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Radiation Conditions and Scattering Theory for N-Particle Hamiltonians

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Abstract. The correct form of the angular part of radiation conditions is found in scattering problem for N-particle quantum systems. The estimates obtained allow us to give an elementary proof of asymptotic completeness for such systems in the framework of the theory of smooth perturbations.

1. Introduction

One of the main problems of scattering theory is a description of asymptotic behaviour of N interacting quantum particles for large times. The complete classification of all possible asymptotics (channels of scattering) is called asymptotic completeness. The final result can easily be formulated in physics terms. Two particles can either form a bound state or are asymptotically free. In the case $N \geqq 3$ a system of N particles can additionally be decomposed for large times into non-trivial subsystems (clusters). Particles of the same cluster form a bound state and different clusters do not interact with each other.

There are two essentially different approaches to a proof of asymptotic completeness for multiparticle ($N \geqq 3$) quantum systems. The first of them, suggested by L.D. Faddeev [1], relies on the detailed study of a set of equations derived by him for the resolvent of the corresponding Hamiltonian. This approach was developed in [1] for the case of three particles and was further elaborated by J. Ginibre and M. Moulin [2] and L. Thomas [3]. The attempts [4, 5] towards a straightforward generalization of Faddeev's method to an arbitrary number of particles meet with numerous difficulties. However, the results of [6] for weak interactions are quite elementary.

Another approach relies on the commutator method [7] of T. Kato. In the theory of N-particle scattering it was introduced by R. Lavine [8, 9] for repulsive potentials. The proof of asymptotic completeness in the general case is much more complicated and is due to I. Sigal and A. Soffer [10] (see also the article [11] by

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J. Derezinski for the proof of intermediary analytical results). In the recent paper [12] G.M. Graf gave an accurate proof of asymptotic completeness in the time-dependent framework. The distinguishing feature of [12] is that all intermediary results are also purely time-dependent and most of them have a direct classical interpretation. Papers [10, 12] were to a large extent inspired by V. Enss (see e.g. [13]) who was the first to apply a time-dependent technique for the proof of asymptotic completeness. Actually, for three-particle systems V. Enss proved asymptotic completeness for arbitrary short-range potentials (fall-off at infinity quicker than that of the Coulomb potential) and without implicit assumptions on subsystems.

The aim of the present paper is to give an elementary proof of asymptotic completeness for N-particle Hamiltonians H with short-range potentials which fits into the theory of smooth perturbations [7, 14]. This proof is quite similar to the one [15] suggested by the author for three-particle Hamiltonians. One of the advantages of the theory of smooth perturbations is that it admits two equivalent formulations. The first of them, time-dependent, is given in terms of unitary groups of the Hamiltonians considered. Another, the stationary one, is based on their resolvents. In particular, the stationary version automatically gives (see e.g. [16]) formulas for basic objects of scattering theory: wave operators, scattering matrix, etc. Properties of these representations, specific for N-particle systems, will hopefully be discussed elsewhere.

Our proof of asymptotic completeness relies on new estimates which establish some kind of radiation conditions for N-particle systems. We are concerned only with the angular part of these conditions. Actually, only this part admits a formulation in terms of H-smoothness and is needed for the proof of asymptotic completeness. Compared to the limiting absorption principle (see below) the radiation conditions-estimates give us additional information on the asymptotic behaviour of a quantum system for large distances and large times. The limiting absorption principle suffices for a proof of asymptotic completeness in the case of two-particle Hamiltonians with short-range potentials. However, radiation conditions-estimates are crucial in scattering for long-range potentials (see e.g. [17]), in scattering by unbounded obstacles [18, 19] and in scattering for anisotropically decreasing potentials [20]. In the latter paper (see also the lecture [21] by J.M. Combes) the role of radiation conditions was advocated for three-particle Hamiltonians.

Our interpretation of radiation conditions is, of course, different from the two-particle case. Before discussing their precise form let us introduce the generalized N-particle Hamiltonians. We consider the self-adjoint Schrödinger operator $H = -\Delta + V(x)$ in the Hilbert space $\mathscr{H} = L_2(\mathbb{R}^d)$. Suppose that some finite number α_0 of subspaces X^α of $X := \mathbb{R}^d$ is given and let x^α, x_α be the orthogonal projections of $x \in X$ on X^α and $X_\alpha = X \ominus X^\alpha$, respectively. We assume that

$$V(x) = \sum_{\alpha=1}^{\alpha_0} V^\alpha(x^\alpha), \tag{1.1}$$

where V^α are decaying real functions of variables x^α. Without loss of generality we can suppose that the linear sum of all subspaces X^α exhausts X. The two-particle Hamiltonian H is recovered if (1.1) consists of only one term with $X^\alpha = X$. The three-particle problem is distinguished from the general situation by the condition $X_\alpha \cap X_\beta = \{0\}$ for $\alpha \neq \beta$. We prove asymptotic completeness under the assumption that V^α are short-range functions of x^α but many intermediary results (in particular, radiation conditions-estimates) are as well true for long-range potentials. Clearly, $V^\alpha(x^\alpha)$ tends to zero as $|x| \to \infty$ outside of any conical neighbourhood of X_α and $V^\alpha(x^\alpha)$ is constant on planes parallel to X_α. Due to this property the structure of the spectrum of H is much more complicated than in the two-particle case. Operators H considered here were inroduced in [22] and are natural generalizations of N-particle Hamiltonians. Consideration of a more general class of operators allows to unravel better the geometry of the problem.

The spectral theory of the operator H starts with the following geometrical construction. Let us introduce the set $\mathscr{X}$ of linear sums

$$X^a = X^{\alpha_1} + X^{\alpha_2} + \ldots + X^{\alpha_k} \tag{1.2}$$

of subspaces X^{α_j}. The zero subspace $X^0 = \{0\}$ is included in the set $\mathscr{X}$ and X itself is excluded. The index a labels all subspaces $X^a \in \mathscr{X}$ and can be interpreted as the collection of all those α_j for which $X^{\alpha_j} \subset X^a$. Let x^a and x_a be the orthogonal projections of $x \in X$ on the subspace X^a and its orthogonal complement

$$X_a := X \ominus X^a = X_{\alpha_1} \cap X_{\alpha_2} \cap \ldots \cap X_{\alpha_k}, \tag{1.3}$$

respectively. Since $X = X_a \oplus X^a$ (the symbol $\oplus$ denotes the orthogonal sum), $\mathscr{H}$ splits into a tensor product

$$L_2(X) = L_2(X_a) \otimes L_2(X^a) . \tag{1.4}$$

In the multiparticle terminology, index a parametrizes decompositions of an N-particle system into noninteracting clusters; x^a is the set of "internal" coordinates of all clusters, x_a describes the relative motion of clusters.

Let us introduce for each a an auxiliary operator $H_a = T + V^a$, $T = -\Delta$, with a potential

$$V^a = \sum_{X^\alpha \subset X^a} V^\alpha , \tag{1.5}$$

which does not depend on x_a. In the representation (1.4),

$$H_a = T_a \otimes I + I \otimes H^a , \tag{1.6}$$

where I is the identity operator (in different spaces), $T_a = -\Delta_a = -\Delta_{x_a}$ acts in the space $\mathscr{H}_a = L_2(X_a)$ and

$$H^a = T^a + V^a, \quad T^a = -\Delta^a = -\Delta_{x^a} \tag{1.7}$$

are the operators in the space $\mathscr{H}^a = L_2(X^a)$. Set $\mathscr{H}^0 = \mathbb{C}$, $V^0 = 0$, $H^0 = 0$. The operator H^a corresponds to the Hamiltonian of clusters with their centers-of-mass fixed at the origin, T_a is the kinetic energy of the center-of-mass motion of these clusters and H_a describes an N-particle system with interactions between different clusters neglected. Eigenvalues of the operators H^a are called thresholds for the Hamiltonian H. Denote $U(t) = \exp(-iHt)$, $U_a(t) = \exp(-iH_a t)$ and let $E(\Lambda)$ and $E_a(\Lambda)$ be the spectral projections of the operators H and H_a corresponding to a Borel set $\Lambda \subset \mathbb{R}$. According to (1.6) the spectral analysis of the operator H_a reduces to that of H^a. In particular, if $\tilde{X} = \sum_\alpha X^\alpha \neq X$ so that $V(x) = V(\tilde{x})$, then we can replace H by the operator $\tilde{H} = -\Delta_{\tilde{x}} + V$ in the space $L_2(\tilde{X})$.

To formulate the limiting absorption principle let us inroduce the operator Q of multiplication by $(x^2 + 1)^{1/2}$. The limiting absorption principle asserts that the operator Q^{-r} is locally H-smooth (in the sense of T. Kato) for any $r > 1/2$. The term "locally" means that actually only the operator $Q^{-r}E(\Lambda)$ is H-smooth for an arbitrary bounded interval Λ which is separated from all thresholds and eigenvalues of H. A definition of H-smoothness of the operators $Q^{-r}E(\Lambda)$ can be given either in terms of the resolvent $(H-z)^{-1}$, $\operatorname{Im} z \neq 0$, of the operator H or of its unitary group $U(t)$. This is discussed e.g. in [16] or [23]. In Sect. 2 we recall the definition in terms of $U(t)$. The limiting absorption principle ensures, in particular, that the singular continuous spectrum of H is empty. Furthermore, in the case $N = 2$ (but not $N > 2$) it suffices for construction of scattering theory. There are many different proofs of the limiting absorption principle for $N = 2$ but the only one applicable for $N > 2$ relies on the Mourre estimate [24, 25, 26] which is formulated in Sect. 2.

The fundamental result of N-particle scattering theory called asymptotic completeness is the assertion that the evolution governed by the Hamiltonian H is decomposed as $t \to \pm\infty$ into a sum of simpler evolutions governed by the Hamiltonians H_a. This means that for every f which is orthogonal to eigenvectors of H there exist $f_a^{\pm}$ such that

$$U(t)f \sim \sum_a U_a(t)f_a^{\pm}, \quad t \to \pm\infty, \tag{1.8}$$

where " $\sim$ " denotes that the difference between left and right sides tends to zero. This relation is also sometimes called asymptotic clustering. More detailed formulation of the scattering problem for N-particle Hamiltonians is given in terms of wave operators (see Theorem 2.7). Using separation of variables (1.6) and applying (1.8) to the Hamiltonians H^a (in place of H) one can describe the asymptotics of $U(t)f$ in terms of the "free" operators T_a and of eigenvalues λ_n^a and eigenvectors ψ_n^a of the operators H^a. Actually, by inductive procedure, (1.8) yields

$$U(t)f \sim \sum_a \sum_n \exp(-i(T_a + \lambda_n^a)t) f_{a,n}^{\pm} \otimes \psi_n^a, \quad f_{a,n}^{\pm} \in \mathscr{H}_a, \tag{1.9}$$

where for every a the tensor product is the same as in (1.4). In particular, in the two-particle case the right side of (1.9) consists of the single term $\exp(-iTt)f^{\pm}$, where $f^{\pm} \in \mathscr{H}$.

As was already mentioned, our proof of the asymptotic completeness is based on radiation conditions-estimates. Actually, there is only one estimate which looks differently in different regions of the configuration space X. Denote by $\chi(M)$ the characteristic function of a set M. Let $\nabla_a = \nabla_{x_a}$ be the gradient in the variable x_a (i.e. $\nabla_a u$ is the orthogonal projection of ∇u on the subspace X_a) and let $\nabla_a^{(s)}$,

$$(\nabla_a^{(s)} u)(x) = (\nabla_a u)(x) - |x_a|^{-2} \langle (\nabla_a u)(x), x_a \rangle x_a, \tag{1.10}$$

be its orthogonal projection in X_a on the plane orthogonal to the vector x_a. We emphasize that the space X and its subspaces X_a are not distinguished in notation from their dual spaces to which ∇u and $\nabla_a u$ belong. The symbol $\langle \cdot, \cdot \rangle$ denotes the scalar product in different Euclidean spaces. Let Γ_a be an arbitrary closed cone in $\mathbb{R}^d$ such that $\Gamma_a \cap X_b = \{0\}$ if $X_a \not\subset X_b$. Our main analytical result is that for every a the operator

$$\mathscr{G}_a = \chi(\Gamma_a) Q^{-1/2} \nabla_a^{(s)} \tag{1.11}$$

is locally H-smooth. This result is formulated as a certain estimate which, by analogy with the two-particle problem, we call the radiation conditions-estimate. It can be expressed either in terms of the resolvent of H or of its unitary group. Actually, it suffices to verify local H-smoothness of the operators $G_a = \chi(\mathbf{Y}_a)Q^{-1/2}\nabla_a^{(s)}$, where $\mathbf{Y}_a$ is the intersection of $\mathbf{\Gamma}_a$ with some conical neighbourhood of X_a. In other words, $\mathbf{Y}_a$ is a neighbourhood of X_a with some neighbourhoods of all X_b, $X_a \not\subset X_b$, removed from it. Considering the collection of operators G_a for all a we obtain H-smoothness of the operators $\mathscr{G}_a E(\Lambda)$.

Let us compare the limiting absorption principle with the radiation conditions-estimates. Note that the operator $Q^{-1/2}$ is definitely not H-smooth even in the free case $H = -\Delta$. Thus the radiation conditions-estimates show that the differential operators $\nabla_a^{(s)}$ improve the fall-off of functions $(U(t)f)(x)$ for large t and $x \in \mathbf{\Gamma}_a$. In particular, the operator $\nabla^{(s)}$ is "improving" in the "free" region $\mathbf{\Gamma}_0$, where all potentials V^α are vanishing. H-smoothness of the operator $\chi(\mathbf{\Gamma}_0)Q^{-1/2}\nabla^{(s)}$ is not very astonishing from the viewpoint of analogy with the classical mechanics. Indeed, for the free motion the vector $x(t)$ of the position of a particle is directed asymptotically as its momentum ξ (corresponding to the operator $-i\nabla$). So the projection of ξ on the plane orthogonal to $x(t)$ tends to zero. According to the conjecture (1.9) in the region $\mathbf{\Gamma}_a$ (for arbitrary a) the evolution in the variable x_a (corresponding to the relative motion of clusters of particles) is also asymptotically free. Therefore one can expect that for every a the operator $\chi(\mathbf{\Gamma}_a)\nabla_a^{(s)}$ is "improving."

We emphasize that the radiation conditions-estimates are to a certain extent similar (at least from the viewpoint of classical interpretation) to propagation estimates of I.M. Sigal, A. Soffer (see Sects. 7 and 8 of [10]) and of G.M. Graf (see Theorems 4.2 and 4.3 of [12]). However, the study of the N-particle problem in [10] is impeded by a difficult analysis of the so-called threshold energies of the operator $-iQ^{-1}\langle x, \nabla\rangle - i\langle\nabla, x\rangle Q^{-1}$. We dispense completely with this analysis. Though in [10] (as in the present paper) the smooth method is adopted, we believe that our approach is closer to that of [12]. The difference with [12] is that in place of the stationary operator $\mathscr{G}_a$ G.M. Graf considers some differential operator with time-dependent coefficients. To construct scattering theory he establishes a stronger estimate which, roughly speaking, corresponds to H-smoothness of the operator $\chi(\mathbf{\Gamma}_a)Q^{-1/2}|\nabla_a^{(s)}|^{1/2}$. We do not possess such an estimate and develop scattering theory relying only on H-smoothness of the operators $\mathscr{G}_a$.

Our proof in Sect. 4 of local H-smoothness of the operators G_a hinges on the commutator method rather than the integration-by-parts machinery which is used (see e.g. [17]) to derive the radiation conditions-estimates in the two-particle case. Actually, we construct such an H-bounded operator M that the commutator $[H, M] := HM - MH$ satisfies locally the estimate

$$i[H, M] \geqq c(G_a^* G_a - Q^{-\rho}), \quad \rho > 1, \quad \forall a\,. \tag{1.12}$$

The arguments of [7] (reproduced in the proof of Theorem 4.5) show that H-smoothness of the operator G_a is a direct consequence of this estimate and of the limiting absorption principle. We look for an operator M in a form of a first-order differential operator

$$M = \sum_{j=1}^{d} (m_j D_j + D_j m_j), \quad m_j = \partial m/\partial x_j, \quad D_j = -i\partial/\partial x_j, \tag{1.13}$$

with a suitably chosen real function m which we call generating for M. This function is constructed in Sect. 3. Note that m is a homogeneous function of degree 1 so that coefficients m_j of the operator M are bounded. Due to the operator $Q^{-\rho}$ in

(1.12) values of m in a compact domain are inessential. The commutator $i[H_0, M]$ yields the leading contribution $G_a^* G_a$ to the right side of (1.12). The commutators

$$i[V^\alpha, M] = -2\langle \nabla m(x), \nabla V^\alpha(x^\alpha)\rangle, \quad \text{where } \nabla V^\alpha = \nabla^\alpha V^\alpha := \nabla_{x^\alpha} V^\alpha , \tag{1.14}$$

are controlled by the operator $Q^{-\rho}$.

To give an idea of the choice of m suppose for a moment that $m(x) = |x|$. Then there is the identity

$$i[H_0, M] = 4\nabla^{(s)}|x|^{-1}\nabla^{(s)}, \quad H_0 = T = -\Delta ,$$

which can be deduced e.g. from the formulas (4.1) and (4.6) below. Furthermore, by (1.14),

$$i[V^\alpha, M] = -2|x|^{-1}\langle x^\alpha, \nabla V^\alpha(x^\alpha)\rangle . \tag{1.15}$$

Thus, under proper assumptions on V^α, we have that in the case $X^\alpha = X$,

$$[V^\alpha, M] = O(|x|^{-\rho}), \quad |x| \to \infty , \tag{1.16}$$

for some $\rho > 1$. This yields the estimate (1.12) and hence smoothness of the operator $Q^{-1/2}\nabla^{(s)}$ with respect to the two-particle Schrödinger operator H.

However, if $X^\alpha \neq X$, then functions (1.15) decrease only as $|x|^{-1}$ at infinity. Actually, one can not expect that the operator $Q^{-1/2}\nabla^{(s)}$ is smooth with respect to the N-particle Hamiltonian H. To prove a weaker result about H-smoothness of the operators G_a the function $m(x)$ should be modified in such a way that the estimate (1.16) with some $\rho > 1$ holds for all α. According to (1.14), this is true if $m(x)$ depends only on the variable x_a in some cone where $V^\alpha(x^\alpha)$ is concentrated. A similar idea was applied by G.M. Graf [12] in the time-dependent context. We emphasize that our requirement on the function $m(x)$ ensures that $m(x) = m(x_a)$ in some conical neighbourhood of every X_a. In other words, a level surface $m(x) = \text{const}$ (which is a sphere for $m(x) = |x|$) should be flattened in an appropriate way in a neighbourhood of each X_a. Another restriction on $m(x)$ is that the commutator $i[H_0, M]$ should be positive (up to an error $O(|x|^{-\rho})$, $\rho > 1$). This demands that $m(x)$ be a convex function. In this case we can neglect the region $X \setminus \mathbf{Y}_a$ by the derivation of the estimate (1.12). It turns out that flattening and convexity are compatible. However, the commutator $i[H_0, M]$ gets smaller compared to the case $m(x) = |x|$ so that radiation conditions-estimates in the N-particle case are weaker for $N > 2$ than for $N = 2$. Note also that due to localization in energy in this estimate we can easily dispense with derivatives of V^α and prove H-smoothness of the operators G_a both in short-range and long-range cases.

In Sect. 5 we introduce wave operators (for the definition, see Sect. 2)

$$W^\pm(H, H_a; M^{(a)}E_a(\Lambda)), \quad W^\pm(H_a, H; M^{(a)}E(\Lambda)) \tag{1.17}$$

with "identifications"

$$M^{(a)} = \sum_{j=1}^{d} (m_j^{(a)} D_j + D_j m_j^{(a)}), \quad m_j^{(a)} = \partial m^{(a)}/\partial x_j , \tag{1.18}$$

which are first-order differential operators with suitably chosen (in Sect. 3) generating functions $m^{(a)}$. The "effective perturbation" equals

$$HM^{(a)} - M^{(a)}H_a = [T, M^{(a)}] + [V^a, M^{(a)}] + V_a M^{(a)} , \tag{1.19}$$

where V^a is defined by (1.5) and

$$V_a = V - V^a = \sum_{X^\alpha \not\subset X^a} V^\alpha . \tag{1.20}$$

According to the results of [14, 27] to prove existence of the wave operators (1.17) it suffices to verify that every term in the right side of (1.19) can be factorized into a product K^*K_a, where K is H-smooth and K_a is H_a-smooth (locally). Functions $m^{(a)}$ are chosen as homogeneous functions of order 1 (for $|x| \geqq 1$). Therefore coefficients of the second-order differential operator $[T, M^{(a)}]$ are decreasing only as $|x|^{-1}$ at infinity. It turns out that this term can be considered with the help of the radiation conditions-estimates. Furthermore, similar to $m(x)$, the function $m^{(a)}(x)$ depends only on x_α in some conical neighbourhood of each X_α. This ensures that $[V^{(a)}, M^{(a)}] = O(|x|^{-\rho})$, where $\rho > 1$. Finally, it is required that $m^{(a)}(x)$ equals zero in some conical neighbourhood of X_α such that $X_a \not\subset X_\alpha$. So coefficients of the operator $V_a M^{(a)}$ also vanish as $O(|x|^{-\rho})$, $\rho > 1$, at infinity. Thus the second and third terms in the right side of (1.19) can be taken into account by *the limiting absorption principle. The obtained representation for the operator* (1.19) ensures existence of both wave operators (1.17) (for all a).

Existence of the second wave operator (1.17) implies that for every vector $f^\pm \in E(\Lambda)\mathscr{H}$ and some $f_a^\pm$

$$M^{(a)} U(t) f^\pm \sim U_a(t) f_a^\pm, \quad t \to \pm\infty . \tag{1.21}$$

If the sum of $M^{(a)}$ over all a were equal to the identity operator I, then summing up, as advised in [28], the relations (1.21) we would have obtained the asymptotic completeness (1.8). However, the equality $\sum_a M^{(a)} = I$ is incompatible with the definition (1.18). We choose functions $m^{(a)}$ in such a way that

$$\sum_a M^{(a)} = M , \tag{1.22}$$

where M is the same operator as in (1.12). Summing up the relations (1.21) we find only that

$$MU(t) f^\pm \sim \sum_a U_a(t) f_a^\pm, \quad t \to \pm\infty . \tag{1.23}$$

Note that in the paper [12] of G.M. Graf the role of $M^{(a)}$ is played by time-dependent identifications such that their sum is essentially equal to I.

At the final step of the proof of the asymptotic completeness we get rid of the operator M in the left side of (1.23). To that end we introduce the observable

$$M^\pm(\Lambda) = W^\pm(H, H; ME(\Lambda)) \tag{1.24}$$

and verify that the range of the operator $M^\pm(\Lambda)$ coincides with the subspace $E(\Lambda)\mathscr{H}$. Actually, we show that the operator $\pm M^\pm(\Lambda)$ is positively definite on $E(\Lambda)\mathscr{H}$. By virtue of the inequality $m(x) \geqq |x|$ for $|x| \geqq 1$, this can be easily derived from the Mourre estimate. Here we shall explain this result by analogy with classical mechanics. Let us consider a particle (of mass 1/2) in an external field. In this case the observable $U^*(t)MU(t)$ corresponds, in the Heisenberg picture of motion, to the projection $\mathscr{M}(t) = |x(t)|^{-1}\langle \xi(t), x(t)\rangle$ of the momentum $\xi(t)$ of a particle on a vector $x(t)$ of its position. For positive energies λ and large t we have that $\xi(t) \sim \xi_\pm$, $\xi_\pm^2 = \lambda$, and $x(t) \sim 2\xi_\pm t + x_\pm$. Therefore $\mathscr{M}(t)$ tends to $\pm\lambda^{1/2}$ as $t \to \pm\infty$.

In Sect. 6 we conclude our proof of the main Theorem 2.7. The asymptotic completeness in the form (1.8) is a direct consequence of the results obtained in previous sections. Actually, every $f \in E(\Lambda)\mathscr{H}$ admits the representation $f = M^{\pm}(\Lambda) f^{\pm}$ so that

$$U(t) f \sim M U(t) f^{\pm}, \quad t \to \pm \infty . \tag{1.25}$$

Comparing (1.25) with (1.23) we arrive immediately at (1.8). It remains to establish existence of the wave operators $W^{\pm}(H, H_a)$. This is deduced from existence of the first set of the wave operators (1.17). At this step we assume validity of Theorem 2.7 for all operators H^a (in place of H). This additional assumption is, finally, removed by an inductive procedure.

2. Basic Notions of Scattering Theory

Let us briefly recall some basic definitions of the scattering theory. For a self-adjoint operator H in a Hilbert space $\mathscr{H}$ we introduce the following standard notation: $\mathscr{D}(H)$ is its domain; $\sigma(H)$ is its spectrum; $E(\Omega; H)$ is the spectral projection of H corresponding to a Borel set $\Omega \subset \mathbb{R}$; $\mathscr{H}^{(\mathrm{ac})}(H)$ is the absolutely continuous subspace of H; $P^{(\mathrm{ac})}(H)$ is the orthogonal projection on $\mathscr{H}^{(\mathrm{ac})}(H)$; $\mathscr{H}^{(\mathrm{p})}(H)$ is the subspace spanned by all eigenvectors of the operator H; $\sigma^{(p)}(H)$ is the spectrum of the restriction of H on $\mathscr{H}^{(\mathrm{p})}(H)$, i.e. $\sigma^{(\mathrm{p})}(H)$ is the closure of the set of all eigenvalues of H. Norms of vectors and operators in different spaces are denoted by the same symbol $\| \cdot \|$; $\mathscr{B}$ and $\mathscr{K}_{\infty}$ are the classes of bounded and compact operators (in different spaces) respectively; $R(B)$ is the range of an operator B; "s-lim" means the strong operator limit; C and c are positive constants whose precise values are of no importance. Note that operators $\exp(-iHt) P^{(\mathrm{ac})}(H)$ converge weakly to zero as $|t| \to \infty$ and hence

$$s - \lim_{|t| \to \infty} K \exp(-iHt) P^{(\mathrm{ac})}(H) = 0, \quad \text{if } K \in \mathscr{K}_{\infty} . \tag{2.1}$$

Let K be H-bounded operator, acting from $\mathscr{H}$ into, possibly, another Hilbert space $\mathscr{H}'$. It is called H-smooth (in the sense of T. Kato) on a Borel set $\Omega \subset \mathbb{R}$ if for every $f = E(\Omega; H) f \in \mathscr{D}(H)$,

$$\int_{-\infty}^{\infty} \| K \exp(-iHt) f \|^2 dt \leqq C \| f \|^2 .$$

Obviously, BK is H-smooth on Ω if K has this property and $B \in \mathscr{B}$.

Let now H_j, $j = 1, 2$, be a couple of self-adjoint operators and let J be a bounded operator in a Hilbert space $\mathscr{H}$. The wave operator for the pair H_1, H_2 and the "identification" J is defined by the relation

$$W^{\pm}(H_2, H_1; J) = s - \lim_{t \to \pm\infty} \exp(iH_2 t) J \exp(-iH_1 t) P^{(\mathrm{ac})}(H_1) \tag{2.2}$$

under the assumption that this limit exists. We emphasize that all definitions and considerations for " + " and " − " are independent of each other. It suffices to verify existence of the limit (2.2) on some set dense in $\mathscr{H}$. If the wave operator (2.2) exists, then the intertwining property

$$E_2(\Omega) W^{\pm}(H_2, H_1; J) = W^{\pm}(H_2, H_1; J) E_1(\Omega) \tag{2.3}$$

($\Omega \subset \mathbb{R}$ is any Borel set and $E_i(\Omega) = E_i(\Omega; H_i)$) holds. It follows that the range of the operator (2.2) is contained in $\mathscr{H}^{(ac)}(H_2)$ and its closure is an unvariant subspace of H_2. Moreover, if the wave operator is isometric on some subspace $\mathscr{H}_1$, then the restrictions of H_1 and H_2 on the subspaces $\mathscr{H}_1$ and $\mathscr{H}_2 = W^{\pm}(H_2, H_1; J)\mathscr{H}_1$, respectively, are unitarily equivalent. This equivalence is realized by the wave operator. Clearly, for every $f_2 = W^{\pm}(H_2, H_1; J)f_1$,

$$\exp(-iH_2t)f_2 \sim J\exp(-iH_1t)f_1, \quad t \to \pm\infty .$$

In the case $J = I$ we omit dependence of wave operators on J. The operator $W^{\pm}(H_2, H_1)$ is obviously isometric on $\mathscr{H}^{(ac)}(H_1)$. The operator $W^{\pm}(H_2, H_1)$ is called complete if $R(W^{\pm}(H_2, H_1)) = \mathscr{H}^{(ac)}(H_2)$. This is equivalent to existence of the wave operator $W^{\pm}(H_1, H_2)$.

We note also the multiplication theorem

$$W^{\pm}(H_3, H_1; \tilde{J}J) = W^{\pm}(H_3, H_2; \tilde{J})W^{\pm}(H_2, H_1; J) . \tag{2.4}$$

More precisely, if both wave operators in the right side exist, then the wave operator in the left side also exists and the equality (2.4) holds.

We need the following sufficient condition of existence of wave operators.

Proposition 2.1. *Let an operator $\mathscr{J}$ be H_1-bounded and let its adjoint $\mathscr{J}^*$ be H_2-bounded. Suppose that for some $N < \infty$,*

$$H_2\mathscr{J} - \mathscr{J}H_1 = \sum_{n=1}^{N} K_{2,n}^* K_{1,n}$$

(in the precise sense this should be understood as an equality of sesquilinear forms on $\mathscr{D}(H_1) \times \mathscr{D}(H_2)$), where the operators $K_{j,n}$ are H_j-bounded and are H_j-smooth on some bounded interval Λ. Then the wave operators

$$W^{\pm}(H_2, H_1; \mathscr{J}E_1(\Lambda)), \quad W^{\pm}(H_1, H_2; \mathscr{J}^*E_2(\Lambda))$$

exist.

This result was obtained in the articles [14, 27]. Proof for the case $\mathscr{J} = I$ can be found e.g. in [23]. For arbitrary $\mathscr{J}$ the proof is practically the same [16]. Unboundedness of $\mathscr{J}$ is inessential because real identifications $\mathscr{J}E_1(\Lambda)$ and $\mathscr{J}^*E_2(\Lambda)$ are bounded operators. We use Proposition 2.1 only in the case $\mathscr{D}(H_1) = \mathscr{D}(H_2)$ and $\mathscr{J} = \mathscr{J}^*$.

Let us return to the N-particle problem. We consider an operator $H = T + V$ in the Hilbert space $\mathscr{H} = L_2(\mathbb{R}^d)$, where $T = -\Delta$ and V is multiplication by a function $V(x)$ defined by (1.1). We usually use the same notation for a function and the operator of multiplication by this function. Dimensions $d^\alpha = \dim X^\alpha \neq 0$ of the subspaces X^α are arbitrary. In particular, we do not exclude that one of the subspaces X^α coincides with the whole space $X = \mathbb{R}^d$.

As was already mentioned, we construct scattering theory for short-range potentials. However, the radiation conditions-estimates hold as well true in the case where potentials contain long-range parts. Therefore we distinguish two types of assumptions on functions V^α. The first of them is related to the short-range case and the second – to the general one. We recall (see (1.7)) that $T^\alpha = -\Delta^\alpha$ in the space $\mathscr{H}^\alpha = L_2(X^\alpha)$. Derivatives of V^α are understood below in the sense of distributions.

Assumption 2.2. *Operators $V^\alpha(T^\alpha + I)^{-1}$ are compact in the space $\mathscr{H}^\alpha$ and operators $(|x^\alpha| + 1)^\rho V^\alpha(T^\alpha + I)^{-1}$ are bounded in $\mathscr{H}^\alpha$ for some $\rho > 1$.*

Assumption 2.3. *Functions V^α admit representations*

$$V^\alpha = V_s^\alpha + V_l^\alpha , \tag{2.5}$$

where the short-range parts V_s^α satisfy Assumption 2.2 and long-range parts V_l^α satisfy the following condition. Operators $V_l^\alpha(T^\alpha + I)^{-1}$, $|\nabla V_l^\alpha|(T^\alpha + I)^{-1}$ are compact in the space $\mathscr{H}^\alpha$ and operators $(|x^\alpha| + 1)^\rho|\nabla V_l^\alpha|(T^\alpha + I)^{-1}$ are bounded in $\mathscr{H}^\alpha$ for some $\rho > 1$.

Compactness of $V^\alpha(T^\alpha + I)^{-1}$ ensures that the operator H is self-adjoint on the domain $\mathscr{D}(H) = \mathscr{D}(T) =: \mathscr{D}$ and H is semi-bounded from below. We have chosen Assumption 2.3 on functions V^α because it provides the limiting absorption principle for the operator H. Practically we use only that for $2r = \rho$ the operators

$$((x^\alpha)^2 + 1)^{r/2}|V_s^\alpha|^{1/2}(T^\alpha + I)^{-1/2} \quad \text{and} \quad ((x^\alpha)^2 + 1)^{r/2}|\nabla V_l^\alpha|^{1/2}(T^\alpha + I)^{-1/2}$$

are bounded in the space $\mathscr{H}^\alpha$. This is a consequence of Assumption 2.3 in virtue of the Heinz inequality. It follows that considered in the space $\mathscr{H}$ the operators $|V_s^\alpha|^{1/2}$ and $|\nabla V_l^\alpha|^{1/2}$ admit the representations

$$|V_s^\alpha|^{1/2} = B_s^\alpha(T + I)^{1/2}((x^\alpha)^2 + 1)^{-r/2}, \quad B_s^\alpha \in \mathscr{B} , \tag{2.6}$$

$$|\nabla V_l^\alpha|^{1/2} = B_l^\alpha(T + I)^{1/2}((x^\alpha)^2 + 1)^{-r/2}, \quad B_l^\alpha \in \mathscr{B} . \tag{2.7}$$

Similarly, under Assumption 2.2 the representation (2.6) is valid for the operator $|V^\alpha|^{1/2}$.

Recall that $\mathscr{X}$ is the set of all linear sums (1.2) with $X^0 = \{0\}$ included in $\mathscr{X}$ and X itself excluded. We define also the set $\mathscr{X}'$ of all orthogonal complements X_a to $X^a \in \mathscr{X}$. By (1.3), $\mathscr{X}'$ consists of all intersections of subspaces $X_\alpha = X \ominus X^\alpha$. The space $X_0 = X$ is included in $\mathscr{X}'$ and the zero-subspace $\{0\}$ is excluded from it. *Below a labels all elements of $\mathscr{X}$ or $\mathscr{X}'$.*

Let the operator H^a in the space $\mathscr{H}^a = L_2(X^a)$ be defined by equalities (1.5) and (1.7). The union over all a of their point spectra

$$\Upsilon_0 = \bigcup \sigma^{(p)}(H^a)$$

is called the set of thresholds for the operator H. We need the following basic result (see [24–26]) of spectral theory of multiparticle Hamiltonians. It is formulated in terms of the auxiliary operator

$$A = \sum_{j=1}^{d} (x_j D_j + D_j x_j), \quad D_j = -i\partial_j, \quad \partial_j = \partial/\partial x_j .$$

Proposition 2.4. *Let Assumption 2.3 hold. Then eigenvalues of H may accumulate only at Υ_0 so that the "exceptional" set $\Upsilon = \Upsilon_0 \cup \sigma^{(p)}(H)$ is closed and countable. Furthermore, for every $\lambda \in \mathbb{R} \setminus \Upsilon$ there exists a small interval $\Lambda_\lambda \ni \lambda$ such that the estimate (the Mourre estimate) for the commutator holds:*

$$i([H, A]u, u) \geqq c\|u\|^2, \quad c = c_\lambda > 0, \quad u \in E(\Lambda_\lambda)\mathscr{H} . \tag{2.8}$$

Let Q be multiplication by $(x^2 + 1)^{1/2}$. *Below Λ is always an arbitrary bounded interval such that $\bar{\Lambda} \cap \Upsilon = \emptyset$,* where $\bar{\Lambda}$ is the closure of Λ. One of the main consequences of (2.8) is the limiting absorption principle.

Proposition 2.5. *Let Assumption* 2.3 *hold. Then for any* $r > 1/2$ *the operator* Q^{-r} *is* H*-smooth on* Λ.

The proof of this assertion can be found in the article [20] which somewhat weakens requirements of [24, 25] on V^α.

Corollary 2.6. *The operator* H *is absolutely continuous on* $E(\Lambda)\mathscr{H}$. *In particular,* $\mathscr{H} = \mathscr{H}^{(\mathrm{p})}(H) \oplus \mathscr{H}^{(\mathrm{ac})}(H)$.

Let us give the precise formulation of the scattering problem for N-particle Hamiltonians. Denote by P^{a} the orthogonal projection in $\mathscr{H}^{\mathrm{a}}$ on the subspace $\mathscr{H}^{(\mathrm{p})}(H^{\mathrm{a}})$ and set $P_{\mathrm{a}} = I \otimes P^{\mathrm{a}}$, where the tensor product is defined by (1.4). According to (1.6) the orthogonal projection P_{a} commutes with the operator $H_a = T + V_a$ and its functions. Set also $H_0 = T$, $P_0 = I$. The basic result of the scattering theory for N-particle Schrödinger operators is the following

Theorem 2.7. *Let Assumption* 2.2 *hold. Then the wave operators*

$$W_a^\pm = W^\pm(H, H_a; P_a) \tag{2.9}$$

exist and are isometric on the subspaces $R(P_a)$. *The subspaces* $R(W_a^\pm)$ *are mutually orthogonal and the asymptotic completeness holds*:

$$\sum_a \bigoplus R(W_a^\pm) = \mathscr{H}^{(\mathrm{ac})}(H)\,. \tag{2.10}$$

Our assumptions on V^α are somewhat larger than those of I.M. Sigal and A. Soffer [10] or G.M. Graf [12] since we do not require anything about derivatives of V^α. Note, however, that the methods of [10] and [12] can also accommodate Assumption 2.2 (see [29] and [30], respectively). We remark that in [29] H. Tamura gave another, somewhat more careful, presentation of the method of [10].

Theorem 2.7 gives the complete spectral analysis of the operator H. Actually, by the relation (2.10), the absolutely continuous part $H^{(\mathrm{ac})}$ of H is the orthogonal sum of its restrictions on different subspaces $R(W_a^\pm)$. By virtue of the intertwining property $HW_a^\pm = W_a^\pm H_a$, each of these restrictions is unitarily equivalent to the operator H_a considered in the space $R(P_a)$. Actually, if $f \in \mathscr{H}^{(\mathrm{ac})}(H)$ and $f_a^\pm = (W_a^\pm)^* f \in R(P_a)$, then

$$f = \sum_a W_a^\pm f_a^\pm \quad \text{and} \quad Hf = \sum_a W_a^\pm H_a f_a^\pm\,. \tag{2.11}$$

Furthermore, according to (1.4), (1.6), for a function ϕ

$$\phi(H_a) f_a^\pm = \sum_a \phi(T_a + \lambda_n^a) f_{a,n}^\pm \otimes \psi_n^a, \quad \text{if} \quad f_a^\pm = \sum_a f_{a,n}^\pm \otimes \psi_n^a\,. \tag{2.12}$$

Thus $H^{(\mathrm{ac})}$ is unitarily equivalent to the orthogonal sum of the "free" operators T_{a} shifted by the eigenvalues of the operators H^{a}. Theorem 2.7 describes also the asymptotics as $t \to \pm\infty$ of the evolution $U(t)f$ governed by the Hamiltonian H. Indeed, the first equality (2.11) implies the relation (1.8) which in virtue of (2.12) can be rewritten as (1.9).

We conclude this section with some standard technicalities.

Lemma 2.8. *For any* $r \in [0, 1]$ *the operator* $[H, Q^r](T + I)^{-1/2} \in \mathscr{B}$.

Proof. Simple computation shows that

$$[H, Q^r] = [T, Q^r] = -2(\nabla q_r)\nabla - (\Delta q_r), \quad q_r(x) = (x^2 + 1)^{r/2}\,.$$

Since $r \leqq 1$, functions ∇q_r and Δq_r are bounded. $\square$

Lemma 2.9. *Let $\psi \in C_0^\infty(\mathbb{R})$ and $r \in [0, 1]$. Then $[\psi(H), Q^r] \in \mathscr{B}$.*

Proof. Note that

$$[U(t), Q^r] = -i \int_0^t U(s)[H, Q^r]U(t-s)ds .$$

Thus by virtue of Lemma 2.8,

$$\|[U(t), Q^r](|H| + I)^{-1/2}\| \leqq C|t| . \tag{2.13}$$

For an arbitrary ψ we have that

$$[\psi(H), Q^r] = \int_{-\infty}^{\infty} [U(t), Q^r]\hat{\psi}(t)dt, \quad 2\pi\hat{\psi}(t) = \int_{-\infty}^{\infty} \exp(i\lambda t)\psi(\lambda)d\lambda .$$

By (2.13), it follows that

$$[\psi(H), Q^r](|H| + I)^{-1/2} \in \mathscr{B}, \quad \text{since} \int_{-\infty}^{\infty} |t\hat{\psi}(t)|dt < \infty . \tag{2.14}$$

Finally, let $\psi_1 \in C_0^\infty(\mathbb{R})$ and $\psi_1(\lambda) = 1$ on support of ψ so that $\psi = \psi\psi_1$. Then $[\psi(H), Q^r] = \psi(H)[\psi_1(H), Q^r] + [\psi(H), Q^r]\psi_1(H)$ and both terms in the right side are bounded in virtue of (2.14). □

Lemma 2.10. *For $r \in [0, 1]$ and arbitrary $z \notin \sigma(H)$ the operator $Q^{-r}(T + I)(H - z)^{-1}Q^r$ is bounded.*

Proof. First we commute the last two factors:

$$(H - z)^{-1}Q^r = Q^r(H - z)^{-1} - (H - z)^{-1}[H, Q^r](H - z)^{-1} .$$

Let us multiply this equality by $Q^{-r}(T + I)$. By Lemma 2.8, $[H, Q^r](H - z)^{-1} \in \mathscr{B}$. Thus it remains to notice that the operator

$$Q^{-r}(T + I)Q^r(H - z)^{-1} = (T + I)(H - z)^{-1} + Q^{-r}[T, Q^r](H - z)^{-1}$$

is bounded according, again, to Lemma 2.8. □

Quite similarly we obtain the following result.

Lemma 2.11. *Suppose that a function v obeys the estimate*

$$|v(x)| + |(\nabla v)(x)| + |(\Delta v)(x)| \leqq C(|x| + 1)^{-r}, \quad r \in [0, 1] . \tag{2.15}$$

Then

$$(T + I)v(T + I)^{-1}Q^r \in \mathscr{B}, \quad (T + I)^{1/2}vD_j(T + I)^{-1}Q^r \in \mathscr{B}, \quad j = 1, \ldots, d .$$

Combining Lemma 2.10 with Proposition 2.5 we immediately obtain

Proposition 2.12. *For every $r > 1/2$ the operator $Q^{-r}(T + I)$ is H-smooth on Λ.*

Proof. For any $z \notin \sigma(H)$,

$$Q^{-r}(T + I)U(t)f = (Q^{-r}(T + I)(H - z)^{-1}Q^r)Q^{-r}U(t)(H - z)f .$$

Since the first factor in the right side is bounded it suffices to apply the definition of H-smoothness to the element $(H - z)f \in E(\Lambda)\mathscr{H}$. □

By Lemma 2.11, Proposition 2.12 ensures H-smoothness of the operators $Q^{-r}D_j$ where $r > 1/2$ and $j = 1, \ldots, d$. It contains also Proposition 2.5. Therefore we usually give references below only to Proposition 2.12.

Of course, all results formulated for the operator H are as well true for every operator H_a.

3. Generating Functions

In this section we construct "generating" functions $m(x)$ and $m^{(a)}(x)$ of the operators (1.13) and (1.18) respectively. We recall that the index a (b, f and g are also frequently used) labels, as always, all subspaces $X_a \subset \mathscr{X}'$ (with $X_0 = X$ included in $\mathscr{X}'$ and the zero subspace $\{0\}$ excluded from $\mathscr{X}'$). Set $d_a = \dim X_a$. For every a define a conical neighbourhood

$$\mathbf{X}_a(\varepsilon) = \{|x_a| > (1-\varepsilon)|x|\}, \quad \varepsilon \in (0, 1)\,, \tag{3.1}$$

of $X_a \backslash \{0\}$. In particular, $\mathbf{X}_0(\varepsilon) = X\backslash\{0\}$ for any ε. In this section we suppose that $x \neq 0$. Note an inequality

$$|x^f| \leqq C(|x^a| + |x^b|), \quad \text{where} \quad X^f = X^a + X^b\,. \tag{3.2}$$

Clearly, relations $X^f = X^a + X^b$ and $X_f = X_a \cap X_b$ are equivalent.

We construct, first, non-smooth generating functions parametrized by $\boldsymbol{\varepsilon} = \{\varepsilon_b\}$ and then average it over all ε_b. *Below we always assume that*

$$\varepsilon_b^{(1)} < \varepsilon_b < \varepsilon_b^{(2)}, \quad \textit{where} \quad \varepsilon_b^{(1)} = 2\epsilon^{d_b}, \quad \varepsilon_b^{(2)} = 3\epsilon^{d_b} \tag{3.3}$$

and $\epsilon > 0$ *is sufficiently small.* Such ε_b are sometimes called admissible. Functions $m(x, \boldsymbol{\varepsilon})$ and $m^{(a)}(x, \boldsymbol{\varepsilon})$ are defined in terms of auxiliary functions $(1 + \varepsilon_b)|x_b|$. An important property of this set of functions is formulated in the following

Lemma 3.1. *Let* $x \in \mathbf{X}_b(\epsilon^{d_b})$ *and* $X_a \not\subset X_b$. *Then*

$$(1 + 3\epsilon^{d_a})|x_a| < \max_{X_f \subset X_b} \{(1 + 2\epsilon^{d_f})|x_f|\} \tag{3.4}$$

(*the maximum in the right side of* (3.4) *is taken over all* f *such that* $X_f \subset X_b$).

Proof. Let us introduce $X_f = X_a \cap X_b$. The assumption $X_a \not\subset X_b$ implies that $X_f \neq X_a$ and hence $d_f < d_a$. We shall establish the inequality

$$(1 + 3\epsilon^{d_a})|x_a| < (1 + 2\epsilon^{d_f})|x_f| \tag{3.5}$$

for $x \in \mathbf{X}_f(\epsilon^{d_f})$ and the inequality

$$(1 + 3\epsilon^{d_a})|x_a| < (1 + 2\epsilon^{d_b})|x_b| \tag{3.6}$$

for

$$x \in \mathbf{X}_b(\epsilon^{d_b}) \backslash \mathbf{X}_f(\epsilon^{d_f})\,. \tag{3.7}$$

Note that in the case $X_f = \{0\}$ we verify (3.6). If $X_b \subset X_a$, then $X_f = X_b$ and the inequalities (3.5), (3.6) coincide with each other. In this case we verify (3.6) for all $x \in \mathbf{X}_b(\epsilon^{d_b})$. Without loss of generality we suppose that $|x| = 1$. To check (3.5) for $|x_f| > 1 - \epsilon^{d_f}$ it suffices to notice that

$$1 + 3\epsilon^{d_a} < (1 + 2\epsilon^{d_f})(1 - \epsilon^{d_f})\,.$$

This is true for sufficiently small ϵ because $d_a > d_f$.

Let now (3.7) be satisfied. We can assume $X_f \neq X_b$ so that $d_b > d_f$. Then

$$|x_b| > 1 - \epsilon^{d_b}, \quad |x_f| \leqq 1 - \epsilon^{d_f}, \tag{3.8}$$

and hence

$$|x^b|^2 < 1 - (1 - \epsilon^{d_b})^2 < 2\epsilon^{d_b}, \quad |x^f|^2 \geqq 1 - (1 - \epsilon^{d_f})^2 \geqq \epsilon^{d_f}, \quad \epsilon \leqq 1 .$$

Applying (3.2) and taking into account that $d_b > d_f$, we obtain the estimate $|x^a|^2 \geqq 2c\epsilon^{d_f}$ or $|x_a|^2 \leqq 1 - 2c\epsilon^{d_f}$ for some $c > 0$. By virtue of this estimate for $|x_a|$ and of the estimate (3.8) for $|x_b|$ the inequality (3.6) reduces to

$$(1 + 3\epsilon^{d_a})(1 - c\epsilon^{d_f}) < (1 + 2\epsilon^{d_b})(1 - \epsilon^{d_b}) .$$

For small ϵ the left side is smaller than 1 because $d_a > d_f$. The right side of this estimate is larger than 1. □

We define the function $m^{(a)}(x, \boldsymbol{\varepsilon})$ by the equality

$$m^{(a)}(x, \boldsymbol{\varepsilon}) = (1 + \varepsilon_a)|x_a| \theta((1 + \varepsilon_a)|x_a| - \max_{f \neq a} \{(1 + \varepsilon_f)|x_f|\}) , \tag{3.9}$$

where $\theta(s) = 1$ for $s \geqq 0$ and $\theta(s) < 0$ for $s < 0$. Recall that $\boldsymbol{\varepsilon}$ obeys always (3.3). The definition (3.9) can be rewritten in an equivalent form using the identity

$$\theta((1 + \varepsilon_a)|x_a| - \max_{f \neq a} \{(1 + \varepsilon_f)|x_f|\}) = \prod_{f \neq a} \theta((1 + \varepsilon_a)|x_a| - (1 + \varepsilon_f)|x_f|) . \tag{3.10}$$

Note that (3.9) is not changed if the maximum in its right side is taken over all f. The product in (3.10) can be taken over all f. Functions $m^{(a)}(x, \boldsymbol{\varepsilon})$ are obviously homogeneous of degree 1. Their important property is formulated in the following

Lemma 3.2. *Let $x \in \mathbf{X}_b(\epsilon^{d_b})$. Then $m^{(a)}(x, \boldsymbol{\varepsilon}) = 0$ if $X_a \not\subset X_b$. If $X_a \subset X_b$, then*

$$m^{(a)}(x, \boldsymbol{\varepsilon}) = (1 + \varepsilon_a)|x_a| \theta((1 + \varepsilon_a)|x_a| - \max_{X_f \subset X_b} \{(1 + \varepsilon_f)|x_f|\})$$

so that $m^{(a)}(x, \boldsymbol{\varepsilon}) = m^{(a)}(x_b, \boldsymbol{\varepsilon})$ does not depend on x^b.

Proof. According to Lemma 3.1 and (3.3) for $x \in \mathbf{X}_b(\epsilon^{d_b})$,

$$(1 + \varepsilon_g)|x_g| < \max_{X_f \subset X_b} \{(1 + \varepsilon_f)|x_f|\}, \quad \text{if } X_g \not\subset X_b . \tag{3.11}$$

In the case $X_a \not\subset X_b$ we apply (3.11) to $g = a$. Since the maximum in (3.11) can only increase if taken over all f, this shows that the argument of the function $\theta(\cdot)$ in (3.9) is negative.

Let $X_a \subset X_b$. One needs to verify that for $x \in \mathbf{X}_b(\varepsilon^{d_b})$,

$$\max_f \{(1 + \varepsilon_f)|x_f|\} = \max_{X_f \subset X_b} \{(1 + \varepsilon_f)|x_f|\} .$$

Clearly, the right side here is bounded by the left side. To prove the opposite inequality it suffices to take into account (3.11). □

Corollary 3.3. *If $x \in \mathbf{X}_a(\epsilon^{d_a})$, then*

$$m^{(a)}(x, \boldsymbol{\varepsilon}) = (1 + \varepsilon_a)|x_a| \theta((1 + \varepsilon_a)|x_a| - \max_{X_f \subset X_a} \{(1 + \varepsilon_f)|x_f|\}) .$$

Let us now average the functions $m^{(a)}(x, \boldsymbol{\varepsilon})$ over all admissible $\boldsymbol{\varepsilon}$. We set

$$m^{(a)}(x) = \int m^{(a)}(x, \boldsymbol{\varepsilon}) \prod_f \varphi_f(\varepsilon_f) d\varepsilon_f , \tag{3.12}$$

where $\varphi_f \geqq 0$,

$$\varphi_f \in C_0^\infty(\mathbb{R}_+), \quad \operatorname{supp} \varphi_f \subset [\varepsilon_f^{(1)}, \varepsilon_f^{(2)}] \quad \text{and} \quad \int_0^\infty \varphi_f(\varepsilon) d\varepsilon = 1 . \tag{3.13}$$

The following properties of functions $m^{(a)}(x)$ are easily derived from their definition.

Lemma 3.4. *If $a \neq 0$ and $x \notin \mathbf{X}_a(3\epsilon^{d_a})$, then $m^{(a)}(x) = 0$.*

Proof. Under our assumptions

$$(1 + \varepsilon_a)|x_a| - (1 + \varepsilon_0)|x| < [(1 + 3\epsilon^{d_a})(1 - 3\epsilon^{d_a}) - 1]|x| < 0 .$$

So the factor in the right side of (3.10) corresponding to $f = 0$ equals zero. □

Lemma 3.5. *Let $x \in \mathbf{X}_a(\epsilon^{d_a})$ and $x \notin \mathbf{X}_f(3\epsilon^{d_f})$ for all $X_f \subset X_a$, $X_f \neq X_a$. Then $m^{(a)}(x) = \mu_a |x_a|$, where*

$$\mu_a = \int (1 + \varepsilon)\varphi_a(\varepsilon) d\varepsilon . \tag{3.14}$$

Proof. According to the definition (3.12) one needs to verify that for x considered $m^{(a)}(x, \boldsymbol{\varepsilon}) = (1 + \varepsilon_a)|x_a|$. By Corollary 3.3, it suffices to check that for every $X_f \subset X_a$,

$$(1 + \varepsilon_a)|x_a| \geqq (1 + \varepsilon_f)|x_f| .$$

Since $x \in \mathbf{X}_a(\epsilon^{d_a})$ and $x \notin \mathbf{X}_f(3\epsilon^{d_f})$, this is an immediate consequence of the obvious inequality

$$(1 + 2\epsilon^{d_a})(1 - \epsilon^{d_a}) \geqq (1 + 3\epsilon^{d_f})(1 - 3\epsilon^{d_f}) . \quad \square$$

To study other properties of $m^{(a)}(x)$ we rewrite its definition (3.9), (3.12) taking into account (3.10):

$$m^{(a)}(x) = |x_a| \int (1 + \varepsilon_a)\varphi_a(\varepsilon_a) d\varepsilon_a \prod_{f \neq a} \int \theta((1 + \varepsilon_a)|x_a| - (1 + \varepsilon_f)|x_f|)\varphi_f(\varepsilon_f) d\varepsilon_f .$$

Calculating integrals in ε_f and denoting

$$\Phi_f(\xi) = \int_0^\xi \varphi_f(\varepsilon) d\varepsilon , \tag{3.15}$$

we obtain the representation

$$m^{(a)}(x) = |x_a| \int (1 + \varepsilon_a)\varphi_a(\varepsilon_a) \prod_{f \neq a} \Phi_f((1 + \varepsilon_a)|x_a| |x_f|^{-1} - 1) d\varepsilon_a . \tag{3.16}$$

Now it is easy to establish smoothness of functions $m^{(a)}(x)$.

Lemma 3.6. *Under the assumptions* (3.13) $m^{(a)} \in C^\infty(\mathbb{R}^d \setminus \{0\})$.

Proof. By virtue of homogeneity we may assume that $|x| = 1$. Furthermore, according to Lemma 3.4, it suffices to consider $m^{(a)}(x)$ in a region $|x_a| \geqq c_a$ for some $c_a > 0$. Let us proceed from the representation (3.16). By (3.15), $\Phi_f \in C^\infty(\mathbb{R})$, $\Phi_f(\xi) = 0$ if $\xi \leqq \varepsilon_f^{(1)}$ and $\Phi_f(\xi) = 1$ if $\xi \geqq \varepsilon_f^{(2)}$. Thus the function

$$\Phi_f((1 + \varepsilon_a)|x_a| |x_f|^{-1} - 1) \tag{3.17}$$

is infinitely differentiable in x_a and x_f if $x_f \neq 0$. Moreover, the function (3.17) equals 1 if $(1+\varepsilon_f^{(2)})|x_f| \leqq (1+\varepsilon_a^{(1)})c_a$. So the integrand in (3.16) is infinitely differentiable in x, $|x|=1$, uniformly with respect to $\varepsilon_a \in (\varepsilon_a^{(1)}, \varepsilon_a^{(2)})$. Its derivatives of arbitrary order are uniformly bounded and supported in $(\varepsilon_a^{(1)}, \varepsilon_a^{(2)})$. It follows that the function (3.16) belongs to the class $C^\infty(\mathbb{R}^d \setminus \{0\})$. $\square$

Let us enumerate properties of the functions (3.12) used in Sect. 5:

1_*) $m^{(a)} \in C^\infty(\mathbb{R}^d \setminus \{0\})$ and $m^{(a)}(x)$ is a real homogeneous function of degree 1.
2_*) Let $X_a \not\subset X_b$ and $x \in \mathbf{X}_b(\epsilon^{d_b})$. Then $m^{(a)}(x) = 0$.
3_*) Let $X_a \subset X_b$ and $x \in \mathbf{X}_b(\epsilon^{d_b})$. Then $m^{(a)}(x) = m^{(a)}(x_b)$, i.e. $m^{(a)}(x)$ does not depend on x^b.

Note that 2_*) and 3_*) are direct consequences of Lemma 3.2 where the same properties of $m^{(a)}(x, \boldsymbol{\varepsilon})$ were verified.

Let us now construct the generating function $m(x)$. Again we introduce first a family of functions

$$m(x, \boldsymbol{\varepsilon}) = \max_a \{(1+\varepsilon_a)|x_a|\}, \quad \boldsymbol{\varepsilon} = \{\varepsilon_a\}, \tag{3.18}$$

satisfying all necessary properties except smoothness and then average $m(x, \boldsymbol{\varepsilon})$ over all ε_a:

$$m(x) = \int m(x, \boldsymbol{\varepsilon}) \prod_f \varphi_f(\varepsilon_f) d\varepsilon_f . \tag{3.19}$$

Here is the list of properties of this function used in next sections:

1) $m \in C^\infty(\mathbb{R}^d \setminus \{0\})$ and $m(x)$ is a real homogeneous function of degree 1.
2) $m(x) \geqq 1$ if $|x| = 1$.
3) $m(x)$ is convex, i.e. for arbitrary $x', x'' \in \mathbb{R}^d$,

$$m(r'x' + r''x'') \leqq r'm(x') + r''m(x''), \quad r', r'' \in [0,1], \quad r' + r'' = 1 .$$

4) Let a be arbitrary. If $x \in \mathbf{X}_a(\epsilon^{d_a})$, then $m(x) = m(x_a)$, i.e. $m(x)$ does not depend on x^a.
5) Let a be arbitrary. If $x \in \mathbf{X}_a(\epsilon^{d_a})$ and $x \notin \mathbf{X}_b(3\epsilon^{d_b})$ for all $X_b \subset X_a$, $X_b \neq X_a$, then

$$m(x) = \mu_a |x_a| , \tag{3.20}$$

where μ_a is defined by (3.14).

Furthermore, functions (3.12) and (3.19) are related by the equality:

$$m(x) = \sum_a m^{(a)}(x) . \tag{3.21}$$

Properties 2) and 3) are easily deduced from the definitions (3.18), (3.19). Actually, by (3.18), $m(x, \boldsymbol{\varepsilon}) \geqq (1+\varepsilon_0)|x| \geqq |x|$ and hence $m(x)$ obeys the same estimate. Being maximum of convex functions, $m(x, \boldsymbol{\varepsilon})$ is convex in x for all $\boldsymbol{\varepsilon}$. So $m(x)$ is convex as an integral over the parameter $\boldsymbol{\varepsilon}$ of convex functions.

Other properties of $m(x)$ will be derived from corresponding properties of functions $m^{(a)}(x)$. Thus we verify first (3.21). It suffices to check that for almost all $\boldsymbol{\varepsilon}$ (the exceptional set of $\boldsymbol{\varepsilon}$ may depend on x)

$$m(x, \boldsymbol{\varepsilon}) = \sum_a m^{(a)}(x, \boldsymbol{\varepsilon}) . \tag{3.22}$$

Suppose that $\boldsymbol{\varepsilon}$ is chosen in such a way that

$$(1+\varepsilon_f)|x_f| \neq (1+\varepsilon_g)|x_g|, \quad \forall f, \forall g, f \neq g .$$

Then there exists exactly one $b = b(x)$ such that

$$(1+\varepsilon_b)|x_b| > \max_{f \neq b} \{(1+\varepsilon_f)|x_f|\} .$$

By the definition (3.18), $m(x, \boldsymbol{\varepsilon}) = (1+\varepsilon_b)|x_b|$. Similarly, by the definition (3.9), $m^{(a)}(x, \boldsymbol{\varepsilon}) = 0$ if $a \neq b$ and $m^{(b)}(x, \boldsymbol{\varepsilon}) = (1+\varepsilon_b)|x_b|$. This proves (3.22).

Given the relation (3.21), property 1) of m is a consequence of the same property of functions $m^{(a)}$. According to properties 2_*) and 3_*) of $m^{(a)}$, the function (3.21) satisfies for $x \in \mathbf{X}_b(\epsilon^{d_b})$ the equality $m(x) = m(x_b)$. This implies property 4).

Let us finally verify 5). By Lemma 3.5 it suffices to prove that for x considered $m(x) = m^{(a)}(x)$. By virtue of (3.21) to that end we need to check that

$$m^{(f)}(x) = 0, \quad f \neq a . \tag{3.23}$$

If $X_f \not\subset X_a$, then (3.23) for $x \in \mathbf{X}_a(\epsilon^{d_a})$ follows from property 2_*) of $m^{(f)}$. If $X_f \subset X_a$, then (3.23) for $x \notin \mathbf{X}_f(3\epsilon^{d_f})$ follows from Lemma 3.4, applied to $m^{(f)}$. This concludes the proof of properties 1)–5) of the function m.

4. Positive Commutators and Radiation Conditions

Our proof of the radiation conditions-estimates relies on consideration of the commutator of H with a first-order differential operator M. Suppose for a moment that in (1.13) m is an arbitrary smooth function. We start with the standard calculation of the commutator $[H_0, M]$.

Lemma 4.1. *Let an operator M be defined by* (1.13). *Then*

$$i[H_0, M] = 4 \sum_{j,k} D_j m_{jk} D_k - (\Delta^2 m), \quad m_{jk} = \partial^2 m / \partial x_j \partial x_k . \tag{4.1}$$

Proof. Let us consider

$$[\partial_j^2, m_k \partial_k] = \partial_j^2 m_k \partial_k - m_k \partial_k \partial_j^2 . \tag{4.2}$$

Commuting ∂_j with m_k we find that the first term in the right side equals $\partial_j^2 m_k \partial_k = \partial_j (m_{jk} + m_k \partial_j) \partial_k$. Similarly, the second term

$$\begin{aligned} m_k \partial_k \partial_j^2 &= (m_k \partial_j)(\partial_k \partial_j) = (-m_{jk} + \partial_j m_k)(\partial_k \partial_j) \\ &= -(\partial_j m_{jk} - m_{jjk}) \partial_k + \partial_j m_k \partial_k \partial_j, \quad m_{jjk} = \partial^3 m / \partial x_j^2 \partial x_k . \end{aligned}$$

Inserting these expressions into (4.2) we obtain that $[\partial_j^2, m_k \partial_k] = 2 \partial_j m_{jk} \partial_k - m_{jjk} \partial_k$. It follows that

$$\begin{aligned} [\partial_j^2, m_k \partial_k + \partial_k m_k] &= [\partial_j^2, m_k \partial_k] + [\partial_j^2, m_k \partial_k]^* \\ &= 2(\partial_j m_{jk} \partial_k + \partial_k m_{jk} \partial_j) - m_{jjk} \partial_k + \partial_k m_{jjk} \\ &= 2(\partial_j m_{jk} \partial_k + \partial_k m_{jk} \partial_j) + m_{jjkk}, \quad m_{jjkk} = \partial^4 m / \partial x_j^2 \partial x_k^2 . \end{aligned}$$

Summing up these relations in j and k we arrive at (4.1). $\square$

Basically, we choose $m(x)$ as a function constructed in Sect. 3. However, we need to get rid of its singularity at $x=0$. This is simple because by virtue of Proposition 2.12 values of $m(x)$ in a bounded domain are inessential. So we can replace $m(x)$ in a neighbourhood of $x=0$ by an arbitrary smooth function. Actually, we replace $m(x)$ by $\tau(x)m(x)$, where $\tau\in C^\infty(\mathbb{R}^d)$, $\tau\geqq 0$, $\tau(x)=0$ for $|x|\leqq 1/2$ and $\tau(x)=1$ for $|x|\geqq 1$. From now on we use notation $m(x)$ for this new function. Clearly, $m(sx)=sm(x)$ if $|x|\geqq 1$ and $s\geqq 1$. We say that m is homogeneous for $|x|\geqq 1$.

Let us subtract from a neighbourhood $\mathbf{X}_a$ of X_a neighbourhoods $\mathbf{X}_b$ of all $X_b\subset X_a$, $X_b\neq X_a$. Namely, using notation (3.1), we set

$$\mathbf{Y}_a(\epsilon)=\mathbf{X}_a(\epsilon^d)\setminus\bigcup_{X_b\subset X_a,\,X_b\neq X_a}\mathbf{X}_b(3\epsilon)\,.$$

Note that $\mathbf{X}_a$ and $\mathbf{X}_b$ are parametrized by ϵ^d and 3ϵ, respectively, in order to accommodate property 5) of Sect. 3. Clearly, $\mathbf{Y}_a(\epsilon)\cap \mathring{X}_b=0$ for sufficiently small $\epsilon>0$ if $X_a\not\subset X_b$. For any cone $\Gamma\subset\mathbb{R}^d$ denote by $\mathring{\Gamma}$ its intersection with the exterior $\mathbf{B}'=\mathbb{R}^d\setminus\mathbf{B}$ of the unit ball $\mathbf{B}=\{|x|<1\}$: $\mathring{\Gamma}=\Gamma\cap\mathbf{B}'$. In particular, we consider below $\mathring{\mathbf{X}}_a(\cdot)$ and $\mathring{\mathbf{Y}}_a(\cdot)$. Set also $\mathbf{S}^{d-1}=\{|x|=1\}$.

Reformulating the conditions 1)–5) of Sect. 3, we obtain the properties of $m(x)$:

1^0 $m(x)$ is a real C^∞-function, which is homogeneous of degree 1 for $|x|\geqq 1$ and $m(x)=0$ for $|x|\leqq 1/2$.

2^0 $m(x)\geqq 1$ if $|x|=1$.

3^0 $m(x)$ is (locally) convex function for $|x|\geqq 1$, i.e.

$$\sum_{j,k} m_{jk}(x)\xi_j\bar{\xi}_k\geqq 0,\quad \forall\xi\in\mathbb{C}^d\,.$$

4^0 Let a be arbitrary. If $x\in\mathring{\mathbf{X}}_a(\epsilon^d)$, then $m(x)=m(x_a)$, i.e. $m(x)$ does not depend on x^a.

5^0 Let a be arbitrary. If $x\in\mathring{\mathbf{Y}}_a(\epsilon)$, then $m(x)$ obeys (3.20) with some $\mu_a\geqq 1$.

Remark that properties 4^0 and 5^0 are formulated here in a slightly weaker form than in Sect. 3. We emphasize that *the parameter* $\epsilon>0$ *should be sufficiently small and it can be chosen arbitrary small.* Actually, the concrete construction of the function $m=m_\epsilon$ is of no importance for us and we always use only its properties 1^0–5^0 listed above.

By property 1^0 derivatives m_j of m are homogeneous functions of degree 0, m_{jk} are homogeneous of degree -1 and m_{jjkk} are homogeneous of degree -3. Therefore

$$(\Delta^2 m)(x)=O(|x|^{-3}),\quad |x|\to\infty\,, \tag{4.3}$$

and the main contribution to the commutator (4.1) is determined by the operator

$$L=L(m)=\sum_{j,k} D_j m_{jk} D_k\,. \tag{4.4}$$

To estimate it we first compute the matrix

$$\mathbf{M}(x)=\{m_{jk}(x)\}=\operatorname{Hess} m(x)$$

in the region where $m(x)=\mu_0|x|$:

$$m_j(x)=\mu_0|x|^{-1}x_j,\quad m_{jk}(x)=\mu_0(|x|^{-1}\delta_{jk}-|x|^{-3}x_jx_k)\,. \tag{4.5}$$

Here $\delta_{jj} = 1$ and $\delta_{jk} = 0$ if $j \neq k$. By definition (1.10), the angular part of the gradient ∇u obeys the identity

$$|\nabla^{(s)} u|^2 = |\nabla u|^2 - |x|^{-2} |\langle \nabla u, x \rangle|^2 = \sum_j |u_j|^2 - |x|^{-2} |\sum_j u_j x_j|^2$$
$$= \sum_j (1 - |x|^{-2} x_j^2) |u_j|^2 - |x|^{-2} \sum_{j \neq k} x_j x_k u_j \bar{u}_k, \quad u_j = \partial u / \partial x_j .$$

According to (4.5) it follows that

$$\sum_{j,k} m_{jk} u_j \bar{u}_k = \mu_0 |x|^{-1} |\nabla^{(s)} u|^2 . \tag{4.6}$$

In the region where $m(x) = \mu_a |x_a|$ all calculations hold true if x is replaced by x_a. By virtue of property 5^0 we obtain the following

Lemma 4.2. *Let $\nabla_a^{(s)} u$ be defined by the equality* (1.10) *and let $x \in \mathring{\mathbf{Y}}_a(\epsilon)$. Then the identity holds:*

$$\sum_{j,k} m_{jk} u_j \bar{u}_k = \mu_a |x_a|^{-1} |\nabla_a^{(s)} u|^2 . \tag{4.7}$$

Note that in the case $\dim X_a = 1$ both sides of (4.7) equal zero.

By property 3^0, the quadratic form of the operator (4.4) satisfies the inequality

$$(Lu, u) = \sum_{j,k} \int m_{jk} u_j \bar{u}_k dx \geqq \sum_{j,k} \int_\Omega m_{jk} u_j \bar{u}_k dx - c \int_{|x|<1} |\nabla u|^2 dx ,$$

where Ω is any region lying outside of the unit ball. Combining this inequality with Lemma 4.2 we obtain

Proposition 4.3. *In notation of Lemma* 4.2 *for every $u \in \mathscr{D}$,*

$$(Lu, u) \geqq \mu_a \int_{\mathring{\mathbf{Y}}_a(\epsilon)} |x|^{-1} |\nabla_a^{(s)} u|^2 dx - c \int_{|x|<1} |\nabla u|^2 dx .$$

It turns out that due to property 4^0 the commutator $[V, M]$ is in some sense small. The precise formulation is given in the following

Proposition 4.4. *Suppose that V^α satisfies Assumption* 2.3. *Let m obey property* 1^0 *and $m(x) = m(x_\alpha)$ if $x \in \mathring{\mathbf{X}}_\alpha(\varepsilon_\alpha)$ for some $\varepsilon_\alpha > 0$. Then*

$$|([V^\alpha, M] u, u)| \leqq C \| Q^{-r} (T + I) u \|^2, \quad u \in \mathscr{D}, \quad 2r = \rho . \tag{4.8}$$

Proof. Suppose first that $X^\alpha \neq X$. Let us introduce a smooth homogeneous (for $|x| \geqq 2$) function ζ_α of degree zero such that $0 \leqq \zeta_\alpha(x) \leqq 1$, $\zeta_\alpha(x) = 1$ if $x \notin \mathring{\mathbf{X}}_\alpha(\varepsilon_\alpha)$ and $\zeta_\alpha(x) = 0$ if $x \in \mathbf{X}_\alpha(\varepsilon)$ for some $\varepsilon \in (0, \varepsilon_\alpha)$ and $|x| \geqq 2$. The long-range part of V^α is differentiable so that

$$i[V_l^\alpha, M] = 2 \left[V_l^\alpha, \sum_{j=1}^d m_j \partial_j \right] = -2 \sum_{j=1}^d m_j \partial V_l^\alpha / \partial x_j = -2 \langle \nabla m(x), \nabla V_l^\alpha(x^\alpha) \rangle .$$

This scalar product equals zero for $x \in \mathring{\mathbf{X}}_\alpha(\varepsilon_\alpha)$ because m depends only on x_α in this region and, consequently, $\nabla m(x) \in X_\alpha$ whereas $\nabla V_l^\alpha(x^\alpha) \in X^\alpha$. Since $|\nabla m(x)|$ is bounded, it follows that

$$|\langle \nabla m(x), \nabla V_l^\alpha(x^\alpha) \rangle| \leqq C \zeta_\alpha(x) |\nabla V_l^\alpha(x^\alpha)| \zeta_\alpha(x) . \tag{4.9}$$

Using the representation (2.7) we find that

$$|([V_l^\alpha, M]u, u)| \leqq C\|(T+I)^{1/2}w_\alpha u\|^2 ,$$

where the function

$$w_\alpha(x) = ((x^\alpha)^2+1)^{-r/2}\zeta_\alpha(x) \tag{4.10}$$

obeys the condition (2.15) because $\zeta_\alpha(x)=0$ if $x \in \mathbf{X}_\alpha(\varepsilon)$ and $|x| \geqq 2$. Therefore, taking into account Lemma 2.11, we obtain the bound (4.8) for V_l^α.

To consider $[V_s^\alpha, M]$ we use again that the function $\zeta_\alpha(x)$ differs from 1 only if $x \in \mathring{\mathbf{X}}_\alpha(\varepsilon_\alpha)$. In this region the function m does not depend on x^α. It follows that the operator

$$i\eta_\alpha M = 2\eta_\alpha(\nabla_\alpha m)\nabla_\alpha + \eta_\alpha(\Delta_\alpha m), \quad \eta_\alpha(x) = 1-\zeta_\alpha^2(x) ,$$

commutes with V_s^α and hence $[V_s^\alpha, M] = [V_s^\alpha, \zeta_\alpha^2 M]$. Clearly,

$$\zeta_\alpha^2 M = \sum_{j=1}^{d} (\xi_{\alpha,j}D_j + D_j\xi_{\alpha,j} + i(\partial\zeta_\alpha^2/\partial x_j)m_j), \quad \xi_{\alpha,j} = \zeta_\alpha^2 m_j ,$$

and therefore

$$[V_s^\alpha, \zeta_\alpha^2 M] = 2\sum_{j=1}^{d} (V_s^\alpha\xi_{\alpha,j}D_j - D_jV_s^\alpha\xi_{\alpha,j} - iV_s^\alpha\partial\xi_{\alpha,j}/\partial x_j) . \tag{4.11}$$

Note that the functions m_j are bounded together with their derivatives and $\xi_{\alpha,j}=0$ if $x \in \mathbf{X}_\alpha(\varepsilon)$ and $|x| \geqq 2$. By virtue of the representation (2.6) for $|V_s^\alpha|^{1/2}$ the last term in (4.11) is estimated exactly as the right side of (4.9). Similarly,

$$|(V_s^\alpha\xi_{\alpha,j}D_ju, u)| \leqq C\|(T+I)^{1/2}w_\alpha D_j u\| \, \|(T+I)^{1/2}w_\alpha u\|$$

with the function w_α defined by (4.10). According to Lemma 2.11 the right side here is bounded by the right side of (4.8). In the case $X^\alpha = X$ the estimates are the same but the cut-off by ζ_α is no longer necessary. □

Given Propositions 4.3 and 4.4 the proof of the main result of this section is quite standard. We formulate it only for the operator H since H_a are its special cases.

Theorem 4.5. *Under Assumption* 2.3 *for all a the operators*

$$G_a(\epsilon) = \chi(\mathbf{Y}_a(\epsilon))Q^{-1/2}\nabla_a^{(s)} ,$$

acting from the space $L_2(\mathbb{R}^d)$ *into the vector-spaces* $L_2(\mathbb{R}^d)\otimes\mathbb{C}^{d_a}$, $d_a = \dim X_a$, *are H-smooth on arbitrary bounded interval* Λ, $\bar{\Lambda}\cap\Upsilon = \emptyset$.

Proof. Let us consider

$$d(MU(t)f, U(t)f)/dt = i([H, M]f_t, f_t) , \tag{4.12}$$

where $f_t = U(t)f, f \in \mathscr{D}$. By (4.1), (4.4),

$$i([H,M]f_t,f_t) = 4(Lf_t,f_t) - ((\Delta^2 m)f_t,f_t) + i([V,M]f_t,f_t) .$$

Taking into account (4.3) and applying Propositions 4.3 and 4.4 to elements $u = f_t$ we find that for any a

$$i([H,M]f_t,f_t) \geqq c_1\|G_a(\epsilon)f_t\|^2 - c_2\|Q^{-r}(T+I)f_t\|^2, \quad 2r = \rho , \tag{4.13}$$

(under the assumption $\rho \leqq 3$). Here we have omitted $\| Q^{-3/2} f_t \|^2$ and the integral of $|\nabla f_t|^2$ over the unit ball because they are estimated by the last term in the right side of (4.13). Integrating (4.12) and (4.13) over $t \in (t_1, t_2)$ we obtain that

$$\int_{t_1}^{t_2} \| G_a(\epsilon) f_t \|^2 dt \leqq C(|(Mf_t, f_t)|^{t_2}_{t_1}| + \int_{t_1}^{t_2} \| Q^{-r}(T+I) f_t \|^2 dt) . \tag{4.14}$$

Suppose now that $f = E(\Lambda) f$. Then the first term in the right side of (4.14) is bounded by $C \| f \|^2$ because $ME(\Lambda) \in \mathscr{B}$ for bounded Λ. The second term admits the same estimate according to Proposition 2.12. It follows that the integral in the left side of (4.14) is bounded by $C \| f \|^2$ so that each of the operators $G_a(\varepsilon)$ is H-smooth on Λ. □

Remark. H-smoothness of the operators G_a: $L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d) \otimes \mathbb{C}^{d_a}$ can be reformulated as H-smoothness of d_a "scalar" operators acting in the space $L_2(\mathbb{R}^d)$. Due to the equality $\langle \nabla_a^{(s)} u, x_a \rangle = 0$ only $d_a - 1$ of these d_a results are independent of each other.

Let us associate with every $x_0 \in \mathbf{S}^{d-1}$ an index $\mathbf{a}_0 = \mathbf{a}(x_0)$ (or the corresponding subspace $X_{\mathbf{a}_0}$) defined by the equality

$$X_{\mathbf{a}_0} = \bigcap_{x_0 \in X_b} X_b, \quad X_b \in \mathscr{X}' . \tag{4.15}$$

Clearly, $X_{\mathbf{a}_0}$ is the smallest subspace $X_b \in \mathscr{X}'$ containing a point x_0. In other words, for any X_b,

$$x_0 \in X_b \quad \text{if and only if} \quad X_{\mathbf{a}_0} \subset X_b . \tag{4.16}$$

Let us introduce also a conical neighbourhood of a point $x_0 \in \mathbf{S}^{d-1}$: $\Gamma(x_0; \varepsilon) = \{x : \langle x, x_0 \rangle > (1-\varepsilon)|x|\}$. In terms of these definitions we can reformulate Theorem 4.5 in a more convenient form.

Theorem 4.6. *Suppose that Assumption* 2.3 *holds. Let* x_0 *be an arbitrary point of* $\mathbf{S}^{d-1}$ *and let* $\mathbf{a}_0 = \mathbf{a}(x_0)$ *be defined by* (4.15). *Then for sufficiently small* $\varepsilon > 0$ *the operator*

$$\chi(\Gamma(x_0; \varepsilon)) Q^{-1/2} \nabla_{\mathbf{a}_0}^{(s)} \tag{4.17}$$

is H-smooth on Λ.

Proof. By (4.16), x_0 is separated from all X_b such that $X_b \subset X_{\mathbf{a}_0}$, $X_b \neq X_{\mathbf{a}_0}$. Therefore $\Gamma(x_0; \varepsilon) \subset \mathbf{Y}_{\mathbf{a}_0}(\epsilon)$ for sufficiently small $\varepsilon > 0$ and H-smoothness of the operator (4.17) is ensured by Theorem 4.5 for $a = \mathbf{a}_0$. □

Our proofs in the next section of existence of the wave operators (1.17) and (1.24) rely on Theorem 4.6. Its formulation is "local" with respect to direction of x. For the sake of completeness we give also the "global" formulation. To deduce it from Theorem 4.6 we need the following simple assertion. Recall that the space X and its dual are always identified.

Lemma 4.7. *If* $X_b \subset X_a$, *then*

$$|(\nabla_b^{(s)} u)(x)| \leqq |(\nabla_a^{(s)} u)(x)| . \tag{4.18}$$

Proof. We can assume that u is real. By definition (1.10) the estimate (4.18) is equivalent to the bound

$$|\xi_b|^2 - |x_b|^{-2} |\langle \xi_b, x_b \rangle|^2 \leqq |\xi_a|^2 - |x_a|^{-2} |\langle \xi_a, x_a \rangle|^2 , \tag{4.19}$$

where ξ ($\xi = (\nabla u)(x)$) is an arbitrary vector of X and ξ_a and ξ_b are its projections on X_a and X_b respectively. Let x_a^b and ξ_a^b be the orthogonal projections of x and ξ on $X_a \ominus X_b$. Then $|x_a|^2 = |x_b|^2 + |x_a^b|^2$, $|\xi_a|^2 = |\xi_b|^2 + |\xi_a^b|^2$ and $|\langle \xi_a, x_a \rangle| \leqq |\langle \xi_b, x_b \rangle| + |\xi_a^b|\,|x_a^b|$. So instead of (4.19) it suffices to prove that

$$(|x_b|^2 + |x_a^b|^2)^{-1}(|\langle \xi_b, x_b \rangle| + |\xi_a^b|\,|x_a^b|)^2 \leqq |\xi_a^b|^2 + |x_b|^{-2}|\langle \xi_b, x_b \rangle|^2 .$$

By identical transformations we reduce this estimate to the obvious inequality

$$2|x_b|^2|\langle \xi_b, x_b \rangle|\,|\xi_a^b|\,|x_a^b| \leqq |\langle \xi_b, x_b \rangle|^2|x_a^b|^2 + |x_b|^4|\xi_a^b|^2 . \quad \square$$

Now we are able to prove the radiation conditions-estimates as they are formulated in Sect. 1. Define

$$\mathbf{\Gamma}_a(\varepsilon) = X \setminus \bigcup_{X_a \not\subset X_b} \mathbf{X}_b(\varepsilon) . \tag{4.20}$$

The following result is equivalent to local H-smoothness of the operator (1.11).

Theorem 4.8. *Suppose that Assumption* 2.3 *holds. Then for every a and every $\varepsilon > 0$ the operators*

$$\chi(\mathbf{\Gamma}_a(\varepsilon))Q^{-1/2}\nabla_a^{(s)} \tag{4.21}$$

are H-smooth on arbitrary bounded interval Λ, $\bar{\Lambda} \cap \Upsilon = \emptyset$.

Proof. For every $x_0 \in \mathbf{\Gamma}_a(\varepsilon) \cap \mathbf{S}^{d-1}$ let us consider the subspace $X_{\mathbf{a}_0}$ defined by (4.15). According to definition (4.20), x_0 may belong only to X_b containing X_a. It follows that $X_a \subset X_{\mathbf{a}_0}$ and hence, by Theorem 4.6 and Lemma 4.7, for sufficiently small $\varepsilon > 0$ the operator $\chi(\Gamma(x_0;\varepsilon))Q^{-1/2}\nabla_a^{(s)}$ is H-smooth. So to conclude the proof of H-smoothness of the operator (4.21) it remains to choose a finite covering of the closed set $\mathbf{\Gamma}_a(\varepsilon) \cap \mathbf{S}^{d-1}$ by open sets $\Gamma(x_0;\varepsilon) \cap \mathbf{S}^{d-1}$. $\square$

Remark. By (4.18), Theorem 4.8 gives us more information about $U(t)f$ in the cone $\mathbf{\Gamma}_a$ than in $\mathbf{\Gamma}_b$ if $X_b \subset X_a$. In particular, the most complete information is obtained in the cose $\mathbf{\Gamma}_0$ which does not intersect any $X_a \neq X$. On the contrary, the result of Theorem 4.8 is trivial for a such that $\dim X_a = 1$.

Remark. In the two-particle case the result of Theorem 4.8 reduces to H-smoothness of the operator $Q^{-1/2}\nabla^{(s)}$ on any bounded positive interval separated from the point 0. This is different from the usual form of the radiation conditions-estimate (see e.g. [17]). First, we consider only the angular part of $\nabla U(t)f$. Second, the estimate of [17] implies that

$$\int_{-\infty}^{\infty} \|Q^{-r}\nabla^{(s)}U(t)f\|^2 dt < \infty . \tag{4.22}$$

Here r is some number smaller than $1/2$ whereas we require that $r = 1/2$ which is less informative. On the other hand, in (4.22) f should belong to some dense (in $\mathscr{H}$) set whereas our estimate is uniform for all $f \in \mathscr{H}$.

Note, finally, that in [31] radiation conditions for the N-particle case were derived in the free region $\mathbf{\Gamma}_0$. In this paper both radial and angular parts of radiation conditions were considered. Results of [31] can probably be used for a proof of asymptotic completeness in the three-particle case. However, information about $U(t)f$ in a free region only is not sufficient for the case of $N > 3$ particles.

5. Modified Wave Operators

Here we shall establish existence of the wave operators (1.17). Generating functions $m^{(a)}(x)$ are basically the same as those constructed in Sect. 3. Similarly to Sect. 4, in order to get rid of singularity of $m^{(a)}(x)$ at $x = 0$ we replace it by $\tau(x)m^{(a)}(x)$, where $\tau \in C^\infty(\mathbb{R}^d)$, $\tau \geqq 0$, $\tau(x) = 0$ for $|x| \leqq 1/2$ and $\tau(x) = 1$ for $|x| \geqq 1$. We keep notation $m^{(a)}(x)$ for these new functions. The conditions 1_*)–3_*) of Sect. 3 ensure the following properties:

1^0_* $m^{(a)}(x)$ is a real C^∞-function, which is homogeneous of degree 1 for $|x| \geqq 1$ and $m^{(a)}(x) = 0$ for $|x| \leqq 1/2$.

2^0_* Let b be arbitrary. If $x \in \mathring{\mathbf{X}}_b(\epsilon^d)$, then $m^{(a)}(x) = m^{(a)}(x_b)$, i.e. $m^{(a)}(x)$ does not depend on x^b.

3^0_* Let $X_a \not\subset X_b$. If $x \in \mathbf{X}_b(\epsilon^d)$, then $m^{(a)}(x) = 0$.

Note that property 2^0_* for b such that $X_a \not\subset X_b$ is contained in property 3^0_*. However, this formulation is preferable (compared to 1_*)–3_*) of Sect. 3) because property 3^0_* is used only once (in Lemma 5.3).

In order to apply Proposition 2.1 we consider the "perturbation" (1.19). By the study of the term $[T, M^{(a)}]$ in the following two assertions we omit dependence of $m^{(a)}$ and $M^{(a)}$ on a.

Lemma 5.1. *Let $m(x)$ be any function satisfying conditions 1^0_* and 2^0_*. Let $\lambda_n(x)$ and $p_n(x)$ be eigenvalues and eigenvectors of the symmetric matrix $\mathbf{M}(x) = \{m_{jk}(x)\}$. Then for any b and $x \in \mathring{\mathbf{X}}_b(\epsilon^d)$ eigenvectors $p_n(x)$, corresponding to $\lambda_n(x) \neq 0$, belong to X_b and are orthogonal to the vector x_b.*

Proof. Choose an orthonormal basis in X with the first d_b elements belonging to the subspace X_b and other d^b elements from X^b. Let $x_1, \ldots, x_d$ be the corresponding coordinates. Since $m(x) = m(x_b)$ for $x \in \mathring{\mathbf{X}}_b(\epsilon^d)$ we have that $m_j(x) = 0$ if $j > d_b$ and $m_{jk}(x) = 0$ if $j > d_b$ or $k > d_b$. Therefore $\mathbf{M}(x)\xi = 0$ if $\xi \in X^b$. Furthermore, differentiating the identity $m(sx_b) = sm(x_b)$ in s and setting $s = 1$ we find that $\sum_j m_j(x_b)x_j = m(x_b)$ (Euler's formula). Differentiation of this relation in x_k shows that $\sum_j m_{kj}(x)x_j = 0$. Thus $\mathbf{M}(x)x_b = 0$ and hence $\langle p_n(x), x_b\rangle = 0$ if $\mathbf{M}(x)p_n(x) = \lambda_n(x)p_n(x)$ with $\lambda_n(x) \neq 0$. □

Remark. If $x \in \mathring{\mathbf{X}}_b(\epsilon^d)$ for several b, then the conclusion of Lemma 5.1 holds for every such b. Among these conclusions there is the strongest one corresponding to b such that X_b is the smallest. Actually, if $p_n(x) \in X_b$ and $\langle p_n(x), x_b\rangle = 0$, then $\langle p_n(x), x_{b'}\rangle = 0$ for every $X_{b'} \supset X_b$. On the other hand, the weakest assertion is true for every $x \in \mathbf{B}'$: $\langle p_n(x), x\rangle = 0$ if $\lambda_n(x) \neq 0$.

Proposition 5.2. *Let $m(x)$ be any function satisfying conditions 1^0_* and 2^0_* and let $L = L(m)$ be defined by* (4.4). *Then $L = K_2^* K_1$, where operators K_1 and K_2 are H-smooth on Λ.*

Proof. Diagonalizing the matrix $\mathbf{M}$ we find in notation of Lemma 5.1 that

$$\begin{aligned}(Lu, v) &= \int_X \sum_{j,k} m_{jk}(x) D_k u(x) \overline{D_j v(x)} dx \\ &= \int_X \sum_n \lambda_n(x) \langle \nabla u(x), p_n(x)\rangle \langle p_n(x), \nabla v(x)\rangle dx = (K_1 u, K_2 v)_{\mathbf{H}} ,\end{aligned}$$

where

$$(K_j u)(x) = \sum_n v_{n,j}(x)\langle \nabla u(x), p_n(x)\rangle p_n(x), \quad j = 1, 2 ,$$

$$v_{n,1}(x) = |\lambda_n(x)|^{1/2}, \quad v_{n,1}(x) v_{n,2}(x) = \lambda_n(x) \tag{5.1}$$

and $\mathbf{H} = L_2(\mathbb{R}^d) \otimes \mathbb{C}^d$. Clearly, $\lambda_n(x)$ are homogeneous (for $|x| \geqq 1$) functions of order -1 and $p_n(x)$ – of order 0. Since $|(K_j u)(x)| \leqq C|\nabla u(x)|$, H-smoothness of the operators $\chi(\mathbf{B})K_j$ is ensured by Proposition 2.12.

To treat the operators $\chi(\mathbf{B}')K_j$ we notice that, by the definition (1.10) and Lemma 5.1,

$$\langle \nabla u(x), p_n(x)\rangle = \langle \nabla_b^{(s)} u(x), p_n(x)\rangle, \quad \lambda_n(x) \neq 0 , \tag{5.2}$$

for every b such that $x \in \mathring{\mathbf{X}}_b(\epsilon^d)$. As in Sect. 4, we associate by the equality (4.15) with every $x_0 \in \mathbf{S}^{d-1}$ a subspace $X_{\mathbf{a}_0}$, $\mathbf{a}_0 = \mathbf{a}(x_0)$. According to Theorem 4.6 for sufficiently small $\varepsilon > 0$ the operator (4.17) is H-smooth. Since $x_0 \in X_{\mathbf{a}_0}$, diminishing $\varepsilon = \varepsilon(x_0)$ we can suppose that $\Gamma(x_0, \varepsilon) \subset \mathring{\mathbf{X}}_{\mathbf{a}_0}(\epsilon^d)$. Then the equalities (5.2) are fulfilled for $x \in \mathring{\Gamma}(x_0; \varepsilon)$ and $b = \mathbf{a}_0$. Therefore, by definition (5.1),

$$|(K_j u)(x)| \leqq \sum_n |v_{n,j}(x)|\,|\langle \nabla_{\mathbf{a}_0}^{(s)} u(x), p_n(x)\rangle|, \quad x \in \mathring{\Gamma}(x_0; \varepsilon) ,$$

and hence for such x

$$|(K_j u)(x)| \leqq C|x|^{-1/2}|\nabla_{\mathbf{a}_0}^{(s)} u(x)|, \quad C = \sup_{|x|=1} \sum_n v_{n,1}(x) .$$

Now H-smoothness of the operator (4.17) ensures that for arbitrary $x_0 \in \mathbf{S}^{d-1}$ and sufficiently small $\varepsilon = \varepsilon(x_0)$ the operator $\chi(\mathring{\Gamma}(x_0; \varepsilon))K_j$ is H-smooth. To conclude the proof of H-smoothness of the operators $\chi(\mathbf{B}')K_j$ it remains to choose a finite covering of the unit sphere by open sets $\Gamma(x_0; \varepsilon) \cap \mathbf{S}^{d-1}$. □

We need short-range assumption on potentials only to treat the last term in (1.19).

Lemma 5.3. *Suppose that V^α satisfies Assumptions 2.2 and a function $m^{(a)}$ has properties 1^0_* and 3^0_*. Then for $X_a \not\subset X_\alpha$*

$$V^\alpha M^{(a)} = (T + I)Q^{-r}BQ^{-r}(T + I), \quad r = \rho/2 ,$$

where $B = B^{(a,\alpha)} \in \mathscr{B}$.

Proof. Suppose first that $X^\alpha \neq X$. Let us take into account that $m^{(a)}(x) = 0$ if $x \in \mathbf{X}_\alpha(\varepsilon^d)$. Therefore $m_j^{(a)}(x) = m_j^{(a)}(x)\zeta_\alpha^2(x)$ and $m_{jj}^{(a)}(x) = m_{jj}^{(a)}(x)\zeta_\alpha^2(x)$ for suitable $\zeta_\alpha \in C^\infty(\mathbb{R}^d)$, homogeneous (for $|x| \geqq 1$) of degree 0, such that $\zeta_\alpha(x) = 0$ if $x \in \mathring{\mathbf{X}}_\alpha(\varepsilon)$ for some $\varepsilon > 0$. By (1.18), (2.6) the operator $V^\alpha M^{(a)}$ consists of terms

$$V^\alpha m_j^{(a)} D_j = w_\alpha (T + I)^{1/2} B_j^{(a,\alpha)} (T + I)^{1/2} w_\alpha D_j$$

and

$$V^\alpha m_{jj}^{(a)} = w_\alpha (T + I)^{1/2} B_{jj}^{(a,\alpha)} (T + I)^{1/2} w_\alpha ,$$

where $2r = \rho$, $B_j^{(a,\alpha)} \in \mathscr{B}$, $B_{jj}^{(a,\alpha)} \in \mathscr{B}$ and the function $w_\alpha(x)$ is defined by (4.10). Since this function obeys the condition (2.15), to conclude the proof it remains to take Lemma 2.11 into account. In the case $X^\alpha = X$ the estimates are the same but the cut-off by ζ_α is no longer necessary. □

Now we collect the results obtained together.

Theorem 5.4. *Let Assumption* 2.2 *hold and let Λ be any bounded interval such that $\bar{\Lambda} \cap \Upsilon = \emptyset$. Then for all a the wave operators* (1.17) *exist.*

Proof. We shall show that the triple $H_a, H, M^{(a)}$ satisfies on Λ the conditions of Proposition 2.1. To that end one needs to verify that each term in the right side of (1.19) admits the factorization $K^* K_a$, where K and K_a are H- and H_a-smooth, respectively, on Λ. Actually, we verify that both K and K_a are H-smooth and, in particular, both of them are H_a-smooth. By Lemma 4.1 and (4.4), the first term in (1.19) equals $[T, M^{(a)}] = 4L(m^{(a)}) - (\Delta^2 m^{(a)})$. The operator $L(m^{(a)})$ was considered in Proposition 5.2. Since $m^{(a)}$ is a homogeneous function of degree 1, it satisfies condition (4.3) and hence $\Delta^2 m^{(a)} = Q^{-3/2} B^{(a)} Q^{-3/2}$, where $B^{(a)}$ is multiplication by a bounded function. The operator $Q^{-3/2}$ is H- (and H_a-) smooth by Proposition 2.5. The commutator $[V^a, M^{(a)}]$ consists of terms $[V^\alpha, M^{(a)}]$, where $X^\alpha \subset X^a$. They were actually already considered in Proposition 4.4. Its assumptions are fulfilled by virtue of properties 1^0_* and 2^0_* of the function $m^{(a)}$. The estimate (4.8) is equivalent to the representation

$$[V^\alpha, M^{(a)}] = (T + I) Q^{-r} B^{(\alpha, a)} Q^{-r} (T + I), \quad 2r = \rho, \quad B^{(\alpha, a)} \in \mathscr{B} ,$$

where $Q^{-r}(T+I)$ is H- (and H_a)-smooth on Λ by virtue of Proposition 2.12. Finally, by (1.20), the product $V_a M^{(a)}$ consists of terms $V^\alpha M^{(a)}$, $X_a \not\subset X_\alpha$, treated in Lemma 5.3. □

Let us now consider the observable (1.24). Existence of these wave operators can be verified similarly to Theorem 5.4. Actually, due to Propositions 4.4 and 5.2 the "perturbation"

$$HM - MH = [T, M] + \sum_\alpha [V^\alpha, M]$$

is a sum of products of H-smooth operators. Note that potentials V^α may contain long-range parts since the short-range assumption was used in Theorem 5.4 only for the estimate of the term $V_a M^{(a)}$, which is absent now. Thus, according to Proposition 2.1, we have

Proposition 5.5. *Let an operator M be defined by* (1.13), *where m is any function obeying the conditions* 1^0 *and* 4^0 *of Sect.* 4. *Then under Assumption* 2.3 *the wave operators* (1.24) *exist.*

The operator $M^\pm(\Lambda)$ is, clearly, self-adjoint, bounded and, by (2.3), commutes with H. It is equal to zero on $E(\mathbb{R} \setminus \Lambda)\mathscr{H}$. Our goal is to show that under the additional assumption 2^0 on m the operator $\pm M^\pm(\Lambda)$ is invertible on the subspace $E(\Lambda)\mathscr{H}$. In fact, we shall see that it is positively definite there.

We shall consider $U(t)$ on elements $f = \varphi(H) g$, where $\varphi \in C_0^\infty(\Lambda)$ and $g \in \mathscr{D}(Q)$. Clearly, for different φ and g such elements f are dense in $E(\Lambda)\mathscr{H}$. By Lemma 2.9 applied to $\psi(\lambda) = \exp(-i\lambda t)\varphi(\lambda)$ and $r = 1$, we have that $U(t) f \in \mathscr{D}(Q)$. Thus $mU(t) f$ are well defined.

Let $f_t = U(t) f$ and $h_t = U(t) h$, where $h \in \mathscr{H}$ is arbitrary. The identity $i[T, m] = M$ shows that

$$d(m f_t, h_t)/dt = i([H, m] f_t, h_t) = i([T, m] f_t, h_t) = (M f_t, h_t) .$$

Integrating this equality we find that

$$(m f_t, h_t) = (m f, h) + \int_0^t (M f_s, h_s) ds . \tag{5.3}$$

According to Proposition 5.5

$$|(M f_s, h_s) - (M^{\pm}(\Lambda) f, h)| \leqq \varepsilon(s) \| h \| , \tag{5.4}$$

where $\varepsilon(s)$ does not depend on h and tends to zero as $s \to \pm \infty$. Comparing (5.3) and (5.4) we obtain

Lemma 5.6. *Let $f = \varphi(H) g$, where $\varphi \in C_0^\infty(\Lambda)$ and $g \in \mathscr{D}(Q)$. Then*

$$U^*(t) m U(t) f = t M^{\pm}(\Lambda) f + o(|t|), \quad t \to \pm \infty .$$

Since $m \geqq 0$, Lemma 5.6 implies that

$$\pm (M^{\pm}(\Lambda) f, f) = \lim_{t \to \pm \infty} |t|^{-1} (m f_t, f_t) \geqq 0 .$$

The inequality $\pm (M^{\pm}(\Lambda) f, f) \geqq 0$ established on the dense set extends by continuity to the whole space $E(\Lambda)\mathscr{H}$. Thus we have

Corollary 5.7. *The operator* $\pm M^{\pm}(\Lambda) \geqq 0$.

To prove that $\pm M^{\pm}(\Lambda)$ is positively definite on $E(\Lambda)\mathscr{H}$ we use Proposition 2.4. Recall that λ is an arbitrary point of $\mathbb{R} \setminus \Upsilon$ and Λ_λ is a small interval containing it. By virtue of the identity $i[H, Q^2] = 2A$, it follows from (2.8) that

$$2^{-1} d^2 (Q f_t, Q f_t)/dt^2 = d(A f_t, f_t)/dt = (i[H, A] f_t, f_t) \geqq c \| f \|^2 ,$$

$$f = \varphi(H) g, \quad \varphi \in C_0^\infty(\Lambda_\lambda), \quad g \in \mathscr{D}(Q) .$$

Integrating twice this inequality we find that for sufficiently large $|t|$,

$$\| Q f_t \| \geqq c |t| \, \| f \| . \tag{5.5}$$

On the other hand, according to Lemma 5.6

$$\| m f_t \| = \| M^{\pm}(\Lambda) f \| \, |t| + o(|t|) . \tag{5.6}$$

By property 2^0, $m(x) \geqq |x|$ for $|x| \geqq 1$ so that $\| Q f_t \|^2 \leqq 2 \| f_t \|^2 + \| m f_t \|^2$. Thus comparing (5.5) with (5.6) we obtain the inequality

$$\| M^{\pm}(\Lambda) f \| \geqq c \| f \| , \tag{5.7}$$

where $f = \varphi(H) g, g \in \mathscr{D}(Q), \varphi \in C_0^\infty(\Lambda_\lambda)$ and $c = c_\lambda$. This inequality extends, of course, to all $f \in E(\Lambda_\lambda)\mathscr{H}$. The compact set $\bar{\Lambda}$ is covered by a finite number of intervals Λ_λ. Since $M^{\pm}(\Lambda)$ commutes with $E(\cdot)$, it follows that (5.7) is true for all $f \in E(\Lambda)\mathscr{H}$. Considering now Corollary 5.7 we obtain

Theorem 5.8. *Let an operator M be defined by* (1.13), *where m is any function obeying the conditions* 1^0, 2^0 *and* 4^0 *of Sect.* 4. *Suppose that functions V^α satisfy Assumption* 2.3. *Then for every $f \in E(\Lambda)\mathscr{H}$,*

$$\pm (M^{\pm}(\Lambda) f, f) \geqq c \| f \|^2, \quad c = c(\Lambda) > 0 .$$

Corollary 5.9. *In the space $E(\Lambda)\mathscr{H}$ the kernel of $M^{\pm}(\Lambda)$ is trivial and its range $R(M^{\pm}(\Lambda)) = E(\Lambda)\mathscr{H}$.*

6. Existence and Completeness of Wave Operators

In this section we give the proof of Theorem 2.7. Its difficult part is, of course, asymptotic completeness. We start with its proof in the form (1.8). Let, as always, Λ be a bounded interval such that $\bar{\Lambda} \cap \Upsilon = \emptyset$ and let M and $M^{(a)}$ be defined by the equalities (1.13) and (1.18) with generating functions $m(x)$ and $m^{(a)}(x)$. Recall that these functions satisfy properties 1^0–5^0 of Sect. 4 and 1^0_*–3^0_* of Sect. 5, respectively. Moreover, the equalities (3.21) and, consequently, (1.22) are fulfilled.

Theorem 6.1. *Under the assumptions of Theorem* 2.7 *for every* $f = E(\Lambda)f$ *and some elements* $f_a^\pm$ *the relation* (1.8) *holds.*

Proof. By Corollary 5.9 every $f \in E(\Lambda)\mathscr{H}$ admits the representation $f = M^\pm(\Lambda)f^\pm, f^\pm \in E(\Lambda)\mathscr{H}$, so that the asymptotic relation (1.25) is true. On the other hand, Theorem 5.4 yields for every a the asymptotics (1.21) where $f_a^\pm = W^\pm(H_a, H; M^{(a)}E(\Lambda))f^\pm$. Summing up the relations (1.21) and taking into account (1.22) we obtain the asymptotics (1.23). Comparing it with (1.25) we arrive at (1.8). $\square$

To complete the proof of Theorem 2.7 we need to establish existence of wave operators (2.9). Actually, we shall prove that the wave operators $W^\pm(H, H_a)$ exist (even without projections P_a). Now we use inductive arguments. We suppose that Theorem 2.7 is true for all operators $H^a, X^a \in \mathscr{X}$, and deduce from it the same statement for H. This is sufficient for justification of induction because, repeating this argument, we can reduce Theorem 2.7 to the case of operators H^b, where $X^b \subset X^a, X^b \neq X^a$. Thus after a finite number of steps we arrive at the operator H^0 for which scattering theory is trivial.

Let us define for every $X^b \subset X^a$ the operator $H_b^a = T^a + V^b$ in the space $\mathscr{H}^a$. Set $X_b^a = X^a \ominus X^b$ and $\mathscr{H}_b^a = L_2(X_b^a)$. Then $\mathscr{H}^a = \mathscr{H}_b^a \otimes \mathscr{H}^b$. We introduce also $P_b^a = I \otimes P^b$, where the tensor product is the same as above. Applied to the operator H^a Theorem 2.7 states that wave operators

$$W^\pm(H^a, H_b^a; P_b^a) \tag{6.1}$$

exist, their ranges are mutually orthogonal and

$$\sum_{X^b \subset X^a} \oplus R(W^\pm(H^a, H_b^a; P_b^a)) = \mathscr{H}^a . \tag{6.2}$$

Note that in the case $a = b$ the operator (6.1) equals $P^a = P_a^a$ and $R(P^a) = \mathscr{H}^{(p)}(H^a)$.

Scattering theory for operators H_a reduces to that for operators H^a. Indeed, by virtue of (1.6)

$$U_a(t)U_b(t)P_b = I \otimes \exp(iH^a t)\exp(-iH_b^a t)P_b^a, \quad X^b \subset X^a ,$$

where the tensor product is defined by (1.4). So wave operators $W^\pm(H_a, H_b; P_b)$ and (6.1) exist simultaneously and

$$W^\pm(H_a, H_b; P_b) = I \otimes W^\pm(H^a, H_b^a, P_b^a), \quad X^b \subset X^a .$$

In particular, (6.2) ensures that

$$\sum_{X^b \subset X^a} \oplus R(W^\pm(H_a, H_b; P_b)) = \mathscr{H} .$$

Thus we have

Proposition 6.2. *Suppose that the statement of Theorem 2.7 is true for an operator H^a. Then for any $X^b \subset X^a$ the wave operators $W^{\pm}(H_a, H_b; P_b)$ exist and for every $f \in \mathscr{H}$ and some elements $f_b^{\pm}$ ($f_b^{\pm} = (W^{\pm}(H_a, H_b; P_b))^* f$) the relation holds*

$$U_a(t) f \sim \sum_{X^b \subset X^a} U_b(t) P_b f_b^{\pm}, \quad t \to \pm \infty . \tag{6.3}$$

Note that in the proof of Theorem 6.1 we have used only existence of the second set of wave operators (1.17). For the proof of existence of $W^{\pm}(H, H_a)$ we rely on existence of $W^{\pm}(H, H_a; M^{(a)} E_a(\Lambda))$. Since elements $f = E_a(\Lambda) f$ are dense in the space $\mathscr{H} = \mathscr{H}^{(\mathrm{ac})}(H_a)$, this is equivalent to existence of the wave operators $W^{\pm}(H, H_a; M^{(a)}(H_a + i)^{-1})$. Here $-i$ can, of course, be replaced by an arbitrary regular point of H_a. Some minor technical complications below are related to unboundedness of the operators $M^{(a)}$. We start with some simple auxiliary assertions. They are basically known but we have not found them in the literature.

Lemma 6.3. *Suppose that ζ_a is a bounded function such that $\zeta_a(x) = 0$ if $x \in \mathbf{X}_a(\varepsilon)$ for some $\varepsilon > 0$. Then*

$$s - \lim_{|t| \to \infty} \zeta_a (T + i)(H_a + i)^{-1} U_a(t) P_a = 0 . \tag{6.4}$$

Proof. It suffices to check (6.4) on elements $f = g \otimes \psi^a$, where ψ^a is an eigenvector of the operator H^a, $H^a \psi^a = \lambda^a \psi^a$, g is an arbitrary element of $\mathscr{H}_a$ and the tensor product is defined by (1.4). Linear combinations of such elements f are dense in the space $P_a \mathscr{H}$. According to (1.6)

$$\begin{aligned}
&(T + i)(H_a + i)^{-1} U_a(t) P_a f \\
&\quad = (T_a + T^a + i)(T_a + \lambda^a + i)^{-1} \exp(-i(T_a + \lambda^a)t) g \otimes \psi^a \\
&\quad = \exp(-i(T_a + \lambda^a)t)(g \otimes \psi^a + \tilde{g} \otimes \tilde{\psi}^a)
\end{aligned} \tag{6.5}$$

with $\tilde{g}$, $\tilde{\psi}^a$ defined by $\tilde{g} = (T_a + \lambda^a + i)^{-1} g$, $\tilde{\psi}^a = (T^a - \lambda^a)\psi^a$. Clearly,

$$\| \zeta_a \exp(-iT_a t)(g \otimes \psi^a) \| = \| \Psi_a \exp(-iT_a t) g \|_{\mathscr{H}_a} , \tag{6.6}$$

where

$$\Psi_a^2(x_a) = \int_{X^a} |\zeta_a(x_a, x^a)|^2 |\psi^a(x^a)|^2 dx^a \leqq C \int_{|x^a| \geq c|x_a|} |\psi^a(x^a)|^2 dx^a ,$$

$c = c(\varepsilon) > 0$, by our assumptions on ζ_a. It follows that $\Psi_a(x_a) \to 0$ as $|x_a| \to \infty$ and hence the operator $\Psi_a(T_a + I)^{-1}$ is compact in the space $\mathscr{H}_a$. Therefore (6.6) tends to zero by virtue of (2.1). Since $\tilde{g} \in \mathscr{H}_a$, $\tilde{\psi}^a \in \mathscr{H}^a$ the second term in the right side of (6.5) can be estimated quite similarly. □

Corollary 6.4. *If $X_b \not\subset X_a$, then*

$$s - \lim_{|t| \to \infty} M^{(b)}(H_a + i)^{-1} U_a(t) P_a = 0 . \tag{6.7}$$

Proof. It is sufficient to take into account that, by property 3^0_* of $m^{(b)}$ (compared to Sect. 5 the roles of a and b are here interchanged), the zero-degree homogeneous function $\nabla m^{(b)}$ vanishes in the cone $\mathbf{X}_a(\varepsilon^d)$. □

Lemma 6.5. *Let $V^\alpha(T^\alpha + I)^{-1}$ be compact in $\mathscr{H}^a$ and $X^\alpha \not\subset X^a$. Then*

$$s - \lim_{|t| \to \infty} V^\alpha (H_a + i)^{-1} U_a(t) P_a = 0 . \tag{6.8}$$

Proof. It suffices to check (6.8) on some dense set so that we can assume that P^a is one-dimensional ($H^a P^a = \lambda^a P^a$). Furthermore, the problem can always be reduced to the case $X^a + X^\alpha = X$. Under this assumption $X_a \cap X_\alpha = \{0\}$ so that $\mathbf{X}_a(\varepsilon) \cap \mathbf{X}_\alpha(\varepsilon) = \{0\}$ for sufficiently small $\varepsilon > 0$. Let us choose C^∞-function ζ_a such that $\zeta_a(x) = 0$ if $x \in \mathbf{X}_a(\varepsilon)$ and $\zeta_a(x) = 1$ if $x \in \mathring{\mathbf{X}}_\alpha(\varepsilon)$. We suppose that ζ_a is homogeneous of degree 0 (for $|x| \geqq 1$). Set $\zeta^a = 1 - \zeta_a$ and split (6.8) into two equalities corresponding to the decomposition $V^\alpha = V^\alpha \zeta_a + V^\alpha \zeta^a$. Since the operator $V^\alpha (T^\alpha + I)^{-1}$ is bounded, the first equality is true by virtue of Lemma 6.3.

To prove (6.8) with V^α replaced by $V^\alpha \zeta^a$ we check that

$$V^\alpha \zeta^a (H_a + i)^{-2} \in \mathscr{K}_\infty . \tag{6.9}$$

Let $\eta_\rho(x) = \eta(\rho^{-1}x)$, where $\eta \in C_0^\infty(\mathbb{R}^d)$, $\eta(x) = 1$ for $|x| \leqq 1$, $\eta(x) = 0$ for $|x| \geqq 2$ and let $\tilde{\eta}_\rho(x) = 1 - \eta_\rho(x)$. Since the operator $V^\alpha (T + I)^{-1}$ is bounded and $Q^{-r}(H_a + i)^{-1} \in \mathscr{K}_\infty$ for $r > 0$, it is easy to verify with the help of Lemmas 2.10 and 2.11 that $V^\alpha \eta_\rho (H_a + i)^{-2} \in \mathscr{K}_\infty$. So in order to check (6.9) it remains to verify that

$$\lim_{\rho \to \infty} \| V^\alpha \zeta^a \tilde{\eta}_\rho (H_a + i)^{-1} \| = 0 . \tag{6.10}$$

We remark that $\zeta^a(x) \tilde{\eta}_\rho(x) = 0$ for $|x^\alpha| \leqq \gamma\rho$ with suitably chosen $\gamma > 0$. Let χ_ρ^α be multiplication by the function which equals 0 for $|x^\alpha| \leqq \gamma\rho$ and equals 1 for $|x^\alpha| > \gamma\rho$. Then (6.10) is a consequence of the relation $\| \chi_\rho^\alpha V^\alpha (T^\alpha + I)^{-1} \| \to 0$ as $\rho \to \infty$ in the space $\mathscr{H}^\alpha$. This is true because the operator $V^\alpha (T^\alpha + I)^{-1}$ is compact and χ_ρ^α converges strongly to zero. □

Now we are able to prove

Lemma 6.6. *Suppose that the statement of Theorem 2.7 is true for all operators H^a. Then the wave operators*

$$W^\pm(H, H_a; M^{(b)}(H_a + i)^{-1}) \tag{6.11}$$

exist for all a, b.

Proof. Let us proceed from the relation (6.3). Applying to it the bounded operator $M^{(b)}(H_a + i)^{-1}$ we find that for every $f \in \mathscr{H}$,

$$M^{(b)}(H_a + i)^{-1} U_a(t) f \sim \sum_{X^c \subset X^a} M^{(b)}(H_a + i)^{-1} U_c(t) P_c f_c^\pm \tag{6.12}$$

(the sum here is, of course, taken over all c such that $X^c \subset X^a$). Note the resolvent identity

$$(H_a + i)^{-1} - (H_c + i)^{-1} = -(H_a + i)^{-1}(V^a - V^c)(H_c + i)^{-1} , \tag{6.13}$$

where, by (1.5), $V^a - V^c$ is the sum of those V^α for which $X^\alpha \subset X^a$ and $X^\alpha \not\subset X^c$. By virtue of Lemma 6.5 $(V^a - V^c)(H_c + i)^{-1} U_c(t) P_c \to 0$ strongly as $|t| \to \infty$ so that we can replace in the right side of (6.12) $(H_a + i)^{-1}$ by $(H_c + i)^{-1}$. Furthermore, according to Corollary 6.4 the terms in the right side of (6.12) corresponding to c such that $X^c \not\subset X^b$ tend to zero. It follows that

$$M^{(b)}(H_a + i)^{-1} U_a(t) f \sim \sum_{X^c \subset X^a \cap X^b} M^{(b)}(H_c + i)^{-1} U_c(t) P_c f_c^\pm .$$

Using again the identity (6.13) (for $a = b$) and Lemma 6.5 (for $a = c$) we can replace $(H_c + i)^{-1}$ in the right side by $(H_b + i)^{-1}$. Thus in order to prove existence of (6.11) it remains to do the same for all wave operators

$$W^{\pm}(H, H_c; M^{(b)}(H_b + i)^{-1} P_c), \quad X^c \subset X^a \cap X^b. \tag{6.14}$$

Let us take into account the multiplication theorem (2.4) which asserts that the wave operator (6.14) exists and equals

$$W^{\pm}(H, H_b; M^{(b)}(H_b + i)^{-1}) W^{\pm}(H_b, H_c; P_c)$$

provided these two wave operators exist. The first of them exists by Theorem 5.4 and the second one – by Proposition 6.2. □

Corollary 6.7. *The wave operators $W^{\pm}(H, H_a; M(H_a + i)^{-1})$ exist for all a.*

Proof. It suffices to "sum up" the wave operators (6.11) over all b and to take into account the relation (1.22). □

Now we can get rid of the identification M.

Proposition 6.8. *The wave operators $W^{\pm}(H, H_a)$ exist for all a.*

Proof. Let us apply Proposition 5.5 and Corollary 5.9 to the operator H_a. According to the first of them for every admissible Λ the wave operator $M_a^{\pm}(\Lambda) = W^{\pm}(H_a, H_a; ME_a(\Lambda))$ exists. According to the second for every $f \in E_a(\Lambda)\mathscr{H}$ there exists a (unique) vector $f_a^{\pm} \in E_a(\Lambda)\mathscr{H}$ such that $f = M_a^{\pm}(\Lambda) f_a^{\pm}$. Since $U_a(t) f \sim M U_a(t) f_a^{\pm}$, $t \to \pm\infty$, Corollary 6.7 ensures existence of $W^{\pm}(H, H_a; E_a(\Lambda))$ and hence of $W^{\pm}(H, H_a)$. □

Since P_a commutes with $U_a(t)$, we have

Corollary 6.9. *The wave operators (2.9) exist and are isometric on $P_a\mathscr{H}$.*

The proof of the following assertion is standard and is given for completeness of exposition.

Proposition 6.10. *Suppose that the wave operators $W^{\pm}(H, H_a; P_a)$ and $W^{\pm}(H, H_b; P_b)$, $a \neq b$, exist. Then their ranges are orthogonal.*

Proof. It suffices to check that $(U_a(t) P_a f_a, U_b(t) P_b f_b) \to 0$ as $|t| \to \infty$ if $a \neq b$. Approximating P^a and P^b by finite-dimensional projections we reduce the problem to the case of one-dimensional P^a and P^b. Let λ^a and λ^b be corresponding eigenvalues of the operators H^a and H^b. Then

$$(U_a(t) P_a f_a, U_b(t) P_b f_b) = (\exp(-i(T_a + \lambda^a)t) P_a f_a, \exp(-i(T_b + \lambda^b)t) P_b f_b)$$

so that it remains to verify that the operator $\exp(i(T_b - T_a)t)$ converges weakly to zero in the space $\mathscr{H}$ as $|t| \to \infty$. To this end we shall show that the operator $T_a - T_b$ is absolutely continuous. In the momentum representation $T_a - T_b$ acts as multiplication by some quadratic form of coordinate functions ξ_j (dual to x_j), $j = 1, \ldots, d$. This form reduces to $\sum \mu_j \xi_j^2$ for suitable choice of a basis in $\mathbb{R}^d$. Due to the condition $a \neq b$ at least one of these numbers μ_j is not zero. Therefore the operator of multiplication by $\sum \mu_j \xi_j^2$ is absolutely continuous. □

Let us finally verify the equality (2.10). Since $R(W_a^{\pm}) \subset \mathscr{H}^{(ac)}(H)$, we need only to establish the inclusion

$$\mathscr{H}^{(ac)}(H) \subset \sum_a \oplus R(W_a^{\pm}). \tag{6.15}$$

Linear combinations of elements $f = E(\Lambda)f$ for all admissible Λ are dense in $\mathscr{H}^{(\mathrm{ac})}(H)$. So it suffices to check that every $f = E(\Lambda)f$ belongs to the right side of (6.15). Let us proceed from Theorem 6.1 and Proposition 6.2. Combining relations (1.8) and (6.3) we find that for every $f \in E(\Lambda)\mathscr{H}$ there exist elements $\tilde{f}_a^{\pm}$ such that $U(t)f \sim \sum_a U_a(t)P_a\tilde{f}_a^{\pm}, t \to \pm\infty$. Since the wave operators (2.9) exist, it follows that $f = \sum_a W_a^{\pm}\tilde{f}_a^{\pm}$. This concludes the proof of (6.15) and consequently of Theorem 2.7.

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Note added in proof. The proof in Sect. 6 of existence of wave operators $W^{\pm}(H, H_a; P_a)$ can be considerably simplified. Actually, existence of $W^{\pm}(H, H_a; M^{(a)}P_a)$ ensures that the limit defining $W^{\pm}(H, H_a; P_a)f$ exists if f belongs to the range of the operator $W^{\pm}(H_a, H_a; M^{(a)}P_a)$. This wave operator can be computed explicitly. It turns out that the union for different admissible $m^{(a)}$ of their ranges is dense in $P_a\mathscr{H}$.

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