Commun. Math. Phys. 153, 579–604 (1993)

Communications in Mathematical Physics

Comment on the Generation Number in Orbifold Compactifications*

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Received August 24, 1992

Abstract. There has been some confusion concerning the number of $(1,1)$-forms in orbifold compactifications of the heterotic string in numerous publications. In this note we point out the relevance of the underlying torus lattice on this number. We answer the question when different lattices mimic the same physics and when this is not the case. As a byproduct we classify all symmetric Z_N-orbifolds with $(2,2)$ world sheet supersymmetry obtaining also some new ones.

1. Introduction

String compactifications on toroidal Z_N-orbifolds [1] are among the most intensively studied ones. They provide us with the simplest string models which have semi-realistic features. Because the one loop partition function and the couplings can be calculated explicitly in dependence of the untwisted moduli, many generic properties concerning the string moduli space and the effective low energy theory can be investigated here in detail [1]. Including all background parameters in the framework of heterotic compactifications and allowing for the most general twists, a rich class of models, with partly phenomenological very attractive features, emerges. The question is still open, whether some standard string model, which can be related in a painless manner to the known phenomenology, is contained in this class.

Toroidal orbifolds have also attracted the attention of the mathematicians, because the partition functions of $(2,2)$ models contain information, which can be interpreted as topological data of a Calabi-Yau manifold. The latter can indeed be constructed by a certain resolving process of the orbifold singularities, which establishes an exciting relation between singularity theory and the theory of modular functions.

* Supported by Deutsche Forschungsgemeinschaft

[1] The situation for more general compactification schemes has improved, as P. Candelas, X. De la Ossa, P. Green, and L. Parkes worked out the modulus dependence explicitly [2] for the quintic threefold in $\mathbb{P}^4$. Other Calabi-Yau manifolds were investigated in [3]

In this situation it comes as a surprise that some of the most fundamental properties from the physical as well as from the mathematical point of view, namely the individual numbers of generations and antigenerations, i.e. $(1,1)$-forms and $(2,1)$-forms respectively, on the orbifold have been reported incorrectly in the literature.

Z_N-orbifolds [1] are unambiguously defined through a twist acting as an automorphism in some torus lattice. Clearly, a given twist matrix determines both the spectrum of twist eigenvalues and the lattices possessing the automorphism. Some properties like the number of chiral generations, the number of space time supersymmetries or the number of untwisted moduli fields only depend on these eigenvalues. On the other hand, however, the role of the underlying lattice has been underestimated in the past. For instance, it determines the modular symmetry group [4–8], which has attracted much attention due to its importance for discussions of low energy effective actions [9].

In this communication we will point out that the lattice has even impact on the number of $(1,1)$-forms ($h_{1,1}$) and thus on the massless spectrum. Since this fact has been overlooked up to now, the true total number of 27's and $\overline{27}$'s depart from those stated in the literature. Especially the Lie-algebra lattices assigned to a stated $h_{1,1}$ differ from the results of this publication. Neglecting the dependence of orbifold properties on the lattice, has also led to an incomplete classification of symmetric Z_N-orbifolds with $(2,2)$ world sheet supersymmetry and vanishing discrete background field. We have found 18 inequivalent orbifolds in this class [2].

One should stress that the correct treatment does not lead to any changes for the two prime orbifolds Z_3 and Z_7, where no fixed tori occur. In the case of non-prime orbifolds we have typically the situation that there exits a lattice to which apply parts of the reasoning that appear in the literature. This lattice is, however, usually not the one the authors refer to.

There is a variety of methods to obtain information about the spectra of Z_N-orbifolds. One is to study the possible resolutions of orbifolds singularities [11–13]. In three (complex) dimensions these are either related to fixed points or to fixed curves. The cases of fixed points are completely understood. In the presence of fixed curves, however, more care is needed and different torus lattices give rise to different resolutions.

Another possibility is to construct the one loop partition function as done in [14]. This is equivalent to knowing all massless and massive states [3]. We will show that those parts of the partition function which are related to sectors, where the corresponding twist matrix leaves fixed tori, have a somewhat different structure.

Finally, one can construct twist invariant vertex operators by using the mode expansions of the untwisted and twisted coordinates [16]. Again we will argue that in sectors with fixed directions different solutions arise whenever the underlying lattice is changed.

Since the main purpose of this note is to emphasize the importance of the compactification lattice, we will construct the complete list of symmetric Z_N-orbifolds of $(2,2)$-type with vanishing discrete background fields $B_{\mu\nu}$. As shown in [7] non-vanishing discrete background fields very much mimic asymmetric orbifolds. It was also shown there that they can sometimes be transformed to orbifolds without such

[2] There exists a more natural and much more efficient way of constructing $(2,2)$ string compactifications using the Landau-Ginzburg approach. A classification of these string vacua was performed in [10]

[3] In more general constructions like $Z_N \times Z_M$ or non-abelian orbifolds one-loop modular invariance might be insufficient for all loop modular invariance and also for defining the model completely [15]

backgrounds. If this is the case the defining torus lattice in the transformed model typically differs from the original one. Thus we are also led to discuss these class of models.

The organization of the paper is as follows: In Sect. 2 we construct in a systematic way all models of the above mentioned type. This includes all Coxeter-twists but also more general ones. Using methods developed in [7], we will also recognize equivalences of models considered as different before [16]. Section 3 is devoted to the proof that $h_{1,1}$ depends on the chosen lattice. It uses the one loop partition function as described above and discusses also some aspects of the mode expansion approach. In Sect. 4 we confirm our results by a completely different method utilizing results of singularity theory. We give an easy prescription how to calculate $h_{1,1}$ and $h_{1,2}$ from the fixed sets.

2. Classification of the Z_N-Orbifolds

In this section we classify (2.2) string theories on orbifolds, which can be obtained by dividing out a Z_N group in a symmetric way from a six dimensional torus T^6 with vanishing discrete B-field.

To be more precise we will classify equivalence classes defined up to modular deformations. We start the discussion with a short extraction of the necessary concepts for toroidal orbifolds, mainly from the physics literature.

2.1. *Concepts in Toroidal- and Orbifold Compactifications*

Let $\Lambda = \left\{ \sum_{i=1}^{6} n^i e_i | n_i \in \mathbb{Z} \right\}$ be a lattice embedded in Euclidean space $\mathbb{R}^6$. Due to the canonical isomorphism we denote the basis of the tangent space also by e_i and define the induced metric as $g_{ij} = \langle e_i, e_j \rangle$, where $\langle\, , \rangle$ is the Euclidean scalar product. We denote the dual lattice w.r.t. $\langle\, , \rangle$ by Λ^*. The six torus is defined as the quotient of $\mathbb{R}^6$ w.r.t. Λ,

$$T^6 := \mathbb{R}^6/\Lambda . \tag{1}$$

One may also introduce an antisymmetric background field $B = b_{ij} e^i \wedge e^j$ as a geometrical data of the torus. The allowed momenta for the left and right movers of strings compactified on the torus with this background field are given by [17]

$$P_L = \tfrac{1}{2}m + gn - bn , \qquad P_R = \tfrac{1}{2}m - gn - bn , \tag{2}$$

where n and m are integer six vectors describing the winding and momentum quantum numbers of the string state. They label the elements of Λ and its dual Λ^* respectively. The geometry of the underlying torus, i.e. the so-called *modular parameters* [4] g_{ij} and b_{ij}, enters string theory only via bilinear forms of (m, n) describing the scaling dimension,

$$\begin{aligned} H_{(g,b)}(m, n; m, n) &= P_L^T g^{-1} P_L + P_R^T g^{-1} P_R \\ &= \tfrac{1}{2} m^T g^{-1} m + 2n^T gn \\ &\quad - m^T g^{-1} bn + n^T bg^{-1} m + 2n^T b^{-1} gn \end{aligned} \tag{3}$$

[4] In the heterotic theory other modular parameters appear in form of Wilson lines A_i^I

and the spin,

$$S(m,n{:}m,n) = P_L^T g^{-1} P_L - P_R^T g^{-1} P_R = 2m^T n\,, \tag{4}$$

of the physical vertex operators, up to geometry independent oscillator contributions. A linear relabelin(g of (n,m) leaving invariant the bilinear form (4) can be accompanied by a relabeling of the modular parameters g and b such that also the scaling dimensions (3) of the vertex operators are invariant. That is, string theory is the same for geometrically different tori and this leads us to following definition:

Definition 1. The bijective linear integer transformations of $(n,m) \mapsto gl(n,m)$, which leave (4) invariant, generate an invariance group G for string theory on the torus.

The requirement that H stays invariant $H_{(g,b)}(n,m;n,m) = H_{(g',b')}(gl(n,m); gl(n,m))$ induces a representation of G as transformations of the moduli $\varrho(gl)$: $(g,b) \mapsto (g',b')$, which is not [5] faithful. The group generated by this transformation is called modular symmetry group $\mathscr{G}$.

Setting the B-field to zero and restricting to a subgroup of G, which do not mix windings and momenta [4] we have the following

Lemma 1. *Let $M \in GL(6,\mathbb{Z})$. The string theory on T^6 remains invariant under the simultaneous transformations $n \to Mn$, $m \to M^{T^{-1}}m$ and $g \to M^{T^{-1}}gM^{-1}$.*

An orbifold is defined by a finite group T generated by elements Θ_i of G, all of which leave $H_{(g_o,b_0)}((g_0,b_0) \neq \emptyset)$ invariant. The latter requirement defines (g_o,b_o) the untwisted moduli space of the orbifold, which is a subspace of the moduli space (g,b) of the torus. (g_o,b_o) might be a point in moduli space, but in general the $\Theta_\imath$ specify the orbifold only up to modular deformations.

The symmetric Z_N orbifold with vanishing discrete B-field, which we want to classify, amount to the simplest possible choice for Θ. Namely we pick an element $\theta \in GL(6,\mathbb{Z})$ with $\theta^N = 1$ and transform simultaneously [6] $n \to \theta n$ and $m \to \theta^{T^{-1}}m$. The spin S is invariant by construction and requiring invariance of H_{g_o} we get

$$\theta^T g_o \theta = g_o\,. \tag{5}$$

Condition (5) ensures that the lattice automorphism acts crystallographically.

The basis transformation on (n,m) of Lemma 1 accompanied with a transformation of $g \mapsto M^{T^{-1}}gM^{-1}$ does not change the string theory on the torus and if we transform a given twist covariantly we also define the same orbifold thereof. We have therefore the following equivalence relation for the twist matrices [7]:

Definition 2. Two twist matrices θ and θ' are equivalent in the sense that they define the same orbifold string theory, if and only if $\exists M \in GL(6,\mathbb{Z})$ such that $\theta = M\theta' M^{-1}$.

The modular group of the orbifold can be described as follows [4, 8]:

Corollary 1. *The $M \in GL(6,\mathbb{Z})$ which fulfill $\theta^n = M\theta M^{-1}$ $(n \in \mathbb{Z}_+)$ generate a symmetry group of the orbifold, which induces a non [7]-faithful representation*

[5] Obviously $gl(n,m) = (-n,-m)$ is in the kernel of ϱ

[6] Note that these symmetries are directly realised as discrete automorphism on the target-space torus, whereas symmetries which mix momenta and windings are specific for string theory. It might be interesting for geometers to consider also the latter ones, as orbifoldizing w.r.t. some of them also give rise to Calabi-Yau manifolds. Moreover the symmetries which lead to the mirrors of the Z_N-orbifolds are also in this class

[7] By (5) all powers of θ are in the kernel of ϱ

on the orbifold moduli $\varrho(M): g_o \mapsto g'_o$ *by the requirement* $H_{g_o}(n, m : n, m) = H_{g'_o}(Mn, M^{T^{-1}}m; Mn, M^{T^{-1}}m)$.

2.2. Description of the Classification. We start the classification of the symmetric Z_N twists by analysing their possible eigenvalues. Let θ be a Z_N twist, i.e. a lattice automorphism with $\theta^N = 1$, in a d-dimensional lattice. By a transformation matrix $B \in GL(d, \mathbb{C})$ we can pass from the lattice basis where the twist θ is integer valued to a basis where it is diagonal

$$\theta_d = B^{-1}\theta B = \mathrm{diag}(\xi^{a_1}, \ldots, \xi^{a_d}) . \tag{6}$$

Here $\xi = e^{\frac{2\pi \imath}{N}}$ is the N^{th} order root of unity and $a_i \in \mathbb{N}_0$, $a_i < N$. Since we wish θ to act as an integer matrix in some lattice we get as a necessary condition on the exponents $\lambda_i = a_i/N$ of θ_d that they define a characteristic polynomial $P(x)$ *over the integers*. Especially the *Lefschetz Fixed Point Theorem* which gives the number of fixed points [8] χ_F of a lattice automorphism θ as

$$\chi_F = \det(1 - \theta) , \tag{7}$$

implies $P(1) \in \mathbb{Z}$.

First we search for θ_d, which fulfill the necessary condition above. Let us denote by (a, b) the greatest common divisor of a and b. We call θ_d *proper*, if $(a_i, N) = 1$ for all i. Obviously all θ_d can be constructed from *proper* subblocks θ'_d. The Euler function is defined $\phi(N)$ as the number of integers $0 < l < N$ with $(l, N) = 1$. The characteristic polynomial for a *proper* twist is a so-called cyclotomic polynomial of degree $\phi(N)$ [21]

$$P_N(x) = \prod_{\substack{0<a<N \\ (a,N)=1}} (x - \xi^a) , \tag{8}$$

which enjoys the following properties:
1. $P_N(x)$ is the polynom ring over the integers: $P_N(x) \in \mathbb{Z}[x]$.
2. $P_N(x)$ is irreducible in $\mathbb{Z}[x]$.
3. $P_N(1) = \begin{cases} p & \text{if } N = p^\tau \text{ with } p \text{ prime} , \\ 1 & \text{otherwise} . \end{cases}$

1. ensures our necessary condition for θ' to act as a lattice automorphism. Due to 2. the minimal dimension in which a Z_N twist can be realised crystallographically and *proper* is given by $\phi(N)$. Finally, 3. provides us with an effective method for computing the number of fixed points. The dimensions of the *proper* blocks are given by

$$\phi(N) = N \prod_i \frac{p_i - 1}{p_i} , \quad \text{where } p_i \text{ are the distinct prime factors of } N , \tag{9}$$

which reduces to $N - 1$ for prime N. From that we see that for $d = 1$ only Z_2 with 2 fixed points is allowed and furthermore that no other block can be defined *properly* in odd dimensions. We can easily construct a table of exponents up to $d = 6$. Because all ξ^{a_i} appear together with $\overline{\xi^{a_i}}$, we only list half of them. Using (9) it is easy to see that no higher orders are possible.

[8] $\det(1 - \theta) = 0$ in (7) signals the occurrence of fixed sublattices

Table 1. Exponents a_i of proper Z_N twists

$d=2$		$d=4$		$d=6$	
Twist	χ_F	Twist	χ_F	Twist	χ_F
$\frac{1}{3}(1)$	3	$\frac{1}{5}(1,2)$	5	$\frac{1}{7}(1,2,3)$	7
$\frac{1}{4}(1)$	2	$\frac{1}{8}(1,3)$	2	$\frac{1}{9}(1,2,4)$	3
$\frac{1}{6}(1)$	1	$\frac{1}{10}(1,3)$	1	$\frac{1}{14}(1,3,5)$	1
		$\frac{1}{12}(1,5)$	1	$\frac{1}{18}(1,5,7)$	1

The allowed sets of exponents in dimensions less than 6 can now be obtained by building all possible combinations of the proper ones. From the table we see e.g. that crystallographic automorphisms in $d=6$ can only exist for $N=2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 18, 20, 24, 30$. Indeed this list agrees with the one given e.g. in [1].

All eigenvalues appear with their complex conjugate so that we can define a complex coordinate system

$$z_i = \frac{1}{\sqrt{2}}(x_i + ix_{i+1}), \qquad i = 1,2,3 \tag{10}$$

on which θ acts holomorphically. The condition for obtaining a supersymmetric orbifold can be formulated in different ways. We follow the geometric approach [11] and require invariance of the holomorphic $(3,0)$ form of the complex torus ([18] II.6)

$$\omega = dz_1 \wedge dz_2 \wedge dz_3\,,$$

which implies $a_1 + a_2 + a_3 = 0 \bmod N$. If we furthermore restrict our attention to models which have exactly ($N=1$)-supersymmetry, i.e. no fixtorus w.r.t. the lattice automorphism θ in the first twisted sector), we get the following 9 sets of allowed exponents:

Table 2. Exponents a_i of Z_N twists leaving $N=1$ supersymmetry

Twist	χ_F	Twist	χ_F	Twist	χ_F
$Z_3 : \frac{1}{3}(1,1,-2)$	27	$Z_6' : \frac{1}{6}(1,2,-3)$	12	$Z_8' : \frac{1}{8}(1,3-4)$	8
$Z_4 : \frac{1}{4}(1,1,-2)$	16	$Z_7 : \frac{1}{7}(1,2,-3)$	7	$Z_{12} : \frac{1}{12}(1,-5,4)$	3
$Z_6 : \frac{1}{6}(1,1,-2)$	3	$Z_8 : \frac{1}{8}(1,-3,2)$	4	$Z_{12}' : \frac{1}{12}(1,5,-6)$	4

One approach to find Z_N orbifolds, which was carried out in [11], is to start with a lattice and consider the compatible Z_N automorphisms. In contrast we will start with the twist matrix and then specify the lattice metric as solution to (5). In virtue of the equivalence relation Definition 2 the action of an irreducible[9] block of a Z_N

[9] In the following irreducibility is to be understood over $\mathbb{Z}$

twist can be brought into the canonical form [10]

$$\theta = \begin{pmatrix} 0 & & & \dots & 0 & v_1 \\ 1 & 0 & & \dots & 0 & v_2 \\ 0 & 1 & 0 & \dots & 0 & v_3 \\ \vdots & & \ddots & & & \vdots \\ 0 & & \dots & & 1 & v_n \end{pmatrix}, \tag{11}$$

where $v_1 = \pm 1$ such that $|\det(\theta)| = 1$. Note that the form of (11) alone does not imply that the block is not further reducible. We calculate the N^{th} power of (11) according to our list of possible orders in the given dimension and search for integer $\vec{v}$ such that this is the unit matrix. The corresponding twist matrices can easily be found by means of a computer program. It turns out that $|v_i| \le 3$. In the following table we list only the vectors $\vec{v}$, which specify irreducible twist matrices relevant for the $N = 1$ supersymmetric Z_N-orbifolds [11]:

Table 3. Irreducible building blocks for Z_N twists relevant for $N = 1$ supersymmetry

$d = 1$	$d = 2$	$d = 3$	$d = 4$
$Z_2^{(1)}:(-1)$	$Z_3^{(2)}:(-1,-1)$	$Z_4^{(3)}:(-1,-1,-1)$	$Z_3^{(4)}:(-1,0,-1,0)$
	$Z_4^{(2)}:(-1,0)$	$Z_6^{(3)}:(-1,0,0)$	$Z_8^{(4)}:(-1,0,0,0)$
	$Z_6^{(2)}:(-1,1)$		$Z_{12}^{(4)}:(-1,0,1,0)$

$d = 5$	d=6
$Z_6^{(5)}:(-1,-1,-1,-1,-1)$	$Z_7^{(6)}:(-1,-1,-1,-1,-1,-1)$
$Z_8^{(5)}:(-1,-1,0,0,-1)$	$Z_8^{(6)}:(-1,0,-1,0,-1,0)$
	$Z_{12}^{(6)}:(-1,-1,0,1,0,-1)$

Now we combine these irreducible blocks to (6×6) twist matrices giving rise to $(N = 1)$ supersymmetric orbifolds.

To make contact with the classification of Coxeter orbifolds in [11, 14], we use the equivalence relation in Definition 2 to rewrite our twists as Coxeter automorphisms, if possible. A Weyl reflection [12] is a reflection on the hyperplane perpendicular to a simple root

$$S_i(x) = x - 2\,\frac{\langle x, e_i\rangle}{\langle e_i, e_i\rangle}\, e_i\,.$$

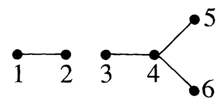
A Coxeter automorphism c in a Lie algebra lattice is defined by successive Weyl reflections w.r.t. all simple roots $c = S_1 \cdot \ldots \cdot S_{\text{rank}}$. Automorphisms are called outer if they cannot be generated by Weyl reflections. They are generated by transpositions of roots which are symmetries of the Dynkin diagram. Generalised

[10] In this form χ_F is simply given by $1 - \sum_i v_i$

[11] It also turned out that the cases with $v_1 = +1$ exactly correspond to fixed tori

[12] For the following definitions concerning Lie algebras [23] is the standard reference

Coxeter automorphisms can be obtained by combining Weyl reflections with outer automorphisms. We denote a transposition which exchange the roots $i \leftrightarrow j$ by P_{ij}. In [11] generalised Coxeter automorphisms were only considered if they act in one semisimple factor, e.g. in the lattice $A_2 \times D_4$ the automorphism $S_1 S_2 S_3 P_{36} P_{35}$ [cyclic permutation of the roots $(3,5,6)$].



There is no reason for this restriction and in fact the full classification involves transpositions between the semisimple factors, e.g. in $A_3 \times A_3$ the automorphism $S_1 S_2 S_3 P_{16} P_{25} P_{34}$. As the result [13] of our classification we have the following

Theorem 1. *There exist* 18 *inequivalent* ($M = 1$) *supersymmetric string theories on symmetric* Z_N *orbifolds of* $(2,2)$*-type without discrete background all having at least one representative in the class of generalised Coxeter orbifolds. More precisely we have* 15 *ordinary Coxeter orbifolds realized on the lattices*

$A_2 \times A_2 \times A_2$,	$A_1 \times A_1 \times B_2 \times B_2$,	$A_1 \times A_3 \times B_2$,	$A_3 \times A_3$,
$A_2 \times G_2 \times G_2$,	$A_1 \times A_1 \times A_2 \times G_2$,	$A_2 \times D_4$,	$A_1 \times A_5$,
A_6,	$B_2 \times B_4$,	$A_1 \times A_1 \times B_4$,	$A_1 \times D_5$,
$A_2 \times F_4$,	E_6,	$D_2 \times F_4$.	

and 3 *involving outer automorphsims, namely* $A_1 \times A_1 \times A_2 \times A_2$ *with* $S_1 S_2 S_3 S_4 P_{36} P_{45}$, $G_2 \times A_2 \times A_2$ *with* $S_1 S_2 S_3 S_4 P_{36} P_{45}$ *and finally* $A_3 \times A_3$ *with* $S_1 S_2 S_3 P_{16} P_{25} P_{34}$.

This result is to be compared with the result of [11], which was the basis for further investigations [14, 16, 22]. Here the authors give in Table 1 of their classification theorem 13 examples. They suggest that only 9 are inequivalent, namely the one which have different twist eigenvalues (cf. Table 2 above). Three pairs of orbifolds which are identified in [11], are inequivalent in the sense of Definition 2. In fact they have different hodge numbers, as we will explain below. We agree on the other hand with the identification of the models $A_2 \times D_4^{[3]}$ (with automorphism $S_1 S_2 S_3 P_{36} P_{35}$) and $A_2 \times F_4$ (with Coxeter automorphism) in Table 1 of [11]. Finally we have found *six new* examples which are inequivalent to the ones appearing in [11], three of them as mentioned involve outer automorphisms between semi-simple factors.

3. Partition Functions

In this section we discuss the construction of one loop partition functions, which allows for a survey of all string states. For the $E_8 \times E_8$ heterotic string before

[13] The relevant combinations of the twist matrices specified in Table 3 and the Hodge numbers for the 18 models can be found in Table 5

compactification it is given by [19]

$$Z(\tau,\bar{\tau}) = \frac{1}{8}\frac{(2\pi\,\mathrm{Im}\,\tau)^{-4}}{|\eta(\tau)|^{16}}\;\frac{[\theta_3^8(\tau)+\theta_4^8(\tau)+\theta_2^8(\tau)]^2}{\eta^{16}(\tau)} \times \frac{[\theta_3^4(\bar{\tau})-\theta_4^4(\bar{\tau})-\theta_2^4(\bar{\tau})]}{\eta^4(\bar{\tau})}, \tag{12}$$

where the first factor refers to ten dimensional Minkowski space in light cone gauge, the part holomorphic in the complex world sheet parameter τ describes the gauge part of the left handed bosonic string and the antiholomorphic part account for the right handed superfermions. We introduced Dedekind's η-function

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) \tag{13}$$

and Jacobi's θ-functions

$$\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau) = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n+\alpha)^2} e^{2\pi i(n+\alpha)\beta} \tag{14}$$

with $q = \varepsilon^{2\pi i\tau}$. In the context of toroidal orbifolds each part can be realized as free world sheet bosons. Thus the θ-functions will be regarded as the instanton part whereas the η-functions describe quantum oscillations. Factors like $(2\pi\,\mathrm{Im}\,\tau)$ arise after integrating out continuous momentum states. After compactifying on T^6 this integration transmutes to a sum over discrete momenta p^μ characterized by elements of the dual torus lattice Λ^*. Modular invariance requires the appearance of an additional sum over elements of Λ itself interpreted as winding states w^μ. These two lattices can be combined to define an even self-dual lattice Λ_N with Lorentzian signature [17]

$$(+,+,+,+,+,+,-,-,-,-,-,-,)$$

and elements $P = (P_L, P_R)$, defined in (2).

Again we used the freedom of turning on an antisymmetric background field b_{ij} which corresponds to considering *all* even, self-dual lattices in $6+6$ dimensions [14]. The partition function now reads

$$Z(\tau,\bar{\tau}) = \frac{\sum_{P\in\Lambda_N} q^{P_L\frac{1}{2g}P_L}\bar{q}^{P_R\frac{1}{2g}P_R}}{8(2\pi\,\mathrm{Im}\,\tau)\,|\eta(\tau)|^{16}} \times \frac{[\theta_3^8(\tau)+\theta_4^8(\tau)+\theta_2^8(\tau)]^2}{\eta^{16}(\tau)}\;\frac{[\theta_3^4(\bar{\tau})-\theta_4^4(\bar{\tau})-\theta_2^4(\bar{\tau})]}{\eta^4(\bar{\tau})}. \tag{15}$$

The orbifold projection can be performed yielding the *untwisted partition function*

$$\sum_{n=0}^{N-1} Z_{(1,\theta^n)}(\tau,\bar{\tau}) = \tfrac{1}{4}\sum_{n=0}^{N-1}\mathrm{Tr}\,\theta^n q^{L_0}\bar{q}^{\bar{L}_0}\,, \tag{16}$$

[14] Similarly, it is possible to add the $E_8\times E_8$ gauge lattice as well and to allow for the most general lattice in $22+6$ dimensions of even and self-dual type, what in turn corresponds to turning on Wilson lines

where the trace is to be taken over all states of the torus theory [15] with conformal dimensions L_0 and $\bar{L}_0$. The projection is such that oscillator states in the quantum part are multiplied by phases $\alpha = \varepsilon^{2\pi i/N}$ and powers thereof, whereas instantons are organized in *orbit sums* with definite twist eigenvalues. In other words, orbit sums are linear combinations of states of toroidal Hilbert space diagonalizing the twist. We conclude that $Z_{(1,1)}$ is simply given by $1/N$ times the torus function and thus clearly contains the full instanton part. If the defining twist matrix θ fixes the orgigin of the lattice *only*, as is the case for all our models, $Z_{(1,\theta)}$ will *not* contain any instantons from the six dimensional part, since the phases of the orbits add up to zero. Clearly, the same is true for all $Z_{(1,\theta^n)}$, where n is not a divisor of N. In all other cases the appearance of an instanton sum precisely depends on the question whether the corresponding power of θ leaves fixed tori or not. Namely, if it does give rise to fixed directions the corresponding instantons are (like the origin) not organized in orbit sums. It is this fact where our disagreement with the literature stems from. For instance, in formulae (3.3a) and (3.3c) of reference [14] there is no such sum. In these formulae there appear correctly the instanton sums coming from the gauge part. The remark to make is simply that instanton sums can appear in parts of $Z_{(1,\theta^n)}$ even if the lattice is not invariant under θ itself.

The twisted sectors of the orbifold can be obtained by successive S and T transformations [16], which then ensure one loop modular invariance by construction [20]. For symmetric Z_N-orbifolds these transformations close after having created $N(N-1)$ new terms labeled by $Z_{(\theta^m,\theta^n)}$, with $1 \leq m < N$ and $0 \leq n < N$. For these world sheet modular transformations we will need the identities

$$\theta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (-1/\tau) = e^{2\pi i \alpha\beta} \sqrt{-i\tau}\, \theta \begin{bmatrix} \beta \\ -\alpha \end{bmatrix} (\tau)\,, \tag{17}$$

$$\theta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (\tau + 1) = e^{i\pi\alpha(1-\alpha)} \theta \begin{bmatrix} \alpha \\ \alpha + \beta - \frac{1}{2} \end{bmatrix} (\tau)\,. \tag{18}$$

The question arises whether the invariant lattice we had argued for in the last paragraph has implications on the spectrum of twisted states and in particular on the *generalized* GSO-*projection* established in [6]. Concerning massive states the answer is certainly yes, since the lattice dual to the invariant lattice contributes. Massless states are never built up by states of this dual lattice, so how can the number of massless matter multiplets depend on it? The important point is that the invariant lattice lowers the degeneracy factors of $Z_{(\theta^m,\theta^n)}$ iff its volume differs from one. As a corollary of our classification in the proceeding section we can state that for any configuration of twist eigenvalues there exist a model where the volume of the invariant lattice is one. In the remaining 9 cases $h_{1,1}$ is reduced.

We don't need to worry whether we really obtain a sensible string theory after dividing the degeneracy factors by the volume of the invariant sublattice, i.e. whether we get an integer number of states. As shown in [24, 25] this is guaranteed due to the fact that Λ_N appearing in (15) is even and self-dual.

In order to illustrate all this, we now discuss as an example the Z_4 case in more detail. In particular, we compare models 2 and 4 in our list. These are also the most explicit ones of reference [22], where for the first time such a comparative study of different lattices in a somewhat different context was undertaken.

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[15] not just the ones subject to *physical conditions*

[16] $S: \tau \to -1/\tau : T: \tau \to \tau + 1$

First we give the partition functions. In order to keep the formulae readable we will now restrict ourselves to the twisted bosons of internal space. I.e., the holomorphic gauge and antiholomorphic superfermionic parts as well as uncompactified space time dimensions are disregarded and can be found in [14].

$$Z^{\text{int}}_{(1,1)} = \frac{\sum\limits_{P\in\Lambda_N} q^{P_L^2/2}\bar{q}^{P_R^2/2}}{|\eta(\tau)|^{12}}, \qquad Z^{\text{int}}_{(1,\theta)} = 16\frac{|\eta(\tau)|^6}{\left|\theta^2\begin{bmatrix}1/2\\3/4\end{bmatrix}\theta\begin{bmatrix}1/2\\0\end{bmatrix}\right|^2},$$

$$Z^{\text{int}}_{(1,\theta^2)} = 16\frac{\sum\limits_{P\in\Lambda_N^\perp} q^{P_L^2/2}\bar{q}^{P_R^2/2}}{\left|\theta^2\begin{bmatrix}1/2\\0\end{bmatrix}\right|^2}, \qquad Z^{\text{int}}_{(1,\theta^3)} = 16\frac{|\eta(\tau)|^6}{\left|\theta^2\begin{bmatrix}1/2\\1/4\end{bmatrix}\theta\begin{bmatrix}1/2\\0\end{bmatrix}\right|^2},$$

$$Z^{\text{int}}_{(\theta,1)} = 16\frac{|\eta(\tau)|^6}{\left|\theta^2\begin{bmatrix}3/4\\1/2\end{bmatrix}\theta\begin{bmatrix}0\\1/2\end{bmatrix}\right|^2}, \qquad Z^{\text{int}}_{(\theta,\theta)} = 16\frac{|\eta(\tau)|^6}{\left|\theta^2\begin{bmatrix}3/4\\3/4\end{bmatrix}\theta\begin{bmatrix}0\\0\end{bmatrix}\right|^2},$$

$$Z^{\text{int}}_{(\theta,\theta^2)} = 16\frac{|\eta(\tau)|^6}{\left|\theta^2\begin{bmatrix}3/4\\0\end{bmatrix}\theta\begin{bmatrix}0\\1/2\end{bmatrix}\right|^2}, \qquad Z^{\text{int}}_{(\theta,\theta^3)} = 16\frac{|\eta(\tau)|^6}{\left|\theta^2\begin{bmatrix}3/4\\1/4\end{bmatrix}\theta\begin{bmatrix}0\\0\end{bmatrix}\right|^2},$$

$$Z^{\text{int}}_{(\theta^2,1)} = 16\frac{\sum\limits_{P\in(\Lambda_N^\perp)^*} q^{P_L^2/2}\bar{q}^{P_R^2/2}}{\operatorname{vol}\Lambda_N^\perp\left|\theta^2\begin{bmatrix}0\\1/2\end{bmatrix}\right|^2}, \qquad Z^{\text{int}}_{(\theta^2,\theta)} = 16\frac{|\eta(\tau)|^6}{\left|\theta^2\begin{bmatrix}0\\1/4\end{bmatrix}\theta\begin{bmatrix}1/2\\0\end{bmatrix}\right|^2},$$

$$Z^{\text{int}}_{(\theta^2,\theta^2)} = 16\frac{\sum\limits_{P\in(\Lambda_N^\perp)^*} e^{\pi i(P_L^2-P_R^2)}q^{P_L^2/2}\bar{q}^{P_R^2/2}}{\operatorname{vol}\Lambda_N^\perp\left|\theta^2\begin{bmatrix}0\\0\end{bmatrix}\right|^2}, \qquad Z^{\text{int}}_{(\theta^2,\theta^3)} = 16\frac{|\eta(\tau)|^6}{\left|\theta^2\begin{bmatrix}0\\3/4\end{bmatrix}\theta\begin{bmatrix}1/2\\0\end{bmatrix}\right|^2},$$

$$Z^{\text{int}}_{(\theta^3,1)} = 16\frac{|\eta(\tau)^6}{\left|\theta^2\begin{bmatrix}1/4\\1/2\end{bmatrix}\theta\begin{bmatrix}0\\1/2\end{bmatrix}\right|^2}, \qquad Z^{\text{int}}_{(\theta^3,\theta)} = 16\frac{|\eta(\tau)|^6}{\left|\theta^2\begin{bmatrix}1/4\\3/4\end{bmatrix}\theta\begin{bmatrix}0\\0\end{bmatrix}\right|^2},$$

$$Z^{\text{int}}_{(\theta^3,\theta^2)} = 16\frac{|\eta(\tau)|^6}{\left|\theta^2\begin{bmatrix}1/4\\0\end{bmatrix}\theta\begin{bmatrix}0\\1/2\end{bmatrix}\right|^2}, \qquad Z^{\text{int}}_{(\theta^3,\theta^3)} = 16\frac{|\eta(\tau)|^6}{\left|\theta^2\begin{bmatrix}1/4\\1/4\end{bmatrix}\theta\begin{bmatrix}0\\0\end{bmatrix}\right|^2}.$$

We denoted the invariant part of the self-dual lattice by $\Lambda_N^\perp$. The numerical factors in the untwisted sector arise when expressing infinite products in terms of θ-functions. They carry over to the twisted parts, but they are implicitly lowered whenever a function $\theta\begin{bmatrix}1/2\\\beta\end{bmatrix}$ appears. In our case the actual degeneracy factor in $Z_{(\theta^2,\theta)}$ and $Z_{(\theta^2,\theta^3)}$ are thus reduced to 4. Recalling that the whole partition function is multiplied by the inverse twist order this yields projectors for *massless twisted states* of the form (compare [14])

$$\frac{1}{4}(16 + 16\Delta_\theta + 16\Delta_\theta^2 + 16\Delta_\theta^3), \tag{19}$$

$$\frac{1}{4}\left(\frac{16}{\mathrm{vol}\,\Lambda_N^\perp} + 4\Delta_{\theta^2} + \frac{16}{\mathrm{vol}\,\Lambda_N^\perp}\,\Delta_{\theta^2}^2 + 4\Delta_{\theta^2}^3\right), \tag{20}$$

$$\frac{1}{4}(16 + 16\Delta_{\theta^3} + 16\Delta_{\theta^3}^2 + 16\Delta_{\theta^3}^3), \tag{21}$$

where $\Delta = \pm 1$ for generations and antignerations, respectively. As mentioned earlier the only difference to the corresponding projector in [14] is the appearance of the volume factors. It differs from one exactly in the cases where $\Lambda_N^\perp$ does not factorize trivially. For model 2 it is just given by a two-dimensional torus lattice and its dual. Being self-dual such a lattice transforms into itself under the Poisson resummation [17] connected with the S transformation and we conclude $\mathrm{vol}\,\Lambda_N^\perp = 1$. Expressed in more formal terms for the torus lattice we have the relation

$$(\Lambda^*)^\perp = (\Lambda^\perp)^* . \tag{22}$$

The same is true for the *first* model of each $Z_N(Z'_N)$ in Table (5), respectively. The corresponding values $h_{(1,1)}$ and $h_{(1,2)}$ are the ones stated in the literature. In contrast, the other cases no longer satisfy (22) as will now be illustrated with help of model 4.

The Z_4-twist matrix and its dual in an $SU(4)$-lattice are

$$\theta = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}, \qquad \theta^* = \begin{pmatrix} -1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{23}$$

In solving the equations

$$\theta^2 n = n\,, \qquad \theta^{T^{-1^2}} m = m\,, \tag{24}$$

we find the fixed directions

$$(n_1, 0, n_1)\,, \qquad (m_1, -m_1, m_1)\,. \tag{25}$$

To find the volume factor of $\Lambda_N^\perp$ in this case, we consider one of the two scalar products in (3) or (4). It is convenient to take the latter one, since it does not depend on the background parameters. We just compute the quantity $2m^T n$ for the sublattices defined in (25) and normalize the result w.r.t. to a (self-dual) circle theory,

$$\mathrm{vol}\,\Lambda_N^\perp = \frac{4m_1^T n_1}{2m_1^T n_1}\,. \tag{26}$$

Since model 4 is a product of two $SU(4)$-lattices, we finally find a volume factor of four and we can read off from the projector in (20), that four generations and no

[17] The Poisson resummation formula reads

$$\sum_{w\in\Lambda} \exp[-\pi(u' + e)^T A(w + e) + 2\pi i\phi^T(w + e)]$$
$$= \frac{1}{\mathrm{vol}\,\Lambda\sqrt{\det A}} \sum_{p\in\Lambda^*} \exp[-\pi(p + \phi)^T A^{-1}(p + \phi) - 2\pi i e^T p]\,,$$

where e and ϕ are constant vectors and A is an invertible matrix antigeneration survive in the double twisted sector of this model. Comparing with model 2, where ten generations and six antigenerations survive the projection, we make two observations which in fact turn out to be general rules. First, the number of *chiral* generations is unchanged when considering different lattices but the same twist eigenvalues. This one indeed expects, since this number can be computed by a formula conjectured by Dixon, Harvey, Vafa, and Witten [1] and proved by Markushevich, Olshanetsky, and Perelomov [11], which only uses twist eigenvalues. Second, the total number of generations is lowered when one considers cases where formula (22) is no longer fulfilled. For the projector of massive states the phase factor $e^{2\pi i P_L^2 - P_R^2}$ in $Z^{\text{int}}_{(\theta^2,\theta^2)}$ has to be taken into account in case of non-trivial instanton contributions.

The list of the basic degeneracy factors for all twist eigenvalue configurations can be found in [16]. Here, we give the volume factors V_i for all our models, where the subscript denotes the sector in which the fixed torus appears.

Table 4. Volume factors V_i for twist sectors with fixed tori

Model	Volume factors
1	–
2	$V_2 = 1$
3	$V_2 = 2$
4	$V_2 = 4$
5	$V_3 = 1$
6	$V_3 = 4$
7	$V_2 = 1,\ V_3 = 1$
8	$V_2 = 1,\ V_3 = 4$
9	$V_2 = 3,\ V_3 = 1$
10	$V_2 = 3,\ V_3 = 4$
11	–
12	$V_4 = 1$
13	$V_4 = 4$
14	$V_2 = 1,\ V_4 = 1$
15	$V_2 = 2,\ V_4 = 2$
16	$V_3 = 1,\ V_6 = 1$
17	$V_3 = 4,\ V_6 = 4$
18	$V_2 = 1,\ V_4 = 1,\ V_6 = 1$

For the resulting generation numbers we refer to Table 5.

4. Geometrical Resolution of the Orbifold Singularities

In this section we will calculate the Hodge numbers for the Calabi-Yau manifold which is constructed by resolving the orbifold singularities. As it was conjectured in [1] there exists a resolution $\widehat{T/G}$ of the toroidal orbifold T/G to a Calabi-Yau manifold, if the group action leave the holomorphic three form of the torus invariant. The prediction for the Euler number was extracted from the partition function [1] and stated in form of the famous orbifold formula

$$\chi(\widehat{T/G}) = \frac{1}{|G|} \sum_{[g,h]=0} \chi_{g,h}\,, \tag{27}$$

Table 5. Hodge numbers of the 18 symmetric Z_N-orbifolds with $(2,2)$ world sheet supersymmetry and vanishing discrete $b_{\mu\nu}$-field. The corresponding twist eigenvalues are given in Table 2. The twist matrix is specified by the irreducible blocks which appear in Table 3. We also specify the Lie-algebra lattice where an equivalent twist can be realised as generalised Coxeter automorphism. If no automorphism is explicitly specified the twist is realized as ordinary Coxeter twist [19]

Case		Twist	Lattice	$h^{1,1}$		$h^{1,2}$		χ
1	Z_3	$(Z_3^{(2)}, Z_3^{(2)}, Z_3^{(2)})$	$A_2 \times A_2 \times A_2$	36	(9)	0	(0)	72
2	Z_4	$(Z_2^{(1)}, Z_2^{(1)}, Z_4^{(2)}, Z_4^{(2)})$	$A_1 \times A_1 \times B_2 \times B_2$	31	(5)	7	(1)	48
3	Z_4	$(Z_2^{(1)}, Z_4^{(2)}, Z_4^{(3)})$	$A_1 \times A_3 \times B_2$	27	(5)	3	(1)	48
4	Z_4	$(Z_4^{(3)}, Z_4^{(3)})$	$A_3 \times A_3$	25	(5)	1	(1)	48
5	Z_6	$(Z_3^{(2)}, Z_6^{(2)}, Z_6^{(2)})$	$A_2 \times G_2 \times G_2$	29	(5)	5	(0)	48
6	Z_6	$Z_6^{(2)}, Z_3^{(4)})$	$G_2 \times A_2 \times A_2$ $S_1S_2S_3S_4P_{36}P_{45}$	25	(5)	1	(0)	48
7	Z_6'	$(Z_2^{(1)}, Z_2^{(1)}, Z_3^{(2)}, Z_6^{(2)})$	$A_1 \times A_1 \times A_2 \times G_2$	35	(3)	11	(1)	48
8	Z_6'	$(Z_2^{(1)}, Z_3^{(2)}, Z_6^{(3)})$	$A_2 \times D_4$	29	(3)	5	(1)	48
9	Z_6'	$Z_2^{(1)}, Z_2^{(1)}, Z_3^{(4)})$	$A_1 \times A_1 \times A_2 \times A_2$ $S_1S_2S_3S_4P_{36}P_{45}$	31	(3)	7	(1)	48
10	Z_6'	$(Z_2^{(1)}, Z_6^{(5)})$	$A_1 \times A_5$	25	(3)	1	(1)	48
11	Z_7	$(Z_7^{(6)})$	A_6	24	(3)	0	(0)	48
12	Z_8	$(Z_4^{(2)}, Z_8^{(4)})$	$B_2 \times B_4$	27	(3)	3	(0)	48
13	Z_8	$(Z_8^{(6)})$	$A_3 \times A_3$ $S_1S_2S_3P_{35}P_{36}P_{45}$	24	(3)	0	(0)	48
14	Z_8'	$(Z_2^{(1)}, Z_2^{(2)}, Z_8^{(4)})$	$B_4 \times D_2$	31	(3)	7	(1)	48
15	Z_8'	$(Z_2^{(1)}, Z_8^{(5)})$	$A_1 \times D_5$	27	(3)	3	(1)	48
16	Z_{12}	$(Z_3^{(2)}, Z_{12}^{(4)})$	$A_2 \times F_4$	29	(3)	5	(0)	48
17	Z_{12}	$(Z_{12}^{(6)})$	E_6	25	(3)	1	(0)	48
18	Z_{12}'	$(Z_2^{(1)}, Z_2^{(1)}, Z_{12}^{(4)})$	$D_2 \times F_4$	31	(3)	7	(1)	48

where the sum is taken over all commuting elements of the group and $\chi_{g,h} = \chi(\mathrm{Fix}(g) \cap \mathrm{Fix}(h))$ is the Euler number of the intersection of the fixed sets under g and h, respectively. For the special case of Z_N actions the corresponding Calabi-Yau spaces were constructed explicitly [11, 12] confirming (27).

4.1. The Fixed Sets

In our examples we have only fixed points $P\,(\chi(P) = 1)$ and fixed tori $T\,(\chi(T) = 0)$. Obviously $\chi_{g,h} = \chi_{h,g}$. The application of (27) is simplified by the fact that

$\chi(\mathrm{Fix}(g)) \subset \chi(\mathrm{Fix}(h))$ if $\exists n \in \mathbb{Z}$ with $g^n = h$. Because $\chi(T) = 0$ the precise number of fixed tori in the higher twisted sectors does not affect the Euler number. Using the numbers of fixed points which depend only on the eigenvalues of the twist matrix (cf. Sect. 2, Table 2) we see that $\chi = 72$ in the Z_3 case and $\chi = 48$ in all other cases.

In contrast to the Euler number the numbers of independent $(1,1)$-forms and $(1,2)$-forms in Hodge cohomology depend on the fixed tori and their intersection pattern and hence on the lattice. Let us illustrate this point with help of a series of examples, namely three Z_6' models with twist exponents $\frac{1}{6}(1,2,3)$. As the first model we consider case 7 of Table 5 which is equivalent to the Coxeter twist in the lattice $A_1 \times A_1 \times A_2 \times G_2$. An explicit twist matrix can be obtained by combining the irreducible blocks $(Z_2^{(1)}, Z_2^{(1)}, Z_6^{(2)}, Z_3^{(2)})$ specified by the vectors in Table 3. This matrix has the following fixed sets in the first, second and third twist sector respectively:

θ: 12 fixed points: $(v_i, 0, 0, w_j)$, $i = 1,2,3,4$; $j = 1,2,3$

$$v_1 = (0,0),\ v_2 = \left(\tfrac{1}{2}, 0\right),\ v_3 = \left(0, \tfrac{1}{2}\right),\ v_4 = \left(\tfrac{1}{2}, \tfrac{1}{2}\right);$$
$$w_1 = (0,0),\ w_2 = \left(\tfrac{1}{3}, \tfrac{2}{3}\right),\ w_3 = \left(\tfrac{2}{3}, \tfrac{1}{3}\right)$$

θ^2: 9 fixed tori: invariant subspace: $n_1 = (1,0,0,0,0,0)$, $n_2 = (0,1,0,0,0,0)$;

$$[m_1 = (1,0,0,0,0,0),\ m_2 = (0,1,0,0,0,0)]$$

base points: $(0,0,v_i,w_j)$, $i,j = 1,2,3$

$$v_1 = (0,0),\ v_2 = \left(\tfrac{1}{3}, \tfrac{1}{3}\right),\ v_3 = \left(\tfrac{2}{3}, \tfrac{2}{3}\right);$$
$$w_1 = (0,0),\ w_2 = \left(\tfrac{1}{3}, \tfrac{2}{3}\right),\ w_3 = \left(\tfrac{2}{3}, \tfrac{1}{3}\right)$$

θ^3: 16 fixed tori: invariant subspace: $n_1 = (0,0,1,0,0,0)$, $n_2 = (0,0,0,1,0,0)$;

$$[m_1 = (0,0,1,0,0,0),\ m_2 = (0,0,0,1,0,0)]$$

base points: $(v_i, 0, 0, v_j)$, $i,j = 1,2,3,4$

$$v_1 = (0,0),\ v_2 = \left(\tfrac{1}{2}, 0\right),\ v_3 = \left(0, \tfrac{1}{2}\right),\ v_4 = \left(\tfrac{1}{2}, \tfrac{1}{2}\right)$$

The vectors n_i span an invariant subspace of the matrix θ^s from the given base points. These data define the corresponding fixerd tori. The vectors m_i span the invariant subspace of $(\theta^{T^{-1}})^s$: their significance is explained below. Note that all base points given, lie in fact on different tori, as their only non-vanishing entries are perpendicular to the n_i. The schematic view of the intersection pattern can be found in Fig. 1. Let us compare this situation with the orbifold in case 8, which may be defined by the Coxeter twist in the lattice $A_2 \times D_4$,

$$\theta = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix} \tag{28}$$

and has the following fixed sets:

θ: 12 fixed points: (v_i, w_j), $i = 1,2,3$; $j = 1,2,3,4$

$$v_1 = (0,0),\ v_2 = \left(\tfrac{1}{3},\tfrac{2}{3}\right),\ v_3 = \left(\tfrac{2}{3},\tfrac{1}{3}\right)$$
$$w_1 = (0,0,0,0),\ w_2 = \left(0,0,\tfrac{1}{2},\tfrac{2}{2}\right),$$
$$w_3 = \left(\tfrac{1}{2},0,0,\tfrac{1}{2}\right),\ w_4 = \left(\tfrac{1}{2},0,\tfrac{1}{2},0\right).$$

θ^2: 3 fixed tori: invariant subspace: $n_1 = (0,0,1,0,0,1)$, $n_2 = (0,0,1,0,1,0)$;

$$[m_1 = (0,0,1,-1,0,1),\ m_2 = (0,0,1,-1,1,0)]$$

base points: $(v_i, 0,0,0,0)$, $i = 1,2,3$

$$v_1 = (0,0),\ v_2 = \left(\tfrac{1}{3},\tfrac{2}{3}\right),\ v_3 = \left(\tfrac{2}{3},\tfrac{1}{3}\right).$$

θ^3: 16 fixed tori: invariant subspace: $n_1 = (1,0,0,0,0,0)$, $n_2 = (0,1,0,0,0,0)$;

$$[m_1 = (1,0,0,0,0,0),\ (0,1,0,0,0,0)]$$

base points: $(0,0,v_i,v_j)$, $i,j = 1,2,3,4$

$$v_1 = (0,0),\ v_2 = \left(\tfrac{1}{2},0\right),\ v_3 = \left(0,\tfrac{1}{2}\right),\ v_4 = \left(\tfrac{1}{2},\tfrac{1}{2}\right).$$

This model was investigated in detail in [13], however here the author claims that there are 9 instead of 3 fixed tori in the second twisted sector and the conclusion about the massless spectrum is therefore not completely correct. The reason for the difference seems to be due to an improper use of the *Lefschetz Fixed Point Theorem.* In the second twist sector the exponents of θ_d^2 are $\frac{1}{3}(1,2,0)$, so that the coordinate plane $\tilde{I}$ spanned by z_3 (comp. eq. 10), is fixed. In order to calculate the multiplicity of the corresponding fixed tori, the authors of [11, 13] apply now the *Lefschetz Fixed Point Theorem* (7) to the action of θ_d^2 on the subspace $\tilde{J}$, spanned by the first and second coordinate plane. The result is $n_F = \det(1-\theta^2)|_{\tilde{J}} = (1-\exp[2\pi i/3])^2(1-\exp[4\pi i/3])^2 = 9$. This is inadequate, because the splitting of $\mathbb{R}^6$ in $\tilde{I}$ and $\tilde{J}$ does not correspond to a splitting of the lattice Λ into sublattices on which θ^2 acts as an automorphism. Let us pass to the lattice basis and denote the sublattice fixed w.r.t. to the lattice automorphism θ^2 by I. The *Lefschetz Point Theorem* could be utilized in the above sense, if there would be a sublattice J invariant under θ^2, which is complementary to I, i.e. $\Lambda = I \bigoplus_{\mathbb{Z}} J$. This is not the case because θ^2 has no block structure w.r.t. I.

In the third twisted sector θ^3 has block structure w.r.t. its invariant sublattice I so the reasoning of the authors [11, 13] yields the correct result. Similarly it applies to the second and third twisted sector of Example 7.

Instead of calculating the fixed sets explicitly the *Fixed Point Theorem* can be modified by taking into account volume factors in the Poisson resummation formula which reduces the multiplicity of the twisted states as it was explained in the previous section. Let Θ be the action of the twist in the Narain lattice labeled by (n,m) and I the invariance subspace in this lattice. The number of connected fixed sets is then given by [25]

$$n_F = \sqrt{\frac{\det'(\mathbf{1}-\Theta)}{\mathrm{Vol}_N[I]}}\,, \tag{29}$$

where the evaluation of $\det'$ is defined by taking the product over the nonzero eigenvalues only. The volume of the invariant lattice is defined as the volume of its fundamental parallelepiped w.r.t. the Narain scalar product (4). In order to cover also the case of fixed points we define Vol_N of any number of discrete points to be

1. In the cases at hand we have

$$\Theta = \begin{bmatrix} \theta & 0 \\ 0 & \theta^{T^{-1}} \end{bmatrix}.$$

Let m_i be the vectors which span the sublattice invariant under $\theta^{T^{-1}}$. The formula (29) simplifies to

$$n_F = \frac{\det'(\mathbf{1} - \theta)}{\det'(n_i^T m_j)}; \tag{30}$$

e.g. for the θ^2 sector of the case above we get indeed the reduction factor $\det'(n_i^T m_j) = 3$. For the other volume factors see Table 4.

As the last example in our series we consider the case 10 which can be realised as the Coxeter twist in $A_1 \times A_5$. Our canonical twist $(Z_2^{(1)}, Z_6^{(5)})$ has the following fixed sets (cf. Fig. 1):

θ: 12 fixed points: (v_i, w_j), $i = 1,2$; $j = 1,2,3,4,5,6$

$v_1 = (0)$, $v_2 = \left(\frac{1}{2}\right)$,

$w_1 = (0,0,0,0,0)$, $w_2 = \frac{1}{6}(1,2,3,4,5)$, $w_3 = \frac{1}{6}(2,4,0,2,4)$,

$w_4 = \frac{1}{6}(3,0,3,0,3)$, $w_5 = \frac{1}{6}(4,2,0,4,2)$, $w_6 = \frac{1}{6}(5,4,3,2,1)$.

θ^2: 3 fixed tori: invariant subspace: $n_1 = (0,1,0,1,0,1)$, $n_2 = (1,0,0,0,0,0,)$;

$[m_1 = (0,1,-1,1,-1,1)$, $m_2 = (1,0,0,0,0,0)]$

base points: $(0,0,v_i)$, $i,j = 1,2,3$

$v_1 = (0,0,0,0)$, $v_2 = \left(\frac{1}{3},\frac{1}{3},\frac{2}{3},\frac{2}{3}\right)$, $v_3 = \left(\frac{1}{3},\frac{1}{3},\frac{2}{3},\frac{2}{3}\right)$.

θ^3: 4 fixed tori: invariant subspace: $n_1 = (0,1,0,0,1,0)$, $n_2 = (0,0,1,0,0,1)$;

$[m_1 = (0,1,0,-1,1,0)$, $m_2 = (0,0,1,-1,0,1)]$

base points: (v_i, w_j), $i,j = 1,2$

$v_1 = (0)$, $v_2 = \left(\frac{1}{2}\right)$, $w_1 = (0,0,0,0,0)$,

$w_2 = \left(\frac{1}{4},\frac{1}{4},\frac{1}{2},\frac{3}{4},\frac{3}{4}\right)$,

The fixed sets were e.g. calculated in the appendix of [22], however here the conclusion was again that there are 9 tori in the second twisted sector and 16 fixed tori in the third twisted sector. The authors have correctly calculated the vectors n_i of the invariant lattice, but as one can check, from the 9(16) base points they give, groups of 3(4) lie on the same torus, respectively.

4.2. *Description of the Desingularisations*

Let us now count the numbers of (1, 1)-form introduced by the resolutions of the fixed points and the fixed tori singularities. By *Poincaré duality* it is equivalent to count the *irreducible* components of the *exceptional divisors*, which are introduced by the resolution process [18]. Below we review the necessary facts about the resolutions for the kind of singularities we encounter.

[18] See e.g. [18] as a general reference for these concepts of algebraic geometry

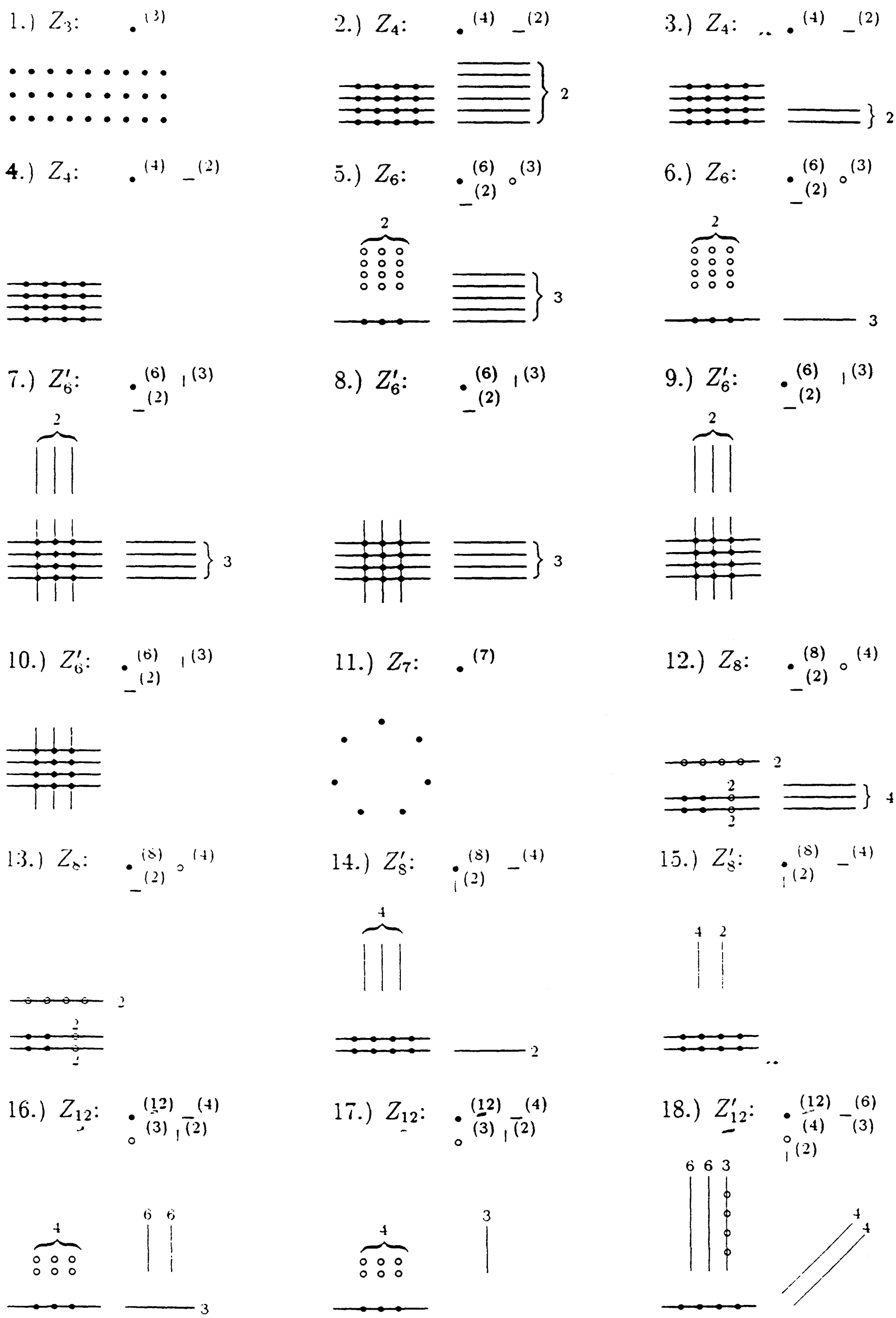


Fig. 1. Schematic configuration of the orbifold singularities. Fixed point singularities are depicted by dots, fixed torus singularities by lines. We indicate the maximal order of the group element under which the sets stay fix in parentheses. The numbers on the sets indicates their multiplicity on the torus

In the case of fixed tori we have singularities due to the action of a discrete Z_k subgroup of G on the $\mathbb{C}^2$ fibres of the bundle normal to the fixed tori [19] T^2. This action can locally always be recasted in the form

$$(z_1, z_2) \mapsto \left(\exp \frac{2\pi i q}{k} z_1, \exp \frac{2\pi i}{k} z_2 \right). \tag{31}$$

The singularity of $\mathbb{C}^2/Z_k$ at the origin can be described by the constraint

$$y^k = xz^{k-q}, \tag{32}$$

in $\mathbb{C}^3$, where in our cases we have always $q = k - 1$. The resolution of this type of singularity, known as the *rational double point* of type A_{k-1}, is given [27] by a Hirzebruch-Jung sphere tree. The number of spheres and their self intersection numbers $-b_i$ in the resolved manifold can be obtained by an Euclidean algorithm [20] [27]. In the cases at hand [21] one gets a sequence of $k-1$ projective spaces $\mathbb{P}_1 \vee \ldots \vee \mathbb{P}_{k-1}$ joined in one point with self intersection numbers $\mathbb{P}_i \cap \mathbb{P}_i = -2$. The resolution replaces the fibers in the normal bundle over a *generic point* on the fixed torus with a sphere tree. It introduces an *exceptional divisor* of the form $T \times (\mathbb{P}_1 \vee \ldots \vee \mathbb{P}_{k-1})$. The new $h_{1,1}$ forms correspond to the number of irreducible components of these exceptional divisors, which is $k-1$.

In the case of fixed points the singularities are locally of the form $\mathbb{C}^3/G$. If G is abelian as in our examples *toric geometry* is a suitable framework to describe $\mathbb{C}^r/G$ singularities and their resolutions. In this sense it allows for a generalisation of the above treatment of the A_{k-1} singularities to higher dimensions. In order to avoid lengthy repetitions we refer to the book of Oda [29] and the appendix of Markushevich in [11] for the precise definitions and proofs of the properties of *convex rational polyhedral cones*, fans and *toric varieties*.

Let $N \simeq \mathbb{Z}^r$ be a lattice in an $\mathbb{R}$ vector space V, which is the completion of N over $\mathbb{R}$, i.e. $V = N \bigotimes_{\mathbb{Z}} \mathbb{R}$, and $n_1, \ldots, n_s$ elements of N. A *strongly convex rational polyhedral cone* with *apex* at the origin O is defined as

$$\tau := \mathbb{R}_0^+ n_1 + \ldots + \mathbb{R}_0^+ n_s = \{a_1 n_1 + \ldots + a_s n_s | a_i \in \mathbb{R}_0^+\}, \tag{33}$$

where $\tau \cap (-\tau) = O$. Let $n^{(1)}, \ldots, n^{(N)}$ be a set of generators of the *semi group* $\mathbf{S}_\tau = (\tau \cap \mathbf{N})$, i.e. every lattice site in $(\tau \cap N)$ can be reached by a linear combination

[19] The resolution of singularities along rational fixed curves was discussed in the context of global actions of finite automorphism on Calabi-Yau spaces defined as complete intersections in (weighted) projective spaces in [26]

[20] The b_i correspond to a representation of k/q as a continued fraction of the following form:

$$\frac{k}{q} = b_1 - \cfrac{1}{b_2 - \ldots - \cfrac{1}{b_s}}$$

[21] The finite subgroups G of $SU(2)$ fall into an ADE classification. For this group the intersection patterns of the spheres in the resolutions of the singularities $\mathbb{C}^2/G$ correspond to the Dynkin diagrams, where points represent $\mathbb{P}'s$ and links represent intersections, i.e. the intersection matrix equals the negative Cartan matrix [28]

of the $n^{(i)}$ with positive integer coefficients and let

$$\begin{aligned} a_{11}n^{(1)} + \ldots + a_{1N}n^{(N)} &= 0\,, \\ \ldots \ \ldots \\ a_{t1}n^{(1)} + \ldots + a_{tN}n^{(N)} &= 0 \end{aligned} \tag{34}$$

be a maximal set of linear relations among the $n^{(1)}, \ldots, n^{(N)}$ over $\mathbb{Z}$. We can define from the above data an *affine toric variety* $X^{(V,N,\tau)}$ as follows:

$$X^{(V,N,\tau)}: = \{(z_1, \ldots, z_N) \in \mathbb{C}^N | F_i(z_1, \ldots, z_N) = 0, i = 1, \ldots, t\}\,, \tag{35}$$

where the $F_i(z_1, \ldots, z_N) = 0$ are *monomial equations* of the form

$$z_1^{a_{i1}} \cdot \ldots \cdot z_N^{a_{iN}} = 1\,. \tag{36}$$

Example 1. $N \simeq \mathbb{Z}^r$ is a lattice in $\mathbb{R}^r$ spanned by n_i, $i = 1, \ldots, r$, $\tau = \mathbb{R}_0^+ n_1 + \ldots + \mathbb{R}_0^+ n_r = (\mathbb{R}_0^+)^r$. Then we have no relation of the type (34) for the generators of the semi-group so $X^{(\mathbb{R}^r,N,\tau)} = \mathbb{C}^r$.

Example 2. Let $N \simeq \mathbb{Z}^2$ be a lattice and $\tau = \mathbb{R}_0^+(kn_1 - n_2) + \mathbb{R}_0^+ n_2$. We have $n^{(1)} = n_1$, $n^{(2)} = n_2$, and $n^{(3)} = (kn_1 - n_2)$ (compare Fig. 2a)), so that $X^{(\mathbb{R}^2,N,\tau)} = \{(x,y,z) \in \mathbb{C}^3 | x^k = yz\}$ is the rational double point of type A_{k-1} encountered above. More generally [11, 29]).

Example 3. Let G be a finite abelian group acting on $\mathbb{C}^r$. Elements $g \in G$ of order k map $(z_1, \ldots, z_r) \mapsto \left(\exp\left[\left(\frac{2\pi i}{k}\right) a_1\right] z_1, \ldots, \exp\left[\left(\frac{2\pi i}{k}\right) a_r\right] z_r \right)$. We have $\mathbb{C}^r/G = X^{(\mathbb{R}^n,N,\tau)}$, with $\tau = \mathbb{R}_0^+)^n$ and the lattice is defined as $N = \bigcap_{g \in G} N_g$, where the lattice N_g is spanned by the minimal set of vectors $n = (n_{(1)}, \ldots, n_{(r)})$ with

$$n_{(1)}a_1 + \ldots + n_{(r)}a_r \equiv 0 \bmod k\,.$$

The constructive power of toric geometry lies in the fact that one can glue together affine toric varieties in a natural way. For this it is convenient to pass first to the dual cone. Denote by $W = V^\vee$ the dual of V and by M the dual lattice to N w.r.t. the bilinear form

$$\langle\,,\rangle : M \times N \to \mathbb{Z}\,. \tag{37}$$

The dual cone $\sigma = \tau^\vee$ is defined by

$$\sigma : \{x \in W | \langle x, y\rangle \geq 0, \forall y \in \tau\}\,. \tag{38}$$

Defining

$$X_\sigma = X_{(W,M,\sigma)}: = X^{(V,N,\tau)} \tag{39}$$

has the advantage that $X_{\sigma \cap \sigma'} = X_\sigma \cap X_{\sigma'}$. This property, which does not hold for $X^\tau = X^{(V,N,\tau)}$, allows to visualize the gluing of toric varieties by the gluing of cones. *Faces* ϕ of a cone σ are subsets which defined via an element of the dual cone n_0,

$$\phi: = \{y \in \sigma | \langle y, n_0\rangle = 0\}\,. \tag{40}$$

Face of strongly convex rational cones are strongly convex rational cones. *Fans* Δ are made by sticking together cones σ such that

1. Every face of any σ is in Δ.

2. Every intersection $\sigma \cap \sigma'$ is a face of σ and σ'.

Let now $X_\sigma = X^{(V,N,\tau)}$ and $X'_\sigma = X^{(V,N,\tau')}$ be affine varieties associated to cones $\sigma, \sigma' \in \Delta$ and $n^{(1)}, \ldots, n^{(N)}$, $n'^{(1)}, \ldots, n'^{(N)}$, generators of the semi-groups $\mathscr{S}_\sigma$ and $\mathscr{S}_{\sigma'}$ respectively. The transition functions between affine coordinates z_i and z'_i are defined by the maximal set of linear relations $j = 1, \ldots, s$ between these generators

$$l_{1j} n^{(1)} + \ldots + l_{Nj} n^{(N)} + l'_{1j} n'^{(1)} + \ldots + l'_{Nj} n'^{(N)} = 0\,,$$

over $\mathbb{Z}$, as

$$z_1^{l_{1j}} \cdot \ldots \cdot z_n^{l_{Nj}} z_1'^{l'_{1j}} \cdot \ldots \cdot z_N'^{l'_{Nj}} = 1\,. \tag{41}$$

Using this transition functions one can associate to a fan Δ a *general toric variety* $X_\Delta := \bigcup_{\sigma \in \Delta} X_\sigma$.

Example 4. Let $M \simeq \mathbb{Z}^2$ be a lattice spanned by $\mathbb{R}^2$. The fan Δ made by sticking together $\sigma_1 = \mathbb{R}_0^+ m_1 + \mathbb{R}_0^+ m_2$, $\sigma_2 = \mathbb{R}_0^+(-m_1, m_2) + \mathbb{R}_0^+ m_2$, and $\sigma_3 = \mathbb{R}_0^+(-m_1 - m_2) + \mathbb{R}_0^+ m_1$ defines $X_\Delta = \mathbb{P}^2$. The affine planes ($\{z_i \neq 0\}$ in homogeneous coordinates) correspond to the X_{σ_i}.

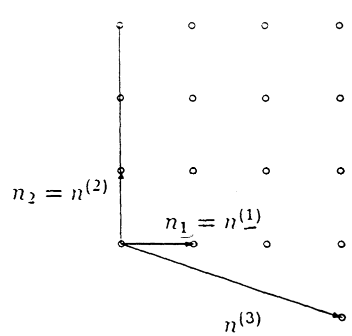
Note that the fan above covers the whole two plane. As shown e.g. in Theorem 1.11 of [29] we have in general that X_Δ is compact if and only if Δ covers W.

Crucial for the resolution of singularities is the following

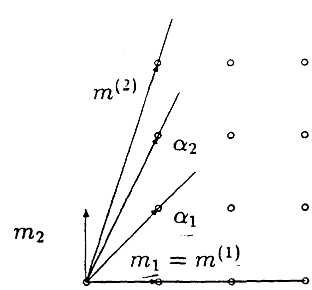
Theorem 1. *The toric variety X_Δ associated to a fan Δ in $M \simeq \mathbb{Z}^r$ is nonsingular if and only if each $\sigma \in \Delta$ is nonsingular in the following sense: $\exists$ a $\mathbb{Z}$ basis $\{m_1, \ldots, m_r\}$ of M such that $\sigma = \mathbb{R}_0^+ m_1 + \ldots + R_0^+ m_s$. Such cones are called basic cones.*

A proof can be found in [29]. If we consider the lattice $M = \mathbb{Z}^2$ and the cone $\sigma = \mathbb{R}_0^+ m_1 + \mathbb{R}_0^+(k m_2 + m_2)$ with $N \doteq M^\vee = M$ and $\tau = \sigma^\vee$ as in Example 2, one sees that one has to *subdivide* σ into a fan of k *basic* cones in order to meet the requirement of Theorem 1. The $k-1$ inner faces spanned by α_i by which this is achieved correpond to the $k-1$ exceptional $\mathbb{P}$ curves necessary to resolve the A_{k-1} singularity, see Fig. 2b) below.

We are interested in the resolutions of singularities of type $\mathbb{C}^3/Z_k$, which have trivial canonical bundle. From this requirement we get the following restriction [11].



a The cone τ



b Subdivision of σ

Fig. 2. Resolution of the A_{k-1} rational double point ($k = 3$)

Theorem 2. *Let $X_{W,M,\sigma}$ be a toric variety. Let σ_i $i = 1, \ldots, r$ a subdivision of σ in the sense of Theorem 1. The canonical bundle of $X_\Delta = X_{W,M,\bigcup_{i=1}^{r}\sigma_i}$ is trivial if and only if the generators of $\mathcal{S}_{\sigma_i}$ lie on a hyperplane H_N in W.*

Moreover the additional lattice vectors α_i, which are needed to define the subdivision, can be associated in a one to one way to the compact divisors on X_Δ.

Now consider the Z_k actions whose exponents, see Table 2, are of the form $\frac{1}{k}(1, a_2, a_3)$ with

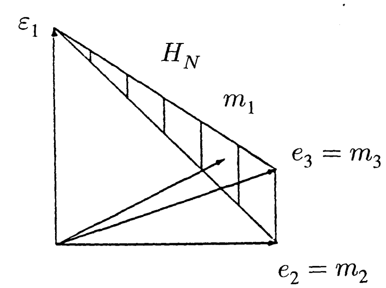
$$1 + a_2 + a_3 = 0 \bmod k\,. \tag{42}$$

In accordance with Example 3 we define the lattices N, M by

$$N = \langle n_1, n_2, n_3 \rangle_{\mathbb{Z}} = \left\langle \begin{matrix} k & -a_2 & -a_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \right\rangle_{\mathbb{Z}},$$

$$M = \langle m_1, m_2, m_3 \rangle_{\mathbb{Z}} = \left\langle \begin{matrix} \frac{1}{k} & 0 & 0 \\ \frac{a_2}{k} & 1 & 0 \\ \frac{a_3}{k} & 0 & 1 \end{matrix} \right\rangle_{\mathbb{Z}}. \tag{43}$$

The cone $\sigma = \langle e_1, e_2, e_3 \rangle_{\mathbb{R}_0^+}$ is spanned by orthonormal vectors as:



Due to (42) the lattice vector $m_1 = \alpha_1$ lies indeed on the hyperplane H_N. The other lattice vectors $\alpha_i = (\alpha_i^{(1)}, \alpha_i^{(2)}, \alpha_i^{(3)})$, which define the resolution according to Theorem 1 can be easily obtained as set of three tuples:

$$\{k(\alpha_{ij}^{(1)}, \alpha_{ij}^{(2)}, |\alpha_{ij}^{(3)}|) = (j \bmod k, ja_2 \bmod k, ja_3 \bmod k) \left| \sum_{i=1}^{3} \alpha_{ij}^{(l)} = k,\ j = 1, \ldots, k-1 \right\}. \tag{44}$$

We have drawn in Fig. 3 the trace of a resolving fan X_Δ in the hyperplane H_N for the 9 different types of fixed point singularities. To obtain a Kählerian manifold we have also to make sure that the corresponding resolutions are *projective algebraic*. Indeed the necessary conditions given in Sect. II.2.3 of [29] are fulfilled.

The dots on the edges and in the interior of the triangles (e_1, e_2, e_3) correspond to α_i and hence to exceptional divisors of X_Δ. In the case of non-prime k the fixed points of the Z_k actions lie always on a fixed torus. Note that the *exceptional divisors* associated to the points on the edges of the triangle coincide with the exceptional

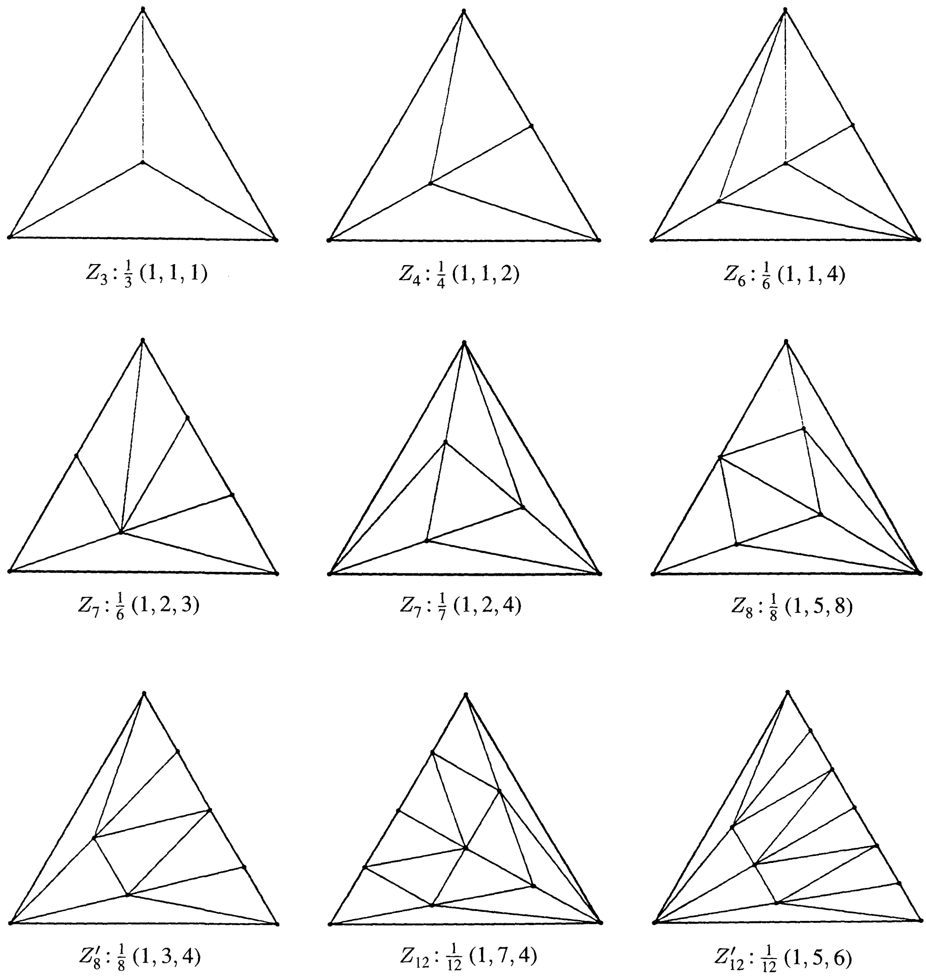


Fig. 3. Desingularisation of the nine types of Z_N singular points

divisors present over the generic points of the fixed torus. From Fig. 1 and 3 we can easily count the exceptional divisors. E.g. consider the Z_{12}' case 18.), here we have two tori fixed under a group action of order three, each gives rise to a divisor $T \times (\mathbb{P} \vee \mathbb{P})$, i.e. we get 4 $(1,1)$-forms from that resolution. The resolutions of three fixed tori of order two and one of order six add 3 and 5 $(1,1)$-forms, respectively. The four fixed points of order four on the Z_2 fixed torus are of type $\frac{1}{4}(1,1,2)$ and hence introduce 4 *new* exceptional divisors (cf. Fig. 3). Likewise the resolution of the four Z_{12}' fixed points on the Z_6 torus give rise to 12 exceptional divisors according to the inner points in the trace of the last fan in Fig. 3.

A basis for the 9 $(1,1)$-forms of the complex torus is given by $dz_j \wedge d\bar{z}_z \; i,j = 1,2,3$ ([18], II.6). From this we see that three $(1,1)$-forms are invariant if all twist exponents are different. If two (three) twist exponents are equal we have five (nine) invariant $(1,1)$-forms. Adding in the above example the three invariant $(1,1)$-forms from the torus we arrive $h_{1,1} = 31$ the number given in Table 5. Similarly the reader might use

Fig. 2 and 3 to count all $(1,1)$-forms of all other Coxeter orbifolds. As $\chi = 2(\chi_{11} - \chi_{21})$ for Calabi-Yau manifolds h_{21} follows. Alternatively the additional $(1,2)$-forms can be obtained using the formulas for the relations between the topological invariants for the singular space T/G, the singular locus and the resolved manifold $\widehat{T/G}$ stated in [30]. Note that in the case of non-prime k these formulas have to be applied in a successive resolution process with one step for each factor subgroup. Concerning the Hodge numbers we disagree in eight of the twelve cases given in Table 1 of [11]. We agree with the Hodge numbers given in [12] for the prime cases Z_3 and Z_7.

While the location of the α_i is fixed by the requirements of Theorem 1 and 2, there is some freedom in the triangulation which lead to the actual choice of the σ_i and hence the resolving space. Namely, whenever there is a quadruple of points $\alpha_{i_1}, \ldots, \alpha_{i_4}$ in general position, the diagonal of the quadrangle can be flipped by an *elementary transformation.*

Given a triangulation the triple intersection numbers of the irreducible hypersurfaces can be calculated [11]. Especially the intersection of three exceptional divisors can be obtained by a simple algebraic prescription [31]. Note that these numbers correspond to the Yukawa couplings in the large radius limit. They can be used to check the conformal field theory results for these couplings. The ambiguity in the triangulation process leads to different couplings. This ambiguity seems to be hard in the sense that it cannot be removed by a field redefinition [13]. From the point of view of conformal field theory it might correspond to an ambiguity in taking the large radius limit in the orbifold moduli space.

5. Conclusions

We have investigated in a systematic way orbifolds with $(2,2)$ world sheet supersymmetry, which can be constructed, by modding out symmetric Z_N actions from the six dimensional torus with vanishing discrete B-field. As a result we give a classification of these types of models. Preceding investigations [11, 14, 16, 22] in this direction have the drawback of being incomplete and more seriously of stating or using incorrect spectra.

The source for the deviations is that properties of the twist, which are *not* directly related to the twist eigenvalues, were not taken into account properly. In the case of the Z_4, Z_6, Z_6', Z_8, Z_8', and Z_{12} actions there exist, for the same twist eigenvalues, inequivalent automorphisms which are realised in different lattices. The complete reducibility of the twist over $\mathbb{C}$, which is used to define the complex planes in the space-time basis, is often confused with the reducibility of the twist over $\mathbb{Z}$, such that the authors erroneously imply that the twist can be made block diagonal in the lattice basis. Consequently their conclusions are only correct if this holds indeed, which is usually *only* for one of the above mentioned inequivalent automorphisms the case. Statements concerning the factorisation properties of the modular group, which have been made in the same spirit, are also wrong in the general cases.

As the impact of the different lattices even on the spectrum was not fully understood, almost all conclusions about these non-prime Z_N orbifolds should be reconsidered. E.g. the couplings stated in [22] should be adapted to the correct spectrum, etc.

Furthermore one should investigate how other phenomenological relevant properties such as the *non-perturbative potential* for the moduli fields or the *threshold* *corrections* to the gauge couplings will change, when the different automorphisms with the same twist eigenvalues are considered.

Acknowledgements. We are grateful to Hans-Peter Nilles and Dimitrios Dais for useful comments and thank Luis Ibáñez for an email note.

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Communicated by S.-T. Yau