = \frac{e^{\mp\frac{\pi}{4}\sqrt{-1}}}{\sqrt{2}}\,.$$

It is easy to show

$$\alpha\cdot\rho_\pm = \rho_\pm\,, \quad \rho_\pm\cdot\alpha = \mathrm{Ad}(U)\cdot\rho_\pm\,. \tag{3.4.7}$$

By direct computation we have the following:

$$\rho_\pm(U) = S_1S_2^* + S_2S_1^* \pm\sqrt{-1}(S_3US_4^* - S_4US_3^*)\,, \tag{3.4.8}$$

$$S_1^*\rho_\pm^2(S_1)S_1 = S_1, \quad S_1^*\rho_\pm^2(S_3)S_1 = S_3, \quad S_1^*\rho_\pm^2(S_4)S_1 = S_4\,.$$

Due to (3.4.8) we obtain

$$\begin{aligned}S_1^*\rho_\pm^2(S_2)S_1 &= S_1^*\rho_\pm(U)\rho_\pm^2(S_1)\rho_\pm(U)S_1 = S_2^*\rho_\pm^2(S_1)S_2\\ &= \alpha(S_1^*\rho_\pm^2(S_1)S_1) = \alpha(S_1) = S_2\,.\end{aligned}$$

Hence, by Lemma 2.6 and (3.4.7) we get the following:

$$S_1\in(\mathrm{id}, \rho_\pm^2), \quad S_2\in(\alpha, \rho_\pm^2)\,. \tag{3.4.9}$$

Let $\hat{T}_{\pm} \equiv \frac{S_3 \pm S_4}{\sqrt{2}}$. Then by direct computation, one can show $\hat{T}^*_{\mp}\rho^2_{\pm}(x)\hat{T}_+ = \rho_{\pm}(x)$. So using $\alpha(\hat{T}_+) = \hat{T}_-$, we obtain the following:

$$S_3, S_4 \in (\rho_{\pm}, \rho^2_{\pm}) . \tag{3.4.10}$$

If we require only (3.4.7) and (3.4.9), we can construct other endomorphisms $\hat{\rho}_{\pm}$ by replacing c_i with $\hat{c}_i$, $i = 1, 2, 3, 4$,

$$\hat{c}_1 = \frac{e^{\mp\frac{5\pi}{12}\sqrt{-1}}}{\sqrt{d}}, \quad \hat{c}_2 = \frac{e^{\pm\frac{\pi}{6}\sqrt{-1}}}{\sqrt{2}}, \quad \hat{c}_3 = \frac{-1}{\sqrt{2}}, \quad \hat{c}_4 = \frac{e^{\mp\frac{\pi}{4}\sqrt{-1}}}{d} .$$

In a similar way as above, one can show the following:

$$S_1 \in (\mathrm{id}, \hat{\rho}^2_{\pm}), \quad S_2 \in (\alpha, \hat{\rho}^2_{\pm}), \quad \hat{T}_+ \in (\tilde{\rho}_{\mp}, \hat{\rho}^2_{\pm}), \quad \hat{T}_- \in (\alpha \cdot \tilde{\rho}_{\mp}, \hat{\rho}^2_{\pm}) , \tag{3.4.11}$$

where $\tilde{\rho}_{\mp} = \theta \cdot \rho_{\mp} \cdot \theta^{-1}$, and $\theta \in \mathrm{Aut}(\mathcal{O}_n)$ is the flip of S_3 and S_4,

$$\theta(S_1) = S_1, \quad \theta(S_2) = S_2, \quad \theta(S_3) = S_4, \quad \theta(S_4) = S_3 .$$

$\rho_{\pm}$ and $\hat{\rho}_{\pm}$ satisfy the assumption of Proposition 2.5 with $S_1 = V$, and we have $\mathrm{Index}\, E_{\rho_{\pm}} = \mathrm{Index}\, E_{\hat{\rho}_{\pm}} = 4 + 2\sqrt{3}$, for $E_{\rho_{\pm}}(x) = \rho_{\pm}(S_1^* \rho_{\pm}(x) S_1)$, $E_{\hat{\rho}_{\pm}}(x) = \hat{\rho}_{\pm}(S_1^* \hat{\rho}_{\pm}(x) S_1)$. Let $\omega = (2, 2, 1, 1)$. Then $\rho_{\pm}$ and $\hat{\rho}_{\pm}$ commute with λ^{ω}. By induction, one can show the following for $\rho = \rho_{\pm}, \hat{\rho}_{\pm}$, $k \geqq 2$.

$$\begin{aligned} \rho^k(S_1S_1^*) = {} & S_1\rho^{k-2}(S_1S_1^*)S_1^* + S_2\alpha \cdot \rho^{k-2}(S_1S_1^*)S_2^* \\ & + S_3\rho^{k-1}(S_1S_1^*)S_3^* + S_4\rho^{k-1}(S_1S_1^*)S_4^* . \end{aligned} \tag{3.4.12}$$

We can generalize Example 3.4 as follows.

Example 3.5. Let G be a finite abelian group with order N. Since any finite abelian group is isomorphic to its dual group, we fix an identification and a dual pairing $\langle , \rangle : G \times G \to \mathbf{T}$. We assume $\langle g, h \rangle = \langle h, g \rangle$, $g, h \in G$. (Such a pairing always exists.) Let us consider functions on G, $a: G \to \mathbf{T}$, $b: G \to \mathbf{C}$, and complex number $c \in \mathbf{T}$ satisfying the following equations:

$$a(0) = 1, \quad a(g) = a(-g), \quad a(g+h)\langle g, h \rangle = a(g)a(h) , \tag{3.5.1}$$

$$a(g)b(-g) = \overline{b(g)} , \tag{3.5.2}$$

$$\frac{c\sqrt{N}}{d} + \sum_g b(g) = 0 , \tag{3.5.3}$$

$$\frac{1}{d} + \sum_g b(g+h)\overline{b(g)} = \delta_{h,e} . \tag{3.5.4}$$

In the above equations $d = \frac{N + \sqrt{N^2 + 4N}}{2}$, which satisfies $d^2 = Nd + N$. We put $n \equiv 2N$, and consider the Cuntz algebra $\mathcal{O}_n$ with the generators $\{S_g, T_g\}_{g \in G}$. We define $\rho \in \mathrm{End}(\mathcal{O}_n)$, a G action α, and a unitary representation of G in $\mathcal{O}_n U$ as follows:

$$\alpha_g(S_h) = S_{g+h}, \quad \alpha_g(T_h) = \langle g, h\rangle T_h \,, \tag{3.5.5}$$

$$U(g) = \sum_h \langle h, g\rangle S_h S_h^* + \sum_h T_{h-g} T_h^* \,, \tag{3.5.6}$$

$$\rho(S_g) = \left[\frac{1}{d}\sum_h \langle h, g\rangle S_h + \frac{1}{\sqrt{d}}\sum_h a(h) T_{h-g} T_{-h}\right] U(g)^* \,, \tag{3.5.7}$$

$$\begin{aligned}\rho(T_g) = {} & \frac{c}{\sqrt{Nd}}\sum_{h,k} \langle k, g\rangle \overline{\langle h, k\rangle} S_h T_k^* \\ & + \frac{\overline{a(g)c}}{\sqrt{N}}\sum_{h,k} \langle h, g\rangle \langle k, h\rangle T_h S_k S_k^* \\ & + \sum_{h,k} a(h) b(g+h) \langle k, g\rangle T_{h+k} T_{-h} T_k^* \,. \end{aligned} \tag{3.5.8}$$

Thanks to (3.5.3), (3.5.4) ρ is well-defined. It is easy to see

$$\alpha_g \cdot \rho = \rho, \quad \mathrm{Ad}(U(g)) \cdot \rho = \rho \cdot \alpha_g \,. \tag{3.5.9}$$

By direct computation as in Example 3.4, we can show the following:

$$\rho(U(g)) = \sum_h S_h S_{g+h}^* + a(g) \sum_h \overline{\langle h, g\rangle} T_h U(g) T_{h-g}^* \,, \tag{3.5.10}$$

$$S_g \in (\alpha_g, \rho^2) \,. \tag{3.5.11}$$

So we have $\mathrm{Index}\, E_\rho = \dfrac{N(N+2+\sqrt{N^2+4N})}{2}$ for $E_\rho(x) = \rho(S_e^* \rho(x) S_e)$. ρ commutes with λ^ω, $\omega = (\overbrace{2, 2, \dots, 2}^{N \text{ times}}, \overbrace{1, 1, \dots, 1}^{N \text{ times}})$. As in the previous cases, the following holds:

$$\rho^k(S_e S_e^*) = \sum_g S_g \alpha_g \cdot \rho^{k-2}(S_e S_e^*) S_g^* + \sum_g T_g \rho^{k-1}(S_e S_e^*) T_g^*, \quad k \geqq 2 \,. \tag{3.5.12}$$

For groups with small order such as $\mathbf{Z}_2$, $\mathbf{Z}_3$, $\mathbf{Z}_4$, $\mathbf{Z}_2 \times \mathbf{Z}_2$, one can explicitly obtain the solutions of (3.5.1)–(3.5.4). There are eight solutions in the case of $\mathbf{Z}_2$, which correspond to $\rho_\pm$, $\lambda_\pi^\omega \cdot \rho_\pm$, $\hat{\rho}_\pm$, $\lambda_\pi^\omega \cdot \hat{\rho}_\pm$ in Example 3.4, up to the change of the relative phase between $\{S_g\}$ and $\{T_g\}$.

Example 3.6. We start with a pair of infinite factors with the principal graph A_7. Then we have the following fusion rules:

$$[\rho_2^2] = [\mathrm{id}] \oplus [\rho_2] \oplus [\alpha \cdot \rho_2], \quad [\alpha \cdot \rho_2] = [\rho_2 \cdot \alpha], \quad \alpha^2 = \mathrm{id} \,,$$

where we use the notations in [I1, Proposition 3.3]. Due to the second equality, we have a unitary U satisfying $\mathrm{Ad}(U) \cdot \alpha \cdot \rho_2 = \rho_2 \cdot \alpha$. Using $\alpha^2 = \mathrm{id}$ and irreducibility of ρ_2, we may assume $U\alpha(U) = 1$. So U is a α-cocycle. Since any outer action of a finite group on a factor is stable [Co], there exists a unitary U_0 satisfying $U_0^*\alpha(U_0) = U$. Hence we have $\alpha\cdot\mathrm{Ad}(U_0)\cdot\rho_2 = \mathrm{Ad}(U_0)\cdot\rho_2\cdot\alpha$. Let $\rho \equiv \mathrm{Ad}(U_0)\cdot\rho_2$. Then we get the following:

$$[\rho^2] = [\mathrm{id}] \oplus [\rho] \oplus [\alpha\cdot\rho]\,, \tag{3.6.1}$$

$$\alpha^2 = \mathrm{id}, \quad \alpha\cdot\rho = \rho\cdot\alpha\,. \tag{3.6.2}$$

Using the above relations, we can obtain the following endomorphisms of $\mathcal{O}_3$:

$$\rho_\pm(S_1) = \frac{S_1}{d} + \frac{S_2S_2 + S_3S_3}{\sqrt{d}}\,, \tag{3.6.3}$$

$$\rho_\pm(S_2) = S_2S_1S_1^* + \left(\frac{S_1}{\sqrt{d}} + \frac{S_2S_2}{\sqrt{2d}} - \frac{S_3S_3}{\sqrt{2}}\right)S_2^* - \frac{\pm S_3S_2 + S_2S_3}{\sqrt{2}}S_3^* \tag{3.6.4}$$

$$\rho_\pm(S_3) = \mp S_3S_1S_1^* + \frac{\pm S_3S_2 - S_2S_3}{\sqrt{2}}S_2^* \mp \left(\frac{S_1}{\sqrt{d}} - \frac{S_2S_2}{\sqrt{2}} + \frac{S_3S_3}{\sqrt{2d}}\right)S_3^*\,, \tag{3.6.5}$$

$$\alpha_\pm(S_1) = S_1, \quad \alpha_\pm(S_2) = \pm S_2, \quad \alpha_\pm(S_3) = \mp S_3\,, \tag{3.6.6}$$

where $d = 1 + \sqrt{2}$. $\alpha_\pm$ and $\rho_\pm$ satisfy (3.6.2). Using Lemma 2.6 one can check the following:

$$S_1 \in (\mathrm{id}, \rho^2), \quad S_2 \in (\rho, \rho^2), \quad S_3 \in (\alpha\cdot\rho, \rho^2)\,. \tag{3.6.7}$$

So we have Index $E_\rho = 3 + 2\sqrt{2}$ for $E_\rho(x) = \rho(S_1^*\rho(x)S_1)$. Let $\omega = (2, 1, 1)$. Then ρ commutes with λ^ω. It is easy to see

$$\rho_\pm^k(S_1S_1^*) = S_1\rho_\pm^{k-2}(S_1S_1^*)S_1^* + S_2\rho_\pm^{k-1}(S_1S_1^*)S_2^* + S_3\rho_\pm^{k-1}(S_1S_1^*)S_3^*, \quad k \geqq 2\,. \tag{3.6.8}$$

The last example is rather exceptional in this article.

Example 3.7. Let G, and $\langle\,,\rangle$ be as in Example 3.5. We put $n = N$, and consider the Cuntz algebra $\mathcal{O}_n$ whose generators are $\{S_g\}_{g\in G}$. We define $\rho \in \mathrm{End}(\mathcal{O}_n)$, a G action α, and a unitary representation of G in $\mathcal{O}_n U$ as follows.

$$\alpha_g(S_h) = S_{g+h}\,, \tag{3.7.1}$$

$$U(g) = \sum_h \langle g, h\rangle S_hS_h^*\,, \tag{3.7.2}$$

$$\rho(S_g) = \frac{1}{\sqrt{n}}\sum_h \langle g, h\rangle S_hU(g)^*\,. \tag{3.7.3}$$

Then the following hold:

$$\alpha_g\cdot\rho = \rho, \quad \mathrm{Ad}(U(g))\cdot\rho = \rho\cdot\alpha_g\,, \tag{3.7.4}$$

$$S_g \in (\alpha_g, \rho^2)\,. \tag{3.7.5}$$

So we have Index $E_\rho = n$ for $E_\rho(x) = \rho(S_e^*\rho(x)S_e)$. In contrast with the other cases, ρ commutes with the usual gauge action λ^1.

The first equation of (3.7.4) means that $\rho(\mathcal{O}_n)$ is a subalgebra of the fixed point algebra $\mathcal{O}_n^\alpha$ of $\mathcal{O}_n$ under α. In fact, these two coincide. Indeed, using (3.7.3), (3.7.5), we have

$$E_\rho(x) = \rho(S_e)^* \left(\sum_g S_g \alpha_g(x) S_g^* \right) \rho(S_e) = \frac{1}{n} \sum_g \alpha_g(x) .$$

So E_ρ is the mean on G, and we get $\rho(\mathcal{O}_n) = \mathcal{O}_n^\alpha$. Let H be the n dimensional Hilbert space generated by $\{S_g\}_{g \in G}$. Then $\alpha|_H$ is equivalent to the regular representation of G. The above fact means that $\mathcal{O}_n^\alpha$ is isomorphic to $\mathcal{O}_n$. So one can consider the same type of problem for non-commutative finite groups and finite dimensional Kac algebras [C2]. The answer is the same as in our case, and in [15] we shall prove it in a similar way. C. Pinzari independently obtained the same result in the case of finite groups [P], and R. Longo in the case of finite dimensional Kac algebras [L5]. For a generalization of this problem to local compact groups, see [CDPR].

4. Representations

In this section we shall construct inclusions of AFD type III_λ $(0 < \lambda < 1)$ factors by representing the examples in Sect. 3. To investigate the Murray–von Neumann–Connes types of the factors, we shall determine the type of the GNS representation of φ^ω.

Let us start with the following lemma, of which R. Longo informed the author as a folklore among specialists.

Lemma 4.1. *Let A be a unital C*-algebra, φ a state of A and $(\pi_\varphi, H_\varphi, \Omega_\varphi)$ the* GNS *triplet of φ. We assume that Ω_φ is separating for $\pi_\varphi(A)''$. Then the following hold:*

(1) *Let B be a unital C*-subalgebra of A and ψ the restriction of φ to B. Then $(\pi_\varphi|_B, H_\varphi)$ is quasi-equivalent to the* GNS *representation of ψ (π_ψ, H_ψ).*
(2) *Let ρ be a unital endomorphism of A which preserves φ. Then ρ can be extended to a normal endomorphism of $\pi_\varphi(A)''$.*

Proof. (1) Let $K = \overline{\pi_\varphi(B)\Omega_\varphi}$. Then (π_ψ, H_ψ) is unitary equivalent to $(\pi_\varphi|_B, K)$, and $(\pi_\varphi|_B, K)$ is quasi-equivalent to $(\pi_\varphi|_B, \overline{\pi_\varphi(B)'K})$. By assumption we have $\overline{\pi_\varphi(B)'K} \supset \overline{\pi_\varphi(A')K} = H_\varphi$. (2) In a similar way as above, we can see that (π_φ, H_φ) is quasi-equivalent to $(\pi_\varphi \cdot \rho, H_\varphi)$. Hence we obtain the result. Q.E.D.

Remark 4.2. The assumption of Lemma 4.1 is automatically satisfied for KMS states [BR, Corollary 5.3.9].

The following proposition shows that our examples in Sect. 3 have "nice" representations.

Proposition 4.3. *Let ρ be an endomorphism of $\mathcal{O}_n$ which commutes with λ^ω. Then ρ can extend to a normal endomorphism of $\pi_{\varphi^\omega}(\mathcal{O}_n)''$.*

Proof. Since ρ commutes with λ^ω, we have $\varphi^\omega \cdot \rho = \varphi^\omega$ due to the uniqueness of the KMS state for λ^ω. Then the statement follows from Lemma 4.1, (2). Q.E.D.

Let ρ be one of the endomorphisms we constructed in Sect. 3. Then there exists λ^ω which commutes with ρ. Let $M = \pi_{\varphi^\omega}(\mathcal{O}_n)''$, $N = \pi_{\varphi^\omega}(\rho(\mathcal{O}_n))''$. Then $M \supset N$ is an inclusion of factors because φ^ω is the unique KMS state for φ^ω [BR, Theorem 5.3.30]. Due to the above proposition, the expectation E_ρ has normal extension. Therefore $M \supset N$ has finite index.

In what follows, we fix $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ and consider the GNS representation $(\pi_{\varphi^\omega}, H_{\varphi^\omega}, \Omega_{\varphi^\omega})$ of φ^ω. We shall omit π_{φ^ω} if no confusion arises. For simplicity, we denote by the same symbol φ^ω the vector state of Ω_{φ^ω} on $\mathcal{O}_n''$. Since λ^1 and λ^ω preserve φ^ω, they can extend to actions on $\mathcal{O}_n''$. We denote by λ^1, λ^ω their extensions too. Let $\mathcal{O}_{U(n)}$ be the fixed point algebra of $\mathcal{O}_n$ under the natural action of $U(n)$. Then the permutation operators we defined in Sect. 2 belong to $\mathcal{O}_{U(n)}$. $\mathcal{O}_{U(n)}$ is a subalgebra of $\mathcal{F}^n \cap \mathcal{O}_n^{\lambda^\omega}$.

Lemma 4.4. *Let $R \in \mathcal{O}_n''$ with $\lambda_t^1(R) = e^{\sqrt{-1}kt}R$, $k \in \mathbf{Z}$. Then,*

$$\sigma(R) = \lim_{r \to \infty} \theta(r + k, 1)R\theta(r, 1)^* \quad \text{(in strong * topology)},$$

where σ is defined by $\sigma(x) = \sum_i S_i x S_i^*$, $x \in \mathcal{O}_n''$.

Proof. It suffices to show strong convergence because of $\lambda_t^1(R^*) = e^{-\sqrt{-1}kt}R^*$. First we assume $k = 0$. Since λ^1 is an action of a compact group, we have $(\mathcal{O}_n'')^{\lambda^1} = (\mathcal{O}_n^{\lambda^1})'' = \mathcal{F}^{n''}$. So we can take a net $\{R_j\} \subset \mathcal{F}^n$ which converges to R in strong topology. Let $A \in {}^0\mathcal{O}_n$. Then,

$$\begin{aligned}\|(\theta(r, 1)R\theta(r, 1)^* - \sigma(R))A\Omega_{\varphi^\omega}\| \leqq{} & \|\theta(r, 1)(R - R_j)\theta(r, 1)^* A\Omega_{\varphi^\omega}\| \\ & + \|A\| \cdot \|\theta(r, 1)(R_j)\theta(r, 1)^* - \sigma(R_j)\| \\ & + \|\sigma(R - R_j)A\Omega_{\varphi^\omega}\| .\end{aligned}$$

Due to [S, Proposition 2.14] and $\theta(r, 1) \in \mathcal{O}_n^{\lambda^\omega}$, we obtain the following estimate of the first term of the right-hand side:

$$\begin{aligned}\|\theta(r, 1)(R - R_j)\theta(r, 1)^* A\Omega_{\varphi^\omega}\| &= \|(R - R_j)\theta(r, 1)^* A\Omega_{\varphi^\omega}\| \\ &= \|\sigma^{\varphi^\omega}_{-\frac{i}{2}}(A^*)\theta(r, 1)J_{\varphi^\omega}(R - R_j)\Omega_{\varphi^\omega}\| \\ &\leqq \|\sigma^{\varphi^\omega}_{-\frac{i}{2}}(A^*)\| \cdot \|(R - R_j)\Omega_{\varphi^\omega}\| ,\end{aligned}$$

where J_{φ^ω} is the modular conjugation with respect to Ω_{φ^ω}. So thanks to Proposition 2.2, $\|(\theta(r, 1)R\theta(r, 1)^* - \sigma(R))A\Omega_{\varphi^\omega}\|$ converges to 0 when r tends to ∞. Since $\{\theta(r, 1)R\theta(r, 1)^*\}$ is a bounded sequence and ${}^0\mathcal{O}_n\Omega_{\varphi^\omega}$ is dense in H_{φ^ω}, we obtain the result.

If $k > 0$ (resp. $k < 0$), then $R = (RS_1^{*k})S_1^k$ (resp. $R = S_1^{*-k}(S_1^{-k}R)$) and $RS_1^{*k} \in \mathcal{O}_n''^{\lambda^1}$ (resp. $(S_1^{-k}R \in \mathcal{O}_n''^{\lambda^1})$. Therefore we get the result from Proposition 2.2 and the above argument. Q.E.D.

The following proposition is a W*-version of [DR, Lemma 3.2] [BE, Theorem 3.2].

Proposition 4.5. $\mathcal{O}_n'' \cap (\mathcal{O}_{U(n)})' = \mathbf{C}1$.

Proof. We shall modify the argument in [DR, Lemma 3.2]. Let $X \in \mathcal{O}_n'' \cap (\mathcal{O}_{U(n)})'$. By using Fourier decomposition, we may assume $\lambda_t^1(X) = e^{\sqrt{-1}kt}X$, $k \in \mathbf{Z}$. If $k = 0$, we have the following by Lemma 4.4:

$$\sigma(X) = s - \lim_{r\to\infty} \theta(r, 1)X\theta(r, 1)^* = X .$$

Let ψ^ω be the product state of $\mathscr{F}^n$ as in Subsect. 2.1. Then $(\pi_{\varphi^\omega}|_{\mathscr{F}^n}, H_{\varphi^\omega})$ is quasiequivalent to $(\pi_{\varphi^\omega}, H_{\psi^\omega})$ due to Lemma 4.1. Since $\sigma|_{\mathscr{F}^n}$ is the one-sided shift of $\mathscr{F}^n$, we get $X \in \mathscr{F}^{n''} \cap \mathscr{F}^{n'} = \mathbf{C}1$. In the general case, due to the above argument, we have $X^*X, XX^* \in \mathbf{C}$. So X is a multiple of a unitary. Suppose $X \neq 0$ and $k > 0$. Then from Lemma 4.4, we have

$$X^{-1}\sigma(X) = \lim_{r\to\infty} \theta(r + k, 1)\theta(r, 1)^* \quad \text{(in strong $*$ topology)} .$$

Note that the left-hand side is a unitary. Let $A \in {}^0\mathcal{O}_n^l$. Then from Lemma 2.2 we have the following for large r:

$$\begin{aligned}\sigma(A)\theta(r + k, 1)\theta(r, 1)^* &= \theta(r + k + l, 1)A\theta(r, 1)^* \\ &= \theta(r + k + l, 1)\theta(r + l, 1)^*\sigma(A) .\end{aligned}$$

Hence we have $X^{-1}\sigma(X) \in \mathcal{O}_n'' \cap \sigma(\mathcal{O}_n'')'$. Since φ^ω is the unique KMS state for λ^ω, $\mathcal{O}_n''$ is a factor [BR, Theorem 5.3.30]. So we obtain the following because σ is the inner endomorphism defined by $H = \text{span}\{S_i\}$.

$$X^{-1}\sigma(X) = \sum_{i,j} c_{i,j}S_iS_j^*, \quad c_{i,j} \in \mathbf{C} .$$

We can determine $c_{i,j}$ as follows.

$$\begin{aligned}c_{i,j} &= \lim_{r\to\infty} \varphi^\omega(S_i^*\theta(r + k, 1)\theta(r, 1)^*S_j) = \lim_{r\to\infty} \varphi^\omega(\sigma^{r+k}(S_i^*)\sigma^r(S_i)) \\ &= \varphi^\omega(\sigma^k(S_i^*)S_j) = \varphi^\omega(S_j\sigma^{k-1}(S_i^*)) = e^{-\beta\omega_j}\varphi^\omega(\sigma^{k-1}(S_i^*)S_j) \\ &= \dots = e^{-(k-1)\beta\omega_j}\varphi^\omega(S_jS_i^*) = \delta_{ij}e^{-k\beta\omega_j} ,\end{aligned}$$

where we use Proposition 2.2, $\varphi^\omega \cdot \sigma = \varphi^\omega$ and the KMS condition of φ^ω. But this contradicts the unitarity of $X^{-1}\sigma(X)$. Q.E.D.

Remark 4.6. Actually, the following holds:

$$w - \lim_{r\to\infty} \theta(r + k, 1)\theta(r, 1)^* = \sum_{j=1}^{n} e^{-k\beta\omega_j}S_jS_j^* .$$

Indeed, since every weak limit point of $\{\sigma^r(\sigma^k(S_i^*)S_j)\}_{r\in N}$ belongs to $\mathcal{O}_n'' \cap \mathscr{F}^{n'} \subset \mathcal{O}_n'' \cap (\mathcal{O}_{U(n)})' = \mathbf{C}1$, we have

$$w - \lim_{r\to\infty} S_i^*\theta(r + k, 1)\theta(r, 1)^*S_j = \varphi^\omega(\sigma^k(S_i^*)S_j)1 = e^{-k\beta\omega_j}\delta_{ij} .$$

Now we determine the type of $\mathcal{O}_n''$.

Theorem 4.7.

(1) *If $\omega_i/\omega_j \notin \mathbf{Q}$ for some i, j, $\mathcal{O}_n''$ is the* AFD *type III_1 factor.*

(2) *If $\omega_i/\omega_j \in \mathbf{Q}$ for all i, j, $\mathcal{O}_n''$ is the* AFD *type III_λ $(0 < \lambda < 1)$ factor, and λ is determined by an explicit algebraic equation.*

Proof. Since $\mathcal{O}_n$ is nuclear $\mathcal{O}_n''$ is AFD. Due to the KMS condition of φ^ω, the modular automorphism group is given by $\sigma_t^{\varphi^\omega} = \lambda_{-\beta t}^\omega$, $t \in \mathbf{R}$. From Proposition 4.5 and $(\mathcal{O}_n'')_{\varphi^\omega} \supset \mathcal{O}_{U(n)}''$, $(\mathcal{O}_n'')_{\varphi^\omega}$ is a type II_1 factor. Then the Connes spectrum $\Gamma(\sigma^{\varphi^\omega})$ coincides with the Arveson spectrum $\mathrm{Sp}(\sigma^{\varphi^\omega})$ [S, 16.1]. Thus we obtain (1) and the first part of (2). To determine λ in the rational case, for simplicity we assume the following:

$$\omega = (\overbrace{m_1, \ldots m_1}^{p_1 \text{ times}}, \overbrace{m_2, \ldots m_2}^{p_2 \text{ times}}, \ldots, \overbrace{m_k, \ldots m_k}^{p_k \text{ times}}),$$

where $\{m_1, m_2, \ldots, m_k\}$ are relatively prime natural numbers. Then the period of σ^{φ^ω} is $\frac{2\pi}{\beta}$ where β is determined by

$$\sum_{i=1}^{n} e^{-\beta\omega_i} = \sum_{l=1}^{k} p_l e^{-m_l\beta} = 1 .$$

So we obtain $\lambda = e^{-\beta}$ and λ satisfies $\sum_l p_l \lambda^{m_l} = 1$. Q.E.D.

Remark 4.8. Let ρ be one of the endomorphisms we constructed in Sect. 3. Then there naturally appeared the following type of ω associated with ρ:

$$\omega = (\overbrace{2, 2, \ldots, 2}^{p \text{ times}}, \overbrace{1, 1, \ldots, 1}^{q \text{ times}}) .$$

We say that such ω is of 2-1 type. In this case one can obtain λ by the above formula, and we have $\lambda^{-1} = \frac{q + \sqrt{q^2 + 4p}}{2}$. Note that this coincides with the square root of Index E_ρ. In next section we shall prove that $\mathcal{O}_n'' \supset \rho(\mathcal{O}_n)'$ is irreducible. So we have $\lambda^{-1} = (\mathrm{Index}\, E_\rho)^{\frac{1}{2}} = d(\rho)$, where $d(\rho)$ is the statistical dimension of ρ [L2]. (We denote by the same ρ its extension.) By the additivity and multiplicativity of the statistical dimension [H, KL, L4], one can see that $d(\rho)$ satisfies the equation $d(\rho)^2 = p + qd(\rho)$ in model cases because of its fusion rules. This is the reason of the above coincidence.

5. The Relation to Ocneanu's Connection

In this section, we shall try giving a conceptional explanation of the fact that only 2-1 type of ω appeared in Sect. 3. We shall prove that the endomorhisms in Sect. 3 come from Ocneanu's connection when restricted to $\mathcal{O}_n^{\lambda^\omega}$, and using this observation we shall show that the pair $\mathcal{O}_n'' \supset \rho(\mathcal{O}_n)''$ is irreducible. One can find basic facts on Ocneanu theory in [Ka, O1, O2, O3].

First we investigate the structure of $\mathcal{O}_n^{\lambda^\omega}$, $\omega = (\overbrace{2, 2, \ldots, 2}^{p \text{ times}}, \overbrace{1, 1, \ldots, 1}^{q \text{ times}})$, $p \neq 0$, $q \neq 0$. Let us consider a bipartite graph $\mathcal{G}_{p,q}$ in Fig. 1 with the distinguished point $*$. For the edges between x and y, we use the numbering from $p+1$ to $p+q=n$. We denote by $\mathrm{Path}^1 \mathcal{G}_{p,q}$ the set of paths in $\mathcal{G}_{p,q}$ with length 1, and $\mathrm{Path}_*^k \mathcal{G}_{p,q}$ the set of paths with length k and source $*$. We define a map

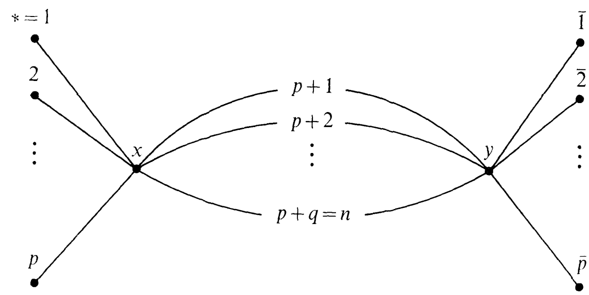


Fig. 1. The graph $\mathcal{G}_{p,q}$

m: Path$^1\,\mathcal{G}_{p,q} \to \{0, 1, 2, \ldots, n\}$ as follows:

$$m(i \to x) = m(\bar{i} \to y) = 0, \quad 1 \leqq i \leqq p, \tag{5.1}$$

$$m(x \to i) = m(y \to \bar{i}) = i, \quad 1 \leqq i \leqq p, \tag{5.2}$$

$$m(x \xrightarrow{j} y) = m(y \xrightarrow{j} x) = j, \quad p+1 \leqq j \leqq n. \tag{5.3}$$

For each path $\xi = \xi_1 \cdot \xi_2 \cdot \ldots \xi_k \in \mathrm{Path}_*^k\,\mathcal{G}_{p,q}$, $\xi_j \in \mathrm{Path}^1\,\mathcal{G}_{p,q}$, we define the following isometry in $\mathcal{O}_n$:

$$S_{m(\xi)} \equiv S_{m(\xi_1)} S_{m(\xi_2)} \ldots S_{m(\xi_k)}, \tag{5.4}$$

where $S_0 = 1$. Let $\mathrm{String}_*^k\,\mathcal{G}_{p,q}$ be the string algebra of $\mathcal{G}_{p,q}$ generated by strings, which are pairs of paths with common source $*$, common ranges and length k, and $\mathrm{String}_*\,\mathcal{G}_{p,q}$ the C*-algebra generated by $\bigcup_{k \geqq 0} \mathrm{String}_*^k\,\mathcal{G}_{p,q}$ [O1, O2].

Proposition 5.1. *In the above notations,* $\mathcal{O}_n^{\lambda^\omega}$ *is isomorphic to* $\mathrm{String}_*\,\mathcal{G}_{p,q}$. *The isomorphism is given by* m: $\mathrm{String}_*\,\mathcal{G}_{p,q} \ni (\xi_+, \xi_-) \mapsto S_{m(\xi_+)} S_{m(\xi_-)}^* \in \mathcal{O}_n^{\lambda^\omega}$.

Proof. We define finite dimensional C*-subalgebras of $\mathcal{O}_n^{\lambda^\omega}$, $A(k)$ ($k \geqq 0$), $A(i, k)$ ($0 \leqq i \leqq p$, $k \geqq 2$) as follows:

$$A(0, k) \equiv C^*\{S_{\mu_+} S_{\mu_-}^*;\, l^\omega(\mu_+) = l^\omega(\mu_-) = k - 1\},$$

$$A(i, k) \equiv C^*\{S_{\mu_+} S_{\mu_-}^*;\, l^\omega(\mu_+) = l^\omega(\mu_-) = k, f(\mu_+) = f(\mu_-) = i\}, \quad i \neq 0,$$

$$A(0) = A(1) \equiv \mathbf{C}1, \quad A(k) \equiv \bigvee_{i=0}^{p} A(i, k), \quad k \geqq 2.$$

Then it is easy to see that $A(i, k)$ is simple and $A(k) \cong \bigoplus_{i=0}^p A(i, k)$ $k \geqq 2$. First we show $A(k) \subset A(k+1)$. Obviously $A(i, k) \subset A(0, k+1)$ holds for $i \neq 0$. Let $S_{\mu_+} S_{\mu_-}^* \in A(0, k)$, which is the matrix unit of $A(0, k)$. Then we have,

$$S_{\mu_+} S_{\mu_-}^* = \sum_{i=1}^{p} S_{\mu_+} S_i S_i^* S_{\mu_-}^* + \sum_{j=p+1}^{n} S_{\mu_+} S_j S_j^* S_{\mu_-}^*,$$

$$S_{\mu_+} S_i S_i^* S_{\mu_-}^* \in A(i, k+1), \quad S_{\mu_+} S_j S_j^* S_{\mu_-}^* \in A(0, k+1). \tag{5.5}$$

This inclusion means the Bratteli diagram of $(A(k))$ is the left-hand side of Fig. 2. Since the conditional expectation $\frac{1}{2\pi}\int_0^{2\pi}\lambda_t^\omega\,dt$ preserves the algebraic part ${}^0\mathcal{O}_n$, $\mathcal{O}_n^{\lambda^\omega}$ is the norm closure of $({}^0\mathcal{O}_n)^{\lambda^\omega}$, which coincides with $\bigcup_{k=0}^\infty A(k)$. So $\bigcup_{k=0}^\infty A(k)$ generates $\mathcal{O}_n^{\lambda^\omega}$. Comparing two Bratteli diagrams in Fig. 2, we can see that the two inductive systems $\{A(k)\}_k$ and $\{\mathrm{String}_*^k\,\mathcal{G}_{p,q}\}_k$ are isomorphic. By induction using (5.5) and the definition of m, we can show that m gives the above isomorphism. Q.E.D.

Remark 5.2. For $\omega=(\omega_1,\omega_2,\ldots,\omega_n)$ with $\omega_i/\omega_j\in\mathbf{Q}$, one can write down the Bratteli diagram of $\mathcal{O}_n^{\lambda^\omega}$ in a similar way. But it is difficult to find string algebra structure except in the case of 2-1 type of ω. In the above proposition, we assume $p\neq 0$, $q\neq 0$. If $p=0$ or $q=0$, i.e. in the case of λ^1, of course the fixed point algebra $\mathcal{F}^n$ is isomorphic to the UHF algebra of type n^∞. Let $\mathcal{G}_n$ be the depth 2 graph as in Fig. 3. Then $\mathcal{F}^n$ is isomorphic to $\mathrm{String}_*\,\mathcal{G}_n$. As in the previous case, we define a map m: $\mathrm{Path}^1\,\mathcal{G}_n\to\{0,1,\ldots,n\}$ by $m(i\to x)=0$, $m(x\to i)=i$, $(1\leq i\leq n)$. For each path $\xi=\xi_1\cdot\xi_2\ldots\xi_k\in\mathrm{Path}_*^k\,\mathcal{G}_n$, we define isometry $S_{m(\xi)}\in\mathcal{O}_n$ by $S_{m(\xi)}=S_{m(1)}S_{m(2)}\ldots S_{m(k)}$, where $S_0=1$. Then m gives the isomorphism as before.

Remark 5.3. Let ω be of 2-1 type. As in [C1], we have the following expansion of a general element $X\in\mathcal{O}_n$,

$$X=\sum_{k>0}S_n^{*k}x_{-k}+x_0+\sum_{k>0}x_kS_n^k,\quad x_k\in\mathcal{O}_n^{\lambda^\omega}.$$

Note that $\Phi\equiv\mathrm{Ad}(S_n)|_{\mathcal{O}_n^{\lambda^\omega}}$ is a trace scaling endomorphism of $\mathcal{O}_n^{\lambda^\omega}\cong\mathrm{String}_*\,\mathcal{G}_{p,q}$. This means that $\mathcal{O}_n$ can be expressed by the "endomorphism crossed product" of $\mathrm{String}_*\,\mathcal{G}_{p,q}$ by Φ, in a similar way as in [C1, Sect. 2]. This is the key observation to generalize our construction to the Cuntz–Krieger algebras [I5].

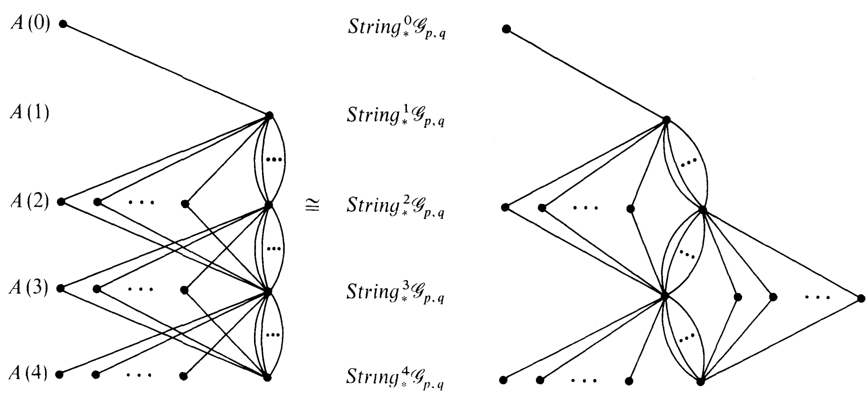


Fig. 2. The Bratteli diagrams of $\{A(k)\}_k$ and $\{\mathrm{String}_*^k\,\mathcal{G}_{p,q}\}_k$

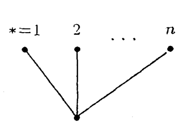


Fig. 3. The graph $\mathcal{G}_n$

Next we discuss the relation between our examples and Ocneanu's connections. Let $\mathscr{G}$, and $\mathscr{H}$ be finite bipartite graphs with distinguished points, and W a connection on them [O1]. Then one can construct two injective morphisms v, $\bar{v}$ from W.

$$v\colon \mathrm{String}_* \mathscr{H} \to \mathrm{String}_* \mathscr{G}, \quad \bar{v}\colon \mathrm{String}_* \mathscr{G} \to \mathrm{String}_* \mathscr{H} .$$

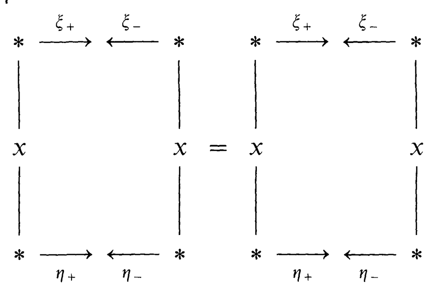
In general, even if $\mathscr{G} \cong \mathscr{H}$ one can not expect $v = \bar{v}$. We call W self-conjugate if $v = \bar{v}$ holds under suitable identification of $\mathscr{G}$ and $\mathscr{H}$.

Lemma 5.4. *Let $\mathscr{G}$ be a finite bipartite graph with distinguished points $*$ and x. We assume that there is only one edge between $*$ and x, and $*$ has no other edges. Let v be a unital endomorphism of* $\mathrm{String}_* \mathscr{G}$ *satisfying the following three conditions*:

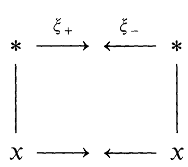
(1) $v(\mathrm{String}_*^k \mathscr{G}) \subset \mathrm{String}_*^{k+1} \mathscr{G}$.
(2) *Let $\{e_k\}_{k \geqq 1}$ be the canonical Jones projections [Ka, Sect.* 1, *O2]. Then $v(e_k) = e_{k+1}$.*
(3) *Let $\xi_0 \equiv * \to x \to *$. Then $e_1 \in v^2(\mathrm{String}_* \mathscr{G})'$ and $v^2((\xi_+, \xi_-))e_1 = (\xi_0 \cdot \xi_+, \xi_0 \cdot \xi_-)$.*

Then, v comes from a self-conjugate connection on $\mathscr{G}$.

Proof. We use the notations in [Ka]. From (1) and (2), v comes from a connection W [I3, Sect. 2, O2]. Thanks to the renormalization rule and the unitarity of connections, we have the following for any connections and possible paths $\xi_+, \xi_-, \eta_+, \eta_-$:


$= \delta_{\xi_+, \eta_+} \delta_{\xi_-, \eta_-}$. (5.6)

Using W, we define $u_{\xi,\sigma} \in \mathbf{C}$, for $\xi = (\xi_+, \xi_-)$, $\sigma = (\sigma_+, \sigma_-)$ as follows:

$u_{\xi,\sigma} \equiv$  .

We also define $u'_{\xi,\sigma} \in \mathbf{C}$ in the same way using the dual connection of W. Then (5.6) is equivalent to $\sum_\sigma u_{\xi,\sigma} \overline{u_{\eta,\sigma}} = \sum_\sigma u'_{\xi,\sigma} \overline{u'_{\eta,\sigma}} = \delta_{\xi,\eta}$. From (3), we have $\sum_\sigma u_{\xi,\sigma} \overline{u'_{\eta,\sigma}} = \delta_{\xi,\eta}$. So we obtain

$$\begin{aligned} \sum_\sigma |u_{\xi,\sigma} - u'_{\xi,\sigma}|^2 &= \sum_\sigma (u_{\xi,\sigma} \overline{u_{\xi,\sigma}} - u_{\xi,\sigma} \overline{u'_{\xi,\sigma}} - u'_{\xi,\sigma} \overline{u_{\xi,\sigma}} + u'_{\xi,\sigma} \overline{u'_{\xi,\sigma}}) \\ &= \delta_{\xi,\xi} - \delta_{\xi,\xi} - \delta_{\xi,\xi} + \delta_{\xi,\xi} = 0 . \end{aligned}$$

Hence $u_{\xi,\eta} = u'_{\xi,\eta}$. This means that W is self-conjugate. Q.E.D.

Let ρ be one of endomorphisms in Sect. 3. As we saw in Sect. 3, there exists 2-1 type of $\omega = (\overbrace{2, 2, \ldots, 2}^{p\text{ times}}, \overbrace{1, 1, \ldots, 1}^{q\text{ times}})$ satisfying $\lambda_t^\omega \cdot \rho = \rho \cdot \lambda_t^\omega$. Since ρ preserves $\mathcal{O}_n^{\lambda^\omega}$, we have an endomorphism of String$_* \mathscr{G}_{p,q}$ which is defined by $v(x) \equiv m^{-1} \cdot \rho \cdot m(x)$. We put $\mathscr{G}_{n,0} \equiv \mathscr{G}_{0,n} \equiv \mathscr{G}_n$.

Proposition 5.5. *In the above notations, v comes from a self-conjugate connection on* $\mathscr{G}_{p,q}$.

Proof. First we assume $p \neq 0$, $q \neq 0$. For Example 3.3, (resp. Example 3.5), we use the following identification:

$$\{S_1, S_2, \ldots, S_p\} = \{S_g\}_{g \in G}, \quad \{S_{p+1}, \ldots, S_n\} = \{T_\tau\}_{\tau \in \hat{G} \setminus \{e\}}, \quad S_1 = S_e .$$

$$(\text{resp. } \{S_1, S_2, \ldots, S_p\} = \{S_g\}_{g \in G}, \{S_{p+1}, \ldots, S_n\} = \{T_g\}_{g \in G}, S_1 = S_e.)$$

Let $\{e_k\}$ be the canonical Jones projections, which are defined by

$$e_k = \frac{1}{\beta} \sum_{\substack{|\xi| = k-1 \\ |v| = |w| = 1}} \frac{\sqrt{\mu(r(v))\mu(r(w))}}{\mu(r(\xi))} (\xi \cdot v \cdot \tilde{v}, \xi \cdot w \cdot \tilde{w}) ,$$

where we use the notations in [Ka]. We define the following paths with length 2.

$$v_i = i \to x \to i, \quad v_{\bar{i}} = \bar{i} \to y \to \bar{i} ,$$

$$w_i = x \to i \to x, \quad w_{\bar{i}} = y \to \bar{i} \to y ,$$

$$u_j = x \xrightarrow{j} y \xrightarrow{j} x, \quad u_{\bar{j}} = y \xrightarrow{j} x \xrightarrow{j} y .$$

Let $d = \dfrac{q + \sqrt{q^2 + 4p}}{2}$. Then the Jones projections are written as follows:

$$\begin{aligned} e_{2k} = & \sum_{1 \leq i \leq p} \sum_{|\xi| = 2k-1} (\xi \cdot v_{\bar{i}}, \xi \cdot v_{\bar{i}}) + \sum_{1 \leq i, i' \leq p} \sum_{|\xi| = 2k-1} \frac{1}{d^2} (\xi \cdot w_i, \xi \cdot w_{i'}) \\ & + \sum_{\substack{1 \leq i \leq p \\ p+1 \leq j \leq n}} \sum_{|\xi| = 2k-1} \frac{1}{d\sqrt{d}} [(\xi \cdot w_i, \xi \cdot u_j) + (\xi \cdot u_j, \xi \cdot w_i)] \\ & + \sum_{p+1 \leq j, j' \leq n} \sum_{|\xi| = 2k-1} \frac{1}{d} (\xi \cdot u_j, \xi \cdot u_{j'}) , \end{aligned}$$

$$\begin{aligned} e_{2k+1} = & \sum_{1 \leq i \leq p} \sum_{|\xi| = 2k} (\xi \cdot v_i, \xi \cdot v_i) + \sum_{1 \leq i, i' \leq p} \sum_{|\xi| = 2k} \frac{1}{d^2} (\xi \cdot w_{\bar{i}}, \xi \cdot w_{\bar{i}'}) \\ & + \sum_{\substack{1 \leq i \leq p \\ p+1 \leq j \leq n}} \sum_{|\xi| = 2k} \frac{1}{d\sqrt{d}} [(\xi \cdot w_{\bar{i}}, \xi \cdot u_{\bar{j}}) + (\xi \cdot u_{\bar{j}}, \xi \cdot w_{\bar{i}})] \\ & + \sum_{p+1 \leq j, j' \leq n} \sum_{|\xi| = 2k} \frac{1}{d} (\xi \cdot u_{\bar{j}}, \xi \cdot u_{\bar{j}'}) . \end{aligned}$$

Thanks to (3.1.5), (3.2.8), (3.3.9), (3.4.12), (3.5.12), (3.6.8) and the above expression of Jones projections, we obtain $m(e_k) = \rho^{k-1}(S_1 S_1^*)$ by induction. (In the case of Example 3.3, Example 3.4 and Example 3.5, we need a slight modification of definition of m.) So v satisfies (1) of Lemma 5.4. Since the depth of our graphs are 4, it suffices to show (2) of Lemma 5.4 for $k \leqq 4$ because $\mathrm{String}_*^{k+1} \mathscr{G}_{p,q}$ is generated by $\mathrm{String}_*^k \mathscr{G}_{p,q}$ and e_k for $k \geqq 4$ [Ka, Sect. 1, O2]. So it is enough to show the following:

$$\rho(S_i S_i^*), \rho(S_j S_{j'}^*) \in A(3), \quad 1 \leqq i \leqq p, \quad p+1 \leqq j, j' \leqq n ,$$

$$\rho(S_j S_i S_i^* S_{j'}^*) \in A(4), \quad 1 \leqq i \leqq p, \quad p+1 \leqq j, j' \leqq n .$$

By direct computation, we can check these. Thanks to $S_1 \in (\mathrm{id}, \rho^2)$ and $m(* \to x \to *) = S_1$, v satisfies (3) of Lemma 5.4.

In the case of Example 3.7, we can do the same thing by using (3.7.5) and Remark 5.2. Q.E.D.

We keep the above notations. Let $M \equiv \pi_{\varphi^\omega}(\mathcal{O}_n)''$ and $R \equiv M_{\varphi^\omega}$. We use the same symbols ρ and E_ρ for their extensions to M.

Theorem 5.6. *In the above notations, the following hold:*

(1) $M \cap \rho(M)' = \mathbf{C}$.
(2) $M \cap \rho^k(M)' \subset R \cap \rho^k(R)' \subset m(\mathrm{String}_*^k \mathscr{G}_{p,q})$.

Proof. Thanks to Proposition 5.5 and Ocneanu's general result, we have $R \cap \rho^k(R)' \subset m(\mathrm{String}_*^k \mathscr{G}_{p,q})$ [O3, II6]. In particular, $R \cap \rho(R)' = \mathbf{C}$ holds because there is only one edge connected to $*$. Let $X \in M \cap \rho(M)'$. Since λ_t^ω commutes with ρ, we may assume $\lambda_t^\omega(X) = e^{\sqrt{-1}kt} X$ by using Fourier decomposition. Suppose $k > 0$ and $X \neq 0$. From $X^*X, XX^* \in R \cap \rho(R)' = \mathbf{C}$, we may assume that X is a unitary. Let $x = XS_n^{*k} \in R$. Then we have the following:

$$x^*x = S_n^k X^* X S_n^{*n} = S_n^k S_n^{*k}, \quad xx^* = XS_n^{*k} S_n^k X^* = XX^* = 1 .$$

Since R is a II_1 factor, this is contradiction, thus proving (1).

To show (2), we need Hiai' minimal expectation [H]. Let $E_\rho(x) = \rho(S_1^* \rho(x) S_1)$. (For Example 3.3, Example 3.5 and Example 3.7, $S_1 = S_e$.) Then E_ρ is minimal because $M \supset \rho(M)$ is irreducible as shown above. Let $E_k \equiv \rho^{k-1} \cdot E_\rho \cdot \rho^{-(k-1)}$: $\rho^{k-1}(M) \to \rho^k(M)$, and $\varepsilon_k \equiv E_k \cdot E_{k-1} \cdot \ldots E_\rho : M \to \rho^k(M)$. Thanks to [KL, L4], ε_k is minimal. If ε_k preserves φ^ω we have the following:

$$M \cap \rho^k(M)' = (M \cap \rho^k(M)')_{\varepsilon_k} = (M \cap \rho^k(M)')_{\varphi^\omega} \subset M_{\varphi^\omega} = R .$$

So we can obtain the result. To prove $\varphi^\omega \cdot \varepsilon_k = \varphi^\omega$, it suffices to show $\varphi^\omega \cdot E_\rho = \varphi^\omega$. Since R is a II_1 factor and $R \supset \rho(R)$ is irreducible, there are a unique normal trace τ on R, which is the restriction of φ^ω, and a unique normal conditional expectation $E_0 : R \to \rho(R)$. Due to the uniqueness of E_0, E_0 preserves τ, and $E_0 = E_\rho|_R$ because E_ρ commutes with λ_t^ω. Let $F^\omega : M \to R$ be the conditional expectation defined by $F^\omega(x) = \frac{1}{2\pi} \int_0^{2\pi} \lambda_t^\omega(x)\,dt$. Then we have $E_0 \cdot F^\omega = F^\omega \cdot E_\rho$. So we obtain the following:

$$\varphi^\omega = \tau \cdot F^\omega = \tau \cdot E_0 \cdot F^\omega = \tau \cdot F^\omega \cdot E_\rho = \varphi^\omega \cdot E_\rho .$$

Q.E.D.

Remark 5.7. For the analysis of inclusions of type III_λ $(0 < \lambda \leqq 1)$ factors with common flow of weights, so-called "type II principal graphs" play crucial roles [KL, I4]. In [I5], we shall show that the type II principal graph of our $M \supset \rho(M)$ coincides with the principal graph of $R \supset \rho(R)$. (Cf. Remark 6.4, 6.8.)

6. Computation of Principal Graphs

In this final section, we shall compute principal graphs for a few examples in Sect. 3. In general, to determine the flat part of a given connection is a difficult problem. But in our case, since we have simple form of endomorphisms, it is possible. Note that we have already known the principal graphs in the case of Example 3.1, 3.2 for $a^3 = 1$, 3.4 for $\rho = \rho_\pm$, 3.6 and 3.7, thanks to the fusion rules of sectors generated by the endomorphisms.

As in the previous section, we use the notation $M = \pi_{\varphi^\omega}(\mathcal{O}_n)''$, $R = M_{\varphi^\omega}$, if no confusion arises.

6.1. Example 3.2. We put $\omega = (2, 2, 1)$. In this case $\mathcal{G}_{2,1}$ is the Coxeter graph $D_5^{(1)}$. We shall determine the principal graphs of $M \supset \rho_a(M)$, and $R \supset \rho_a(R)$. Let $p = S_1S_1^* + S_2S_2^* \in \mathcal{O}_3$. For $a \in \mathbf{T}$, we define non-unital endomorphism μ_a as follows:

$$\mu_a(S_1) = S_1S_1S_1^* + S_2S_2S_2^* , \tag{6.11}$$

$$\mu_a(S_2) = S_1S_2S_1^* + S_2S_1S_2^* , \tag{6.1.2}$$

$$\mu_a(S_3) = aS_+S_3S_-^* + \bar{a}S_-S_3S_+^* , \tag{6.1.3}$$

where $S_+ \equiv \dfrac{S_1 + S_2}{\sqrt{2}}$, $S_- \equiv \dfrac{S_1 - S_2}{\sqrt{2}}$. Note that $\mu_a(1) = p$. Direct computation shows

$$\rho_b \cdot \rho_a(x) = \mu_{\bar{a}b}(x) + S_3\rho_{\overline{ab}}(x)S_3^* , \tag{6.1.4}$$

$$\mu_b \cdot \rho_a(x) = S_+\rho_{ab}(x)S_+^* + S_-\rho_{a\bar{b}}(x)S_-^* , \tag{6.1.5}$$

$$(S_1S_1^* - S_2S_2^*)\mu_a(x)(S_1S_1^* - S_2S_2^*) = \mu_{\bar{a}}(x) . \tag{6.1.6}$$

Thanks to (3.2.7) and Theorem 5.6, we have the following:

$$M \cap \rho_a^2(M)' = R \cap \rho_a^2(R)' = m(\mathrm{String}_*^2\, D_5^{(1)}) .$$

From (3.2.7) and (6.1.4),

$$\rho_a^3(x) = S_1\rho_a(x)S_1^* + S_2\rho_a(x)S_2^* + S_3S_3\rho_a(x)S_3^*S_3^* + S_3\mu_{\bar{a}^3}(x)S_3^* . \tag{6.1.7}$$

So μ_a has a normal extension to M, and we use the same symbol μ_a for its extension. Due to Theorem 5.6 and (6.1.7), we have the following:

$$pMp \cap \mu_{\bar{a}^3}(M)' \subset pRp \cap \mu_{\bar{a}^3}(R)' \subset \mathbf{C}S_1S_1^* + \mathbf{C}S_2S_2^* . \tag{6.1.8}$$

Lemma 6.1.
(1) *$pMp \supset \mu_a(M)'$ is reducible if and only if $a^2 = 1$.*
(2) *$pRp \supset \mu_a(R)'$ is reducible if and only if $a^4 = 1$.*

Proof. (1) follows from (6.1.1)–(6.1.3) and (6.1.8). Since the depth of $D_5^{(1)}$ is 4, $\mathrm{String}_* D_5^{(1)}$ is generated by Jones projections and the following two elements.

$$f = (* \to x \to 2, * \to x \to 2), \quad g = (* \to x \to y \to \bar{1}, * \to x \to y \to \bar{1}) .$$

By the definition of the map m, we have $m(f) = S_2 S_2^*$, $m(g) = S_3 S_1 S_1^* S_3^*$. It is easy to see $\mu_a(m(e_k))$, $\mu_a(S_2 S_2^*) \in (\mathbf{C}S_1 S_1^* + \mathbf{C}S_2 S_2^*)'$. From the definition of μ_a we have $S_1^* \mu_a(m(g)) S_2 = \frac{\bar{a}^2 - a^2}{4} S_3(S_1 S_1^* - S_2 S_2^*) S_3^*$. This vanishes if and only if $a^4 = 1$. Q.E.D.

From (6.1.4)–(6.1.6), we have the following:

$$\rho_{\bar{a}^{3k-1}} \cdot \rho_a(x) = S_3 \rho_{a^{3k-2}}(x) S_3^* + \mu_{\bar{a}^{3k}}(x) ,$$

$$\mu_{\bar{a}^{3k}} \cdot \rho_a(x) = S_+ \rho_{\bar{a}^{3k-1}}(x) S_+^* + S_- \rho_{a^{3k+1}}(x) S_-^* ,$$

$$\rho_{a^{3k+1}} \cdot \rho_a(x) = (S_1 S_1^* - S_2 S_2^*) \mu_{\bar{a}^{3k}}(x) (S_1 S_1^* - S_2 S_2^*) + S_3 \rho_{\bar{a}^{3k+2}}(x) S_3^* .$$

So starting from ρ_a, we obtain the following sequence of endomorphisms:

$$\rho_{\bar{a}^2}, \mu_{\bar{a}^3}, \rho_{a^4}, \ldots, \rho_{\bar{a}^{3k-1}}, \mu_{\bar{a}^{3k}}, \rho_{a^{3k+1}}, \rho_{\bar{a}^{3k+2}}, \mu_{\bar{a}^{3k+3}}, \rho_{a^{3k+4}}, \ldots .$$

Since $M \supset \rho_b(M)$ and $R \supset \rho_b(R)$ are irreducible, the principal graphs are determined by the level where $\mu_{\bar{a}^{3k}}$ turns reducible. So we have the following proposition.

Proposition 6.2.
(1) *If* $a^{6k} = 1$, $k \in \mathbf{N}$ *and* $a^{6l} \neq 0$ *for* $0 < l < k$, $l \in \mathbf{N}$, *the principal graph of* $M \supset \rho_a(M)$ *is* $D_{2+3k}^{(1)}$. *Otherwise it is* D_∞.
(2) *If* $a^{12k} = 1$, $k \in \mathbf{N}$ *and* $a^{12l} \neq 0$ *for* $0 < l < k$, $l \in \mathbf{N}$, *the principal graph of* $R \supset \rho_a(R)$ *is* $D_{2+3k}^{(1)}$. *Otherwise it is* D_∞.

Remark 6.3. In [IK], we determined the flat connections of $D_k^{(1)}$. By computing $\rho_a(m(f))$, and $\rho_a(m(g))$, one can see that the parameter c of the connection in [IK, Sect. 1] corresponds to our a^2.

Remark 6.4. If $a^{12k} = 1$, $k \in \mathbf{N}$ and $a^{6l} \neq 1$ for $0 < l < 2k$, $l \in \mathbf{N}$, the principal graphs of $M \supset \rho_a(M)$ and $R \supset \rho_a(R)$ are $D_{2+6k}^{(1)}$ and $D_{2+3k}^{(1)}$. So due to the Remark 5.7, the type II and type III principal graphs of $M \supset \rho_a(M)$ do not coincide. This implies that for some $j \in \mathbf{N}$, $[\rho_a^{2j}]$ contains the modular automorphism $[\sigma_t^{\lambda^\omega}]$, for some $t \notin T(M)$ [I4, Theorem 3.5]. We can see this phenomenon directly. Indeed, from (6.1.1)–(6.1.3) we have the following:

$$\mu_{\bar{a}^{6k}}(x) = \mu_{-1}(x) = S_1 \lambda_\pi^\omega(x) S_1^* + S_2 \lambda_\pi^\omega \cdot \alpha(x) S_2^* .$$

This means that $[\rho_a^{6k}]$ contains $[\lambda_\pi^\omega] = [\sigma_{\frac{T}{2}}^{\varphi^\omega}]$, where T is the period of σ^{φ^ω}.

6.2. *Example 3.4, 3.5.* Let ρ be one of the endomorphisms in Example 3.5, and A be the C*-subalgebra of $\mathcal{O}_n$ generated by $\rho(\mathcal{O}_n)$ and $\{U(g)\}_{g \in G}$. Thanks to (3.5.9), A is the norm closure of the following $*$-algebra A_0,

$$A_0 \equiv \left\{ \sum_{g \in G} \rho(x_g) U(g); x_g \in \mathcal{O}_n \right\} .$$

We define a linear map $F_\rho \colon \mathcal{O}_n \to A$ by $F_\rho(x) \equiv \sum_{g \in G} E_\rho(x U(g)^*) U(g)$.

Proposition 6.5. *F_ρ is a conditional expectation with*

$$\operatorname{Index} F_\rho = \frac{\operatorname{Index} E_\rho}{N} = \frac{N + 2 + \sqrt{N^2 + 4N}}{2}.$$

Proof. First we shall show that F_ρ is a unital $*$-map which enjoys bimodule property. Thanks to (3.5.10), the following holds:

$$\rho(U(g)^*)S_h = S_{h+g}. \tag{6.2.1}$$

Hence, $F_\rho(1) = 1$ is obvious. Using (3.5.9) and (6.2.1), we have

$$\begin{aligned} F_\rho(x)^* &= \sum_g U(g)^* \rho(S_e^* \rho(U(g)x^*)S_e) = \sum_g \rho \cdot \alpha_{-g}(S_g^* \rho(x^*)S_e)U(g)^* \\ &= \sum_g \rho(S_e^* \rho(x^*)S_{-g})U(g)^* = \sum_g \rho(S_e^* \rho(x^* U(g))S_e)U(g)^* = F_\rho(x^*). \end{aligned}$$

To show the bimodule property, it suffices to show $F_\rho(xa) = F_\rho(x)a$ for $a \in A_0$, $x \in \mathcal{O}_n$, and this easily follows from (3.5.9). Next we show that $\left(\frac{d}{\sqrt{N}} S_e^*, \frac{d}{\sqrt{N}} S_e\right)$ is a quasi-basis for F_ρ. Since $(d \cdot S_e^*, d \cdot S_e)$ is a quasi-basis for E_ρ, we have,

$$\frac{d^2}{N} S_e^* F_\rho(S_e x) = \frac{d^2}{N} \sum_g S_e^* E_\rho(S_e x U(g)^*)U(g) = \frac{1}{N} \sum_g x U(g)^* U(g) = x.$$

What remains is to show the positivity of F_ρ. Using the above formula, we have the following:

$$F_\rho(x^* x) = \frac{d^4}{N^2} F_\rho(F_\rho(S_e x)^* S_e S_e^* F_\rho(S_e x)) = \frac{d^4}{N^2} F_\rho(S_e x)^* F_\rho(S_e S_e^*) F_\rho(S_e x).$$

So it is enough to show $F_\rho(S_e S_e^*) \geqq 0$, and this holds as follows:

$$\begin{aligned} F_\rho(S_e S_e^*) &= \sum_g E_\rho(S_e S_e^* U(g)^*)U(g) = E_\rho(S_e S_e^*) \sum_g U(g) \\ &= \frac{N}{d^2}(S_e S_e^* + \hat{T}_e \hat{T}_e^*), \end{aligned}$$

where $\hat{T}_e \equiv \frac{1}{\sqrt{N}} \sum_g T_g$. Q.E.D.

Remark 6.6. F_ρ has the following explicit form.

$$\begin{aligned} F_\rho(x) &= \sum_g \rho(S_e)^* \rho^2(x) \rho(S_{-g}) U(g) = \sum_g \rho(S_e)^* \rho^2(x) U(g) \rho(S_e) \\ &= N\rho(S_e)^* \rho^2(x)(S_e S_e^* + \hat{T}_e \hat{T}_e^*)\rho(S_e) = \frac{N}{d^2} x + N\rho(S_e)^* \rho^2(x) \hat{T}_e \hat{T}_e^* \rho(S_e). \end{aligned}$$

In particular, we have the following for $\rho = \rho_\pm, \hat{\rho}_\pm$ in Example 3.4:

$$F_{\rho_\pm}(x) = \frac{2}{d^2} x + 2\rho_\pm(S_e)^* \hat{T}_+ \rho_\pm(x) \hat{T}_+^* \rho_\pm(S_e) = \frac{2}{d^2} x + \frac{2}{d} S_\mp^* \rho_\pm(x) S_\mp,$$

$$F_{\hat{\rho}_\pm}(x) = \frac{2}{d^2} x + 2\hat{\rho}_\pm(S_e)^* \hat{T}_+ \tilde{\rho}_\mp(x) \hat{T}_+^* \hat{\rho}_\pm(S_e) = \frac{2}{d^2} x + \frac{2}{d} S_\mp^* \tilde{\rho}_\mp(x) S_\mp,$$

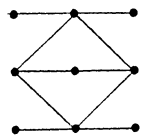


Fig. 4. The principal graph of $M \supset \hat{\rho}_{\pm}(M)$

where $S_{\mp} \equiv \dfrac{S_3 \mp \sqrt{-1}S_4}{\sqrt{2}}$, $d = 1 + \sqrt{3}$. So $F_{\hat{\rho}_{\pm}} = \theta \cdot F_{\rho_{\mp}} \cdot \theta^{-1}$.

We put $\omega = (2, 2, 1, 1)$ and $M = \pi_{\varphi^\omega}(\mathcal{O}_4)''$. By definition $F_{\rho_{\pm}}$ has normal extension, and we also use the same symbol $F_{\rho_{\pm}}$ for its extension. Let us consider $M \supset F_{\rho_{\pm}}(M)$. From Index $F_{\rho_{\pm}} = 2 + \sqrt{3} = 4\cos^2\dfrac{\pi}{12}$, the principal graph of this inclusion is one of A_{11} and E_6 [I1, Ka, O1, SV].

Proposition 6.7. *The principal graph of* $M \supset F_{\rho_{\pm}}(M)$ *is* E_6.

Proof. Let $\gamma: M \to F_{\rho_{\pm}}(M)$ be the canonical endomorphism of R. Longo [L3]. Then $[\gamma]$ can be read out of $F_{\rho_{\pm}}$ [L2, I1, I2]. Thanks to Remark 6.6, we obtain $[\gamma] = [\mathrm{id}] \oplus [\rho_{\pm}]$. Equations (3.4.9) and (3.4.10) mean that the fusion rule of $[\rho_{\pm}]$ is as follows:

$$[\rho_{\pm}]^2 = [\mathrm{id}] \oplus [\alpha] \oplus 2[\rho_{\pm}]\,.$$

This is not the fusion rule for A_{11} [I1, Subsect. 3.1]. Hence we obtain the result. Q.E.D.

Remark 6.8. One can show that the principal graph of $M \supset \hat{\rho}_{\pm}(M)$ is as in Fig. 4, and $[\hat{\rho}_{\pm}]^4$ contains $[\sigma^{\varphi^\omega}_{\frac{T}{2}}]$. We omit the details.

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