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Crystal Base and a Generalization of the Littlewood–Richardson Rule for the Classical Lie Algebras

Toshiki Nakashima

Department of Mathematical Science, Faculty of Engineering Science, Osaka University, Toyonaka Osaka 560, Japan

Abstract. We shall give a generalization of the Littlewood–Richardson rule for $U_q(\mathfrak{g})$ associated with the classical Lie algebras by use of crystal base. This rule describes explicitly the decomposition of tensor products of given representations.

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0. Introduction

In representation theory, it is one of the most fundamental problems to decompose a given representation into the irreducible components. For the Lie algebra $\mathfrak{gl}(n)$, we know a very famous rule called the Littlewood–Richardson rule, which gives the irreducible decomposition of the tensor product of two finite-dimensional irreducible representations. There are various generalizations of this rule to other Lie algebras (e.g. cf. [B-Z, L, T]). The purpose of this paper is to give an explicit description of irreducible decomposition of tensor products of finite-dimensional representations of the q-analogue of universal enveloping algebra associated with the classical Lie algebras by a new tool "crystal base."

The notion of the q-analogue of universal enveloping algebras was introduced by V.G. Drinfeld ([D]) and M. Jimbo ([J]) in 1985 independently. In 1990, the theory of crystal base was constructed by M. Kashiwara ([K1, K2]). Roughly speaking, this is the representation theory of the universal enveloping algebra $U_q(\mathfrak{g})$ at $q = 0$. In the world at $q = 0$, various phenomena become much simpler. In particular, the crystal base has a nice property with respect to tensor products of given representations (Theorem 1.1.5 and Proposition 3.2.1).

The crystal base has a colored and oriented graph structure, called a crystal graph (Definition 1.1.7). In [K-N], we describe all the crystal graphs of finite-dimensional representations of $U_q(\mathfrak{g})$ ($\mathfrak{g} = A_n, B_n, C_n, D_n$). That gives an explicit parametrization of vertices of the crystal graphs in terms of analogues of semi-standard tableaux. The description of the rule of irreducible decompositions depends on the nice property for tensor products and these combinatorial parametrizations of crystal base.

The contents of this paper are as follows. In Sect. 1, the definition and several properties of crystal base are given, in particular, Theorem 1.1.3, Corollary 1.1.4 and Theorem 1.1.5 guarantee the validity of arguments in the later sections. In Sect. 2, we summarize the results of [K-N], which describes all the crystal graphs of finite-dimensional irreducible representations of $U_q(\mathfrak{g})$. First, the crystal graphs of the vector representation and the spin representation are given. Next, the crystal graphs of the fundamental representations are realized in tensor products of the vector representations or the spin representations. Finally, by tensor products of the fundamental weight cases, the crystal graphs in the general case are described. Namely, we embed the crystal graph $B(\lambda)$ of the irreducible module $V(\lambda)$ with highest weight λ into $B(V_\Box)^{\otimes m}$ or $B(\lambda)$ into $B(V_\Box)^{\otimes m} \otimes B(V_{\mathrm{sp}})$, where $V_\Box$ is the vector representation and V_{sp} is the spin representation. Then we describe $B(\lambda)$ as its image. In Sect. 3, first, we introduce generalized Young diagrams of type $\mathfrak{g}$, which parametrize all finite-dimensional irreducible representations of $U_q(\mathfrak{g})$. Next, the following proposition is proved.

Proposition 3.2.1. *Let λ and μ be dominant integral weights of $\mathfrak{g}$. For $u \in B(\lambda)$ and $v \in B(\mu)$, the following two conditions are equivalent;*

(a) $\tilde{e}_i(u \otimes v) = 0$ *for any* i.
(b) $\tilde{e}_i u = 0$ *and* $\tilde{e}_i^{\langle h_i, \lambda\rangle + 1} v = 0$ *for any* i.

By Corollary 1.1.4, we have that $V(\lambda) \otimes V(\mu)$ is decomposed into $V(wt(u \otimes v))$ where $u \otimes v$ ranges over $B(\lambda) \otimes B(\mu)$ satisfying the condition (a). By the condition (b), we know that if $u \otimes v$ is a highest weight element of $B(\lambda) \otimes B(\mu)$, then $u = u_\lambda$. In Sects. 4 and 5, the second condition of Proposition 3.2.1(b) is translated in terms of generalized Young diagrams for the special cases $V_Y \otimes V_\Box$ and $V_Y \otimes V_{\mathrm{sp}}$ and the rules of decomposition of those representations are given. Those rules are the steps in the procedure for general cases. In Sect. 6, as the consequence of Sects. 1–5, the procedure to obtain the decomposition of $V_Y \otimes V_W$ is given, where Y and W are generalized Young diagrams of type $\mathfrak{g}$ (Theorem 6.3.1 and Corollary 6.3.2).

The rule of decompositions given in this paper coincides with the corresponding rule in the classical case, namely the case at $q = 1$. So, in the appendix, we shall give the 1-1 correspondence between the original Littlewood–Richardson rule and the rule given in this paper for $\mathfrak{gl}(n + 1)$.

After submitting this paper the author received the preprint "Crystal graphs and Young tableaux" from Littelmann. In that preprint, he also give descriptions of crystal graphs and the Littlewood–Richardson rule for A_n, B_n, C_n, D_n, E_6 and G_2 in terms of the generalized Young tableaux.

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1. Basic Notions of Crystal Base

In this section, we give the basic notion of crystal base.

1.1. Definitions. *Let* $\mathfrak{g}$ *be a finite-dimensional simple Lie algebra with a Cartan subalgebra* $\mathfrak{t}$, *the set of simple roots* $\{\alpha_i \in \mathfrak{t}^*\}_{i\in I}$ *and the set of simple coroots* $\{h_i \in \mathfrak{t}\}_{i\in I}$, *where* I *is a finite index set. We take an inner product* $(\,,)$ *on* $\mathfrak{t}^*$ *such that* $(\alpha_i, \alpha_i)\in \mathbf{Z}_{>0}$ *and* $\langle h_i, \lambda\rangle = 2(\alpha_i, \lambda)/(\alpha_i, \alpha_i)$ *for* $\lambda \in \mathfrak{t}^*$. *Let* $\{\Lambda_i\}_{i\in I}$ *be the dual base of* $\{h_i\}$ *and set* $P = \sum \mathbf{Z}\Lambda_i$ *and* $P^* = \sum \mathbf{Z}h_i$. *Then the q-analogue* $U_q(\mathfrak{g})$ *is the algebra over* $\mathbf{Q}(q)$ *generated by* e_i, f_i *and* q^h $(h\in P^*)$ *satisfying the following relations*:

$$q^h = 1 \text{ if } h = 0 \text{ and } q^h q^{h'} = q^{h+h'}\,, \tag{1.1.1}$$

$$q^h e_j q^{-h} = q^{\langle h, \alpha_j\rangle} e_j \text{ and } q^h f_j q^{-h} = q^{-\langle h, \alpha_j\rangle} f_j\,, \tag{1.1.2}$$

$$[e_i, f_j] = \delta_{i,j}\frac{t_i - t_i^{-1}}{q_i - q_i^{-1}}, \text{ where } q_i = q^{(\alpha_i, \alpha_i)} \text{ and } t_i = q^{(\alpha_i, \alpha_i)h_i}\,, \tag{1.1.3}$$

$$\sum_{\mu=0}^{b} e_i^{(\mu)} e_j e_i^{(b-\mu)} = \sum_{\mu=0}^{b} f_i^{(\mu)} f_j f_i^{(b-\mu)} = 0.\ (i \neq j \text{ and } b = 1 - \langle h_i, \alpha_j\rangle)\,, \tag{1.1.4}$$

where $e_i^{(k)} = e_i^k/[k]_i!$ $f_i^{(k)} = f_i^k/[k]_i!$, $[n]_i = (q_i^n - q_i^{-n})/(q_i - q_i^{-1})$ *and* $[k]_i! = \prod_{n=1}^{k}[n]_i$.

The comultiplication $\Delta\colon U_q(\mathfrak{g}) \to U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ is given by $\Delta(q^h) = q^h \otimes q^h$, $\Delta(e_i) = e_i \otimes t_i^{-1} + 1 \otimes e_i$ and $\Delta(f_i) = f_i \otimes 1 + t_i \otimes f_i$. If M and N are $U_q(\mathfrak{g})$-modules, then by this comultiplication $M \otimes N$ is also $U_q(\mathfrak{g})$-module.

For a finite-dimensional $U_q(\mathfrak{g})$-module M and $\lambda \in P$, we set $M_\lambda = \{u\in M;\ t_i u = q^{2(\alpha_i, \lambda)}u\}$. We call M integrable if $M = \bigoplus M_\lambda$. Then we have

$$M_\lambda = \bigoplus_{k\geq 0,\ -\langle h_i, \lambda\rangle} f_i^{(k)}(M_{\lambda+k\alpha_i} \cap \operatorname{Ker} e_i)\,. \tag{1.1.5}$$

We define the operators $\tilde{e}_i$, $\tilde{f}_i$ acting on M by $\tilde{e}_i f_i^{(k)}u = f_i^{(k-1)}u$ and $\tilde{f}_i f_i^{(k)}u = f_i^{(k+1)}u$, for $u \in M_{\lambda+k\alpha_i} \cap \operatorname{Ker} e_i$ and (λ, k) as above.

Definition 1.1.1. *Let* A *be the ring of rational functions regular at* $q = 0$. *A pair* (L, B) *is called a crystal base of a finite-dimensional integrable representation* M *if the following conditions are satisfied*;

(1) L *is a free sub-A-module of* M *such that* $\mathbf{Q}(q)\otimes_A L \cong M$.
(2) B *is a base of the* $\mathbf{Q}$*-vector space* L/qL.
(3) $L = \bigoplus L_\lambda$, $B = \coprod B_\lambda$, *where* $L_\lambda = L \cap M_\lambda$ *and* $B_\lambda = B \cap (L_\lambda/qL_\lambda)$.
(4) $\tilde{f}_i L \subset L$, *and* $\tilde{e}_i L \subset L$.
(5) $\tilde{f}_i B \subset B \cup \{0\}$ *and* $\tilde{e}_i B \subset B \cup \{0\}$.
(6) *For* $u, v \in B$ *and* $i \in I$, $u = \tilde{e}_i v$ *if and only if* $v = \tilde{f}_i u$.

We call L *crystal lattice* and B *crystal.*

Then the following results are proved in [K1] for $\mathfrak{g} = A_n, B_n, C_n$ and D_n and in [K2] in the general case. Let $\lambda \in P_+ = \{\lambda \in t^*;\ \langle h_i, \lambda\rangle \in \mathbf{Z}_{\geq 0}\}$ and $V(\lambda)$ the irreducible integrable $U_q(\mathfrak{g})$-module generated by the highest weight vector u_λ with weight λ. We set $L(\lambda) = \sum A\tilde{f}_{i_1} \ldots \tilde{f}_{i_k} u_\lambda$ and $B(\lambda) = \{\tilde{f}_{i_1} \ldots \tilde{f}_{i_k} u_\lambda \bmod qL(\lambda)\} \backslash \{0\} \subset L(\lambda)/qL(\lambda)$, where i_j runs over I.

Theorem 1.1.2. *$(L(\lambda), B(\lambda))$ is a crystal base of $V(\lambda)$.*

Theorem 1.1.3. *If (L, B) is a crystal base of an integrable $U_q(\mathfrak{g})$-module M, then there is an isomorphism: $M \cong \bigoplus_j V(\lambda_j)$ by which $(L, B) \cong \bigoplus_j (L(\lambda_j), B(\lambda_j))$.*

Corollary 1.1.4. *Let M and (L, B) be as above. Then we obtain*

$$M \cong \bigoplus_{b \in B^h} V(wt(b)), \quad \textit{where } B^h = \{b \in B; \tilde{e}_i b = 0 \text{ for any } i\} .$$

We call an element of B^h a *highest weight element* of B.

Theorem 1.1.5. *Let (L_j, B_j) be a crystal base of an integrable $U_q(\mathfrak{g})$-module M_j $(j = 1, 2)$. Set $L = L_1 \otimes_A L_2 \subset M_1 \otimes M_2$ and $B = \{b_1 \otimes b_2;\ b_j \in B_j\ (j = 1, 2)\} \subset L/qL$. Then we have*

(1) $$(L, B) \textit{ is a crystal base of } M_1 \otimes M_2 .$$

(2) $$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \textit{if } \varphi_i(b_1) > \varepsilon_i(b_2) , \\ b_1 \otimes \tilde{f}_i b_2 & \textit{if } \varphi_i(b_1) \leqq \varepsilon_i(b_2) . \end{cases}$$

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} b_1 \otimes \tilde{e}_i b_2 & \textit{if } \varphi_i(b_1) < \varepsilon_i(b_2) , \\ \tilde{e}_i b_1 \otimes b_2 & \textit{if } \varphi_i(b_1) \geqq \varepsilon_i(b_2) . \end{cases}$$

Here, $\varepsilon_i(b) = \max\{k \geqq 0;\ \tilde{e}_i^k b \neq 0\}$ and $\varphi_i(b) = \max\{k \geqq 0;\ \tilde{f}_i^k b \neq 0\}$.

Corollary 1.1.6. *For $b_j \in B_j$ $(j = 1, 2)$, we have*

$$\varepsilon_i(b_1 \otimes b_2) = \max(\varepsilon_i(b_1), \varepsilon_i(b_1) + \varepsilon_i(b_2) - \varphi_i(b_1)) ,$$

$$\varphi_i(b_1 \otimes b_2) = \max(\varphi_i(b_2), \varphi_i(b_1) + \varphi_i(b_2) - \varepsilon_i(b_2)) .$$

Definition 1.1.7. *A crystal graph of a crystal base (L, B) is the colored and oriented graph B, with the arrows: $u \xrightarrow{i} v$ if and only if $v = \tilde{f}_i u$.*

Let M_i $(i = 1, 2)$ and B be as in Theorem 1.1.5, and B^h be the set of highest weight elements of B. By Corollary 1.1.4 and Theorem 1.1.5, we get

$$M_1 \otimes M_2 \cong \bigoplus_{b_1 \otimes b_2 \in B^h} V(wt(b_1 \otimes b_2)) .$$

The aim of this paper is to give the explicit procedure for obtaining such b_1 and b_2.

2. Review of Crystal Graphs

In this section, we shall summarize the results of [K-N], which gives an explicit description of crystal graphs of $U_q(\mathfrak{g})$-modules ($\mathfrak{g} = A_n, B_n, C_n, D_n$). We omit the rule of arrows.

2.1. Crystals for $U_q(A_n)$-modules. We shall treat the A_n-case. We set $\varepsilon_1 = \Lambda_1$, $\varepsilon_i = \Lambda_{i+1} - \Lambda_i$ for $2 \leqq i \leqq n$ and $\varepsilon_{n+1} = -(\varepsilon_1 + \cdots + \varepsilon_n)$. Define $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$.

The crystal graph $B(V_\square)$ of the vector representation $V_\square$ is easily obtained by explicit construction. It is labelled by $\{\boxed{i}\,;\ i=1,\dots,n+1\}$ and the crystal graph is

$$\boxed{1}\xrightarrow{1}\boxed{2}\xrightarrow{2}\cdots\xrightarrow{n-1}\boxed{n}\xrightarrow{n}\boxed{n+1}. \tag{2.1.1}$$

Here, $\boxed{i}$ has weight ε_i. Graph (2.1.1) implies

$$\tilde{e}_i\boxed{j}=\delta_{i,j-1}\boxed{j-1},\quad \tilde{f}_i\boxed{j}=\delta_{i,j}\boxed{j+1}. \tag{2.1.2}$$

Let $V(\Lambda_N)$ be the irreducible representation with the fundamental weight Λ_N $(1\leqq N\leqq n)$ as a highest weight. We embed $V(\Lambda_N)$ into $V_\square^{\otimes N}$. Accordingly, $B(\Lambda_N)$ is embedded into $B(V_\square)^{\otimes N}$. We have that $B(\Lambda_N)$ consists of $\boxed{i_1}\otimes\cdots\otimes\boxed{i_N}$ with $1\leqq i_1<\cdots<i_N\leqq n+1$. The base $u_{\Lambda_N}=\boxed{1}\otimes\boxed{2}\otimes\cdots\otimes\boxed{N}$ is annihilated by all $\tilde{e}_i$ and it has weight $\Lambda_N=\varepsilon_1+\cdots+\varepsilon_N$, then the crystal graph $B(\Lambda_N)$ is the connected component of $B(V_\square)^{\otimes N}$ containing $\boxed{1}\otimes\cdots\otimes\boxed{N}$. We write $\begin{array}{|c|}\hline i_1\\\hline \vdots\\\hline i_N\\\hline\end{array}$ for $\boxed{i_1}\otimes\cdots\otimes\boxed{i_N}\in B(V_\square)^{\otimes N}$.

Let $\lambda=\sum_{i=1}^{p}\Lambda_{l_i}$ $(1\leqq l_1\leqq l_2\leqq\cdots\leqq l_p\leqq n)$ be a dominant integral weight. For $u_k=\begin{array}{|c|}\hline t_1^k\\\hline \vdots\\\hline t_{l_k}^k\\\hline\end{array}\in B(\Lambda_{l_k})$, we denote $u_1\otimes\cdots\otimes u_p=\begin{array}{|c|c|c|c|c|}\hline t^p_{l''} & \vdots & t^k_l & \vdots & t^1_{l'}\\\hline\end{array}\in B(\Lambda_{l_1})\otimes\cdots\otimes B(\Lambda_{l_p})$. We obtain the following:

$$B(\lambda)=\left\{\begin{array}{|c|c|c|c|c|}\hline t^p_{l''} & \vdots & t^k_l & \vdots & t^1_{l'}\\\hline\end{array}\in B(\Lambda_{l_1})\otimes\cdots\otimes B(\Lambda_{l_p});\ \begin{array}{l}\text{For } 1\leqq k\leqq p,\ 1\leqq l\leqq l_k,\\ t_l^k\in\{1,\dots,n+1\}\text{ satisfies}\\ t_l^{k+1}\leqq t_l^k \text{ and } t_l^k<t_{l+1}^k.\end{array}\right\}. \tag{2.1.3}$$

An element of $B(\lambda)$ is called a *semi-standard A-tableau* of shape λ. Note that *semi-standard A-tableaux* coincide with usual *semi-standard tableaux*, so in the case $\mathfrak{g}=\mathfrak{gl}(n+1)$, $B(\lambda)$ is also given by the same rule as the A_n-case.

2.2. Crystals for $U_q(C_n)$-modules. Let $(\varepsilon_1,\dots,\varepsilon_n)$ be the orthonormal base of the dual of the Cartan subalgebra of C_n such that $\alpha_i=\varepsilon_i-\varepsilon_{i+1}(1\leqq i<n)$ and $\alpha_n=2\varepsilon_n$ are the simple roots. Hence, α_n is the long root and $\alpha_1,\dots,\alpha_{n-1}$ are short roots. Let $\{\Lambda_i\}_{1\leqq i\leqq n}$ be the dual base of $\{h_i\}_{1\leqq i\leqq n}$. Hence $\Lambda_i=\varepsilon_1+\cdots+\varepsilon_i(1\leqq i\leqq n)$.

The crystal graph $B(V_\square)$ of the vector representation $V_\square$ is described as follows. It is labelled by $\{\boxed{i},\boxed{\bar{i}};\ 1\leqq i\leqq n\}$, where $\boxed{i}$ has weight ε_i and has $\boxed{\bar{i}}$ has weight $-\varepsilon_i$. Its crystal graph is

$$\boxed{1}\xrightarrow{1}\boxed{2}\xrightarrow{2}\cdots\xrightarrow{n-1}\boxed{n}\xrightarrow{n}\boxed{\bar{n}}\xrightarrow{n-1}\cdots\xrightarrow{2}\boxed{\bar{2}}\xrightarrow{1}\boxed{\bar{1}}. \tag{2.2.1}$$

Similarly to the A_n-case, the connected component of the crystal graph $B(V_\square)^{\otimes N}$ containing $u_{\Lambda_N} = \boxed{1} \otimes \boxed{2} \otimes \cdots \otimes \boxed{N}$ is isomorphic to $B(\Lambda_N)$.

We give the linear order on $\{i, \bar{i};\ 1 \leqq i \leqq n\}$ by $1 \prec 2 \prec \cdots n \prec \bar{n} \prec \cdots \prec \bar{2} \prec \bar{1}$. By using the same notation as in 2.1, we have

$$B(\Lambda_N) = \left\{ \begin{array}{|c|} \hline i_1 \\ \hline \vdots \\ \hline i_N \\ \hline \end{array} \in B(V_\square)^{\otimes N};\ \begin{array}{l} (1)\ 1 \leqq i_1 \prec \cdots \prec i_N \leqq \bar{1}, \\ (2)\ \text{if } i_k = p \text{ and } i_l = \bar{p}, \\ \quad\ \text{then } k + (N - l + 1) \leqq p \end{array} \right\}. \tag{2.2.2}$$

Next, we shall give the crystal of $V(\Lambda_M + \Lambda_N)$ with $1 \leqq M \leqq N \leqq n$. By embedding $V(\Lambda_M + \Lambda_N)$ into $V(\Lambda_M) \otimes V(\Lambda_N)$, $B(\Lambda_M + \Lambda_N)$ is the connected component of $B(\Lambda_M) \otimes B(\Lambda_N)$ containing $u_{\Lambda_M} \otimes u_{\Lambda_N}$.

For $u = \begin{array}{|c|} \hline j_1 \\ \hline \vdots \\ \hline j_M \\ \hline \end{array} \in B(\Lambda_M)$ and $v = \begin{array}{|c|} \hline i_1 \\ \hline \vdots \\ \hline i_N \\ \hline \end{array} \in B(\Lambda_N)$, $u \otimes v$ will be denoted by $\begin{array}{|c|c|} \hline i_1 & j_1 \\ \hline \vdots & \vdots \\ \hline & j_M \\ \hline i_N & \\ \hline \end{array}$.

Definition 2.2.1. *For $1 \leqq a \leqq b \leqq n$ and u, v as above, we say that $u \otimes v \in B(\Lambda_M) \otimes B(\Lambda_N)$ is in the (a, b)-configuration if $u \otimes v$ satisfies the following; there exist $1 \leqq p \leqq q < r \leqq s \leqq M$ such that $i_p = a$, $j_q = b$, $j_r = \bar{b}$, $j_s = \bar{a}$ or $i_p = a$, $i_q = b$, $i_r = \bar{b}$, $j_s = \bar{a}$.*

This definition includes the case $a = b$, $p = q$ and $r = s$. Now, we define $p(a, b; u \otimes v) = (q - p) + (s - r)$. We obtain the following;

$$B(\Lambda_M + \Lambda_N) =$$

$$\left\{ w = \begin{array}{|c|c|} \hline i_1 & j_1 \\ \hline \vdots & \vdots \\ \hline & j_M \\ \hline i_N & \\ \hline \end{array} \in B(\Lambda_M) \otimes B(\Lambda_N);\ \begin{array}{l} (1)\ i_k \leqq j_k \text{ for } 1 \leqq k \leqq M. \\ (2)\ \text{If } w \text{ is in the } (a, b)\text{-configuration,} \\ \quad\ \text{then } p(a, b; w) < b - a\,. \end{array} \right\} \tag{2.2.3}$$

Let $\lambda = \sum_{i=1}^{p} \Lambda_{l_i}$ $(1 \leqq l_1 \leqq \cdots \leqq l_p \leqq n)$ be a dominant integral weight. Using the same notation as in (2.1.3), we obtain;

$$B(\lambda) = \left\{ \begin{array}{|c|c|c|c|c|} \hline & & & & t_l^1 \\ \hline t_{l'}^p & \vdots & \vdots & \vdots & \\ \hline \end{array} \in B(\Lambda_{l_1}) \otimes \cdots \otimes B(\Lambda_{l_p});\ \begin{array}{l} u_k \otimes u_{k+1} \in B(\Lambda_{l_k} + \Lambda_{k+1}) \\ \text{for any } k. \end{array} \right\}. \tag{2.2.4}$$

An element of $B(\lambda)$ is called a *semi-standard C-tableau* of shape λ.

2.3. Crystals for $U_q(B_n)$-modules. Let $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ be the orthonormal base of the dual of the Cartan subalgebra of B_n such that $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ $(1 \leqq i < n)$ and

$\alpha_n = \varepsilon_n$ are simple roots. Hence, $\alpha_1, \dots, \alpha_{n-1}$ are long roots and α_n is the short root. Let $\{\Lambda_i\}_{1 \leqq i \leqq n}$ be the dual base of $\{h_i\}_{1 \leqq i \leqq n}$. Hence $\Lambda_i = \varepsilon_1 + \cdots + \varepsilon_i$ $(i = 1, \dots, n-1)$ and $\Lambda_n = (\varepsilon_1 + \cdots + \varepsilon_n)/2$.

The crystal graph $B(V_\square)$ of the vector representation $V_\square$ is described as follows. It is labelled by $\{\boxed{i}, \boxed{\bar{i}};\ 1 \leqq i \leqq n\} \cup \{\boxed{0}\}$, where $\boxed{i}$ has weight ε_i, $\boxed{\bar{i}}$ has weight $-\varepsilon_i$ and $\boxed{0}$ has weight 0. The crystal graph of $V_\square$ is

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{n-1} \boxed{n} \xrightarrow{n} \boxed{0} \xrightarrow{n} \boxed{\bar{n}} \xrightarrow{n-1} \cdots \xrightarrow{2} \boxed{\bar{2}} \xrightarrow{1} \boxed{\bar{1}}\,. \tag{2.3.1}$$

Next, we set $\omega_i = \varepsilon_1 + \cdots + \varepsilon_i$ $(1 \leqq i \leqq n)$. Note that $\omega_i = \Lambda_i$ $(1 \leqq i < n)$ and $\omega_n = 2\Lambda_n$. The representation $V(\omega_N)$ can be embedded into $V_\square^{\otimes N}$ $(1 \leqq N \leqq n)$.

We give the linear order on $\{i, \bar{i};\ 1 \leqq i \leqq n\} \cup \{0\}$ by

$$1 \prec 2 \prec \cdots \prec n \prec 0 \prec \bar{n} \prec \cdots \prec \bar{2} \prec \bar{1}\,. \tag{2.3.2}$$

Then we obtain

$$B(\omega_N) = \left\{ \begin{array}{|c|} \hline i_1 \\ \hline \vdots \\ \hline i_N \\ \hline \end{array} \in B(V_\square)^{\otimes N};\ \begin{array}{l} (1)\ 1 \leqq i_1 \leqq \cdots \leqq i_N \leqq \bar{1}, \\ \text{but any element other than } 0 \\ \text{cannot appear more than once.} \\ (2)\ \text{if } i_k = p \text{ and } i_l = \bar{p}\ (1 \leqq p \leqq n), \\ \text{then } k + (N - l + 1) \leqq p \end{array} \right\}. \tag{2.3.3}$$

Next, we introduce the spin representation which is denoted V_{sp}. This is the finite-dimensional irreducible representation with highest weight Λ_n. This is described explicitly by use of the following half-size tableaux. With the linear order on $\{1, 2, \dots, n, \bar{n}, \dots, \bar{2}, \bar{1}\}$ as before, we set

$$B_{\mathrm{sp}} = \left\{ \begin{array}{|c|} \hline i_1 \\ \hline \vdots \\ \hline i_n \\ \hline \end{array};\ \begin{array}{l} (1)\ i_j \in \{1, 2, \dots, n, \bar{n}, \dots, \bar{2}, \bar{1}\}, \\ (2)\ i_1 \prec \cdots \prec i_n, \\ (3)\ i \text{ and } \bar{i} \text{ do not appear simultaneously} \end{array} \right\}, \tag{2.3.4}$$

$V_{\mathrm{sp}} = \bigoplus_{v \in B_{\mathrm{sp}}} \mathbf{Q}(q)v$ and $L_{\mathrm{sp}} = \bigoplus_{v \in B_{\mathrm{sp}}} Av$. If we define the actions of generators as in [K-N], $V_{\mathrm{sp}} \cong V(\Lambda_n)$ and $(L_{\mathrm{sp}}, B_{\mathrm{sp}})$ is a crystal base of V_{sp}.

Next, we shall give the crystal of $V(\omega_M + \omega_N)$ $(1 \leqq M \leqq N \leqq n)$. $V(\omega_M + \omega_N)$ can be embedded into $V(\omega_M) \otimes V(\omega_N)$ uniquely.

Definition 2.3.1.

(1) *For $1 \leqq a \leqq b < n$, we say that $w = u \otimes v \in B(\omega_M) \otimes B(\omega_N)$ is in the (a, b)-configuration if $w = u \otimes v$ satisfies the same condition as Definition* 2.2.1.

(2) *For $1 \leqq a < n$, we say that $w = u \otimes v \in B(\omega_M) \otimes B(\omega_N)$ is in the (a, n)-configuration if $w = u \otimes v$ holds the following;*
there exist $1 \leqq p \leqq q < r = q + 1 \leqq s \leqq M$ such that $i_p = a$, $j_s = \bar{a}$ and one of the following conditions is satisfied:
(i) *i_q and $i_r (= i_{q+1})$ are n, 0 or $\bar{n}$.*
(ii) *j_q and $j_r (= j_{q+1})$ are n, 0 or $\bar{n}$.*

(3) *We say that* $w = u \otimes v \in B(\omega_M) \otimes B(\omega_N)$ *is in the* (n, n)*-configuration if there exist* $1 \leqq p < q \leqq M$ *such that* $i_p = n$ *or* 0 *and* $j_q = 0$ *or* $\bar{n}$.

This definition includes the case $a = b$, $p = q$ and $r = s$.

Now, for w in the (a, b)-configuration, we define $p(a, b; w) = (q - p) + (s - r)$. (If $a = b = n$, we set $p(a, b; w) = 0$.)

We obtain

$$B(\omega_M + \omega_N) = \left\{ w = u \otimes v = \begin{array}{|c|c|} \hline i_1 & j_1 \\ \hline \vdots & \vdots \\ \hline & j_M \\ \hline i_N & \\ \cline{1-1} \end{array} \in B(\omega_M) \otimes B(\omega_N);\ \begin{array}{c} w \text{ satisfies following} \\ \text{(M.N.1) and (M.N.2)} \end{array} \right\}. \qquad (2.3.5)$$

(M.N.1) $\quad i_k \leqq j_k$ for $1 \leqq k \leqq M$, but if $i_k = 0$ or $j_k = 0$, then $i_k \prec j_k$.

(M.N.2) $\quad$ If w is in the (a, b)-configuration, then $p(a, b; w) < b - a$.

We shall consider the crystal of $V(\omega_M + \Lambda_n)$ $(1 \leqq M \leqq n)$.

For $u = \begin{array}{|c|} \hline j_1 \\ \hline \vdots \\ \hline j_M \\ \hline \end{array} \in B(\omega_M)$ and $v = \begin{array}{|c|} \hline i_1 \\ \hline \vdots \\ \hline i_n \\ \hline \end{array} \in B_{\mathrm{sp}} = B(\Lambda_n)$, $u \otimes v$ will be denoted by $\begin{array}{|c|c|} \hline i_1 & j_1 \\ \hline \vdots & \vdots \\ \hline & j_M \\ \hline i_n & \\ \cline{1-1} \end{array}$

Definition 2.3.2. *When* $u \in B(\omega_M)$ *and* $v \in B_{\mathrm{sp}}$ *have the above expression, for* $1 \leqq a \leqq b \leqq n$ *we say that* $w = u \otimes v$ *is in the* (a, b)*-configuration if* w *satisfies the same condition as Definition* 2.3.1. *We define* $p(a, b; w) = (q - p) + (s - r)$. *We have*

$$B(\omega_M + \Lambda_n) = \left\{ w = \begin{array}{|c|c|} \hline i_1 & j_1 \\ \hline \vdots & \vdots \\ \hline & j_M \\ \hline i_n & \\ \cline{1-1} \end{array} \in B(\omega_M) \otimes B_{\mathrm{sp}};\ \begin{array}{l} (1)\ i_k \leqq j_k \textit{ for } 1 \leqq k \leqq M. \\ (2)\ \textit{If } w \textit{ is in the } (a, b)\textit{-configuration,} \\ \qquad \textit{then } p(a, b; w) < b - a. \end{array} \right\}.$$

Finally, let λ be a dominant integral weight.

Theorem 2.3.3. (i) *Suppose* $\langle h_n, \lambda \rangle$ *is even. We can write* $\lambda = \sum_{k=1}^{p} \omega_{l_k}$ *with* $l_1 \leqq \cdots \leqq l_p$. *We use the same notation as in* (2.1.3). *Then*

$$B(\lambda) = \left\{ \begin{array}{|c|c|c|c|c|} \hline t_l^p & \vdots & t_{l'}^k & \vdots & t_{l'}^1 \\ \hline \end{array} \in B(\omega_{l_1}) \otimes \cdots \otimes B(\omega_{l_p});\ \begin{array}{c} u_k \otimes u_{k+1} \in B(\omega_{l_k} + \omega_{l_{k+1}}) \\ \textit{for } k = 1, \ldots, p-1 \end{array} \right\}.$$

(2.3.6)

(ii) *Suppose $\langle h_n, \lambda\rangle$ is odd. We can write $\lambda = \sum_{k=1}^{p-1}\omega_{l_k} + \Lambda_n$ with $l_1 \leqq \cdots \leqq l_{p-1}$. For $u_k \in B(\omega_{l_k})$ $(1 \leqq k < p)$ (the same expression as above) and $u_p = \begin{array}{|c|}\hline t^p_1 \\ \hline \vdots \\ \hline t^p_n \\ \hline\end{array} \in B_{\mathrm{sp}}$, denote*

$$u_1 \otimes \cdots \otimes u_p = \begin{array}{|c|c|c|c|c|}\hline t^p_l & \vdots & t^k_{l'} & \vdots & t^1_{l''} \\ \hline\end{array} \in B(\omega_{l_1}) \otimes \cdots \otimes B(\omega_{l_{p-1}}) \otimes B_{\mathrm{sp}},$$

then we obtain

$$B(\lambda) = \left\{ \begin{array}{|c|c|c|c|c|}\hline t^p_l & \vdots & t^k_{l'} & \vdots & t^1_{l''} \\ \hline\end{array} \in B(\omega_{l_1}) \otimes \cdots \otimes B(\omega_{l_{p-1}}) \otimes B_{\mathrm{sp}}; \quad \begin{array}{l} \text{For } k = 1, \ldots, p-1 \\ u_k \otimes u_{k+1} \in B(\omega_{l_k} + \omega_{l_{k+1}}) \\ u_{p-1} \otimes u_p \in B(\omega_{l_{p-1}} + \Lambda_n) \end{array} \right\}. \tag{2.3.7}$$

An element of $B(\lambda)$ is called a semi-standard B-tableau of shape λ.

2.4. *Crystals for $U_q(D_n)$-modules.* Let $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ be the orthonormal base of the dual of the Cartan subalgebra of D_n such that $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ $(1 \leqq i < n)$ and $\alpha_n = \varepsilon_{n-1} + \varepsilon_n$ form the set of simple roots. Let $\{\Lambda_i\}_{1 \leqq i \leqq n}$ be the dual base of $\{h_i\}_{1 \leqq i \leqq n}$. Hence $\Lambda_i = \varepsilon_1 + \cdots + \varepsilon_i$ $(i = 1, \ldots, n-2)$ and $\Lambda_{n-1} = (\varepsilon_1 + \cdots + \varepsilon_{n-1} - \varepsilon_n)/2$. $\Lambda_n = (\varepsilon_1 + \cdots + \varepsilon_{n-1} + \varepsilon_n)/2$.

The crystal graph $B(V_\square)$ of the vector representation $V_\square$ is the following. It is labelled by $\{\boxed{i}, \boxed{\bar{i}}; 1 \leqq i \leqq n\}$, where $\boxed{i}$ has weight ε_i and $\boxed{\bar{i}}$ has weight $-\varepsilon_i$. The crystal graph of $V_\square$ is

$$\begin{tikzcd} & & & & \boxed{n} \arrow[dr, "n"] & & & & \\ \boxed{1} \arrow[r, "1"] & \boxed{2} \arrow[r, "2"] & \cdots \arrow[r, "n-2"] & \boxed{n-1} \arrow[ur, "n-1"] \arrow[dr, "n"'] & & \boxed{\overline{n-1}} \arrow[r, "n-2"] & \cdots \arrow[r, "2"] & \boxed{\bar{2}} \arrow[r, "1"] & \boxed{\bar{1}} \\ & & & & \boxed{\bar{n}} \arrow[ur, "n-1"'] & & & & \end{tikzcd} . \tag{2.4.1}$$

Next, we set $\omega_i = \varepsilon_1 + \cdots + \varepsilon_i$ $(1 \leqq i \leqq n)$ and $\omega_{n+1} = \varepsilon_1 + \cdots + \varepsilon_{n-1} - \varepsilon_n$. Note that $\omega_i = \Lambda_i$ $(1 \leqq i \leqq n-2)$, $\omega_{n-1} = \Lambda_{n-1} + \Lambda_n$, $\omega_n = 2\Lambda_n$ and $\omega_{n+1} = 2\Lambda_{n-1}$.

The representation $V(\omega_N)$ $(1 \leqq N \leqq n)$ (respectively $V(\omega_{n+1})$) is embedded into $V_\square^{\otimes N}$ (respectively $V_\square^{\otimes n}$). We give the ordering on $\{i, \bar{i}; 1 \leqq i \leqq n\}$ by;

$$1 \prec 2 \prec \cdots n-1 \prec \begin{matrix} n \\ \bar{n} \end{matrix} \prec \overline{n-1} \prec \cdots \prec \bar{2} \prec \bar{1}. \tag{2.4.2}$$

Note that there is no order between n and $\bar{n}$. Using the same expression as before, we obtain for $1 \leqq N \leqq n$,

$$B(\omega_N) = \left\{ \begin{array}{|c|} \hline i_1 \\ \hline \vdots \\ \hline i_N \\ \hline \end{array} \in B(V_\square)^{\otimes N};\ \begin{array}{l} (1)\ i_\nu \not\succeq i_{\nu+1} \text{ for } 1 \leqq \nu < N, \\ (2) \text{ if } i_k = p \text{ and } i_l = \bar{p}\ (1 \leqq p \leqq n) \\ \text{then } k + (N - l + 1) \leqq p \end{array} \right\}, \quad (2.4.3)$$

$$\begin{array}{l} B(\omega_n) = \\ (\text{respectively } B(\omega_{n+1})) \end{array} \left\{ \begin{array}{|c|} \hline i_1 \\ \hline \vdots \\ \hline i_n \\ \hline \end{array} \in B(V_\square)^{\otimes n};\ \begin{array}{l} (1) \text{ and } (2) \text{ are the same} \\ \text{conditionsm as in (2.4.3)} \\ (3) \text{ If } i_k = n, \text{ then } n-k \text{ is even} \\ (\text{respectively odd}) \\ \text{and if } i_k = \bar{n}, \text{ then } n-k \text{ is odd} \\ (\text{respectively even}) \end{array} \right\}. \quad (2.4.4)$$

Remark 2.4.1. The condition (1) in (2.4.3) is equivalent to saying that for any ν either $i_\nu \prec i_{\nu+1}$, or $(i_\nu, i_{\nu+1}) = (n, \bar{n})$ or $(\bar{n}, n)$.

For the algebra $U_q(D_n)$, there are two spin representations $V_{\mathrm{sp}}^{(+)}$ and $V_{\mathrm{sp}}^{(-)}$. They are the finite-dimensional irreducible representations with highest weight Λ_n and Λ_{n-1} respectively. They have the explicit description as follows. First we give the order on $P = \{1, 2, \dots, n, \bar{n}, \dots, \bar{2}, \bar{1}\}$ as (2.4.2). Next, we use the half-size tableaux and set

$B_{\mathrm{sp}}^{(+)}$(respectively $B_{\mathrm{sp}}^{(-)}$) =

$$\left\{ v = \begin{array}{|c|} \hline i_1 \\ \hline \vdots \\ \hline i_n \\ \hline \end{array};\ \begin{array}{l} (1)\ i_j \in P, \\ (2)\ i_1 \prec i_2 \prec \cdots \prec i_{n-1} \prec i_n, \\ (3)\ i \text{ and } \bar{i} \text{ cannot appear simultaneously}, \\ (4) \text{ If } i_k = n, \text{ then } n-k \text{ is even (respectively odd)} \\ (5) \text{ If } i_k = \bar{n}, \text{ then } n-k \text{ is odd (respectively even)} \end{array} \right\}, \quad (2.4.5)$$

$V_{\mathrm{sp}}^{(\pm)} = \bigoplus_{v \in B_{\mathrm{sp}}^{(\pm)}} \mathbf{Q}(q)v$ and $L_{\mathrm{sp}}^{(\pm)} = \bigoplus_{v \in B_{\mathrm{sp}}^{(\pm)}} Av$. If we give the actions of generators as in [K-N], then $V_{\mathrm{sp}}^{(+)} \cong V(\Lambda_n)$, $V_{\mathrm{sp}}^{(-)} \cong V(\Lambda_{n-1})$ and $(L_{\mathrm{sp}}^{(\pm)}, B_{\mathrm{sp}}^{(\pm)})$ is the crystal base of $V_{\mathrm{sp}}^{(\pm)}$.

Next, we shall describe the crystal of $V(\omega_M + \omega_N)$ $(1 \leqq M \leqq N \leqq n+1)$. For $u \otimes v \in B(\omega_M) \otimes B(\omega_N)$ we use the same expression as before.

Definition 2.4.2. *Let $1 \leqq M \leqq N \leqq n+1$ such that $(M, N) \neq (n, n+1)$.*

(1) *For $1 \leqq a \leqq b < n$, we say that $u \otimes v \in B(\omega_M) \otimes B(\omega_N)$ is in the (a, b)-configuration if $u \otimes v$ satisfies the same condition as Definition 2.2.1.*
(2) *For $1 \leqq a < n$, we say that $u \otimes v \in B(\omega_M) \otimes B(\omega_N)$ is in the (a, n)-configuration if $u \otimes v$ satisfies the following;*

there exist $1 \leqq p \leqq q < r = q+1 \leqq s \leqq M$ *such that* $(i_p, j_s) = (a, \bar{a})$ *or* $(\bar{a}, a)$ *and one of the following conditions is satisfied:*

(i) i_q *and* $i_r (= i_{q+1})$ *are* n *or* $\bar{n}$,

(ii) j_q *and* $j_r (= j_{q+1})$ *are* n *or* $\bar{n}$.

If w *is in the* (a, b)*-configuration, we define* $p(a, b; w) = (q-p) + (s-r)$.

(3) *For* $1 \leqq a < n$, *we say that* $u \otimes v \in B(\omega_M) \otimes B(\omega_N)$ *is in the a-odd-configuration if* $u \otimes v$ *satisfies the following; there exist* $1 \leqq p \leqq q < r \leqq s \leqq M$ *such that* (a) $r - q + 1$ *is odd,* (b) $i_p = a$ *and* $j_s = \bar{a}$, (c) $j_q = n$ *and* $i_r = \bar{n}$, *or* $j_q = \bar{n}$ *and* $i_r = n$.

(4) *For* $1 \leqq a < n$, *we say that* $u \otimes v \in B(\omega_M) \otimes B(\omega_N)$ *is in the a-even-configuration if* $u \otimes v$ *satisfies the following; there exist* $1 \leqq p \leqq q < r \leqq s \leqq M$ *such that (a)* $r - q + 1$ *is even, (b)* $i_p = a$ *and* $j_s = \bar{a}$, *(c)* $j_q = n$ *and* $i_r = n$, *or* $j_q = \bar{n}$ *and* $i_r = \bar{n}$.

If $w \in B(\omega_M) \otimes B(\omega_N)$ is in the a-odd or even-configuration, we define $q(a; w) = s - p$. Then we have, for $1 \leqq M \leqq N \leqq n+1$, with $(M, N) \neq (n, n+1)$.

$$B(\omega_M + \omega_N) = \left\{ w = \begin{array}{|c|c|} \hline i_1 & j_1 \\ \hline \vdots & \vdots \\ \hline & j_M \\ \hline i_N & \\ \hline \end{array} \in B(\omega_M) \otimes B(\omega_N); \quad \begin{array}{l} w \text{ satisfies the following} \\ \text{(M.N.1)–(M.N.3).} \end{array} \right\}, \tag{2.4.6}$$

(M.N.1) $i_k \preceq j_k$ for $1 \leqq k \leqq M$.

(M.N.2) If w is in the (a, b)-configuration, then $p(a, b; w) < b - a$.

(M.N.3) If w is in the a-odd-configuration or the a-even-configuration, then $q(a; w) < n - a$.

Next, we shall treat the representations $V(\omega_M + \Lambda_n)$ $(1 \leqq M \leqq n)$ and $V(\omega_N + \Lambda_{n-1})$ $(1 \leqq N \leqq n+1,\ N \neq n)$. They can be embedded into respectively $V(\omega_M) \otimes V_{\mathrm{sp}}^{(+)}$ and $V(\omega_N) \otimes V_{\mathrm{sp}}^{(-)}$ with multiplicity free. For $u \otimes v \in B(\omega_M) \otimes B_{\mathrm{sp}}^{(+)}$ and $B(\omega_N) \otimes B_{\mathrm{sp}}^{(-)}$, we use the same expression as in 2.3.

Definition 2.4.3.

(1) $u \otimes v \in B(\omega_M) \otimes B_{\mathrm{sp}}^{(+)}$ $(1 \leqq M \leqq n)$ *or* $B(\omega_N) \otimes B_{\mathrm{sp}}^{(-)}$ $(1 \leqq N \leqq n+1,\ N \neq n)$ *is in the* (a, b)*-configuration* $(1 \leqq a \leqq b \leqq n)$ *if* $u \otimes v$ *satisfies the same condition as Definition* 2.4.2. (1), (2).

(2) $u \otimes v \in B(\omega_M) \otimes B_{\mathrm{sp}}^{(+)}$ $(1 \leqq M \leqq n)$ *or* $B(\omega_N) \otimes B_{\mathrm{sp}}^{(-)}$ $(1 \leqq N < n)$ *is in the a-odd (respectively even)-configuration* $(1 \leqq a \leqq n)$ *if* $u \otimes v$ *satisfies the same condition as Definition* 2.4.2. (3) *(respectively* (4)).

$p(a, b; w)$ *and* $q(a; w)$ *are the ones defined in Definition* 2.4.2. *Then we obtain,*

$B(\omega_M + \Lambda_n) =$

(respectively $B(\omega_M + \Lambda_{n-1})$)

$$\left\{ w = \begin{array}{|c|c|} \hline i_1 & j_1 \\ \hline \vdots & \vdots \\ \hline & j_M \\ \hline i_n & \\ \hline \end{array} \begin{array}{l} \in B(\omega_M) \otimes B_{\mathrm{sp}}^{(+)} \\ \text{(respectively } B(\omega_M) \otimes B_{\mathrm{sp}}^{(-)}) \end{array} \quad \begin{array}{r} w \text{ satisfies} \\ ;\text{(M.N.1)} \\ -\text{(M.N.3)} \end{array} \right\}. \tag{2.4.7}$$

Let $\lambda = \sum_{i=1}^{n} m_i \Lambda_i$ ($m_i \in \mathbf{Z}_{\geqq 0}$) be a dominant integral weight of D_n. Now, we shall rewrite λ by use of ω_M, Λ_n and Λ_{n-1}. By the definition of ω_M, we have $\Lambda_i = \omega_i$ ($1 \leqq i \leqq n-2$), $\Lambda_{n-1} + \Lambda_n = \omega_{n-1}$, $2\Lambda_n = \omega_n$ and $2\Lambda_{n-1} = \omega_{n+1}$. Hence, any $\lambda \in P_+$ can be written in one of the following forms;

(W1) $\lambda = \sum_{i=1}^{n} m_i \omega_i$ ($m_i \in \mathbf{Z}_{\geqq 0}$).

(W2) $\lambda = \sum_{i=1}^{n} m_i \omega_i + \Lambda_n$ ($m_i \in \mathbf{Z}_{\geqq 0}$).

(W3) $\lambda = \sum_{i=1}^{n-1} m_i \omega_i + m_{n+1}\omega_{n+1}$ ($m_i \in \mathbf{Z}_{\geqq 0}$).

(W4) $\lambda = \sum_{i=1}^{n-1} m_i \omega_i + m_{n+1}\omega_{n+1} + \Lambda_{n-1}$ ($m_i \in \mathbf{Z}_{\geqq 0}$).

If λ is of type (W4), we can write $\lambda = \omega_{l_1} + \cdots + \omega_{l_p} + \omega_{l_{p+1}} + \cdots + \omega_{l_{q-1}} + \Lambda_{n-1}$ with $1 \leqq l_1 \leqq \cdots \leqq l_p < n < n+1 = l_{p+1} = \cdots = l_{q-1}$. Then for $u_k = \begin{array}{|c|}\hline t_1^k \\ \hline \vdots \\ \hline t_{l_k}^k \\ \hline \end{array} \in B(\omega_{l_k})$ ($1 \leqq k < q$) and $u_q = \begin{array}{|c|}\hline t_1^q \\ \hline \vdots \\ \hline t_n^q \\ \hline \end{array} \in B_{\mathrm{sp}}^{(-)}$, we denote $u_1 \otimes \cdots \otimes u_q = \begin{array}{|c|c|c|c|c|}\hline t_l^q & \vdots & t_{l'}^k & \vdots & t_{l''}^1 \\ \hline \end{array} \in (\bigotimes_{k=1}^{q-1} B(\omega_{l_k})) \otimes B_{\mathrm{sp}}^{(-)}$. Then we obtain,

$$B(\lambda) = \left\{ w = \begin{array}{|c|c|c|c|c|}\hline t_l^q & \vdots & t_{l'}^k & \vdots & t_{l''}^1 \\ \hline \end{array} \in \left(\bigotimes_{k=1}^{q-1} B(\omega_{l_k}) \right) \otimes B_{\mathrm{sp}}^{(-)};\ \begin{array}{l} w \text{ satisfies the} \\ \text{following (1)–(4)} \end{array} \right\}, \tag{2.4.8}$$

(1) $u_k \otimes u_{k+1} \in B(\omega_{l_k} + \omega_{l_{k+1}})$ for any $k = 1, \ldots, p-1$,
(2) $u_p \otimes u_{p+1} \in B(\omega_{l_p} + \omega_{n+1})$,
(3) $u_k \otimes u_{k+1} \in B(2\omega_{n+1})$, for any $k = p+1, \ldots, q-2$,
(4) $u_{q-1} \otimes u_q \in B(\omega_{n+1} + \Lambda_{n-1})$.

For λ of type (W1)–(W3), we can obtain $B(\lambda)$ similarly. An element of $B(\lambda)$ is called a *semi-standard D-tableau* of shape λ.

3. Generalized Young Diagrams

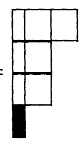
It is well known that arbitrary finite-dimensional irreducible representations of A_n and C_n are characterized by Young diagrams. But it is not true for the B_n and D_n cases. In this section we shall introduce a "generalized Young diagram of type $\mathfrak{g}$($\mathfrak{g} = A_n, B_n, C_n, D_n$)" and characterize arbitrary finite-dimensional irreducible representations of $U_q(\mathfrak{g})$ in terms of "generalized Young diagrams."

3.1. Definition of generalized Young diagrams. For a sequence of half integers l_j ($j = 1, 2, \ldots, n =$ rank of $\mathfrak{g}$), such that $l_j - l_{j+1} \in \mathbf{Z}_{\geqq 0}$, let $Y = (l_1, l_2, \ldots, l_n)$ mean a diagram which has n rows and a length of the j-th row is l_j (includes "–" length). We associate it with a weight $\sum_{j=1}^{n} l_j \varepsilon_j$.

For example, if $Y=(l_1, l_2, \ldots, l_n)$ satisfies $l_j \in \mathbf{Z}_{\geqq 0}$, then Y is an ordinary Young diagram. Now we define a "generalized Young diagram of type $\mathfrak{g}$."

Definition 3.1.1. *Let $Y=(l_1, l_2, \ldots, l_n)$ be a diagram such that $l_j - l_{j+1} \in \mathbf{Z}_{\geqq 0}$.*
(1) $\mathfrak{g}=A_n$, C_n-*case: $Y=(l_1, l_2, \ldots, l_n)$ is a generalized Young diagram of type A_n and C_n if all l_j are non-negative integers respectively.*
(2) $\mathfrak{g}=B_n$-*case: $Y=(l_1, l_2, \ldots, l_n)$ is a generalized Young diagram of type B_n if all l_j are non-negative half integers.*
(3) $\mathfrak{g}=D_n$-*case: $Y=(l_1, l_2, \ldots, l_n)$ is a generalized Young diagram of type D_n if all l_j are half integers and $l_1 \geqq l_2 \geqq \cdots \geqq l_{n-1} \geqq |l_n|$.*

Example 3.1.2. In the case of $\mathfrak{g}=D_4$, the generalized Young diagram $(\frac{5}{2}, \frac{3}{2}, \frac{3}{2}, -\frac{1}{2})$ of type D_4 is visualized as follows;

$$\left(\frac{5}{2}, \frac{3}{2}, \frac{3}{2}, -\frac{1}{2}\right)=$$

.

Here, "black box" means "–".

The following proposition is straightforward.

Proposition 3.1.3. *Let $\mathscr{Y}$ be the set of generalized Young diagrams of type $\mathfrak{g}=A_n, B_n, C_n$ and D_n and P_+ the set of dominant integral weights of $\mathfrak{g}$. The map $\Psi\colon \mathscr{Y} \to \mathscr{P}_+$ defined by $Y=(l_1, \ldots, l_n) \mapsto \sum_{j=1}^{n} l_j \varepsilon_j$ gives an isomorphism from $\mathscr{Y}$ to P_+.*

Remark 3.1.4. For $Y=(l_1, \ldots, l_n)$, the image by Ψ is described as follows;

$$\Psi(Y)=\begin{cases} \sum_{k=1}^{n-1}(l_k - l_{k+1})\Lambda_k + l_n \Lambda_n & \text{if } \mathfrak{g}=A_n, C_n, \\ \sum_{k=1}^{n-1}(l_k - l_{k+1})\Lambda_k + 2l_n \Lambda_n & \text{if } \mathfrak{g}=B_n, \\ \sum_{k=1}^{n-1}(l_k - l_{k+1})\Lambda_k + (l_{n-1}+l_n)\Lambda_n & \text{if } \mathfrak{g}=D_n. \end{cases} \tag{3.1.1}$$

Example 3.1.5. The image by Ψ of the generalized Young diagram $(\frac{5}{2}, \frac{3}{2}, \frac{3}{2}, -\frac{1}{2})$ of type D_4 in Example 3.1.2 is

$$\frac{5}{2}\varepsilon_1 + \frac{3}{2}\varepsilon_2 + \frac{3}{2}\varepsilon_3 - \frac{1}{2}\varepsilon_4 = \Lambda_1 + 2\Lambda_3 + \Lambda_4 \in P_+ .$$

3.2. Property of highest weight elements of $B(\lambda) \otimes B(\mu)$. The following proposition is a key for descriptions of a generalization of the Littlewood–Richardson rule.

Proposition 3.2.1. *Let λ and μ be dominant integral weights of $\mathfrak{g}$. For $u \in B(\lambda)$ and $v \in B(\mu)$, the following two conditions are equivalent;*

(a) $\tilde{e}_i(u \otimes v)=0$ *for any i.*
(b) $\tilde{e}_i u = 0$ *and* $\tilde{e}_i^{\langle h_i, \lambda\rangle+1} v = 0$ *for any i.*

Proof. By Corollary 1.1.6, the condition (a) is equivalent to

$$\varepsilon_i(u \otimes v)=\max(\varepsilon_i(u), \varepsilon_i(u)+\varepsilon_i(v)-\varphi_i(u))=0 \quad \text{for any } i . \tag{3.2.1}$$

The condition (b) is equivalent to

$$\varepsilon_i(u)=0 \quad \text{and} \quad \varepsilon_i(v) \leqq \langle h_i, \lambda\rangle \quad \text{for any } i . \tag{3.2.2}$$

Note that by the definition of $\varphi_i(u)$ and $\varepsilon_i(u)$, we have

$$\varphi_i(u) - \varepsilon_i(u) = \langle h_i, wt(u) \rangle . \tag{3.2.3}$$

If $\varepsilon_i(u) = 0$ for any i, then $wt(u) = \lambda$. Therefore, from (3.2.3), we obtain the equivalence of (3.2.1) and (3.2.2). Q.E.D.

4. Decomposition of $V_Y \otimes V_\square$

In this section, for $\mathfrak{g} = A_n$, B_n, C_n and D_n, we shall give combinatorial descriptions for irreducible decomposition of $U_q(\mathfrak{g})$-module $V_Y \otimes V_\square$ with the help of crystal bases. Here, Y is a generalized Young diagram of type $\mathfrak{g}$ and V_Y is the irreducible $U_q(\mathfrak{g})$-module with highest weight $\Psi(Y)$. Of course, this result is well-known, but it is important that we can explain this in terms of crystal bases.

4.1. Lemmas. Let $Y = (l_1, l_2, \ldots, l_n)$ be a generalized Young diagram of type $\mathfrak{g}$ and u_Y the highest weight element of $B(V_Y)$. The following lemmas play an important role in this section.

Lemma 4.1.1. *Let $\mathfrak{g}$ be of type A_n. For a Young diagram $Y = (l_1, l_2, \ldots, l_n)$ and $\boxed{j} \in B(V_\square)$ $(j = 1, 2, \ldots, n+1)$, the following* (i) *and* (ii) *are equivalent*;

(i) *$u_Y \otimes \boxed{j}$ is a highest weight element of $B(V_Y \otimes V_\square)$.*
(ii) $l_{j-1} - l_j > 0$.

Here, we set $l_0 = \infty$ and $l_{n+1} = 0$. (If $\mathfrak{g} = \mathfrak{gl}(n+1)$, we consider Y has the $n+1$-th row.)

Proof. Let λ be the weight of Y, i.e. $\lambda = \Psi(Y)$. By Proposition 3.2.1,

$$u_Y \otimes \boxed{j} \text{ is a highest weight element if and only if } \varepsilon_i(\boxed{j}) \leqq \langle h_i, \lambda \rangle \text{ for any } i. \tag{4.1.1}$$

By the crystal graph (2.1.1) of the $U_q(A_n)$-module $V_\square$, we obtain,

$$\varepsilon_i(\boxed{j}) = \delta_{i,j-1} . \tag{4.1.2}$$

From (4.1.1) and (4.1.2), we obtain that $u_Y \otimes \boxed{1}$ is always a highest weight element of $B(V_Y \otimes V_\square)$ and for $j = 2, \ldots, n+1$,

$$\begin{aligned} \varepsilon_i(\boxed{j}) \leqq \langle h_i, \lambda \rangle \text{ for any } i &\Leftrightarrow \varepsilon_{j-1}(\boxed{j}) \leqq \langle h_{j-1}, \lambda \rangle \\ &\Leftrightarrow \langle h_{j-1}, \lambda \rangle \text{ is positive} . \end{aligned} \tag{4.1.3}$$

Since $\langle h_{j-1}, \lambda \rangle = l_{j-1} - l_j$, we obtain the equivalence of (i) and (ii). Q.E.D.

The proofs of the following lemmas are quite similar to that of Lemma 4.1.1, so we omit them.

Lemma 4.1.2. *Let $\mathfrak{g}$ be of type C_n and $Y = (l_1, l_2, \ldots, l_n)$ be a Young diagram.*
(1) *For $\boxed{j} \in B(V_\square)$ $(j = 1, 2, \ldots, n)$, $u_Y \otimes \boxed{j}$ is a highest weight element of $B(V_Y \otimes V_\square)$ if and only if $l_{j-1} - l_j > 0$ $(l_0 = \infty)$.*
(2) *For $\boxed{\bar{j}} \in B(V_\square)$ $(j = 1, 2, \ldots, n)$, $u_Y \otimes \boxed{\bar{j}}$ is a highest weight element of $B(V_Y \otimes V_\square)$ if and only if $l_j - l_{j+1} > 0$ $(l_{n+1} = 0)$.*

Lemma 4.1.3. *Let $\mathfrak{g}$ be of type B_n and $Y = (l_1, l_2, \ldots, l_n)$ a generalized Young diagram of type B_n.*

(1) *For $\boxed{j} \in B(V_\square)$ $(j = 1, 2, \ldots, n)$, $u_Y \otimes \boxed{j}$ is a highest weight element of $B(V_Y \otimes V_\square)$ if and only if $l_{j-1} - l_j > 0$.*

(2) *For $\boxed{\bar{j}} \in B(V_\square)$ $(j = 1, 2, \ldots, n-1)$, $u_Y \otimes \boxed{\bar{j}}$ is a highest weight element of $B(V_Y \otimes V_\square)$ if and only if $l_j - l_{j+1} > 0$.*

(3) *For $\boxed{0} \in B(V_\square)$, $u_Y \otimes \boxed{0}$ is a highest weight element of $B(V_Y \otimes V_\square)$ if and only if $l_n > 0$.*

(4) *For $\boxed{\bar{n}} \in B(V_\square)$, $u_Y \otimes \boxed{\bar{n}}$ is a highest weight element of $B(V_Y \otimes V_\square)$ if and only if $l_n \geqq 1$.*

Note that for a generalized Young diagram of type B_n, the condition $l_n > 0$ in (3) is not equivalent to the condition $l_n \geqq 1$ in (4).

Lemma 4.1.4. *Let $\mathfrak{g}$ be of type D_n and $Y = (l_1, l_2, \ldots, l_n)$ a generalized Young diagram of type D_n.*

(1) *For $\boxed{j} \in B(V_\square)$ $(j = 1, 2, \ldots, n)$, $u_Y \otimes \boxed{j}$ is a highest weight element of $B(V_Y \otimes V_\square)$ if and only if $l_{j-1} - l_j > 0$.*

(2) *For $\boxed{\bar{j}} \in B(V_\square)$ $(j = 1, 2, \ldots, n-2)$, $u_Y \otimes \boxed{\bar{j}}$ is a highest weight element of $B(V_Y \otimes V_\square)$ if and only if $l_j - l_{j+1} > 0$.*

(3) *For $\boxed{\bar{n}} \in B(V_\square)$, $u_Y \otimes \boxed{\bar{n}}$ is a highest weight element of $B(V_Y \otimes V_\square)$ if and only if $l_{n-1} \neq -l_n$.*

(4) *For $\boxed{\overline{n-1}} \in B(V_\square)$, $u_Y \otimes \boxed{\overline{n-1}}$ is a highest weight element of $B(V_Y \otimes V_\square)$ if and only if $l_{n-1} > |l_n|$.*

4.2. For a generalized Young diagram $Y = (l_1, l_2, \ldots, l_n)$ of type $\mathfrak{g}$, we shall define the following notations.

$$(Y \leftarrow j) := (l_1, \ldots, l_j + 1, \ldots, l_n) \quad \text{for } j = 1, \ldots, n\,, \tag{4.2.1}$$

$$(Y \leftarrow \bar{j}) := (l_1, \ldots, l_j - 1, \ldots, l_n) \quad \text{for } j = 1, \ldots, n\,, \tag{4.2.2}$$

$$(Y \leftarrow n+1) := \begin{cases} (l_1 - 1, l_2 - 1, \ldots, l_n - 1) & \mathfrak{g} = A_n\,, \\ (l_1, l_2, \ldots, l_n, l_{n+1} + 1) & \mathfrak{g} = \mathfrak{gl}(n+1)\,, \end{cases} \tag{4.2.3}$$

$$(Y \leftarrow 0) := \begin{cases} (l_1, \ldots, l_n) & \text{if } l_n > 0\,, \\ (l_1, \ldots, l_{n-1}, -\infty) & \text{if } l_n = 0\,. \end{cases} \tag{4.2.4}$$

Note that the case $l_n = 0$ in (4.2.4) implies that $(Y \leftarrow 0)$ is not an element of $\mathscr{Y}$ and for $\mathfrak{g} = \mathfrak{gl}(n+1)$, we consider that Y has $n+1$ rows.

Proposition 4.2.1. *Let $Y = (l_1, l_2, \ldots, l_n)$ be a generalized Young diagram of type $\mathfrak{g}$ and V_Y a finite-dimensional irreducible $U_q(\mathfrak{g})$-module associated to Y. Then we have*

$$V_Y \otimes V_\square \cong \begin{cases} \bigoplus_{j=1}^{n+1} V_{(Y \leftarrow j)} & \text{if } \mathfrak{g} = A_n \text{ or } \mathfrak{gl}(n+1)\,, \\ \bigoplus_{j=0}^{n} V_{(Y \leftarrow j)} \oplus \bigoplus_{j=1}^{n} V_{(Y \leftarrow \bar{j})} & \text{if } \mathfrak{g} = B_n\,, \\ \bigoplus_{j=1}^{n} V_{(Y \leftarrow j)} \oplus \bigoplus_{j=1}^{n} V_{(Y \leftarrow \bar{j})} & \text{if } \mathfrak{g} = C_n, D_n\,. \end{cases} \tag{4.2.5}$$

Here if Y is not a generalized Young diagram, then V_Y means the 0-dimensional vector space.

Proof. By Corollary 1.1.4, it suffices to determine all the highest weight elements of $B(V_Y \otimes V_\square)$. By Proposition 3.2.1, if $u \otimes v \in B(V_Y \otimes V_\square)$ is a highest weight element, then we have $u = u_Y$. Hence, by Lemma 4.1.1–4.1.4 we have already known all the highest weight elements of $B(V_Y \otimes V_\square)$ in terms of l_j's. We shall restate this in terms of generalized Young diagrams.

(i) $\mathfrak{g} = A_n$ ($\mathfrak{gl}(n+1)$)-case

The condition $l_{j-1} - l_j > 0$ in Lemma 4.1.1 is equivalent to the condition that $(Y \leftarrow j)$ remains a Young diagram. Hence $u_Y \otimes \boxed{j}$ is a highest weight element of $B(V_Y \otimes V_\square)$ if and only if $(Y \leftarrow j)$ is a Young diagram. It remains to note

$$wt(u_Y \otimes \boxed{j}) = \left(\sum_{k=1}^{n} l_k \varepsilon_k\right) + \varepsilon_j = wt(Y \leftarrow j) .$$

(If $\mathfrak{g} = A_n$, since $\varepsilon_1 + \cdots + \varepsilon_{n+1} = 0$, it is true for the case $j = n+1$.) Hence we obtain the case A_n ($\mathfrak{gl}(n+1)$)-case.

(ii) $\mathfrak{g} = C_n$-case

Similarly to the A_n-case, by Lemma 4.1.2.(1), $u_Y \otimes \boxed{j}$ is a highest weight element of $B(V_Y \otimes V_\square)$ if and only if $(Y \leftarrow j)$ remains a Young diagram. It remains to note that $wt(Y \leftarrow j) = wt(u_Y \otimes \boxed{j})$.

The condition $l_j - l_{j+1} > 0$ in Lemma 4.1.2.(2) is equivalent to the condition that $(Y \leftarrow \bar{j})$ remains a Young diagram. Hence $u_Y \otimes \boxed{\bar{j}}$ is a highest weight element of $B(V_Y \otimes V_\square)$ if and only if $(Y \leftarrow \bar{j})$ is a Young diagram. It remains to note that

$$wt(u_Y \otimes \boxed{\bar{j}}) = \left(\sum_{k=1}^{n} l_k \varepsilon_k\right) - \varepsilon_j = wt(Y \leftarrow \bar{j}) .$$

Therefore we get the C_n-case.

(iii) $\mathfrak{g} = B_n$-case

Similarly to the C_n-case, by Lemma 4.1.3(1) and (2), $u_Y \otimes \boxed{j} \in B(V_Y \otimes V_\square)$ $(j = 1, \ldots, n)$ satisfies the highest weight condition if and only if $(Y \leftarrow j)$ is a generalized Young diagram of type B_n and $u_Y \otimes \boxed{\bar{j}} \in B(V_Y \otimes V_\square)$ $(j = 1, \ldots, n-1)$ satisfies the highest condition if and only if $(Y \leftarrow \bar{j})$ is a generalized Young diagram of type B_n. We have $wt(u_Y \otimes \boxed{j}) = wt(Y \leftarrow j)$ and $wt(u_Y \otimes \boxed{\bar{j}}) = wt(Y \leftarrow \bar{j})$.

By Lemma 4.1.3.(3), $u_Y \otimes \boxed{0}$ is a highest weight element of $B(V_Y \otimes V_\square)$ if and only if $(Y \leftarrow 0)$ remains a generalized Young diagram of type B_n. Both $Y \otimes \boxed{0}$ and Y have the same weight $\sum_{k=1}^{n} l_k \varepsilon_k$.

By Lemma 4.1.3.(4), $u_Y \otimes \boxed{\bar{n}} \in B(V_Y \otimes V_\square)$ is a highest weight element if and only if $l_n \geqq 1$. Here, the condition $l_n \geqq 1$ is equivalent to the condition that $(Y \leftarrow \bar{n})$ remains a generalized Young diagram. We have

$$wt(Y \otimes \boxed{\bar{n}}) = \left(\sum_{k=1}^{n} l_k \varepsilon_k\right) - \varepsilon_n = wt(Y \leftarrow \bar{n}) .$$

By these results, we obtain the B_n-case.

(iv) $\mathfrak{g} = D_n$-case

Similarly to the B_n-case and the C_n-case, by Lemma 4.1.4.(1) and (2), if $u \otimes \boxed{j}$ $(j = 1, \ldots, n)$ (respectively $u \otimes \boxed{\bar{j}}$ $(j = 1, \ldots, n-2)) \in B(V_Y \otimes V_\square)$ satisfies the highest weight condition if and only if $(Y \leftarrow j)$ (respectively $(Y \leftarrow \bar{j})$) is a generalized

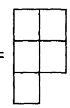
Young diagram of type D_n. We have $wt(u_Y \otimes \boxed{j}) = wt(Y \leftarrow j)$ and $wt(u_Y \otimes \boxed{\bar{j}}) = wt(Y \leftarrow \bar{j})$. By Lemma 4.1.4.(3), $u_Y \otimes \boxed{\bar{n}}$ is a highest weight element of $B(V_Y \otimes V_\square)$ if and only if $l_n \neq -l_{n-1}$. By the definition of generalized Young diagram of type D_n, the condition $l_n \neq -l_{n-1}$ is equivalent to the condition that $(Y \leftarrow \bar{n}) = (l_1, \ldots, l_{n-1}, l_n - 1)$ remains a generalized Young diagram. We have

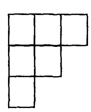
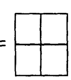
$$wt(Y \otimes \boxed{\bar{n}}) = \left(\sum_{k=1}^{n} l_k \varepsilon_k \right) - \varepsilon_n = wt(Y \leftarrow \bar{n}) .$$

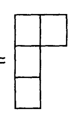
By Lemma 4.1.4.(4), $u_Y \otimes \boxed{\overline{n-1}}$ is a highest weight element of $B(V_Y \otimes V_\square)$ if and only if $l_{n-1} > |l_n|$. By the definition of generalized Young diagram of type D_n, the condition $l_{n-1} > |l_n|$ is equivalent to the condition that $(Y \leftarrow \overline{n-1})$ remains a generalized Young diagram of type D_n. It remains to note

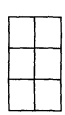
$$wt(Y \otimes \boxed{\overline{n-1}}) = \left(\sum_{k=1}^{n} l_k \varepsilon_k \right) - \varepsilon_{n-1} = wt(Y \leftarrow \overline{n-1}) .$$

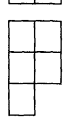
Thus, we obtain the D_n-case. Q.E.D.

Example 4.2.2. For $\mathfrak{g} = B_3$ and $Y = (2, 2, 1) =$ , we consider $V_Y \otimes V_\square$. We have $\Psi(Y) = 2\varepsilon_1 + 2\varepsilon_2 + \varepsilon_3 = \Lambda_2 + 2\Lambda_3$ and $B(V_\square) = \{\boxed{1}, \boxed{2}, \boxed{3}, \boxed{0}, \boxed{\bar{3}}, \boxed{\bar{2}}, \boxed{\bar{1}}\}$.

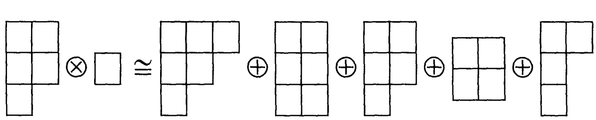
$u_Y \otimes \boxed{1} \leftrightarrow (Y \leftarrow 1) = (3, 2, 1) =$ , $u_Y \otimes \boxed{\bar{3}} \leftrightarrow (Y \leftarrow \bar{3}) = (2, 2, 0) =$ ,

$u_Y \otimes \boxed{2} = (Y \leftarrow 2) = (2, 3, 1)\times$, $u_Y \otimes \boxed{\bar{2}} \leftrightarrow (Y \leftarrow \bar{2}) = (2, 1, 1) =$ ,

$u_Y \otimes \boxed{3} \leftrightarrow (Y \leftarrow 3) = (2, 2, 2) =$ , $u_Y \otimes \boxed{\bar{1}} \leftrightarrow (Y \leftarrow \bar{1}) = (1, 2, 1)\times$,

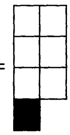
$u_Y \otimes \boxed{0} \leftrightarrow (Y \leftarrow 0) = (2, 2, 1) =$ .

Therefore, we get

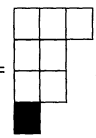


Hence,

$$V(\Lambda_2 + 2\Lambda_3) \otimes V_\square \cong V(\Lambda_1 + \Lambda_2 + 2\Lambda_3) \oplus V(4\Lambda_3) \oplus V(\Lambda_2 + 2\Lambda_3)$$
$$\oplus V(2\Lambda_2) \oplus V(\Lambda_1 + 2\Lambda_3).$$

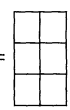
Example 4.2.3. For $\mathfrak{g} = D_4$ and $Y = (2, 2, 2, -1) =$  ($\blacksquare$ means "–"), we consider $V_Y \otimes V_\square$. We have $\Psi(Y) = 2\varepsilon_1 + 2\varepsilon_2 + 2\varepsilon_3 - \varepsilon_4 = 3\Lambda_3 + \Lambda_4$ and

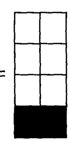
$$B(V_\square) = \{\boxed{1}, \boxed{2}, \boxed{3}, \boxed{4}, \boxed{\bar{4}}, \boxed{\bar{3}}, \boxed{\bar{2}}, \boxed{\bar{1}}\}$$

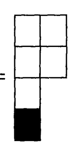
$u_Y \otimes \boxed{1} \leftrightarrow (Y \leftarrow 1) = (3, 2, 2, -1) =$  ,

$u_Y \otimes \boxed{2} \leftrightarrow (Y \leftarrow 2) = (2, 3, 2, -1) \times,$

$u_Y \otimes \boxed{3} \leftrightarrow (Y \leftarrow 3) = (2, 2, 3, -1) \times,$

$u_Y \otimes \boxed{4} \leftrightarrow (Y \leftarrow 4) = (2, 2, 2, 0) =$  ,

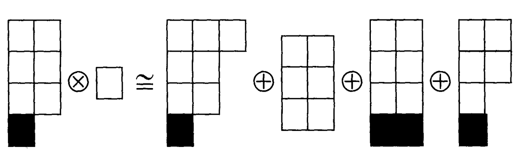
$u_Y \otimes \boxed{\bar{4}} \leftrightarrow (Y \leftarrow \bar{4}) = (2, 2, 2, -2) =$  ,

$u_Y \otimes \boxed{\bar{3}} \leftrightarrow (Y \leftarrow \bar{3}) = (2, 2, 1, -1) =$  ,

$u_Y \otimes \boxed{\bar{2}} \leftrightarrow (Y \leftarrow \bar{2}) = (2, 1, 2, -1) \times,$

$u_Y \otimes \boxed{\bar{1}} \leftrightarrow (Y \leftarrow \bar{1}) = (1, 2, 2, -1) \times.$

Then, we obtain (we omit V)



Hence,

$$V(3\Lambda_3 + \Lambda_4) \otimes V_\square \cong V(\Lambda_1 + 3\Lambda_3 + \Lambda_4) \oplus V(2\Lambda_3 + 2\Lambda_4) \oplus V(4\Lambda_3) \oplus V(\Lambda_2 + 2\Lambda_3).$$

5. Decomposition of $V_Y \otimes V_{\mathrm{sp}}$

For $\mathfrak{g} = B_n$ and D_n, there exist "spin representations." In this section we shall give combinatorial descriptions for irreducible decomposition of $U_q(\mathfrak{g})$-module $V_Y \otimes V_{\mathrm{sp}}$ by use of crystal bases.

5.1. Decomposition of $U_q(B_n)$-module $V_Y \otimes V_{\mathrm{sp}}$. We have already introduced the spin representation V_{sp} of $U_q(B_n)$ in 2.3. In this section we introduce another description of V_{sp} ([Re]).

First we set $B_{\mathrm{sp}} = \{v = (i_1, i_2, \ldots, i_n);\ i_j = \pm\}$, $V_{\mathrm{sp}} = \bigoplus_{v \in B_{\mathrm{sp}}} \mathbf{Q}(q)v$ and $L_{\mathrm{sp}} = \bigoplus_{v \in B_{\mathrm{sp}}} Av$. For $v = (i_1, i_2, \ldots, i_n) \in B_{\mathrm{sp}}$, if we define the actions of generators and the operators $\tilde{e}_i$ and $\tilde{f}_i$ as follows:

$$q^h v = q^{\langle h, wt(v)\rangle} v, \text{ where } wt(v) = \frac{1}{2}\sum_{j=1}^{n} i_j \varepsilon_j \text{ for } v = (i_1, \ldots, i_n)\,, \tag{5.1.1}$$

$$e_j v = \tilde{e}_j v = \begin{cases} (i_1, \ldots, \overset{j}{+}, \overset{j+1}{-}, \ldots, i_n) & i_j = - \text{ and } i_{j+1} = +, \\ 0 & \text{otherwise}\,, \end{cases} \tag{5.1.2}$$

$$f_j v = \tilde{f}_j v = \begin{cases} (i_1, \ldots, \overset{j}{-}, \overset{j+1}{+}, \ldots, i_n) & i_j = + \text{ and } i_{j+1} = -, \\ 0 & \text{otherwise}\,, \end{cases} \tag{5.1.3}$$

for $j = 1, \ldots, n-1$, and

$$e_n v = \tilde{e}_n v = \begin{cases} (i_1, \ldots, i_{n-1}, \overset{n}{+}) & i_n = -\,, \\ 0 & \text{otherwise}\,, \end{cases} \tag{5.1.4}$$

$$f_n v = \tilde{f}_n v = \begin{cases} (i_1, \ldots, i_{n-1}, \overset{n}{-}) & i_n = +\,, \\ 0 & \text{otherwise}\,, \end{cases} \tag{5.1.5}$$

then $V_{\mathrm{sp}} \cong V(\Lambda_n)$ and $(L_{\mathrm{sp}}, B_{\mathrm{sp}})$ is a crystal base of V_{sp}.

Note that the correspondence of this description and the description in 2.3 is given by:

$$\text{“}+\text{” in the } j\text{-th row} \leftrightarrow j \text{ and “}-\text{” in the } j\text{-th row} \leftrightarrow \bar{j}\,.$$

Now, for a generalized Young diagram $Y = (l_1, \ldots, l_n)$ and $v = (i_1, \ldots, i_n) \in B_{\mathrm{sp}}$ we define

$$(Y + v) := \left(l_1 + \frac{1}{2} i_1, l_2 + \frac{1}{2} i_2, \ldots, l_n + \frac{1}{2} i_n\right).$$

The following lemma plays a similar role to Lemmas 4.1.1–4.1.4 in Sect. 4.

Lemma 5.1.1. *Let $Y = (l_1, l_2, \ldots, l_n)$ be a generalized Young diagram of type B_n and u_Y the highest weight element of $B(V_Y)$. For $v = (i_1, i_2, \ldots, i_n) \in B_{\mathrm{sp}}$ the following two statements are equivalent,*

(a) *$u_Y \otimes v$ is a highest weight element of $B(V_Y \otimes V_{\mathrm{sp}})$.*
(b) *$(Y + v)$ is a generalized Young diagram of type B_n.*

Proof. Let $\lambda = \sum_{k=1}^{n} l_k \varepsilon_k$ be the weight of Y. By Proposition 3.2.1, $u_Y \otimes v$ is a highest weight element of $B(V_Y \otimes V_{sp})$ if and only if $\varepsilon_j(v) \leqq \langle h_j, \lambda \rangle$ for any j. Therefore it is enough to show that $(Y + v) = (l_1 + \frac{1}{2}i_1, l_2 + \frac{1}{2}i_2, \ldots, l_n + \frac{1}{2}i_n)$ is not a generalized Young diagram of type B_n if and only if $\varepsilon_j(v) > \langle h_j, \lambda \rangle$ for some j. By (5.1.2) and (5.1.4), we have $\varepsilon_j(v) \leqq 1$ for any j.

First we assume that $\varepsilon_j(v) > \langle h_j, \lambda \rangle$ for some j. Then we have

$$\langle h_j, \lambda \rangle = 0 \quad \text{and} \quad \varepsilon_j(v) = 1 . \tag{5.1.6}$$

If $j \neq n$, then from (5.1.2) and (5.1.6) we have

$$\langle h_j, \lambda \rangle = 0, \quad i_j = - \text{ and } i_{j+1} = + . \tag{5.1.7}$$

Since $\langle h_j, \lambda \rangle = l_j - l_{j+1} = 0$, we obtain

$$l_j + \frac{1}{2} i_j = l_j - \frac{1}{2} < l_{j+1} + \frac{1}{2} = l_{j+1} + \frac{1}{2} i_{j+1} . \tag{5.1.8}$$

This implies that $(Y + v)$ is not a generalized Young diagram of type B_n.

If $j = n$, then (5.1.4) and (5.1.6) imply

$$\langle h_n, \lambda \rangle = 0 \quad \text{and} \quad i_n = - . \tag{5.1.9}$$

The condition $\langle h_n, \lambda \rangle = 0$ is equivalent to $l_n = 0$. Therefore we have

$$l_n + \frac{1}{2} i_n = -\frac{1}{2} < 0 . \tag{5.1.10}$$

This implies that $(Y + v)$ does not satisfy the condition of a generalized Young diagram of type B_n in Definition 3.3.1 (2).

Next we assume that for a generalized Young diagram $Y = (l_1, l_2, \ldots, l_n)$ of type B_n and $v = (i_1, i_2, \ldots, i_n) \in B_{sp}$, $(Y + v) = (l_1 + \frac{1}{2}i_1, l_2 + \frac{1}{2}i_2, \ldots, l_n + \frac{1}{2}i_n)$ is not a generalized Young diagram of type B_n. One of the following cases can occur,

$$l_j + \frac{1}{2} i_j < l_{j+1} + \frac{1}{2} i_{j+1} \quad \text{for some } j \neq n , \tag{5.1.11}$$

$$l_n + \frac{1}{2} i_n < 0 . \tag{5.1.12}$$

Note that $(Y + v)$ always satisfies $(l_j + \frac{1}{2}i_j) - (l_{j+1} + \frac{1}{2}i_{j+1}) \in \mathbf{Z}$.

In the case (5.1.11), from the condition $l_j - l_{j+1} \in \mathbf{Z}_{\geqq 0}$ in Definition 3.3.1.(2), we get

$$l_j = l_{j+1}, \quad i_j = - \text{ and } i_{j+1} = + . \tag{5.1.13}$$

This implies that $\langle h_j, \lambda \rangle = 0$ and $\varepsilon_j(v) = 1$. Hence, $\tilde{e}_j(u_Y \otimes v) \neq 0$.

In the case (5.1.12), we have

$$l_n = 0 \quad \text{and} \quad i_n = - . \tag{5.1.14}$$

This implies that $\langle h_n, \lambda \rangle = 0$ and $\varepsilon_n(v) = 1$. Hence, $\tilde{e}_n(u_Y \otimes v) \neq 0$. Thus we get the equivalence of (a) and (b). Q.E.D.

Proposition 5.1.2. *Let $Y = (l_1, l_2, \ldots, l_n)$ be a generalized Young diagram of type B_n. For $U_q(B_n)$-modules V_Y and V_{sp}, we obtain;*

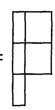
$$V_Y \otimes V_{\mathrm{sp}} \cong \bigoplus_{v=(i_1, \ldots, i_n)\in B_{sp}} V_{(Y+v)} \, . \tag{5.1.15}$$

Here if Y is not a generalized Young diagram, V_Y means the 0*-dimensional vector space.*

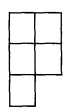
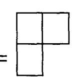
Proof. By Proposition 3.2.1 and Lemma 5.1.1, $u \otimes v \in B(V_Y \otimes V_{\mathrm{sp}})$ satisfies the highest weight condition if and only if $u = u_Y$ and $(Y + v)$ is a generalized Young diagram of type B_n. Since $v = (i_1, i_2, \ldots, i_n)$ has weight $\frac{1}{2}\sum_{k=1}^{n} i_k \varepsilon_k$,

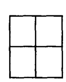
$$wt(u_Y \otimes v) = \sum_{k=1}^{n} \left(l_k + \frac{1}{2} i_k \right) \varepsilon_k = wt(Y + v) \, . \tag{5.1.16}$$

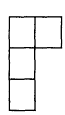
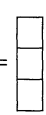
By Corollary 1.1.4, we obtain the desired result. Q.E.D.

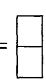
Example 5.1.3. For $\mathfrak{g} = B_3$ and $Y = (\frac{3}{2}, \frac{3}{2}, \frac{1}{2}) =$  $(= \Lambda_2 + \Lambda_3)$

$$B_{\mathrm{sp}} = B(\Lambda_3) = \left\{ \begin{matrix} (+,+,+), (+,+,-), (+,-,+), (-,+,+), \\ (+,-,-), (-,+,-), (-,-,+), (-,-,-) . \end{matrix} \right\}$$

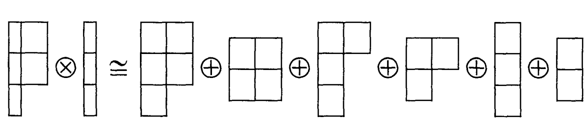
$u_Y \otimes (+,+,+) \leftrightarrow (2, 2, 1) =$  $u_Y \otimes (+,-,-) \leftrightarrow (2, 1, 0) =$ 

$u_Y \otimes (+,+,-) \leftrightarrow (2, 2, 0) =$  $u_Y \otimes (-,+,-) \leftrightarrow (1, 2, 0) \times$

$u_Y \otimes (+,-,+) \leftrightarrow (2, 1, 1) =$  $u_Y \otimes (-,-,+) \leftrightarrow (1, 1, 1) =$ 

$u_Y \otimes (-,+,+) \leftrightarrow (1, 2, 1) = \times$ $u_Y \otimes (-,-,-) \leftrightarrow (1, 1, 0) =$ 

Then we get (we omit V)



Hence,

$$V(\Lambda_2 + \Lambda_3) \otimes V_{\mathrm{sp}} \cong V(\Lambda_2 + 2\Lambda_3) \oplus V(2\Lambda_2) \oplus V(\Lambda_1 + 2\Lambda_3)$$
$$V(\Lambda_1 + \Lambda_2) \oplus V(2\Lambda_3) \oplus V(\Lambda_2) \, .$$

5.2. Decomposition of $U_q(D_n)$ module $V_Y \otimes V_{\mathrm{sp}}^{(+)}$ and $V_Y \otimes V_{\mathrm{sp}}^{(-)}$. We have already introduced the two spin representations $V_{\mathrm{sp}}^{(\pm)}$ of $U_q(D_n)$ in 2.4.

Similarly to the B_n-case we shall give another description of $V_{sp}^{(\pm)}$ ([Re]). First we set

$$B_{sp}^{(+)}(\text{respectively } B_{sp}^{(-)}) = \{v = (i_1, \ldots, i_n);$$
$$i_j = \pm, i_1 \ldots i_n = +(\text{respectively } -)\}, \tag{5.2.1}$$
$$V_{sp}^{(\pm)} = \bigoplus_{v \in B_{sp}^{(\pm)}} \mathbf{Q}(q)v \quad \text{and } L_{sp}^{(\pm)} = \bigoplus_{v \in B_{sp}^{(\pm)}} Av, \tag{5.2.2}$$

and if we define the actions of generators on $V_{sp}^{(\pm)}$ and the operators $\tilde{e}_i$ and $\tilde{f}_i$ on $L_{sp}^{(\pm)}$ as follows, for $v = (i_1, \ldots, i_n)$,

$$q^h v = q^{\langle h, wt(v)\rangle} v, \text{ where } wt(v) = \frac{1}{2}\sum_{j=1}^{n} i_j \varepsilon_j, \tag{5.2.3}$$

$$e_j v = \tilde{e}_i v = \begin{cases} (i_1, \ldots, \overset{j}{+}, \overset{j+1}{-}, \ldots, i_n) & i_j = - \text{ and } i_{j+1} = +, \\ 0 & \text{otherwise}, \end{cases} \tag{5.2.4}$$

$$f_j v = \tilde{f}_i v = \begin{cases} (i_1, \ldots, \overset{j}{-}, \overset{j+1}{+}, \ldots, i_n) & i_j = + \text{ and } i_{j+1} = -, \\ 0 & \text{otherwise}, \end{cases} \tag{5.2.5}$$
$$(j = 1, \ldots, n-1)$$

$$e_n v = \tilde{e}_n v = \begin{cases} (i_1, \ldots, \overset{n-1}{+}, \overset{n}{+}) & i_{n-1} = - \text{ and } i_n = -, \\ 0 & \text{otherwise}, \end{cases} \tag{5.2.6}$$

$$f_n v = \tilde{f}_n v = \begin{cases} (i_1, \ldots, \overset{n-1}{-}, \overset{n}{-}) & i_{n-1} = + \text{ and } i_n = +, \\ 0 & \text{otherwise}, \end{cases} \tag{5.2.7}$$

then $V_{sp}^{(+)} \cong V(\Lambda_n)$, $V_{sp}^{(-)} \cong V(\Lambda_{n-1})$ and $(L_{sp}^{(\pm)}, B_{sp}^{(\pm)})$ is a crystal base of $V_{sp}^{(\pm)}$ respectively. Note that the correspondence between this description and the description in 2.4 is given by,

"$+$" in the j-th row $\leftrightarrow j$ and "$-$" in the j-th row $\leftrightarrow \bar{j}$.

Lemma 5.2.1. *Let $Y = (l_1, \ldots, l_n)$ be a generalized Young diagram of type D_n and u_Y the highest weight element of $B(V_Y)$. For u_Y and $v = (i_1, \ldots, i_n) \in B_{sp}^{(\pm)}$, the following two statements are equivalent;*
(a) *$u_Y \otimes v$ is a highest weight element of $B(V_Y \otimes V_{sp}^{(\pm)})$.*
(b) *$(Y + v) = (l_1 + \frac{1}{2}i_1, l_2 + \frac{1}{2}i_2, \ldots, l_n + \frac{1}{2}i_n)$ is a generalized Young diagram of type D_n.*

The proof of this lemma is quite similar to that of Lemma 5.1.1, so we omit it.

Proposition 5.2.2. *Let Y be a generalized Young diagram of type D_n. For $U_q(D_n)$-modules V_Y and $V_{sp}^{(\pm)}$, we obtain;*

$$V_Y \otimes V_{sp}^{(+)} \cong \bigoplus_{v=(i_1, \ldots, i_n) \in B_{sp}^{(+)}} V_{(Y+v)}, \tag{5.2.8}$$
$$V_Y \otimes V_{sp}^{(-)} \cong \bigoplus_{v=(i_1, \ldots, i_n) \in B_{sp}^{(-)}} V_{(Y+v)}, \tag{5.2.9}$$

where if Y is not a generalized Young diagram of type D_n, V_Y means the 0-dimensional vector space.

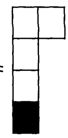
Proof. By Proposition 3.2.1 and Lemma 5.2.1, $u \otimes v \in B(V_Y \otimes V_{\mathrm{sp}}^{(\pm)})$ satisfies the highest weight condition if and only if $u = u_Y$ and $(Y + v)$ is a generalized Young diagram of type D_n. The weight of $v = (i_1, i_2, \ldots, i_n)$ is $\frac{1}{2}\sum_{j=1}^{n} i_j \varepsilon_j$. Hence

$$wt(u_Y \otimes v) = \sum_{k=1}^{n} \left(l_k + \frac{1}{2} i_k \right) \varepsilon_k = wt(Y + v)\,. \tag{5.2.10}$$

Thus, by Corollary 1.1.4, we obtain the desired result. Q.E.D.

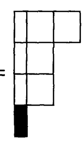
Example 5.2.3. For $\mathfrak{g} = D_4$, we consider the decomposition $V(\Lambda_1 + 2\Lambda_3) \otimes V_{\mathrm{sp}}^{(+)}$. $\Lambda_1 + 2\Lambda_3$ corresponds to the generalized Young diagram $(2, 1, 1, -1)$.

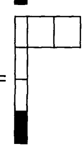
We set

$Y = (2, 1, 1, -1) =$  and $B_{\mathrm{sp}}^{(+)} = B(\Lambda_4)$

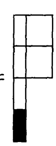
$$= \left\{ \begin{array}{ll} (+, +, +, +), & (+, -, -, +) \\ (+, +, -, -), & (-, +, -, +) \\ (+, -, +, -), & (-, -, +, +) \\ (-, +, +, -), & (-, -, -, -) \end{array} \right\}.$$

We have

$u_Y \otimes (+, +, +, +) \leftrightarrow \left(\frac{5}{2}, \frac{3}{2}, \frac{3}{2}, -\frac{1}{2}\right) =$  ,

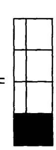
$u_Y \otimes (+, -, -, +) \leftrightarrow \left(\frac{5}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right) =$  ,

$u_Y \otimes (+, +, -, -) \leftrightarrow \left(\frac{5}{2}, \frac{3}{2}, \frac{1}{2}, -\frac{3}{2}\right) \times$

$u_Y \otimes (-, +, -, +) \leftrightarrow \left(\frac{3}{2}, \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}\right) =$  ,

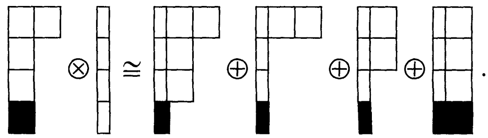
$u_Y \otimes (+, -, +, -) = \left(\frac{5}{2}, \frac{1}{2}, \frac{3}{2}, -\frac{3}{2}\right) \times,$

$u_Y \otimes (-, -, +, +) \leftrightarrow \left(\frac{3}{2}, \frac{1}{2}, \frac{3}{2}, -\frac{1}{2}\right) \times,$

$u_Y \otimes (-, +, +, -) \leftrightarrow \left(\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, -\frac{3}{2}\right) =$  ,

$u_Y \otimes (-, -, -, -) \leftrightarrow \left(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{3}{2}\right) = \times.$

Then for D_4 we get. (We omit "V".)



Hence,

$$V(\Lambda_1 + 2\Lambda_3) \otimes V_{\mathrm{sp}}^{(+)} \cong V(\Lambda_1 + 2\Lambda_3 + \Lambda_4) \oplus V(2\Lambda_1 + \Lambda_3) \oplus V(\Lambda_2 + \Lambda_3) \oplus V(3\Lambda_3)\,.$$

6. Decomposition of $V_Y \otimes V_W$

6.1. In this section we treat general cases. Let Y and W be generalized Young diagrams of type $\mathfrak{g}(=A_n, B_n, C_n, D_n)$. We shall give a combinatorial description for the irreducible decomposition of $V_Y \otimes V_W$. In Sects. 4 and 5, we treated special cases; V_W is the vector representation or the spin representation. By the following lemma and the way of the construction of the crystal graph, we will know that these elementary cases play a significant role in general cases.

Lemma 6.1.1. *Let V_j $(j = 1, \ldots, p)$ be a finite dimensional irreducible representation of $U_q(\mathfrak{g})$ $(\mathfrak{g} = A_n, B_n, C_n, D_n)$. For $u_1 \otimes u_2 \otimes \cdots \otimes u_p \in B(V_1 \otimes \cdots \otimes V_p)$, the following two statements are equivalent;*
(a) $u_1 \otimes u_2 \otimes \cdots \otimes u_p$ is a highest weight element of $B(V_1 \otimes \cdots \otimes V_p)$.
(b) $u_1 \otimes u_2 \otimes \cdots \otimes u_j$ is a highest weight element of $B(V_1 \otimes \cdots \otimes V_j)$ for any $j = 1, \ldots, p$.

Proof. (a) follows trivially from (b). Next, we assume (a). For any j we can consider $u_1 \otimes u_2 \otimes \cdots \otimes u_p = (u_1 \otimes \cdots \otimes u_j) \otimes (u_{j+1} \otimes \cdots \otimes u_p) \in B(V_1 \otimes \cdots \otimes V_j) \otimes B(V_{j+1} \otimes \cdots V_p)$. By Proposition 3.2.1, if $(u_1 \otimes \cdots \otimes u_j) \otimes (u_{j+1} \otimes \cdots \otimes u_p)$ satisfies the highest weight condition, $u_1 \otimes \cdots \otimes u_j$ also satisfies the highest weight condition. Hence, we obtain (b). Q.E.D.

6.2. Remarks. Now, we give some remarks on the crystal graphs.

Remark 6.2.1.

(a) Let $W = (l_1, l_2, \ldots, l_n)$ be a generalized Young diagram of type $\mathfrak{g}(=A_n, B_n, C_n, D_n)$ with $l_j \in \mathbf{Z}$ for any j and set $m := l_1 + \cdots + l_{n-1} + |l_n|$ (= the number of squares in W). By the way of the construction of $B(V_W)$ in Sect. 2, any $u \in B(V_W)$ can be written in the following form,

$$u = \boxed{i_1} \otimes \boxed{i_2} \otimes \cdots \otimes \boxed{i_m}\,, \tag{6.2.1}$$

where $\boxed{i_j}$ is an element of $B(V_\square)$.

(b) Let $W = (l_1, l_2, \ldots, l_n)$ be a generalized Young diagram of type $\mathfrak{g}(\mathfrak{g} = B_n, D_n)$ with $l_j \in \mathbf{Z} + \frac{1}{2}$ for any j and set $m := (l_1 - \frac{1}{2}) + \cdots + (l_{n-1} - \frac{1}{2}) + (|l_n| - \frac{1}{2})$ (= the number of squares in W). By the way of the construction of $B(V_W)$ in Sect. 2, any $u \in B(V_W)$ can be written in the following form,

$$u = \boxed{i_1} \otimes \boxed{i_2} \otimes \cdots \otimes \boxed{i_m} \otimes v\,, \tag{6.2.2}$$

where $\boxed{i_j}$ is an element of $B(V_\square)$ and v is an element of B_{sp} or $B_{\mathrm{sp}}^{(\pm)}$.

6.3. Main theorem. Here, by Proposition 4.2.1, Proposition 5.1.2, Proposition 5.2.2, Lemma 6.1.1 and Remark 6.2.1, we obtain the following theorem.

Theorem 6.3.1. (1) *Let W and m be as in Remark* 6.2.1(a) *and Y a generalized Young diagram of type $\mathfrak{g}$ ($= A_n, B_n, C_n, D_n$). Then we obtain*

$$V_Y \otimes V_W \cong \bigoplus_{\boxed{i_1} \otimes \boxed{i_2} \otimes \cdots \otimes \boxed{i_m} \in B(V_W)} V_{(((Y \leftarrow i_1) \leftarrow i_2) \ldots) \ldots \leftarrow i_m)}, \tag{6.3.1}$$

where $V_{(((Y \leftarrow i_1) \leftarrow i_2) \ldots) \ldots \leftarrow i_m)}$ is the 0*-dimensional vector space if there exists $k \in \{1, \ldots, m\}$ such that $((((Y \leftarrow i_1) \leftarrow i_2) \ldots) \ldots \leftarrow i_k)$ is not a generalized Young diagram of type $\mathfrak{g}$.*

(2) *Let W and m be as in Remark* 6.2.1(b) *and Y a generalized Young diagram of type $\mathfrak{g}$ ($= B_n, D_n$). Then we obtain*

$$V_Y \otimes V_W \cong \bigoplus_{\boxed{i_1} \otimes \boxed{i_2} \otimes \cdots \otimes \boxed{i_m} \otimes v \in B(V_W)} V_{((((Y \leftarrow i_1) \leftarrow i_2) \ldots) \ldots \leftarrow i_m) + v)}, \tag{6.3.2}$$

where $V_{((((Y \leftarrow i_1) \leftarrow i_2) \ldots) \ldots \leftarrow i_m) + v)}$ is the 0*-dimensional vector space if there exists $k \in \{1, \ldots, m\}$ such that $((((Y \leftarrow i_1) \leftarrow i_2) \ldots) \ldots \leftarrow i_k)$ is not a generalized Young diagram of type $\mathfrak{g}$ or $(((((Y \leftarrow i_1) \leftarrow i_2) \ldots) \ldots \leftarrow i_m) + v)$ is not a generalized Young diagram of type $\mathfrak{g}$.*

Corollary 6.3.2. *Let $J = \{1, 2, \ldots, p\}$ be a finite index set and Y and W_j ($j \in J$) be generalized Young diagrams of type $\mathfrak{g}$. We obtain*

$$V_Y \otimes V_{W_1} \otimes \cdots \otimes V_{W_p} \cong \bigoplus_{u_j \in B(V_{W_j})\,(j \in J)} V_{(((Y \Leftarrow u_1) \Leftarrow u_2) \Leftarrow \ldots) \ldots \Leftarrow u_p)}, \tag{6.3.3}$$

where for Y and $u_j \in B(V_{W_j})$, if u_j is in the form (6.2.1), *we define*

$$(Y \Leftarrow u_j) := ((((Y \leftarrow i_1) \leftarrow i_2) \ldots) \ldots \leftarrow i_m), \tag{6.3.4}$$

and if u_j is in the form (6.2.2), *we define*

$$(Y \Leftarrow u_j) := (((((Y \leftarrow i_1) \leftarrow i_2) \ldots) \ldots \leftarrow i_m) + v) \tag{6.3.5}$$

and $V_{(((Y \Leftarrow u_1) \Leftarrow u_2) \Leftarrow \ldots) \ldots \Leftarrow u_p)}$ is the 0*-dimensional vector space if there exists $j \in J$ such that $((((Y \Leftarrow u_1) \Leftarrow u_2) \Leftarrow \cdots) \cdots \Leftarrow u_j)$ is not a generalized Young diagram of type $\mathfrak{g}$.*

Example 6.3.3. For $\mathfrak{g} = \mathfrak{gl}(3)$, $Y = \ydiagram{2,1}$ and $W = \ydiagram{2,2}$, we consider $V_Y \otimes V_W$.

$$^{\mathrm{n}}B(V_W) = \left\{ \begin{ytableau} 1 & 1 \\ 2 & 2 \end{ytableau}, \begin{ytableau} 1 & 1 \\ 2 & 3 \end{ytableau}, \begin{ytableau} 1 & 2 \\ 2 & 3 \end{ytableau}, \begin{ytableau} 1 & 1 \\ 3 & 3 \end{ytableau}, \begin{ytableau} 1 & 2 \\ 3 & 3 \end{ytableau}, \begin{ytableau} 2 & 2 \\ 3 & 3 \end{ytableau}. \right\},$$

$$u_Y \otimes \begin{ytableau} 1 & 1 \\ 2 & 2 \end{ytableau} \leftrightarrow \left(\left(\left(\left(\ydiagram{2,1} \leftarrow 1\right) \leftarrow 2\right) \leftarrow 1\right) \leftarrow 2\right) = \ydiagram{4,3},$$

$$u_Y \otimes \begin{ytableau} 1 & 1 \\ 2 & 3 \end{ytableau} \leftrightarrow \left(\left(\left(\left(\ydiagram{2,1} \leftarrow 1\right) \leftarrow 3\right) \leftarrow 1\right) \leftarrow 2\right) = \ydiagram{4,2,1},$$

$$u_Y \otimes \begin{ytableau} 1 & 2 \\ 2 & 3 \end{ytableau} \leftrightarrow \left(\left(\left(\left(\ydiagram{2,1} \leftarrow 2\right) \leftarrow 3\right) \leftarrow 1\right) \leftarrow 2\right) = \ydiagram{3,3,1},$$

$$u_Y \otimes \begin{array}{|c|c|}\hline 1 & 1 \\\hline 3 & 3 \\\hline\end{array} \leftrightarrow \left(\left(\left(\left(\begin{array}{|c|c|}\hline \ & \ \\\hline \ \\\cline{1-1}\end{array} \leftarrow 1\right) \leftarrow 3\right) \leftarrow 1\right) \leftarrow 3\right) \times ,$$

$$u_Y \otimes \begin{array}{|c|c|}\hline 1 & 2 \\\hline 3 & 3 \\\hline\end{array} \leftrightarrow \left(\left(\left(\left(\begin{array}{|c|c|}\hline \ & \ \\\hline \ \\\cline{1-1}\end{array} \leftarrow 2\right) \leftarrow 3\right) \leftarrow 1\right) \leftarrow 3\right) = \begin{array}{|c|c|c|}\hline \ & \ & \ \\\hline \ & \ \\\cline{1-2} \ & \ \\\cline{1-2}\end{array} ,$$

$$u_Y \otimes \begin{array}{|c|c|}\hline 2 & 2 \\\hline 3 & 3 \\\hline\end{array} \leftrightarrow \left(\left(\left(\left(\begin{array}{|c|c|}\hline \ & \ \\\hline \ \\\cline{1-1}\end{array} \leftarrow 2\right) \leftarrow 3\right) \leftarrow 2\right) \leftarrow 3\right) \times .$$

Hence by Theorem 6.3.1.(1), for $\mathfrak{gl}(3)$ we obtain (we omit "V")

$$\begin{array}{|c|c|}\hline \ & \ \\\hline \ \\\cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline \ & \ \\\hline \ & \ \\\hline\end{array} \cong \begin{array}{|c|c|c|c|}\hline \ & \ & \ & \ \\\hline \ & \ & \ \\\cline{1-3}\end{array} \oplus \begin{array}{|c|c|c|c|}\hline \ & \ & \ & \ \\\hline \ & \ \\\cline{1-2} \ \\\cline{1-1}\end{array} \oplus \begin{array}{|c|c|c|}\hline \ & \ & \ \\\hline \ & \ & \ \\\hline \ \\\cline{1-1}\end{array} \oplus \begin{array}{|c|c|c|}\hline \ & \ & \ \\\hline \ & \ \\\cline{1-2} \ & \ \\\cline{1-2}\end{array} .$$

Example 6.3.4. For $\mathfrak{g} = B_2$, we consider $V(4\Lambda_2) \otimes V(\Lambda_1 + \Lambda_2)$, where

$$4\Lambda_2 \Leftrightarrow (2, 2) = \begin{array}{|c|c|}\hline \ & \ \\\hline \ & \ \\\hline\end{array}, \quad \Lambda_1 + \Lambda_2 \Leftrightarrow \left(\frac{3}{2}, \frac{1}{2}\right) = \begin{array}{|c|c|}\hline \ & \ \\\hline \ \\\cline{1-1}\end{array} .$$

We set $Y = \begin{array}{|c|c|}\hline \ & \ \\\hline \ & \ \\\hline\end{array}$ and $W = \begin{array}{|c|c|}\hline \ & \ \\\hline \ \\\cline{1-1}\end{array}$. By Theorem 2.3.3 (ii), we have

$$B(V_W) = \left\{ \begin{array}{l} \begin{array}{|c|c|}\hline 1 & 1 \\\hline 2 \\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\\hline 2 \\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 1 \\\hline \bar{2} \\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\\hline 2 \\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 2 & 2 \\\hline \bar{1} \\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 0 \\\hline 2 \\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 0 \\\hline \bar{2} \\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & \bar{2} \\\hline 2 \\\cline{1-1}\end{array}, \\ \begin{array}{|c|c|}\hline 2 & 0 \\\hline \bar{1} \\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & \bar{1} \\\hline 2 \\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & \bar{2} \\\hline \bar{2} \\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 2 & \bar{2} \\\hline \bar{1} \\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & \bar{1} \\\hline \bar{2} \\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 2 & \bar{1} \\\hline \bar{1} \\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline \bar{2} & \bar{2} \\\hline \bar{1} \\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline \bar{2} & \bar{1} \\\hline \bar{1} \\\cline{1-1}\end{array}, \end{array} \right\} .$$

Hence,

$$u_Y \otimes \begin{array}{|c|c|}\hline 1 & 1 \\\hline 2 \\\cline{1-1}\end{array} = u_Y \otimes \boxed{1} \otimes (+,\ +) \leftrightarrow ((Y \leftarrow 1) + (+,\ +)) = \left(\frac{7}{2}, \frac{5}{2}\right) = \begin{array}{|c|c|c|c|}\hline \ & \ & \ & \ \\\hline \ & \ & \ \\\cline{1-3}\end{array} ,$$

$$u_Y \otimes \begin{array}{|c|c|}\hline 1 & 2 \\\hline 2 \\\cline{1-1}\end{array} = u_Y \otimes \boxed{2} \otimes (+,\ +) \leftrightarrow ((Y \leftarrow 2) + (+, +)) \times ,$$

$$u_Y \otimes \begin{array}{|c|c|}\hline 1 & 1 \\\hline 2 \\\cline{1-1}\end{array} = u_Y \otimes \boxed{1} \otimes (+,\ -) \leftrightarrow ((Y \leftarrow 1) + (+,\ -)) = \left(\frac{7}{2}, \frac{3}{2}\right) = \begin{array}{|c|c|c|c|}\hline \ & \ & \ & \ \\\hline \ & \ \\\cline{1-2}\end{array} ,$$

$$u_Y \otimes \begin{array}{|c|c|}\hline 1 & 2 \\\hline 2 \\\cline{1-1}\end{array} = u_Y \otimes \boxed{2} \otimes (+,\ -) \leftrightarrow ((Y \leftarrow 2) + (+,\ -)) \times ,$$

$$u_Y \otimes \begin{array}{|c|c|}\hline 2 & 2 \\\hline 1 \\\cline{1-1}\end{array} = u_Y \otimes \boxed{2} \otimes (-,\ +) \leftrightarrow ((Y \leftarrow 2) + (-,\ +)) \times ,$$

$$u_Y \otimes \begin{array}{|c|c|}\hline 1 & 0 \\\hline 2 \\\cline{1-1}\end{array} = u_Y \otimes \boxed{0} \otimes (+,\ +) \leftrightarrow ((Y \leftarrow 0) + (+,\ +)) = \left(\frac{5}{2}, \frac{5}{2}\right) = \begin{array}{|c|c|c|}\hline \ & \ & \ \\\hline \ & \ & \ \\\hline\end{array} ,$$

$$ {}^{D}u_Y \otimes \begin{ytableau} 1 & 0 \\ 2 \end{ytableau} = u_Y \otimes \boxed{0} \otimes (+,\ -) \leftrightarrow ((Y \leftarrow 0) + (+,\ -)) = \left(\frac{5}{2}, \frac{3}{2}\right) = \ydiagram{3,2}\,, $$

$$ u_Y \otimes \begin{ytableau} 1 & \bar{2} \\ 2 \end{ytableau} = u_Y \otimes \boxed{\bar{2}} \otimes (+,\ +) \leftrightarrow ((Y \leftarrow \bar{2}) + (+,\ +)) = \left(\frac{5}{2}, \frac{3}{2}\right) = \ydiagram{3,2}\,, $$

$$ u_Y \otimes \begin{ytableau} 2 & 0 \\ \bar{1} \end{ytableau} = u_Y \otimes \boxed{0} \otimes (-,\ +) \leftrightarrow ((Y \leftarrow 0) + (-,\ +)) = \times\,, $$

$$ u_Y \otimes \begin{ytableau} 1 & \bar{1} \\ 2 \end{ytableau} = u_Y \otimes \boxed{\bar{1}} \otimes (+,\ +) \leftrightarrow ((Y \leftarrow \bar{1}) + (+,\ +)) = \times\,, $$

$$ u_Y \otimes \begin{ytableau} 1 & \bar{2} \\ \bar{2} \end{ytableau} = u_Y \otimes \boxed{\bar{2}} \otimes (+,\ -) \leftrightarrow ((Y \leftarrow \bar{2}) + (+,\ -)) = \left(\frac{5}{2}, \frac{1}{2}\right) = \ydiagram{3,1}\,, $$

$$ u_Y \otimes \begin{ytableau} 2 & \bar{2} \\ \bar{1} \end{ytableau} = u_Y \otimes \boxed{\bar{2}} \otimes (-,\ +) \leftrightarrow ((Y \leftarrow \bar{2}) + (-,\ +)) = \left(\frac{3}{2}, \frac{3}{2}\right) = \ydiagram{2,2}\,, $$

$$ u_Y \otimes \begin{ytableau} 1 & \bar{1} \\ \bar{2} \end{ytableau} = u_Y \otimes \boxed{\bar{1}} \otimes (+,\ -) \leftrightarrow ((Y \leftarrow \bar{1}) + (+,\ -)) \times\,, $$

$$ u_Y \otimes \begin{ytableau} 2 & \bar{1} \\ \bar{1} \end{ytableau} = u_Y \otimes \boxed{\bar{1}} \otimes (-,\ +) \leftrightarrow ((Y \leftarrow \bar{1}) + (-,\ +)) \times\,, $$

$$ u_Y \otimes \begin{ytableau} \bar{2} & \bar{2} \\ \bar{1} \end{ytableau} = u_Y \otimes \boxed{\bar{2}} \otimes (-,\ -) \leftrightarrow ((Y \leftarrow \bar{2}) + (-,\ -)) = \left(\frac{3}{2}, \frac{1}{2}\right) = \ydiagram{2,1}\,, $$

$$ u_Y \otimes \begin{ytableau} \bar{2} & \bar{1} \\ \bar{1} \end{ytableau} = u_Y \otimes \boxed{\bar{1}} \otimes (-,\ -) \leftrightarrow ((Y \leftarrow \bar{1}) + (-,\ -)) \times\,, $$

By Theorem 6.3.1.(2), for B_2 we obtain (we omit "V").

$$ \ydiagram{2,2} \otimes \ydiagram{2,1} \cong \ydiagram{4,2} \oplus \ydiagram{4,2} \oplus \ydiagram{3,3} \oplus $$
$$ \left(\ydiagram{3,2}\right)^{\oplus 2} \oplus \ydiagram{3,1} \oplus \ydiagram{2,2} \oplus \ydiagram{2,1}\,. $$

Hence,

$$ V(4\Lambda_2) \otimes V(\Lambda_1 + \Lambda_2) \cong V(\Lambda_1 + 5\Lambda_2) \oplus V(2\Lambda_1 + 3\Lambda_2) \oplus V(5\Lambda_2) $$
$$ \oplus\, V(\Lambda_1 + 3\Lambda_2)^{\oplus 2} \oplus V(2\Lambda_1 + \Lambda_2) \oplus V(3\Lambda_2) \oplus V(\Lambda_1 + \Lambda_2)\,. $$

Appendix. Relation to the Original Littlewood–Richardson Rule

The original description of the Littlewood–Richardson rule for the $\mathfrak{g} = \mathfrak{gl}(n+1)$-case is different from ours. In this appendix, we shall give their relation. This relation is well-known to the specialists (e.g. cf. [W]).

First, we explain the original Littlewood–Richardson rule ([M]).

For Young diagrams $Y=(l_1,\dots,l_{n+1})$ and $Z=(m_1,\dots,m_{n+1})$, $Y \geqq Z$ if $l_j \geqq m_j$ for any j. If $Y \geqq Z$, the set-theoretic difference $W = Y - Z$ is called a skew diagram.

Definition A. *Let $J=\{1,2,\dots,n+1\}$ be a finite index set. A finite sequence $s_1 s_2 \cdots s_p$ $(s_j \in J)$ is said to be a lattice permutation if the following condition is satisfied;*

$$\#\{k \mid s_k = i \text{ and } 1 \leqq k \leqq r\} \geqq \#\{k \mid s_k = i+1 \text{ and } 1 \leqq k \leqq r\}$$
$$\text{for any } 1 \leqq i \leqq n \text{ and } 1 \leqq r \leqq p\,.$$

Let $W = Y - Z$ be a skew diagram and T_W a semi-standard skew tableau of shape W with symbols J (see [M]). For a Young diagram $Y'=(l'_1,\dots,l'_{n+1})$, we say that $wt(T_W) = wt(Y')$ if the number of symbols j in W is equal to l'_j for any j. From T_W we derive a sequence $s(T_W)$ by reading the symbols in T_W from the right to the left in successive row.

Theorem B (*the Littlewood–Richardson rule*). *Let V_Y and V_W be irreducible* $\mathfrak{gl}(n+1)$*-modules associated with Young diagrams Y and W with $n+1$ rows. Then we obtain*

$$V_Y \otimes V_W \cong \bigoplus_{\substack{T_{Z-Y} \text{ is semi-standard,} \\ wt(T_{Z-Y}) = wt(W) \text{ and} \\ s(T_{Z-Y}) \text{ is a lattice permutation.}}} V_Z\,. \tag{$*$}$$

For Young diagrams Y, W and Z in $(*)$, let us define the map Φ which associates a semi-standard skew tableau T_{Z-Y} as in $(*)$ with a semi-standard tableau of shape W.

Φ: If there is a symbol k $(1 \leqq k \leqq n+1)$ in the m-th row $(1 \leqq m \leqq n+1)$ in T_{Z-Y}, then a symbol m is written in the k-th row in the diagram W.

Theorem C. *For Young diagrams Y and W, we set*

$$\mathcal{T}(Y,W) = \left\{ T_{Z-Y};\ \begin{array}{l} Z \geqq Y,\ wt(T_{Z-Y}) = wt(W) \text{ and },\\ s(T_{Z-Y}) \text{ is a lattice permutation}\,. \end{array} \right\},$$

$$B^h(Y,W) = \{ b \in B(V_W);\ u_Y \otimes b \text{ is a highest weight element of } B(V_Y \otimes V_W)\}\,.$$

Then Φ gives a $1-1$ correspondence between $\mathcal{T}(Y,W)$ and $B^h(Y,W)$.

Example D. For $\mathfrak{g} = \mathfrak{gl}(3)$, $Y=(2,1,0) = \begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$ and $W=(2,2,0) = \begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \ \\ \hline\end{array}$, which is introduced in Sect. 6, we have

$$\mathcal{T}(Y,W) = \left\{ \begin{array}{|c|c|c|c|}\hline \ & \ & 1 & 1 \\ \hline \ & 2 & 2 & \multicolumn{1}{c}{} \\ \cline{1-3}\end{array},\ \begin{array}{|c|c|c|c|}\hline \ & \ & 1 & 1 \\ \hline \ & 2 & \multicolumn{2}{c}{} \\ \cline{1-2} 2 & \multicolumn{3}{c}{} \\ \cline{1-1}\end{array},\ \begin{array}{|c|c|c|}\hline \ & \ & 1 \\ \hline \ & 1 & 2 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array},\ \begin{array}{|c|c|c|}\hline \ & \ & 1 \\ \hline \ & 1 & \multicolumn{1}{c}{} \\ \cline{1-2} 2 & 2 & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array} \right\},$$

$$B^h(Y,W) = \left\{ \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline\end{array},\ \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 3 \\ \hline\end{array},\ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 3 \\ \hline\end{array},\ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 3 \\ \hline\end{array} \right\}.$$

Hence, we obtain

$$
\Phi:\quad
\begin{aligned}
&\begin{ytableau} \, & \, & 1 & 1 \\ \, & 2 & 2 \end{ytableau} \;\mapsto\; \begin{ytableau} 1 & 1 \\ 2 & 2 \end{ytableau}\,, \\
&\begin{ytableau} \, & \, & 1 & 1 \\ \, & 2 \\ 2 \end{ytableau} \;\mapsto\; \begin{ytableau} 1 & 1 \\ 2 & 3 \end{ytableau}\,, \\
&\begin{ytableau} \, & \, & 1 \\ \, & 1 & 2 \\ 2 \end{ytableau} \;\mapsto\; \begin{ytableau} 1 & 2 \\ 2 & 3 \end{ytableau}\,, \\
&\begin{ytableau} \, & \, & 1 \\ \, & 1 \\ 2 & 2 \end{ytableau} \;\mapsto\; \begin{ytableau} 1 & 2 \\ 3 & 3 \end{ytableau}\,.
\end{aligned}
$$

$$k \text{ in the } m\text{-th row} \mapsto m \text{ in the } k\text{-th row}$$

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