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Crystal Base and q-Vertex Operators

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Abstract. The q-deformed vertex operators of Frenkel and Reshetikhin are studied in the framework of Kashiwara's crystal base theory. It is shown that the vertex operators preserve the crystal structure, and are naturally labeled by the global crystal base. As an application the one point functions are calculated for the associated elliptic RSOS models, following the scheme of Kang et al. developed for the trigonometric vertex models.

1. Introduction

The integrable RSOS models of Andrews-Baxter-Forrester (ABF) [1] and their generalizations [2–4] are built upon elliptic solutions of the Yang-Baxter equation (YBE) in the interaction-round-a-face (IRF) formulation [5]. The one point functions in these models are known to be given in terms of branching functions for some coset pair of affine Lie algebras. (To be precise, this is so in one region of the parameter space of the model, called "regime III.") Similar results hold also for the vertex models corresponding to trigonometric solutions of YBE. As shown by Kang et al. [6, 7], the theory of crystal base [8, 9] offers in the latter case a powerful and systematic method for computing one point functions on the combinatorial level (i.e. assuming the validity of the corner transfer matrix method [5]).

In a recent work [10] Frenkel and Reshetikhin studied the q-deformation of the vertex operators à la Tsuchiya-Kanie [11] in conformal field theory. They showed that the correlation functions satisfy a q-difference analog of the Knizhnik-Zamolodchikov equation, and that the resulting connection matrices give rise to elliptic solutions of YBE of IRF type. It seems quite likely that the previously known models mentioned above are special cases of their construction. This has been confirmed in [10] in the simplest case including the ABF model.

The purpose of the present article is to study the q-vertex operators of [10] in the framework of the crystal base theory [6, 8]. As an application we show that the computation of the one point functions in the elliptic RSOS models can be treated in much the same way as is done in [6].

Throughout this paper we follow the formulations in [6]. In Sect. 2 we recall some basic facts about the crystal base theory following Kashiwara [8, 9]. The discussions about the vertex operators begin with Sect. 3. The vertex operators we consider are of the form $\Phi: V(\lambda) \to \hat{V}(\mu) \otimes V$, where $V(\lambda)$ is an integrable highest weight module, $\hat{V}(\mu)$ is a completion of $V(\mu)$, and V is a finite dimensional module of the quantized enveloping algebra $U_q(\mathfrak{g})$. This is an equivalent of the vertex operators $\Phi(z)$ in the formulation of [10]. Unlike Frenkel and Reshetikhin who treat general highest weights, we restrict ourselves to the case of dominant integral weights since the crystal base theory is specific to the latter situation. Our basic observation is that, provided V has a crystal base, the vertex operators preserve the crystal structure (Theorem 3.4). Assuming that V has a global crystal base [9, 12] we are led to a natural basis of the space of vertex operators labeled by "admissible triples" (Proposition 3.3). In Sect. 4 we consider compositions of vertex operators. We prove in particular that the composition $\Phi(z_1) \circ \Psi(z_2)$ with another vertex operator $\Psi(z)$ is well defined at $z_1 = z_2$ (Lemma 4.1). Section 5 is devoted to the description of the connection (or braiding) matrices relating the compositions of vertex operators in different order. As shown in [10] these connection matrices provide elliptic solutions of the Yang-Baxter equation in the face formulation. From the observation above it follows that these solutions share the same energy function with the corresponding trigonometric R matrix [Sect. 5.3, Eq. (5.11)]. We shall prove the second inversion relation (Proposition 5.2) for the connection matrices, which is necessary in order to apply the corner transfer matrix method. In Sect. 6 we show that the highest weight vectors in the tensor product module $V(\xi) \otimes V(\eta)$ are labeled by "restricted paths" (cf. [13]). Finally we relate these facts to the one point functions of the lattice model defined by the connection matrices.

2. Preliminaries

2.1. Notations. We fix an affine Lie algebra $\mathfrak{g}$. Let Λ_i, $h_i = \alpha_i^\vee$, α_i, $\delta = \sum_{i=0}^{l} a_i \alpha_i$, and d have the same meaning as in [14], except that for the type $A_{2l}^{(2)}$ we reverse the ordering of vertices from [14]. Thus we have $a_0 = 1$ in all cases. The canonical central element will be denoted by $c = \sum_{i=0}^{l} a_i^\vee h_i$. Set $I = \{0, 1, \dots, l\}$, $P = \mathbf{Z}\Lambda_0 \oplus \dots \oplus \mathbf{Z}\Lambda_l \oplus \mathbf{Z}\delta$, $P^* = \mathbf{Z}h_0 \oplus \dots \oplus \mathbf{Z}h_l \oplus \mathbf{Z}d$, and $Q_+ = \mathbf{Z}_{\geq 0}\alpha_0 \oplus \dots \oplus \mathbf{Z}_{\geq 0}\alpha_l$. We normalize the invariant form on P so that $(\alpha_i, \alpha_i) = 1$ for a short simple root α_i. It is related with the normalized form $(\,|\,)$ in [14] via $(\lambda, \mu) = r(\lambda|\mu)/2$, where the number r is such that the dual algebra $\mathfrak{g}^\vee$ (the one obtained by reversing arrows of the Dynkin diagram of $\mathfrak{g}$) is of type $X_l^{(r)}$. Setting $\varrho = \sum_{i=0}^{l} \Lambda_i$ we have $2(\varrho, \delta) = rh^\vee$, $h^\vee = \sum_{i=0}^{l} a_i^\vee$ being the dual Coxeter number.

Throughout this paper we shall mostly follow the notations of [6] unless otherwise stated. In particular we use $q_i = q^{(\alpha_i, \alpha_i)}$, $[k]_i = (q_i^k - q_i^{-k})/(q_i - q_i^{-1})$, and $[k]_i! = [k]_i [k-1]_i \dots [1]_i$. Set $P_{cl} = P/\mathbf{Z}\delta$, $(P_{cl})^* = \bigoplus_{i=0}^{l} \mathbf{Z}h_i \subset P^*$, and let $cl: P \to P_{cl}$ denote the canonical map. We fix $af: P_{cl} \to P$ by $af(cl(\alpha_i)) = \alpha_i$ $(i \neq i_0 = 0)$ and $af(cl(\Lambda_0)) = \Lambda_0$ so that $cl \circ af = \mathrm{id}$ and $af(cl(\alpha_0)) = \alpha_0 - \delta$. With the data $\mathfrak{g}, P, I$ above is associated the quantized affine algebra $U = U_q(\mathfrak{g}; P, I)$ defined over

$\mathbf{Q}(q)$ (q an indeterminate). Its presentation is those given in (2.1.7)–(2.1.12) in [6]. The subalgebra of U generated by e_i, f_i ($i \in I$) and q^h ($h \in (P_{cl})^*$) is denoted by $U' = U'_q(\mathfrak{g}; P_{cl}, I)$. We shall use the coproduct $\Delta = \Delta_+$ and the antipode $a = a_+$ given by

$$\Delta_+(e_i) = e_i \otimes 1 + t_i \otimes e_i, \quad \Delta_+(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i, \quad \Delta_+(q^h) = q^h \otimes q^h, \tag{2.1}$$

$$a_+(e_i) = -t_i^{-1} e_i, \quad a_+(f_i) = -f_i t_i, \quad a_+(q^h) = q^{-h}. \tag{2.2}$$

This differs from the Hopf algebra structure adopted in [6]

$$\Delta_-(e_i) = e_i \otimes t_i^{-1} + 1 \otimes e_i, \quad \Delta_-(f_i) = f_i \otimes 1 + t_i \otimes f_i, \quad \Delta_-(q^h) = q^h \otimes q^h, \tag{2.3}$$

$$a_-(e_i) = -e_i t_i, \quad a_-(f_i) = -t_i^{-1} f_i, \quad a_-(q^h) = q^{-h}. \tag{2.4}$$

The formulations based on these two structures will be compared in Sect. 2.6.

As in [6] we put

$$A = \{ f \in \mathbf{Q}(q) | f \text{ has no pole at } q = 0 \} .$$

2.2. U- and U'-modules. For a positive integer k we set $(P_+)_k = \{\lambda \in P | \langle h_i, \lambda \rangle \in \mathbf{Z}_{\geq 0} \ \forall i \in I, \langle c, \lambda \rangle = k\}$, $(P_+^0)_k = \{\lambda \in (P_+)_k | \langle d, \lambda \rangle = 0\}$, and likewise for $(P_{cl+})_k$. As in [6] $V(\lambda)$ denotes the irreducible highest weight U-module with highest weight λ. We fix a nonzero highest weight vector u_λ of $V(\lambda)$ throughout. In general a weight space of a U-module M is denoted by M_ν ($\nu \in P$), and likewise for U'-modules. We write $\mathrm{wt}\, v = \nu$ for $v \in M_\nu$. Let $V(\lambda) = \bigoplus_{\nu \in \lambda - Q_+} V(\lambda)_\nu$ be the weight space decomposition. We set

$$\hat{V}(\lambda) = \prod_{\nu \in \lambda - Q_+} V(\lambda)_\nu . \tag{2.5}$$

Let Mod^f denote the set of finite dimensional U'-modules V such that

$$\mathrm{wt}(V) \subset \lambda_0 + \sum_{i=0}^{l} \mathbf{Z} cl(\alpha_i) \quad \text{for some } \lambda_0 \in P_{cl},\ \langle c, af(\lambda_0) \rangle = 0 . \tag{2.6}$$

For $V \in \mathrm{Mod}^f$ we shall identify the affinization $\mathrm{Aff}(V)$ ([6], Sect. 3.2) with the U-module structure on $V[z, z^{-1}] = \mathbf{Q}(q)[z, z^{-1}] \otimes V$ defined as follows:

$$\begin{gathered} e_i(z^n \otimes v) = z^{\delta_{i0}+n} \otimes e_i v , \qquad f_i(z^n \otimes v) = z^{-\delta_{i0}+n} \otimes f_i v , \\ \mathrm{wt}(z^n \otimes v) = n\delta + af(\mathrm{wt}\, v) , \end{gathered} \tag{2.7}$$

where $n \in \mathbf{Z}$, $v \in V$ is a weight vector and $\mathrm{wt}\, v$ signifies its weight. We shall often write $z^n \otimes v$ as vz^n.

Analogously, for an invertible element $x \in \mathbf{Q}(q)$, let V_x denote the U'-module whose underlying space is V, equipped with the structure map $\pi_x : U' \to \mathrm{End}(V)$

$$\pi_x(e_i) = x^{\delta_{i0}} \pi(e_i) , \qquad \pi_x(f_i) = x^{-\delta_{i0}} \pi(f_i) , \qquad \pi_x(q^h) = \pi(q^h) ,$$

where π signifies the original structure map. (The notation V_x conflicts with that of weight spaces, but the meaning will be clear from the context.)

2.3. Crystal base. We recall from [8, 9] some basic notions concerning the crystal base.

Let M be an integrable U-module. For each $i \in I$, any weight vector $u \in M$ can be uniquely decomposed as

$$u = \sum_{k=0}^{N} f_i^{(k)} u_k , \qquad e_i u_k = 0, \ \mathrm{wt}\, u_k = \mathrm{wt}\, u + k\alpha_i , \tag{2.8}$$

where $f_i^{(k)} = f_i^k/[k]_i!$. Using (2.8) one defines the linear maps $\tilde{e}_i^{\mathrm{up}}$, $\tilde{f}_i^{\mathrm{up}}$, $\tilde{e}_i^{\mathrm{low}}$, $\tilde{f}_i^{\mathrm{low}} \in \mathrm{End}(M)$ as follows [9]:

$$\tilde{e}_i^{\mathrm{low}} u = \sum_{k=1}^{N} f_i^{(k-1)} u_k , \qquad \tilde{f}_i^{\mathrm{low}} u = \sum_{k=0}^{N} f_i^{(k+1)} u_k ,$$

$$\tilde{e}_i^{\mathrm{up}} u = \sum_{k=1}^{N} \frac{[\langle h_i, \mathrm{wt}\, u\rangle + k + 1]_i}{[k]_i} f_i^{(k-1)} u_k ,$$

$$\tilde{f}_i^{\mathrm{up}} u = \sum_{k=0}^{N} \frac{[k+1]_i}{[\langle h_i, \mathrm{wt}\, u\rangle + k]_i} f_i^{(k+1)} u_k .$$

The notions of upper (resp. lower) crystal, crystal lattice and crystal base at $q=0$ are defined in [8, 9] using $\tilde{e}_i^{\mathrm{up}}, \tilde{f}_i^{\mathrm{up}}$ (resp. $\tilde{e}_i^{\mathrm{low}}, \tilde{f}_i^{\mathrm{low}}$). Those at $q=\infty$ are defined similarly replacing A by $\bar{A} = \{f \in \mathbf{Q}(q) | f \text{ has no pole at } q=\infty\}$.

It is known [8] that an integrable highest weight module $V(\lambda)$ has the standard crystal base at $q=0$ described as follows ([9], (3.3.1–2), (4.2.9)):

$$L^{\mathrm{low}}(\lambda) = \sum A \tilde{f}_{i_1}^{\mathrm{low}} \dots \tilde{f}_{i_k}^{\mathrm{low}} u_\lambda , \tag{2.9}$$

$$B^{\mathrm{low}}(\lambda) = \{\tilde{f}_{i_1}^{\mathrm{low}} \dots \tilde{f}_{i_k}^{\mathrm{low}} u_\lambda \bmod qL^{\mathrm{low}}(\lambda)\} \backslash \{0\} , \tag{2.10}$$

$$L^{\mathrm{up}}(\lambda)_\nu = q^{(\lambda,\lambda)-(\nu,\nu)} L^{\mathrm{low}}(\lambda)_\nu , \qquad B^{\mathrm{up}}(\lambda)_\nu = q^{(\lambda,\lambda)-(\nu,\nu)} B^{\mathrm{low}}(\lambda)_\nu . \tag{2.11}$$

Let φ denote the anti-automorphism of U given by

$$\varphi(e_i) = f_i , \qquad \varphi(f_i) = e_i , \qquad \varphi(q^h) = q^h .$$

Then $V(\lambda)$ carries a unique symmetric bilinear form $(\,,)_\varphi$ such that

$$(u_\lambda, u_\lambda)_\varphi = 1 , \quad (xu, v)_\varphi = (u, \varphi(x)v)_\varphi \quad \text{for all } u, v \in V(\lambda) \text{ and } x \in U . \tag{2.12}$$

The upper crystal lattice can also be characterized as ([9], (4.2.7))

$$L^{\mathrm{up}}(\lambda) = \{u \in V(\lambda) | (u, L^{\mathrm{low}}(\lambda))_\lambda \subset A\} . \tag{2.13}$$

Crystal lattice/base can be formulated also for U'-modules, but for finite dimensional modules the existence of a crystal base is not guaranteed in general. A family of finite dimensional modules having "pseudo-crystal base" have been studied extensively in [7].

2.4. Dual modules. In general, let H be a Hopf algebra, ϕ an anti-automorphism of H, and M an H-module. We shall regard the linear dual $M^* = \mathrm{Hom}_{\mathbf{Q}(q)}(M, \mathbf{Q}(q))$ as equipped with an H-module structure via ϕ:

$$\langle xv^*, v\rangle = \langle v^*, \phi(x)v\rangle \qquad v^* \in M^*, v \in M, x \in H , \tag{2.14}$$

where $\langle\,,\rangle$ denotes the canonical pairing of M^* and M. This module structure is denoted by $M^{*\phi}$. If M is finite dimensional then $M \cong (M^{*\phi})^{*\phi^{-1}}$ (canonically). Taking ϕ to be the antipode a we have the canonical identification

$$\mathrm{Hom}_H(L, M \otimes N) = \mathrm{Hom}_H(M^{*a} \otimes L, N)\,, \tag{2.15a}$$
$$\mathrm{Hom}_H(L \otimes N, M) = \mathrm{Hom}_H(L, M \otimes N^{*a})\,, \tag{2.15b}$$

We remark that the dual of (2.5),

$$V^{*a}(\lambda) = (\hat{V}(\lambda))^{*a} = \bigoplus_{\nu\in\lambda-Q_+} (V(\lambda)_\nu)^*$$

is a lowest weight module with lowest weight $-\lambda$.

Now let ι denote the anti-automorphism of U given by

$$\iota(e_i) = e_i\,, \qquad \iota(f_i) = f_i\,, \qquad \iota(q^h) = q^{-h}\,.$$

(We have changed the sign of ι from [6]). Let M be an integrable U-(or U'-)module, $\langle\,,\rangle$ the canonical pairing of M^* and M. It can be verified directly that

$$\langle \tilde{e}_i^{\mathrm{low}} v^*, v\rangle = \langle v^*, \tilde{e}_i^{\mathrm{up}} v\rangle\,, \qquad \langle \tilde{f}_i^{\mathrm{low}} v^*, v\rangle = \langle v^*, \tilde{f}_i^{\mathrm{up}} v\rangle\,, \qquad v^* \in M^{*\iota},\ v \in M\,.$$

We have also the same relations with *up* and *low* interchanged. Suppose that M has an upper (resp. lower) crystal base (L, B). Then $(L^\perp, B^\perp)$ with

$$L^\perp = \{v^* \in M^* | \langle v^*, L\rangle \subset A\}\,, \tag{2.16a}$$
$$B^\perp = \text{the base of } L^\perp/qL^\perp \text{ dual to } B \text{ with respect to } \langle\,,\rangle\,, \tag{2.16b}$$

is a lower (resp. upper) crystal base of $M^{*\iota}$ [9].

2.5. Global base. In Sect. 3 we need the global crystal base for finite dimensional U'-modules. Let us recall this notion briefly from [9].

Let $U'^{\mathbf{Q}}$ be the subalgebra over $\mathbf{Q}[q, q^{-1}]$ of U generated by $e_i^{(n)}, f_i^{(n)}$ ($i \in I$, $n \in \mathbf{Z}_{\geq 0}$) and q^h, $\left\{ \begin{matrix} q^h \\ n \end{matrix} \right\} = \prod_{k=1}^{n} (q^{h+1-k} - q^{-h-1+k})/(q^k - q^{-k})$ ($h \in (P_{cl})^*$, $n \in \mathbf{Z}_{\geq 0}$). Suppose $V \in \mathrm{Mod}^f$ possesses

$$\text{an upper crystal base } (L_0, B_0) \text{ at } q = 0\,, \tag{2.17a}$$
$$\text{an upper crystal base } (L_\infty, B_\infty) \text{ at } q = \infty\,, \tag{2.17b}$$
$$\text{a } U'^{\mathbf{Q}}\text{-submodule } V_{\mathbf{Q}} \text{ such that } V_{\mathbf{Q}} \bigotimes_{\mathbf{Q}[q,q^{-1}]} \mathbf{Q}(q) \cong V\,. \tag{2.17c}$$

Assume further that the natural map $L_0 \to L_0/qL_0$ induces an isomorphism

$$V_{\mathbf{Q}} \cap L_0 \cap L_\infty \xrightarrow{\sim} L_0/qL_0\,. \tag{2.17d}$$

Let G^{up} denote the inverse map of (2.17d). Under these conditions, $\{G^{\mathrm{up}}(b)\}_{b\in B_0}$ is a base of V, called the upper global base. We say simply that V has a global base if (2.17) are satisfied.

The following fact will be used later ([9], Lemma 5.1.1).

$$\text{For } b \in L_0/qL_0 \text{ and } n \geq 0 \text{ we have } e_i^{n+1} G^{\mathrm{up}}(b) = 0 \Leftrightarrow \tilde{e}_i^{n+1} b = 0\,. \tag{2.18}$$

We remark that the lower global base can be defined analogously [9], but it does not have the property (2.18).

2.6. Intertwiners. In this paper we shall deal with intertwiners of U-modules of the form

$$\Phi: M_1 \to M_2 \hat{\otimes} M_3 \,,$$

where $M_2 \hat{\otimes} M_3 = \bigoplus_{\nu} \prod_{\xi} (M_2)_\xi \otimes (M_3)_{\nu-\xi}$. The coalgebra structure Δ_+ (resp. Δ_-) is adapted to the upper (resp. lower) crystal base at $q=0$ in the sense discussed below.

Let M, N be integrable U-modules such that $\mathrm{wt}(M) \subset \lambda_0 + \sum \mathbf{Z}\alpha_i$, $\mathrm{wt}(N) \subset \mu_0 + \sum \mathbf{Z}\alpha_i$ for some $\lambda_0, \mu_0 \in P$. We define operators β_M, γ_{MN} by

$$\beta_M(u) = q^{-(\lambda,\lambda)+(\lambda_0,\lambda_0)} u \,, \qquad u \in M_\lambda \,, \tag{2.19}$$

$$\gamma_{MN}(u \otimes v) = q^{2(\lambda,\mu)-2(\lambda_0,\mu_0)} u \otimes v \,, \qquad u \in M_\lambda,\, v \in N_\mu \,. \tag{2.20}$$

Then $\Delta_+(x) = \gamma_{MN} \circ \Delta_-(x) \circ \gamma_{MN}^{-1}$ $(x \in U)$ and $\beta_M \otimes \beta_N = \beta_{M \otimes N} \circ \gamma_{MN} = \gamma_{MN} \circ \beta_{M \otimes N}$. We extend γ_{MN} also to $M \hat{\otimes} N$. It is known [8] that

(i) (L, B) is a lower crystal base of M at $q = 0$ if and only if $(\beta_M(L), \beta_M(B))$ is an upper crystal base of M at $q = 0$.

Suppose M_i has a lower crystal base $(L_i^{\mathrm{low}}, B_i^{\mathrm{low}})$ $(i = 1, 2, 3)$, and set $L_i^{\mathrm{up}} = \beta_{M_i}(L_i^{\mathrm{low}})$, $B_i^{\mathrm{up}} = \beta_{M_i}(B_i^{\mathrm{low}})$. For a linear map $\Phi^{\mathrm{low}}: M_1 \to M_2 \hat{\otimes} M_3$ we put $\Phi^{\mathrm{up}} = \gamma_{M_2 M_3} \circ \Phi^{\mathrm{low}}$ and vice versa. Then we have

(ii) $\Phi^{\mathrm{low}} \circ x = \Delta_-(x) \circ \Phi^{\mathrm{low}}$ if and only if $\Phi^{\mathrm{up}} \circ x = \Delta_+(x) \circ \Phi^{\mathrm{up}}$ $(x \in U)$,

(iii) $\Phi^{\mathrm{up}} \circ \beta_{M_1} = (\beta_{M_2} \otimes \beta_{M_2}) \circ \Phi^{\mathrm{low}}$. Hence $\Phi^{\mathrm{low}}(L_1^{\mathrm{low}}) \subset L_2^{\mathrm{low}} \hat{\bigotimes}_A L_3^{\mathrm{low}}$ if and only if $\Phi^{\mathrm{up}}(L_1^{\mathrm{up}}) \subset L_2^{\mathrm{up}} \hat{\bigotimes}_A L_3^{\mathrm{up}}$.

In the rest of this paper, except in Sect. 3.4, crystal lattice/base will always mean upper crystal lattice/base at $q = 0$.

3. Vertex Operators

3.1. Formulation. Fix $\lambda, \mu \in (P_+^0)_k$ and $V \in \mathrm{Mod}^f$. In [10] Frenkel and Reshetikhin studied the vertex operators (VOs). By definition they are operators of the form

$$\Phi(z) = z^{\Delta_\mu - \Delta_\lambda} \tilde{\Phi}(z) \,, \qquad \Delta_\lambda = \frac{(\lambda, \lambda + 2\varrho)}{r(k + h^\vee)} \,, \tag{3.1}$$

where $\tilde{\Phi}(z)$ is an intertwiner of U-modules

$$\tilde{\Phi}(z): V(\lambda) \to V(\mu) \hat{\otimes} V[z, z^{-1}] \,. \tag{3.2}$$

Fixing a weight basis $\{v_j\}$ of V, we define the weight components Φ_{jn} of $\Phi(z)$ by

$$\tilde{\Phi}(z) v = \sum_j \sum_{n \in \mathbf{Z}} \Phi_{jn} v \otimes v_j z^{-n} \,, \qquad \Phi_{jn}: V(\lambda)_\nu \to V(\mu)_{\nu - af(\mathrm{wt}\, v_j) + n\delta} \,. \tag{3.3}$$

Note that for each v we have $\Phi_{jn} v = 0$ for $n \gg 0$, since the weights of $V(\mu)$ are bounded from above. If we set $\Phi = \sum_j \left(\sum_{n \in \mathbf{Z}} \Phi_{jn} \right) \otimes v_j$ we obtain an intertwiner of U'-modules

$$\Phi: V(\lambda) \to \hat{V}(\mu) \otimes V \,. \tag{3.4}$$

The weight components (3.3) are recovered uniquely from Φ by using the weight decomposition of $V(\mu)$. Thus there is a bijective correspondence $\tilde{\Phi}(z) \leftrightarrow \Phi$ between intertwiners (3.2) and (3.4), and the two formulations are equivalent. In the following discussions we often find it more convenient to deal with (3.4), which we also refer to as a VO.

3.2. Existence of vertex operators. Let us examine the conditions for the existence of VOs.

Lemma 3.1. *Let Φ be a VO (3.4). Then for each $v \in V(\lambda)$ there exists an $N \in \mathbf{Z}_{>0}$ such that*

$$e_i^N \Phi_{jn} v = 0\,, \quad f_i^N \Phi_{jn} v = 0 \quad \textit{for all } i, j, n\,.$$

Proof. First note the following simple fact. Let W be an arbitrary $U_q(\mathfrak{sl}_2)$-module, and let $V_l = \bigoplus_{0 \le k \le l} \mathbf{Q}(q) v_k$ be the $l+1$ dimensional irreducible module with basis $v_k = f_1^{(k)} v_0$, $e_1 v_0 = 0$. If $u = \sum_{k=0}^{l} w_k \otimes v_k \in W \otimes V_l$ satisfies $\Delta(e_1^m) u = 0$, then $e_1^{m+l-k} w_k = 0$ for all k. This can be shown inductively for $k = l, l-1, \dots$ by comparing the coefficients of v_k in $\Delta(e_1^m) u$.

To show the lemma we may assume that v is a weight vector. Since $V(\lambda)$ is integrable, we have $e_i^m v = 0$ ($\forall i \in I$) for some m. From the remark above it follows that $e_i^{m+M} \Phi_{jn} v = 0$ for all i, j, n, where

$$M = \max_{i,j} \dim U'_q(\mathfrak{g}_i) v_j - 1 \tag{3.5}$$

with $U'_q(\mathfrak{g}_i)$ denoting the subalgebra generated by e_i, f_i, and q^h ($h \in (P_{cl})^*$). This implies that $f_i^{m+M+s} \Phi_{jn} v = 0$, where $s = \langle h_i, \mathrm{wt}\, \Phi_{jn} v\rangle = \langle h_i, \mathrm{wt}\, v - \mathrm{wt}\, v_j\rangle$ is independent of n. The proof is over. □

Definition. For a VO (3.4) Φ, let the image of the highest weight vector be

$$\Phi u_\lambda = u_\mu \otimes v_{lt} + \dots\,, \tag{3.6}$$

where $\dots$ is a sum of terms of the form $u \otimes v$, $u \in V(\mu)_\nu$ with $\nu \ne \mu$. We call $v_{lt} \in V$ the leading term of Φ.

The following tells that Φ is determined by its leading term (communicated by Kashiwara).

Proposition. *Notations being as above, let*

$$V_\lambda^\mu = \{v \in V \,|\, \mathrm{wt}\, v = cl(\lambda - \mu),\ e_i^{\langle h_i, \mu\rangle + 1} v = 0 \ \forall i \in I\}\,.$$

Then the map sending Φ to its leading term gives an isomorphism of vector spaces

$$\mathrm{Hom}_{U'}(V(\lambda), \hat{V}(\mu) \otimes V) \xrightarrow{\sim} V_\lambda^\mu \subset V\,.$$

Proof. Let $U'(\mathfrak{b}_+)$ be the Hopf subalgebra of U' generated by e_i ($i \in I$) and q^h ($h \in (P_{cl})^*$). Then u_λ generates a one-dimensional $U'(\mathfrak{b}_+)$-submodule $\mathbf{Q}(q) u_\lambda$ with the defining relations $e_i u_\lambda = 0$, $q^h u_\lambda = q^{\langle h, \lambda\rangle} u_\lambda$. We have

$$\mathrm{Hom}_{U'}(V(\lambda), \hat{V}(\mu) \otimes V) \xrightarrow{\sim} \mathrm{Hom}_{U'(\mathfrak{b}_+)}(\mathbf{Q}(q) u_\lambda, \hat{V}(\mu) \otimes V)\,. \tag{3.7}$$

In fact it is clear that the canonical map (3.7) is well defined and injective. To see that it is surjective, pick a $v \in \hat{V}(\mu) \otimes V$ such that $\mathrm{wt}\, v = cl(\lambda)$ and $e_i v = 0$ for all $i \in I$.

Then from the proof of Lemma 3.1 there exists an $N \in \mathbf{Z}_{>0}$ such that $f_i^N v = 0$ for $i \in I$, hence v generates an integrable U'-module isomorphic to $V(\lambda)$.

Noting (2.15a) we can rewrite (3.7) further as

$$\text{the right hand side of (3.7)} = \mathrm{Hom}_{U'(\mathfrak{b}_+)}(V^{*a}(\mu) \otimes \mathbf{Q}(q)u_\lambda, V) \xrightarrow{\sim} V_\lambda^\mu .$$

The last isomorphism follows from the presentation of $V^{*a}(\mu)$ as $U'(\mathfrak{b}_+)$-module

$$U'(\mathfrak{b}_+) \Big/ \left(\sum_i U'(\mathfrak{b}_+) e_i^{\langle h_i, \mu \rangle + 1} + \sum_i U'(\mathfrak{b}_+)(t_i - q_i^{-\langle h_i, \mu \rangle}) \right). \quad \square$$

When V has an upper global base, the space of intertwiners admits a description in terms of crystals as follows.

Definition. Let (L, B) be a crystal base of $V \in \mathrm{Mod}^f$. We say that a triple (μ, b, λ) $(\lambda, \mu \in (P_+^0)_k, b \in B)$ is admissible if $u_\mu \otimes b \in B(\mu) \otimes B$ is a highest weigh vector of weight $cl(\lambda)$; or equivalently if

$$\mathrm{wt}\, b = cl(\lambda - \mu), \quad \tilde{e}_i^{\langle h_i, \mu \rangle + 1} b = 0 \quad \text{for any } i \in I .$$

Let

$$B_\lambda^\mu = \{ b \in B | (\mu, b, \lambda) : \text{admissible} \} .$$

From (2.18) it follows that $\{G^{\mathrm{up}}(b)\}_{b \in B_\lambda^\mu}$ is a base of V_λ^μ. Hence we have

Proposition 3.3. *Assume that V has a global base in the sense of Sect. 2.5 with the crystal base (L, B). Then the space $\mathrm{Hom}_{U'}(V(\lambda), \hat{V}(\mu) \otimes V)$ of VOs has a basis $\{\Phi_\lambda^{\mu b}\}_{b \in B_\lambda^\mu}$, such that $\Phi_\lambda^{\mu b}$ has the leading term $G^{\mathrm{up}}(b)$ $(b \in B_\lambda^\mu)$:*

$$\Phi_\lambda^{\mu b} u_\lambda = u_\mu \otimes G^{\mathrm{up}}(b) + \dots .$$

3.3. Stability of crystal lattice. Let $V \in \mathrm{Mod}^f$. In this subsection we assume that V has a crystal base (L, B). We fix a weight basis $\{v_j\}$ of V such that $v_j \bmod qL \in B$.

We say that a VO Φ (3.4) preserves the crystal lattice if

$$\Phi(L(\lambda)) \subset \hat{L}(\mu) \otimes L ,$$

where $\hat{L}(\mu) = \prod_\nu L(\mu)_\nu$. Our goal is to show the following.

Theorem 3.4. *Let Φ be a* VO (3.4) *with the leading term $v_{lt} \in V$.*
(a) *If $v_{lt} \in L$, then Φ preserves the crystal lattice.*
(b) *In addition if $v_{lt} \bmod qL$ belongs to B, then Φ induces a morphism of crystals*

$$\Phi : B(\lambda) \to B(\mu) \otimes B .$$

(c) *There exists an $m > 0$ such that for any $v \in L(\lambda)_{\lambda - \xi}$ we have*

$$\Phi_{jn} v \in q^{-nrh^\vee - (2\varrho, M\xi) - m} L(\mu) \quad \forall j, n ,$$

where M is given in (3.5). *In particular, for any fixed v and N, $\Phi_{jn} v \in q^N L(\mu)$ holds for all but a finite number of (j, n).*

A proof of Theorem 3.4 will be given in the next subsection.

Remark 1. In the same way as (3.4) one may also consider intertwiners of the form

$$\Psi : V(\lambda) \to V \otimes \hat{V}(\mu) . \tag{3.8}$$

Under the assumption of Proposition 3.3 it can be shown that the space of intertwiners (3.8) has a basis $\{\Psi_\lambda^{b\mu}\}$ indexed by $b \in B_\lambda^\mu$, such that $\Psi_\lambda^{b\mu} u_\lambda = G^{\mathrm{up}}(b) \otimes u_\mu +$ terms $v' \otimes u'$, $u' \notin V(\mu)_\mu$.

Remark 2. Let $\overline{}$ be the automorphism of the algebra U over $\mathbf{Q}$ defined by $\overline{e_i} = e_i$, $\overline{f_i} = f_i$, $\overline{q^h} = q^{-h}$, and $\bar{q} = q^{-1}$. We have $\Delta_+(x) = \overline{\sigma \circ \Delta_-(\bar{x})}$ for $x \in U$, where $\sigma(a \otimes b) = b \otimes a$. There exists a linear automorphism $\overline{} \in \mathrm{End}(V(\lambda))$ such that $\overline{u_\lambda} = u_\lambda$ and

$$\overline{xu} = \bar{x}\bar{u} \quad \text{for } x \in U,\, u \in V(\lambda)\,.$$

Suppose the finite dimensional module V also admits $\overline{} \in \mathrm{End}(V)$ with this property, and let

$$\Phi^{(\pm)} : V(\lambda) \to \hat{V}(\mu) \underset{\pm}{\bigotimes} V$$

be an intertwiner with respect to the coproduct $\Delta_\pm$. Setting $\Psi^{(\mp)} v = \overline{\sigma \circ \Phi^{(\pm)} \bar{v}}$ we obtain an intertwiner

$$\Psi^{(\mp)} : V(\lambda) \to V \underset{\mp}{\bigotimes} \hat{V}(\mu)\,.$$

Moreover $\Psi^{(-)}$ (resp. $\Psi^{(+)}$) preserves the upper (resp. lower) crystal lattice at $q = \infty$, but not the one at $q = 0$ in general.

3.4. Proof of Theorem 3.4. In view of the remarks in Sect. 2.6 it is enough to prove the theorem in the setting of lower crystal lattice. *In this subsection only, a crystal lattice/base will mean a lower crystal lattice/base. We put $\Delta = \Delta_-$, $\tilde{e}_i = \tilde{e}_i^{\mathrm{low}}$, $\tilde{f}_i = \tilde{f}_i^{\mathrm{low}}$, and assume that Φ is an intertwiner with respect to Δ_-.*

Let ψ be the anti-automorphism of U given by

$$\psi(e_i) = q_i f_i t_i^{-1}\,, \qquad \psi(f_i) = q_i^{-1} t_i e_i\,, \qquad \psi(q^h) = q^h\,.$$

Define a new bilinear form $(\,,)_\psi$ on $V(\mu)$ by setting $(u, v)_\psi = (\beta^{-1} u, v)_\varphi$, where $\beta u = q^{(\nu,\nu)-(\mu,\mu)} u$ for $u \in V(\mu)_\nu$. Then $(\,,)_\psi$ is nondegenerate, symmetric, and satisfies [cf. (2.12, 2.13)]

$$\begin{gathered}(u_\mu, u_\mu)_\psi = 1\,, \quad (xu, v)_\psi = (u, \psi(x)v)_\psi\,, \quad (u, v \in V(\mu), x \in U)\,, \\ L(\mu) = \{u \in V(\mu) | (L(\mu), u)_\psi \subset A\}\,.\end{gathered} \tag{3.9}$$

Using this we define $\Phi^\vee : V(\mu) \otimes V(\lambda) \to V$ by $\Phi^\vee(u \otimes v) = \sum (u, \Phi_{jn} v)_\varphi v_j$, where $\Phi = \sum \Phi_{jn} \otimes v_j$. In terms of $\Phi^\vee$ the intertwining properties of Φ translate as follows:

$$\Phi^\vee(u \otimes e_i v) = e_i \Phi^\vee(u \otimes v) + q^{-1-\langle h_i, \mathrm{wt}\, v\rangle} \Phi^\vee(f_i u \otimes v)\,, \tag{3.10}$$

$$\Phi^\vee(u \otimes f_i v) = q_i^{\langle h_i, \mathrm{wt}\, u\rangle} f_i \Phi^\vee(u \otimes v) + q_i^{1+\langle h_i, \mathrm{wt}\, u\rangle} \Phi^\vee(e_i u \otimes v)\,, \tag{3.11}$$

$$\mathrm{wt}\, \Phi^\vee(u \otimes v) = cl(\mathrm{wt}\, v - \mathrm{wt}\, u)\,. \tag{3.12}$$

By the definition of M (3.5) the following hold:

$$e_i^m v = 0 \Rightarrow \Phi^\vee(f_i^{m+M} u \otimes v) = 0 \quad (\forall u \in V(\mu))\,. \tag{3.13}$$

We note also that

$$f_i w \in A q_i^{\langle h_i, \nu\rangle - M} \tilde{f}_i w \qquad (\forall w \in V_\nu)\,. \tag{3.14}$$

Proof of (a). Thanks to (3.9), the statement (a) is equivalent to

$$\Phi^{\vee}(u \otimes v) \in L \qquad (\forall u \in L(\mu)_{\mu-\eta}, \forall v \in L(\lambda)_{\lambda-\xi}) \tag{3.15}$$

for any $\xi, \eta \in Q_{+}$. It is true in the case $\xi = \eta = 0$ by the assumption. We show (3.15) by induction on $ht(\xi) + ht(\eta)$ where $ht(\xi) = \sum_{i \in I} n_i$ for $\xi = \sum_{i \in I} n_i \alpha_i$.

Suppose $ht(\eta) > 0$. In view of (2.9) we may assume that $u = \tilde{f}_i u'$ for some $i \in I$ and $u' \in L(\mu)_{\mu-\eta+\alpha_i}$. Consider the decomposition of (2.8): $u' = \sum_{0 \le j} f_i^{(j)} u_j$, $v = \sum_{0 \le k} f_i^{(k)} v_k$, $e_i u_j = e_i v_k = 0$. Then $u' \in L(\mu)$ [resp. $v \in L(\lambda)$] implies $u_j \in L(\mu)$ [resp. $v_k \in L(\lambda)$] ([8], Proposition 2.3.2). Moreover we have $u = \sum_{0 \le j} f_i^{(j+1)} u_j$. By the induction hypothesis we know that $\Phi^{\vee}(u_j \otimes v_k) \in L$. Using the result for $U_q(\mathfrak{sl}_2)$ in Appendix (Corollary A.13) we get $\Phi^{\vee}(u \otimes v) = \sum \Phi^{\vee}(\tilde{f}_i^{j+1} u_j \otimes \tilde{f}_i^{k} v_k) \in L$.

The case $ht(\xi) > 0$ is similar. □

Proof of (b). Since Φ commutes with $\tilde{e}_i$ and $\tilde{f}_i$, it is enough to verify that $\Phi(B(\lambda)) \subset B(\mu) \otimes B$. In view of the description (2.10) of $B(\lambda)$ it suffices to prove the following statement by induction: If $b = v \bmod qL(\lambda) \in B(\lambda)$, $\tilde{f}_i b \neq 0$ and $\Phi v \bmod (q\hat{L}(\mu) \otimes L) \in B(\mu) \otimes B$, then $\Phi \tilde{f}_i v \bmod (q\hat{L}(\mu) \otimes L) \in B(\mu) \otimes B$. Again this is a consequence of Corollary A1.3. □

Proof of (c). We may assume that the leading term v_{lt} belongs to L. We shall show

$$\Phi^{\vee}(u \otimes v) \in q^{(2\varrho, \eta-(M+1)\xi)} L \qquad (\forall u \in L(\mu)_{\mu-\eta}, \ \forall v \in L(\lambda)_{\lambda-\xi}) \tag{3.16}$$

for any ξ, $\eta \in Q_{+}$. The assertion (c) follows from this with the choice $m = \max_j (2\varrho, \lambda - \mu - af(\mathrm{wt}\, v_j))$.

First let us prove (3.16) for $\xi = 0$, $v = u_\lambda$ by induction on $ht(\eta)$. The case $ht(\eta) = 0$ being trivial, suppose $ht(\eta) > 0$. We may assume that $u = \tilde{f}_i^j u_j$ for some i, $j > 0$ and $u_j \in L(\mu)_{\mu-\eta+j\alpha_i}$, $e_i u_j = 0$. From the estimate of powers of q in the case of $U_q(\mathfrak{sl}_2)$ (Proposition A1.2) we see that $\Phi^{\vee}(u_j \otimes u_\lambda) \in q^{\kappa} L$ implies $\Phi^{\vee}(\tilde{f}_i^j u_j \otimes u_\lambda) \in q^{\kappa + (2\varrho, j\alpha_i)} L$. Here we used $(2\varrho, \alpha_i) = (\alpha_i, \alpha_i)$. The assertion follows from this.

Next let us consider the general case by induction on $ht(\xi)$. We may assume $v = \tilde{f}_i v'$ for some i and $v' \in L(\lambda)_{\lambda-\xi+\alpha_i}$. Let k be such that $e_i^k v' \neq 0$ and $e_i^{k+1} v' = 0$, so that $v = f_i v'/[k+1]_i$. Then (3.11) implies

$$\Phi^{\vee}(u \otimes v) = [k+1]_i^{-1} q_i^{\langle h_i, \mathrm{wt}\, u \rangle} (f_i \Phi^{\vee}(u \otimes v') + q_i \Phi^{\vee}(e_i u \otimes v')) . \tag{3.17}$$

Consider the first term of (3.17). If j, n are such that $(u, \Phi_{jn} v')_{\psi} v_j \neq 0$, then $\mathrm{wt}\, u = \mathrm{wt}\, v' - af(\mathrm{wt}\, v_j) + n\delta$. Hence together with (3.14) we find

$$[k+1]_i^{-1} q_i^{\langle h_i, \mathrm{wt}\, u \rangle} (u, \Phi_{jn} v')_{\psi} f_i v_j \in A q_i^{k + \langle h_i, \mathrm{wt}\, v' \rangle - M} (u, \Phi_{jn} v')_{\psi} \tilde{f}_i v_j \, .$$

Since $k + \langle h_i, \mathrm{wt}\, v' \rangle \ge 0$ we have by the induction hypothesis

$$\text{the first term of (3.17)} \in A q_i^{-M} q^{(2\varrho, \eta - (M+1)(\xi - \alpha_i))} L \, . \tag{3.18}$$

As for the second term, let $u = \sum_{j\geq 0} \tilde{f}_i^j u_j$, $e_i u_j = 0$. Then

$$\begin{aligned} \Phi^\vee(e_i u \otimes v') &= \sum_{k+M+1\geq j\geq 1} [\langle h_i, \mathrm{wt}\, u\rangle + j + 1]_i \Phi^\vee(\tilde{f}_i^{j-1} u_j \otimes v') \\ &\in Aq^{-\langle h_i, \mathrm{wt}\, u\rangle - k - M - 1} \sum \Phi^\vee(\tilde{f}_i^{j-1} u_j \otimes v')\,, \end{aligned}$$

where we have used (3.13) to restrict the sum to $k+M+1 \geq j$. Using again the induction hypothesis we find

$$\text{the second term of (3.17)} \in Aq_i^{-M} q^{(2\varrho, \eta - \alpha_i - (M+1)(\xi - \alpha_i))} L \qquad (3.19)$$

Both of the right-hand sides of (3.18, 3.19) belong to $Aq^{(2\varrho, \eta-(M+1)\xi)}L$ as desired. □

4. Compositions of Vertex Operators

4.1. Convergence of composition. Let $V, W \in \mathrm{Mod}^f$ have crystal bases. We shall consider the intertwiners

$$\begin{aligned} \Psi &: V(\lambda) \to \hat{V}(\mu) \otimes W\,, \\ \Phi &: V(\mu) \to \hat{V}(\nu) \otimes V\,. \end{aligned}$$

Fixing bases $\{v_j\} \subset V$, $\{w_k\} \subset W$ we denote by Φ_{jm}, Ψ_{kn} their weight components respectively.

If we set $\Phi_x = \sum \Phi_{jm} \otimes v_j x^{-m}$ ($x \in \mathbf{Q}(q)^\times$), then it gives rise to an intertwiner $V(\mu) \to \hat{V}(\nu) \otimes V_x$ with V_x being the U'-module in Sect. 2.2. We would like to define the composition

$$(\Phi_x \otimes \mathrm{id}) \circ \Psi_y = \sum_{j,k} \sum_{m,n\in\mathbf{Z}} (x^{-m} y^{-n} \Phi_{jm} \circ \Psi_{kn}) \otimes v_j \otimes w_k\,. \qquad (4.1)$$

For this purpose we need to extend the base field to $K = \mathbf{Q}((q))$, the field of formal Laurent series in q. We set $V^K = V \bigotimes_{\mathbf{Q}(q)} K$, $U'^K = U' \bigotimes_{\mathbf{Q}(q)} K$, etc.

Lemma 4.1. *If $x/y = q^{-s}$ with $s \in \mathbf{Z}_{\geq 0}$, then the composition* (4.1) *gives rise to a well defined intertwiner of U'^K-modules*

$$V^K(\lambda) \to \hat{V}^K(\nu) \otimes V^K \otimes W^K\,.$$

Proof. We may assume that the leading terms of Φ, Ψ belong to the crystal lattices. Fix $u \in L(\lambda)$ and $l \in \mathbf{Z}$. It suffices to show that for each $N > 0$ the sum $\sum_{m+n=l} q^{sm} \Phi_{jm} \circ \Psi_{kn} u$ comprises only finitely many non-zero terms $\bmod q^N L(\nu)$. Since $\Psi_{kn} u = 0$ for $n \gg 0$, the sum is restricted to $n \leq n_0$ for some n_0. Theorem 3.4 (c) states that $\Psi_{kn} u \equiv 0 \bmod q^{M_n} L(\mu)$ where $\lim_{n\to-\infty} M_n = \infty$. Since Φ_{jm}, Ψ_{kn} both preserve the crystal lattice, the assertion is clear. □

4.2 Dual crystal base. For $V \in \mathrm{Mod}^f$ we shall consider the dual U'-module $V^{*\iota}$ with respect to ι (see Sect. 2.4). We have $(V_x)^{*\iota} = (V^{*\iota})_x$ for $x \in \mathbf{Q}(q)^\times$.

As in (2.6) fix a reference weight $\lambda_0 \in \mathrm{wt}(V)$. As the reference weight of the dual module we shall always take $\lambda_0^* = -\lambda_0 \in \mathrm{wt}(V^{*\iota})$. For $v \in V_\nu$ we set

$$\beta_V(v) = q^{(af(\nu),af(\nu))-(af(\lambda_0),af(\lambda_0))}v\,,$$
$$T_V(v) = q^{(2\varrho,af(\lambda_0)-af(\nu))}v\,, \qquad s_V(v) = (-1)^{ht(\lambda_0-\nu)}v\,. \tag{4.2}$$

Likewise define β_{V^*}, etc. It is known ([6], Proposition 5.1.8) that the following give isomorphisms of U'-modules:

$$F_{V^*}: V^{*\iota}_{q^{-rh^\vee}} \xrightarrow{\sim} V^{*a}\,, \qquad v^* \mapsto s_{V^*} \circ \beta_{V^*}^{-1} \circ T_{V^*}(v^*)\,, \tag{4.3}$$
$$\tilde{F}_V: V_{q^{-2rh^\vee}} \xrightarrow{\sim} (V^{*a})^{*a}\,, \qquad v \mapsto T_V^2(v)\,, \tag{4.4}$$

where we identify V with $(V^*)^*$.

Define

$$\langle v^*, v\rangle_\iota = \langle \beta_{V^*}^{-1} v^*, v\rangle\,, \tag{4.5a}$$
$$L^{*\iota} = \{v^* \in V^* | \langle v^*, L\rangle_\iota \subset A\}\,, \tag{4.5b}$$
$$B^{*\iota} = \text{the base of } L^{*\iota}/qL^{*\iota} \text{ dual to } B \text{ with respect to } \langle\,,\rangle_\iota\,. \tag{4.5c}$$

From (2.16) and the remark (i) in Sect. 2.6, $(L^{*\iota}, B^{*\iota})$ is an (upper) crystal base of $V^{*\iota}$. Note that $\langle \tilde{e}_i b^*, b'\rangle_\iota = \langle b^*, \tilde{e}_i b'\rangle_\iota$ and $\langle \tilde{f}_i b^*, b'\rangle_\iota = \langle b^*, \tilde{f}_i b'\rangle_\iota$ hold for $b^* \in B^{*\iota}$, $b' \in B$.

In this section we shall assume that

$$V \text{ has a global base}\,, \tag{4.6a}$$
$$V^{*\iota} \text{ has a global base}\,, \tag{4.6b}$$
$$\text{Their crystal bases } (L, B), (L^*, B^*) \text{ are related via (4.5b), (4.5c)}\,. \tag{4.6c}$$

Lemma 4.2. *Let* $\lambda, \mu \in (P_+^0)$, $b \in B$, $b^* \in B^*$ *so that* $\langle b, b^*\rangle_\iota = 1$. *Then*

$$(\mu, b, \lambda) \text{ is admissible} \Leftrightarrow (\lambda, b^*, \mu) \text{ is admissible}\,.$$

Proof. It suffices to check that the following are equivalent for each $i \in I$:

$$\text{(i) } \tilde{e}_i^{\langle h_i,\mu\rangle+1} b = 0\,, \qquad \text{(ii) } \tilde{e}_i^{\langle h_i,\lambda\rangle+1} b^* = 0\,.$$

Let $\varepsilon_i(b) = \max\{k | \tilde{e}_i^k b \neq 0\}$ and $\phi_i(b) = \max\{k | \tilde{f}_i^k b \neq 0\}$. Then (i) states $\varepsilon_i(b) \leq \langle h_i, \mu\rangle$. The condition (ii) is equivalent to

$$\langle \tilde{e}_i^{\langle h_i,\lambda\rangle+1} b^*, b'\rangle_\iota = \langle b^*, \tilde{e}_i^{\langle h_i,\lambda\rangle+1} b'\rangle_\iota = 0 \quad \text{for any } b' \in B\,. \tag{4.7}$$

Equation (4.7) means that there is no $b' \in B$ satisfying $b = \tilde{e}_i^{\langle h_i,\lambda\rangle+1} b'$, i.e. that $\phi_i(b) \leq \langle h_i, \lambda\rangle$. The assertion now follows from the relation $\phi_i(b) - \varepsilon_i(b) = \langle h_i, \lambda - \mu\rangle$. □

4.3 A lemma. By virtue of Lemma 4.2 the set

$$B^{*\lambda}_{\ \mu} = \{b^* \in B^{*\iota} | (\lambda, b^*, \mu)\text{: admissible}\}$$

is in one to one correspondence with B^{μ}_{λ}. In fact they are dual bases to each other with respect to $\langle\, ,\rangle_{\iota}$. In this subsection we assume that they are non-empty.

Let

$$\Phi^{\mu b}_{\lambda V}: V(\lambda) \to \hat{V}(\mu) \otimes V\,, \qquad b \in B^{\mu}_{\lambda}\,, \tag{4.8}$$

$$\Phi^{\lambda b^*}_{\mu V^{*\iota}}: V(\mu) \to \hat{V}(\lambda) \otimes V^{*\iota}\,, \qquad b^* \in B^{*\lambda}_{\mu} \tag{4.9}$$

be the bases of VOs normalized as in Proposition 3.3. For definiteness we have exhibited the spaces $V, V^{*\iota}$ explicitly. Using (4.9) we now define

$$\Phi^{\lambda b^*}_{\mu V^{*a}} = (\mathrm{id} \otimes F_{V^*}) \circ (\Phi^{\lambda b^*}_{\mu V^{*\iota}})_{q^{-rh^\vee}}\,. \tag{4.10}$$

In view of (4.3), (4.10) gives an intertwiner of U'-modules

$$\Phi^{\lambda b^*}_{\mu V^{*a}}: V(\mu) \to \hat{V}(\lambda) \otimes V^{*a}\,.$$

Lemma 4.3. *Let $\lambda, \lambda' \in (P^0_+)_k$. Then*

$$\begin{aligned}\mathrm{Hom}_{U'}(V^K(\lambda), \hat{V}^K(\lambda')) &= K \quad \textit{if } \lambda = \lambda'\,,\\ &= 0 \quad \textit{otherwise}\,.\end{aligned}$$

Proof. Let $\phi: V^K(\lambda) \to \hat{V}^K(\lambda')$ be a U'-linear map, and let ϕ_n be its weight components, so that ϕ_n maps $V^K(\lambda)_\nu$ to $V^K(\lambda')_{\nu+n\delta}$. Each ϕ_n is a U'-linear map from $V^K(\lambda)$ to $V^K(\lambda')$ sending u_λ to a highest weight vector of weight $\lambda + n\delta$ in $V^K(\lambda')$. Its image is a U^K-submodule.

From Theorem 4.12 b) in [15] it follows that the integrable highest weight U-module $V(\lambda)$ is absolutely irreducible, hence in particular $V^K(\lambda)$ is an irreducible U^K module. Hence we find that $\phi_n = 0$ $(n \neq 0)$ and that ϕ_0 is a scalar which can be nonzero only for $\lambda = \lambda'$. □

Proposition 4.4. *Let $b' \in B^{\mu}_{\lambda'}$, $b^* \in B^{*\lambda}_{\mu}$. Then the composition of U'^K-linear maps* ($K = \mathbf{Q}((q))$)

$$V^K(\lambda') \xrightarrow{\Phi^{\mu b'}_{\lambda' V}} \hat{V}^K(\mu) \otimes V^K \xrightarrow{\Phi^{\lambda b^*}_{\mu V^* a} \otimes \mathrm{id}} \hat{V}^K(\lambda) \otimes (V^{*a})^K \otimes V^K \xrightarrow{\mathrm{id} \otimes \langle\, ,\rangle} \hat{V}^K(\lambda) \tag{4.11}$$

is a scalar $\tilde{g}^{\mu b' b^*}_{\lambda}$ id. *Here* $\tilde{g}^{\mu b' b^*}_{\lambda} \in K$ *enjoys the property*

$$C\tilde{g}^{\mu b' b^*}_{\lambda} \equiv \langle b', b^* \rangle_{\iota} \bmod q\mathbf{Q}[[q]]\,, \qquad C = (-1)^{ht(\lambda_0^* - \mathrm{wt}\, b^*)} q^{-\kappa}\,, \tag{4.12}$$

$\kappa = (2\varrho, \lambda + \lambda_0^* - \mu)$ *In particular the matrix* $(\tilde{g}^{\mu b' b^*}_{\lambda})_{b', b^*}$ *is invertible.*

Proof. By virtue of Lemma 4.1, (4.11) is well defined. Hence by Lemma 4.3, it must be a scalar map $\tilde{g}^{\mu b' b^*}_{\lambda}$ id. To see (4.12), let $\lambda = \lambda'$. We set $\kappa = (2\varrho, \lambda + \lambda_0^* - \mu) \in \mathbf{Z}$,

$$T_\lambda(u) = q^{(2\varrho, \lambda - \nu)} u \quad \text{for } u \in V(\lambda)_\nu$$

and likewise for T_μ. It can be checked that

$$\Phi^{\lambda b^*}_{\mu V^{*a}} = q^{\kappa}(T_\lambda^{-1} \otimes (s_{V^*} \circ \beta^{-1}_{V^*})) \circ \Phi^{\lambda b^*}_{\mu V^{*\iota}} \circ T_\mu\,. \tag{4.13}$$

Now take bases $\{v_k\} \subset V$, $\{v_k^*\} \subset V^*$ such that $v_k \bmod qL \in B$, $v_k^* \bmod qL^{*\iota} \in B^{*\iota}$. Setting $\Phi_{\lambda V}^{\mu b'} = \sum \Phi'_k \otimes v_k$, $\Phi_{\mu V^*a}^{\lambda b^*} = \sum \Phi_j^* \otimes v_j^*$ and using (4.13), we write down the image of u_λ as follows:

$$\tilde{g}_\lambda^{\mu b' b^*} u_\lambda = q^\kappa T_\lambda^{-1} \sum \Phi_j^* \circ T_\mu \Phi'_k(u_\lambda) \times \langle s(v_j^*), v_k \rangle_\iota .$$

Note that $T_\lambda u_\lambda = u_\lambda$ and that $T_\mu(L(\mu)_\nu) \in qL(\mu)_\nu$ unless $\nu = \mu$ since $(2\varrho, \alpha) \in \mathbf{Z}_{>0}$ for $\alpha \in Q_+ \backslash \{0\}$, together with Theorem 3.4(a). In view of the normalization (4.8, 4.9) of VOs, we obtain (4.12). □

5. Connection Matrices

5.1. R matrix. Throughout this and the next sections, we deal with only those modules $V \in \mathrm{Mod}^f$ such that

$$V \text{ satisfies the conditions (4.6a)–(4.6c)}, \tag{5.1a}$$
$$\text{Its crystal } B_V \text{ is perfect in the sense of [6]}. \tag{5.1b}$$

Under the assumptions above, there exists an intertwiner of U-modules

$$\check{R}_{VW}(z_1/z_2): V[z_1, z_1^{-1}] \otimes W[z_2, z_2^{-1}] \to W[z_2, z_2^{-1}] \otimes V[z_1, z_1^{-1}],$$

which commutes with the multiplication by z_1, z_2 and depends rationally on $z = z_1/z_2$ (cf. [6]). Set $R_{VW}(z) = P\check{R}_{VW}(z)$, $Pw \otimes v = v \otimes w$. The condition (5.1b) implies in particular that,

$$\mathrm{wt}\, V \subset \lambda_0 - \sum_{\imath \neq 0} \mathbf{Z}_{\geq 0}\alpha_\imath , \qquad \dim V_{\lambda_0} = 1 \text{ for some } \lambda_0 \in P_{cl} .$$

From this section on we take such weight as a reference weight. Pick a nonzero vector $v_0 \in V_{\lambda_0}$, and let $w_0 \in W_{\mu_0}$, μ_0 be the counterpart for W. We normalize $R_{VW}(z)$ by

$$R_{VW}(z) v_0 \otimes w_0 = v_0 \otimes w_0 . \tag{5.2}$$

We have then (cf. [10])

$$\check{R}_{VW}(z)\check{R}_{WV}(z^{-1}) = 1 , \tag{5.3}$$
$$(R_{VW}(z)^{-1})^{t_1} = \beta_{VW}(z) R_{V^*W}(z) , \tag{5.4}$$

with some rational function $\beta_{VW}(z) \in \mathbf{Q}(q)(z)$.

5.2. Connection matrix. Fix $\lambda, \nu \in (P_+^0)_k$. Suppose that triples (ν, b_1, μ), (μ, b_2, λ) are admissible for some $\mu \in (P_+^0)_k$, $b_1 \in B_V$, and $b_2 \in B_W$. In this section we use the VO in the formulation (3.1, 3.2) $\Phi_{\mu V}^{\nu b_1}(z)$, $\Phi_{\lambda W}^{\mu b_2}(z)$. Note that they have the overall fractional powers of z. Correspondingly we must extend the base field $\mathbf{Q}((q))$ to include the fractional powers q^{Δ_λ}, etc.

A result of [10] says that

$$\begin{aligned}&(\mathrm{id} \otimes \check{R}_{VW}(z_1/z_2))\,(\Phi^{\nu b_1}_{\mu V}(z_1) \otimes \mathrm{id})\Phi^{\mu b_2}_{\lambda W}(z_2) \\ &= \sum_{b'_1, b'_2, \mu'} (\Phi^{\nu b'_1}_{\mu' W}(z_2) \otimes \mathrm{id})\Phi^{\mu' b'_2}_{\lambda V}(z_1) C_{VW}\begin{pmatrix} \lambda & b_2 & \mu \\ b'_2 & & b_1 \\ \mu' & b'_1 & \nu \end{pmatrix}(z_1/z_2)\end{aligned} \tag{5.5}$$

holds with some scalar functions

$$C_{VW}\begin{pmatrix} \lambda & b_2 & \mu \\ b'_2 & & b_1 \\ \mu' & b'_1 & \nu \end{pmatrix}(z)\,. \tag{5.6}$$

Here (5.6) is understood to be zero unless (ν, b_1, μ), (μ, b_2, λ), (ν, b'_1, μ'), and (μ', b'_2, λ) are all admissible triples. We denote by $C_{VW}(z)$ the matrix with (5.6) as entries where the matrix indices are (b_2, μ, b_1) and (b'_2, μ', b'_1).

The composition of vertex operators (in the sense of matrix elements)

$$(\Phi^{\nu b_1}_{\mu V}(z_1) \otimes \mathrm{id})\Phi^{\mu b_2}_{\lambda W}(z_2)$$

are absolutely convergent when $|z_1| \gg |z_2|$, and can be continued meromorphically (apart from the overall powers of $z_\imath$) to $(\mathbf{C}^\times)^2$. The right-hand side of (5.5) should be understood as a result of analytic continuation. The matrix $C_{VW}(z)$ satisfies the Yang-Baxter equation ([10] Theorem 6.3.)

$$\begin{aligned}&\sum_{\substack{b_7, b_8, b_9 \\ \nu}} C_{V_1V_2}\begin{pmatrix} \lambda_6 & b_9 & \nu \\ b_5 & & b_8 \\ \lambda_5 & b_4 & \lambda_4 \end{pmatrix}(x)\, C_{V_1V_3}\begin{pmatrix} \lambda_1 & b_1 & \lambda_2 \\ b_6 & & b_7 \\ \lambda_6 & b_9 & \nu \end{pmatrix}(xy) \\ &\quad \times C_{V_2V_3}\begin{pmatrix} \lambda_2 & b_2 & \lambda_3 \\ b_7 & & b_3 \\ \nu & b_8 & \lambda_4 \end{pmatrix}(y) \\ &= \sum_{\substack{b_7, b_8, b_9 \\ \nu}} C_{V_2V_3}\begin{pmatrix} \lambda_1 & b_7 & \nu \\ b_6 & & b_9 \\ \lambda_6 & b_5 & \lambda_5 \end{pmatrix}(y)\, C_{V_1V_3}\begin{pmatrix} \nu & b_8 & \lambda_3 \\ b_9 & & b_3 \\ \lambda_5 & b_4 & \lambda_4 \end{pmatrix}(xy) \\ &\quad \times C_{V_1V_2}\begin{pmatrix} \lambda_1 & b_1 & \lambda_2 \\ b_7 & & b_2 \\ \nu & b_8 & \lambda_3 \end{pmatrix}(x)\,.\end{aligned} \tag{5.7}$$

We note that the compositions of intertwiners $\{(\phi^{\nu b_1}_{\mu V}(z_1) \otimes \mathrm{id})\Phi^{\mu b_2}_{\lambda W}(z_2) | (\nu, b_1, \mu), (\mu, b_2, \lambda) :$ admissible$\}$ are linearly independent since

$$(\Phi^{\nu b_1}_{\mu V} \otimes \mathrm{id})\Phi^{\mu b_2}_{\lambda V} u_\lambda \equiv u_\nu \otimes b_1 \otimes b_2 \bmod q\hat{L}(\nu) \otimes L \otimes L \otimes \mathbf{Q}[[q]]\,.$$

As a direct consequence of (5.3) and the above, $C_{VW}(z)$ satisfies the first inversion relation:

$$\sum_{b'_1,b'_2,\mu'} C_{VW}\begin{pmatrix}\lambda & b_1 & \mu\\ b'_1 & & b_2\\ \mu' & b'_2 & \nu\end{pmatrix}(z)\, C_{WV}\begin{pmatrix}\lambda & b'_1 & \mu'\\ \tilde{b}_1 & & b'_2\\ \tilde{\mu} & \tilde{b}_2 & \nu\end{pmatrix}(z^{-1})$$
$$= \delta_{b_1,\tilde{b}_1}\delta_{b_2,\tilde{b}_2}\delta_{\mu,\tilde{\mu}}\,. \tag{5.8}$$

Remark. Assume that the left (resp. right) hand side of (5.5) is absolutely convergent for $|z_2/z_1| \le 1$ (resp. $|z_2/z_1| \ge 1$). Setting $z_1 = z_2$ in (5.5) and noting that $\check{R}_{VV}(1) = \mathrm{id}$ we get

$$C_{VV}\begin{pmatrix}\lambda & b_1 & \mu\\ b'_1 & & b_2\\ \mu' & b'_2 & \nu\end{pmatrix}(1) = \delta_{b_1 b'_1}\delta_{b_2,b'_2}\delta_{\mu,\mu'}\,. \tag{5.9}$$

In view of Lemma 4.1 it seems likely that the assumption is valid. A rigorous proof would require the knowledge of poles of the coefficients of the qKZ equation, which is beyond the scope of the present paper.

5.3. Energy function. Now let us consider the limit $q \to 0$. By the construction we have

$$\Phi^{\mu b}_{\lambda V}(z)u_\lambda \bmod qL(\mu)\otimes L = z^{\Delta_\mu-\Delta_\lambda}u_\mu \otimes b\,.$$

By the assumption (5.1b) and Proposition 4.3.2 of [6], we have

$$\check{R}_{VV}(z)|_{q=0}(b_1\otimes b_2) = z^{-H(b_1\otimes b_2)}b_1\otimes b_2\,. \tag{5.10}$$

Here H denotes the energy function of $\check{R}_{VV}$ (see [6], Sect. 4).

Therefore at $q = 0$ Eq. (5.5) gives

$$C_{VV}\begin{pmatrix}\lambda & b_2 & \mu\\ b'_2 & & b_1\\ \mu' & b'_1 & \nu\end{pmatrix}(z)\Bigg|_{q=0}$$
$$= \delta_{b_1,b'_1}\delta_{b_2,b'_2}\delta_{\mu\mu'}z^{\Delta_\lambda+\Delta_\nu-2\Delta_\mu-H(b_1\otimes b_2)}\,\mathrm{id}\,. \tag{5.11}$$

In this sense we find that the energy function for the connection matrix C coincides with that of vertex model in the sense of [6].

5.4. Second inversion relation. Applying (5.4) twice together with the isomorphism (4.4) we obtain the second inversion relation for the R matrix

$$\alpha_{VW}(z)\left(((R_{VW}(z)^{-1})^{t_1})^{-1}\right)^{t_1} = (\tilde{F}_V\otimes \mathrm{id})R_{VW}(zq^{-2rh^\vee})(\tilde{F}_V^{-1}\otimes\mathrm{id})\,, \tag{5.12}$$
$$\alpha_{VW}(z) = \beta_{VW}(z)/\beta_{V^*W}(z)\,. \tag{5.13}$$

We shall give its counterpart for the connection matrices. For this purpose we need to prepare a lemma.

Let $\{v_j\}\subset V$, $\{v_j^*\}\subset V^*$ be the dual bases with respect to *the canonical pairing*. Rewriting Proposition 4.4 in terms of VOs (3.1) we have

$$\sum_j \Phi^{\lambda' b^*}_{\mu V^* a}(z)_j \circ \Phi^{\mu b}_{\lambda V}(z)_j = \delta_{\lambda\lambda'}g^{\mu b b^*}_{\lambda}\,\mathrm{id}_{V(\lambda)}\,, \tag{5.14}$$

$$\sum_j q^{-4\varrho_j}\Phi^{\mu' b}_{\lambda V}(q^{-2rh^\vee}z)_j \circ \Phi^{\lambda b^*}_{\mu V^* a}(z)_j = \delta_{\mu\mu'}g^{*\lambda b^* b}_{\mu}\,\mathrm{id}_{V(\mu)}\,. \tag{5.15}$$

Here we have set $\varrho_j = (\varrho, af(\mathrm{wt}(v_j)))$, $g_\lambda^{\mu b b^*} = q^{rh^\vee(\Delta_\mu - \Delta_\lambda)} \tilde{g}_\lambda^{\mu b b^*}$. The weight components $\Phi(z)_j$ of a vertex operator is defined by $\Phi(z)v = \sum_j \Phi(z)_j v \otimes v_j$. The second formula is obtained from the first by replacing V by $V^{*\iota}$ and using (4.3), with a suitable choice of $g^{*\lambda b^* b}_{\ \mu} = q^{rh^\vee(\Delta_\lambda - \Delta_\mu)} \tilde{g}^{*\lambda b^* b}_{\ \mu}$, $\tilde{g}^{*\lambda b^* b}_{\ \mu} \in \mathbf{Q}((q))$.

Now set

$$G_\lambda = q^{2rh^\vee \Delta_\lambda} \chi_\lambda, \qquad \chi_\lambda = q^{-4(\varrho,\lambda)} \mathrm{tr}_{V(\lambda)}(T_\lambda^2) \in \mathbf{Q}((q)). \tag{5.16}$$

Note that χ_λ is the principally specialized character of the irreducible $\mathfrak{g}^\vee$-module with highest weight λ, where $\mathfrak{g}^\vee$ is the dual Kac-Moody algebra.

Lemma 5.1.

$$G_\lambda g_\lambda^{\mu b b^*} = G_\mu g^{*\lambda b^* b}_{\ \mu}.$$

Proof. We are to show that

$$\begin{aligned} &q^{2rh^\vee \Delta_\lambda - 4(\varrho,\lambda)} \mathrm{tr}_{V(\lambda)} \left(T_\lambda^2 \circ \sum_j \Phi_{\mu V^{*a}}^{\lambda' b^*}(z)_j \circ \Phi_{\lambda V}^{\mu b}(z)_j \right) \\ &= q^{2rh^\vee \Delta_\mu - 4(\varrho,\mu)} \mathrm{tr}_{V(\mu)} \left(T_\mu^2 \circ \sum_j q^{-4\varrho_j} \Phi_{\lambda V}^{\mu' b}(q^{-2rh^\vee} z)_j \circ \Phi_{\mu V^{*a}}^{\lambda b^*}(z)_j \right). \end{aligned}$$

From the intertwining property of the VO we have

$$q^{-4(\varrho,\mu)} T_\mu^2 \circ q^{-4\varrho_j - 2rh^\vee(\Delta_\lambda - \Delta_\mu)} \Phi_{\lambda V}^{\mu b}(q^{-2rh^\vee} z) = \Phi_{\lambda V}^{\mu b}(z)_j \circ q^{-4(\varrho,\lambda)} T_\lambda^2.$$

By the cyclic property of the trace and the fact that each hand side is convergent in q-adic topology, the assertion is clear. □

Proposition.

$$\begin{aligned} &\sum_{b_1, b_1', \lambda} \frac{G_\lambda G_\nu}{G_\mu G_{\mu'}} C_{VV} \begin{pmatrix} \lambda & b_1 & \mu \\ b_1' & & b_2 \\ \mu' & b_2' & \nu \end{pmatrix} (z^{-1}) C_{VV} \begin{pmatrix} \lambda & b_1' & \mu' \\ b_1 & & \tilde{b}_2' \\ \mu' & \tilde{b}_2 & \nu' \end{pmatrix} (q^{-2rh^\vee} z) \\ &= \alpha_{VV}(z) \delta_{b_2 \tilde{b}_2} \delta_{b_2' \tilde{b}_2'} \delta_{\nu\nu'}, \end{aligned}$$

where G_λ and α_{VV} are given in (5.16) *and* (5.13), *respectively.*

Proof. Let $R_{VV}(z)_{ij}^{kl}$ be the matrix elements of R_{VV}, that is,

$$R_{VV}(z) v_i \otimes v_j = \sum_{k,l} R_{VV}(z)_{ij}^{kl} v_k \otimes v_l.$$

Similar convention is used for $R_{VV^*}(z)$ and $R_{V^*V}(z)$ by taking the dual base $\{v_j^*\}$ of $\{v_j\}$. From (5.5) we have

$$\begin{aligned} &\sum_{i,j} \Phi_{\mu V}^{\nu b_2}(z_1)_i \circ \Phi_{\lambda V}^{\mu b_1}(z_2)_j R_{VV}(z)_{ij}^{lk} \\ &= \sum_{b_1', b_2', \mu'} \Phi_{\mu' V}^{\nu b_2'}(z_2)_k \circ \Phi_{\lambda V}^{\mu' b_1'}(z_1)_l C_{VV} \begin{pmatrix} \lambda & b_1 & \mu \\ b_1' & & b_2 \\ \mu' & b_2' & \nu \end{pmatrix} (z). \end{aligned} \tag{5.17}$$

Here we put $z_1/z_2 = z$. Operate $\Phi^{\mu' b_2^*}_{\nu V^*}(z_2)_k$ from the left to the both hand sides of (5.17) and sum over k. Setting $W = V^*$ in (5.5) and using (5.3) we have

$$\Phi^{\mu' b_2^*}_{\nu V^*}(z_2)_k \circ \Phi^{\nu b_2}_{\mu V}(z_1)_i = \sum_{\substack{b_1', b_1^*, \lambda' \\ j', l'}} \Phi^{\mu' b_1'}_{\lambda' V}(z_1)_{l'} \circ \Phi^{\lambda' b_1^*}_{\mu V^*}(z_2)_{j'} C_{V^* V}\begin{pmatrix} \mu & b_2 & \nu \\ b_1^* & & b_2^* \\ \lambda' & b_1' & \mu' \end{pmatrix}(z^{-1}) R_{VV^*}(z)^{ik}_{l'j'}\,.$$

From the relation

$$R_{VV}(z)^{lk}_{ij} = (R_{VV}(z)^{t_2})^{lj}_{ik} = \beta(z^{-1})(R_{VV^*}(z)^{-1})^{lj}_{ik}$$

together with (5.14)–(5.15), we get

$$\beta_{VV}(z^{-1}) \sum_{b_1', b_1^*, \lambda} g^{\mu b_1 b_1^*}_{\lambda} \Phi^{\mu' b_1'}_{\lambda V}(z_1)_l C_{V^* V}\begin{pmatrix} \mu & b_2 & \nu \\ b_1^* & & b_2^* \\ \lambda & b_1' & \mu' \end{pmatrix}(z^{-1}) = \sum_{b_1', b_2', \mu'} g^{\nu b_2' b_2^*}_{\mu'} \Phi^{\mu' b_1'}_{\lambda V}(z_1)_l C_{VV}\begin{pmatrix} \lambda & b_1 & \mu \\ b_1' & & b_2 \\ \mu' & b_2' & \nu \end{pmatrix}(z)\,.$$

From Proposition 4.4 there exist inverse matrices $(\gamma^{\mu b^* b}_{\lambda})_{b^* b}$ and $(\gamma^{*\lambda b b^*}_{\mu})_{bb^*}$ of g^{μ}_{λ} and $g^{*\lambda}_{\mu}$. Consequently we get

$$\beta_{VV}(z^{-1}) C_{V^* V}\begin{pmatrix} \mu & b_2 & \nu \\ b_1^* & & b_2^* \\ \lambda & b_1 & \mu' \end{pmatrix}(z^{-1}) = \sum_{b_1', b_2'} \gamma^{\mu b_1^* b_1'}_{\lambda} g^{\nu b_2' b_2^*}_{\mu'} C_{VV}\begin{pmatrix} \lambda & b_1' & \mu \\ b_1 & & b_2 \\ \mu' & b_2' & \nu \end{pmatrix}(z)\,.$$

Similarly we can derive

$$\beta_{V^* V}(z)^{-1} C_{VV^*}\begin{pmatrix} \mu & b_1^* & \lambda \\ \tilde{b}_2 & & b_1 \\ \nu' & \tilde{b}_2^* & \mu' \end{pmatrix}(z^{-1}) = \sum_{\tilde{b}_1, \tilde{b}_2'} g^{*\lambda b_1^* \tilde{b}_1}_{\mu} \gamma^{*\mu' \tilde{b}_2' \tilde{b}_2^*}_{\nu'} C_{VV}\begin{pmatrix} \lambda & b_1 & \mu' \\ \tilde{b}_1 & & \tilde{b}_2' \\ \mu & \tilde{b}_2 & \nu' \end{pmatrix}(q^{-2rh^\vee} z)\,.$$

Using the first inversion relation (5.8) with $W = V^*$ and Lemma 5.1, we obtain the desired result. □

6. Restricted Paths and One Point Functions

6.1. *Restricted paths.* Recall that we consider only $V \in \mathrm{Mod}^f$ whose crystal B is perfect of some level $N \in \mathbf{Z}_{\geq 0}$. Hence for any $\eta \in (P_+^0)_N$ there exists sequence of weights $\eta_0 = \eta, \eta_1, \eta_2, \ldots \in (P_+^0)_N$ and a path $p_{gr} = (p_{gr}(n))_{n\geq 1}$, $p_{gr}(n) \in B$, such that for any n the following isomorphism of crystals holds:

$$B(\eta) \xrightarrow{\sim} \mathscr{P}(\eta_n; B) \otimes B^{\otimes n}.$$

Here u_η is sent to $u_{\eta_n} \otimes p_{gr}(n) \otimes \ldots \otimes p_{gr}(1)$, and $\mathscr{P}(\eta; B)$ denotes the set of η paths (see [6] Sect. 4). It is known that if $b \in B(\eta)$ corresponds to an η path $p = (p(n))_{n\geq 1}$ then

$$\begin{aligned} \mathrm{wt}\, b &= \eta + \sum_{k=1}^{\infty} (af(\mathrm{wt}\, p(k)) - af(\mathrm{wt}\, p_{gr}(k))) - \omega(p)\delta\,, \\ \omega(p) &= \sum_{k=1}^{\infty} k(H(p(k+1) \otimes p(k)) - H(p_{gr}(k+1) \otimes p_{gr}(k)))\,, \end{aligned} \tag{6.1}$$

where H is the energy function (5.10) of the corresponding $\check{R}_{VV}$. We shall identify $B(\eta)$ with $\mathscr{P}(\eta; B)$.

Let k be a positive integer with $k > N$. Fix $\xi \in (P_+^0)_{k-N}$ and $\eta \in (P_+^0)_N$. We set

$$\mathrm{High}^0(\xi, \eta) = \{u_\xi \otimes b \in B(\xi) \otimes B(\eta) | \tilde{e}_i(u_\xi \otimes b) = 0 \ \forall i \in I\}.$$

Definition. We say that $\tilde{p} = (a, p)$ is a restricted (ξ, η)-path in B if the following hold.

(1) $a = (a(n))_{n\geq 0}$, $a(n) \in (P_+^0)_k$,
(2) $p = (p(n))_{n\geq 1} \in \mathscr{P}(\eta; B)$,
(3) the triple $(a(n), p(n), a(n-1))$ is admissible for all $n \geq 1$,
(4) $a(0) = \xi + \eta + \sum\limits_{k=1}^{\infty} (af(\mathrm{wt}\, p(k)) - af(\mathrm{wt}\, p_{gr}(k)))$.

Note that the $a(n)$ are uniquely fixed from ξ and p by (3), (4). We let $\mathscr{P}_{\mathrm{res}}(\xi, \eta; B)$ denote the set of restricted (ξ, η)-path in B.

Proposition 6.1. *The following is a bijection:*

$$\mathscr{P}_{\mathrm{res}}(\xi, \eta; B) \to \mathrm{High}^0(\xi, \eta)\,, \qquad (a, p) \mapsto u_\xi \otimes p\,.$$

The weight of $(a, p) \in \mathscr{P}_{\mathrm{res}}(\xi, \eta; B)$ *is given by* $a(0) - \omega(p)\delta$ *with* $\omega(p)$ *given in* (6.1).

Proof. For $n \geq 0$ we define $v(n) \in B(\xi) \otimes B(\eta_n)$ by the following map induced from $B(\eta) \xrightarrow{\sim} B(\eta_n) \otimes B^{\otimes n}$:

$$\begin{gathered} B(\xi) \otimes B(\eta) \xrightarrow{\sim} (B(\xi) \otimes B(\eta_n)) \otimes B^{\otimes n}\,, \\ u_\xi \otimes p \mapsto v(n) \otimes p(n) \otimes \ldots p(1)\,. \end{gathered}$$

First let $(a, p) \in \mathscr{P}_{\mathrm{res}}(\xi, \eta; B)$. Then we have $\mathrm{wt}(n) = \mathrm{wt}\, v(0) - \sum\limits_{j=1}^{n} \mathrm{wt}\, p(j) = a(n) \in P$. Let us show $v(n) \in \mathrm{High}^0(\xi, \eta_n)$ for all $n \geq 0$ by the induction on n. For $n \gg 0$, we have $v(n) = u_\xi \otimes u_{\eta_n}$, so we get $v(n) \in \mathrm{High}^0(\xi, \eta_n)$. Assume that $v(n) \in \mathrm{High}^0(\xi, \eta_n)$. From the admissibility of the triple $(a(n), p(n), a(n-1))$, $v(n) \otimes p(n) = v(n-1)$ is a highest weight vectorof $B(a(n)) \otimes B$. Setting $n = 0$, we find $v(0) = u_\xi \otimes p \in \mathrm{High}^0(\xi, \eta)$.

Conversely if $u_\xi \otimes p \in \mathrm{High}^0(\xi,\eta)$ then setting $a(n) = \mathrm{wt}\, v(n)$ we have $(a,p) \in \mathscr{P}_{\mathrm{res}}(\xi,\eta;B)$. □

6.2. IRF *models and their one point functions.* Here we define IRF models whose Boltzmann weights are given by the connection matrices, and state results on their one point functions.

As before we fix $\mathfrak{g}$, $V \in \mathrm{Mod}^f$ and $k > N$. Take a two dimensional square lattice $\mathscr{L}$. Place variables $\lambda, \mu, \ldots$ (resp. $b, b', \ldots$) on vertices (resp. bonds) with values in $(P^0_+)_k$ (resp. B). For a configuration of variables around a face

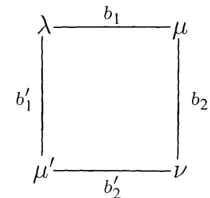

Fig. 1

we associate the Boltzmann weight

$$C_{VV}\begin{pmatrix} \lambda & b_1 & \mu \\ b'_1 & & b_2 \\ \mu' & b'_2 & \nu \end{pmatrix}(z).$$

Recall that it is zero unless the triples (μ, b_1, λ), (ν, b_2, μ), (μ', b'_1, λ), (ν, b'_2, μ') are admissible.

Although in our consideration we treated q to be an indeterminate, the matrix elements of C_{VV} have meaning as functions of q and z. Under such identification we restrict q to $0 < q < 1$ and z to $1 < z < q^{-rh^\vee}$.

Next we explain the ground states of our IRF model. Fix a particular site i. Consider the horizontal half infinite line l on $\mathscr{L}$ having i as the left end. The ground states are labeled by the pair (ξ,η) ($\xi \in (P^0_+)_{k-N}, \eta \in (P^0_+)_N$). The ground state corresponding to (ξ,η) is described as follows. Define the (ξ,η)-path a_{gr} such that $cl(a_{gr}(n-1) - a_{gr}(n)) = \mathrm{wt}(p_{gr}(n))$. Note that the sequences $\{a_{gr}(n)\}_{n\geq 0}$ and $\{p_{gr}(n)\}_{n\geq 0}$ are periodic. Place $a_{gr}(0)$, $a_{gr}(1), \ldots$ [resp. $p_{gr}(1)$, $p_{gr}(2), \ldots$) on every site (resp. edge) on l starting from i. The ground state is uniquely determined by the condition that it is periodic in the horizontal direction and constant along the NE-SW direction.

Take a dominant integral weight $\lambda \in (P^0_+)_k$. We consider the probability of finding the variable on i being the value λ, and denote it by $P(\lambda|\xi,\eta)$. Here (ξ,η) signifies the choice of a boundary condition. Assume that the initial condition (5.9) is valid for the C_{VV}. Thanks to the Yang-Baxter equation (5.7) and the second inversion relation (Proposition 5.2), Baxter's corner transfer matrix method [5] is applicable. We have the following expression for the one point function.

$$P(\lambda|\xi,\eta) = \frac{\chi_\lambda F(\lambda|\xi,\eta;q^{2rh^\vee})}{Z},$$

where

$$F(\lambda|\xi,\eta;q) = \sum_{\tilde{p}\in\mathscr{P}_{\mathrm{res}}(\xi,\eta;B)(\lambda)} q^{\omega(p)},$$

$$\mathscr{P}_{\mathrm{res}}(\xi,\eta;B)(\lambda) = \{\tilde{p} = (a,p) \in \mathscr{P}_{\mathrm{res}}(\xi,\eta;B) | a(0) = \lambda\},$$

$$Z = \sum_{\lambda'\in(P^0_+)_k} \chi_{\lambda'} F(\lambda'|\xi,\eta;q^{2rh^\vee}).$$

Here $\omega(p)$ and χ_λ are given in (6.1) and (5.11), respectively. Then we have the following. Let

$$\mathrm{High}(\xi,\eta)_\nu = \{v \in V(\xi)\otimes V(\eta) |\, \mathrm{wt}\, v = \nu,\ e_i v = 0\, \forall i \in I\}\,.$$

Proposition 6.2.

$$F(\lambda|\xi,\eta;q) = \sum_n \dim \mathrm{High}(\xi,\eta)_{\lambda-n\delta} q^n\,.$$

Moreover if the generalized Cartan matrix of $\mathfrak{g}$ *is symmetric,*

$$P(\lambda|\xi,\eta) = \frac{\chi_\lambda F(\lambda|\xi,\eta;q^{2rh^\vee})}{\chi_\xi\chi_\eta}\,.$$

Proof. From the theory of crystal base we have

$$\dim \mathrm{High}(\xi,\eta)_\nu = \#\,\mathrm{High}^0(\xi,\eta)_\nu\,,$$

where $\mathrm{High}^0(\xi,\eta)_\nu$ denotes the weight space of $\mathrm{High}^0(\xi,\eta)$ of weight ν. Hence the first statement is a direct consequence of Proposition 6.1.

If the generalized Cartan matrix of $\mathfrak{g}$ is symmetric, then, as noted before, χ_λ gives the principally specialized character of the $\mathfrak{g}$-module $V(\lambda)$. Therefore we have the following identity of specialized characters:

$$\chi_\xi\chi_\eta = \sum_{\lambda'\in(P^0_+)_k} \chi_{\lambda'} F(\lambda'|\xi,\eta;q^{2rh^\vee})$$

which provides us with the way of calculating the normalizing factor Z. □

Remark 1. The quantity $q^{s_\xi+s_\eta-s_\lambda}F(\lambda|\xi,\eta;q)$ is called the branching coefficient [16], where $s_\lambda = \dfrac{(\lambda+\varrho,\lambda+\varrho)}{r(k+h^\vee)} - \dfrac{(\varrho,\varrho)}{rh^\vee}$ for $\lambda \in (P^0_+)_k$. The transformation property of the branching coefficients under the modular transformation enables us to analyze the critical behavior of our one point function.

Remark 2. This type of results have been established by direct methods for higher spin representations of $U_q(\widehat{\mathfrak{sl}}_2)$ [17] and the vector representation of $U_q(\mathfrak{g})$ of classical types A, B, D [18, 19]. (There are problems for the type C since the vector representation is not perfect.) Proposition 6.2 covers and generalizes these results, on the assumption (yet to be verified) that the connection matrices coincide with the Boltzmann weights constructed in [3, 4].

Appendix. Vertex Operators for $U_q(\mathfrak{sl}_2)$

In this Appendix we study the vertex operators for integrable modules over $U_q(\mathfrak{sl}_2)$. Dropping indices we write the Chevalley generators as e, f, t. Let V_m denote the $m+1$ dimensional irreducible $U_q(\mathfrak{sl}_2)$-module with highest weight vector u^m_0. We set

$$u^m_k = \begin{bmatrix} m \\ k \end{bmatrix}^{-1} f^{(k)} u^m_0\,, \qquad f^{(k)} = \frac{f^k}{[k]!}\,.$$

The upper crystal base (L_m, B_m) of V_m at $q = 0$ is given by

$$L_m = \bigoplus_{k=0}^{m} A u^m_k\,, \qquad B_m = \{u^m_k\}_{0\le k\le m}\,.$$

Let now l, m, n be non-negative integers such that $l = m + n - 2s$, $s = 0, 1, \ldots, \min(m, n)$. We consider the intertwiners of the form

$$\Phi : V_l \to V_m \otimes V_n \,,$$

where the tensor product is taken with respect to $\Delta = \Delta_+$. Define c_{kj} by

$$\Phi(u_k^l) = \sum_{\max(0,k-n+s)\leq j\leq \min(m,k+s)} c_{kj} u_j^m \otimes u_{s+k-j}^n \,.$$

We are interested in the behavior of c_{kj} as $q \to 0$. Set $\bar{c}_{kj} = c_{l-k\,m-(k+s-j)}$. Explicitly the coefficients $c_{kj}, \bar{c}_{kj}$ are given by

Proposition A1.1.

$$\frac{c_{kj}}{c_{00}} = \sum_{\max(j-k,0)\leq \nu \leq \min(j,s)} (-1)^\nu q^{\nu(1+l-k)+j(k-j+m-l)} \times \begin{bmatrix} s \\ \nu \end{bmatrix} \begin{bmatrix} n-s+\nu \\ k-j+\nu \end{bmatrix} \begin{bmatrix} m-\nu \\ m-j \end{bmatrix} \Big/ \begin{bmatrix} l \\ k \end{bmatrix} . \tag{A1.1}$$

$$\frac{\bar{c}_{kj}}{\bar{c}_{00}} = \textit{the same formula with } m \textit{ and } n \textit{ interchanged}, \tag{A1.2}$$

$$\bar{c}_{00} = c_{00} \,. \tag{A1.3}$$

Proof. Solving $\Delta(e)w = 0$ for $w = \Phi(u_0^l) \in V_m \otimes V_n$ and applying $\Delta(f)$ to w we get (A1.1). Likewise starting from $\Phi(u_l^l)$ and applying $\Delta(e)$ we find (A1.2). We omit the details. □

Proposition A1.2. *We have*

$$\frac{c_{jk}}{c_{00}} = q^{(k-j)(m-s-j)}(1 + \ldots) \,, \qquad (k \geq j,\, j \leq m-s) \tag{A1.4a}$$

$$= (-1)^{j-k} q^{(j-k)(m-s-k+1)}(1 + \ldots) \,, \qquad (k \leq j,\, k \leq m-s) \tag{A1.4b}$$

$$= (-1)^{j-m+s} q^{(j-m+s)(k-m+s+1)}(1 + \ldots) \,, \qquad (k \geq m-s,\, j \geq m-s) \,, \tag{A1.4c}$$

where $\ldots$ *means terms in* qA.

Proof. A direct computation shows that in the case $k \leq m - s$ the right-hand side of (A1.1) contains a unique term which gives the lowest power of q. The estimates (A1.4a) for $k \leq m - s$ and (A1.4b) are derived in this way. The other case can be treated similarly by using (A1.2, A1.3). □

The following is an immediate consequence of Proposition A1.2.

Corollary A1.3.

(i) *Let* $\Phi(u_0^l) = u_0^m \otimes v + \ldots$, *where* $\ldots$ *stands for a sum of terms* $u' \otimes v'$, $u' \in \mathbf{Q}(q)u_j^m$ *with* $j \neq 0$. *If* $v \in L_n$, *then* $\Phi(L_l) \subset L_m \otimes L_n$.

(ii) *Suppose moreover that* $v \bmod qL_n \in B_n$. *Then* Φ *induces a morphism of crystals*

$$\Phi : B_l \to B_m \otimes B_n \,.$$

Remark. From Sect. 2.6 we can deduce analogous results for lower crystal lattices, replacing u_k^m by $v_k^m = f^{(k)}u_0^m$ and Δ_+ by Δ_-. Proposition A1.2 and Corollary A1.3 are both valid in this setting.

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