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Meromorphic $c = 24$ Conformal Field Theories

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Abstract. Modular invariant conformal field theories with just one primary field and central charge $c = 24$ are considered. It has been shown previously that if the chiral algebra of such a theory contains spin-1 currents, it is either the Leech lattice CFT, or it contains a Kac-Moody sub-algebra with total central charge 24. In this paper all meromorphic modular invariant combinations of the allowed Kac-Moody combinations are obtained. The result suggests the existence of 71 meromorphic $c = 24$ theories, including the 41 that were already known.

1. Introduction

A conformal field theory is characterized by two algebraic structures: the chiral algebra and the fusion algebra. The chiral algebra consists at least of the Virasoro algebra, which in general is extended by other operators of integer conformal weight. The representations or primary fields of the chiral algebra obey a set of fusion rules, determining which primary fields can appear in the operator product of two such fields. In general, both the chiral algebra and the set of fusion rules are non-trivial.

If one is interested in classifying conformal field theories, it seems natural to start with the simplest ones. For example, one might consider theories in which either the chiral algebra or the fusion algebra is as simple as possible. Theories of the former kind form the "minimal series" [1] (whose chiral algebra consists a priori only of the Virasoro algebra) and have been classified completely [2–4]. The theories with the simplest possible set of fusion rules are those with only one primary field **1**, and a fusion rule $\mathbf{1} \times \mathbf{1} = \mathbf{1}$. In such theories the entire non-trivial structure resides obviously in the chiral algebra.

Theories of this kind have extremely simple modular transformation properties [5]. The identity is self-conjugate, and hence the charge conjugation matrix C must be equal to 1. Therefore $S = \pm 1$ and the identity character $\chi(\tau)$ satisfies $\chi\left(-\frac{1}{\tau}\right) = \pm\chi(\tau)$. Choosing $\tau = i$, and noting that the character is a polynomial with positive coefficients in $q = e^{2\pi i \tau}$ so that $\chi(i) \neq 0$, we see that $S = 1$. Furthermore, since $(ST)^3 = C$, we find $T = e^{\frac{2k\pi i}{3}}$, $k \in \mathbf{Z}$. Since $T = e^{2\pi i\left(h-\frac{c}{24}\right)}$ and $h = 0$ it follows that c must be a multiple of 8. The one-loop partition function of such a theory is simply $\chi\chi^*$, where χ is the character of the only representation of the theory. If c is a multiple of 24 the character itself is a modular invariant partition function, and one can consider a corresponding purely chiral conformal field theory. In such a theory all correlation functions are meromorphic, and hence these theories have been called *meromorphic* conformal field theories in [6].[1]

Examples of conformal field theories with just one character are easy to construct. Consider N free chiral bosons whose momenta lie on an even self-dual Euclidean lattice. The one-loop character $\chi(\tau)$ is simply the lattice partition function divided by N η-functions. Using Poisson resummation it is easy to show that this function transforms into itself (up to phases) under both S and T. Hence this character transforms as a one-dimensional representation of the modular group, and $c = N$ must be a multiple of 8, which indeed is necessary in order to have an even self-dual Euclidean lattice.

Even self-dual Euclidean lattices have been classified for dimensions 8 (the root lattice of the Lie algebra E_8), 16 (the root lattice of $E_8 \times E_8$ and that of D_{16} with the addition of a spinor weight), and 24 (the 24 Niemeier lattices [7]). Any of these lattices defines a distinct conformal field theory, and it is natural to ask whether this exhausts the list of meromorphic conformal field theories in these dimensions.

For $c = 8$ and $c = 16$ it is easy to see that this must be true. If there were any meromorphic theories one could use them instead of E_8 or $E_8 \times E_8$ in the construction of the heterotic string. In particular one could construct new modular invariant supersymmetric heterotic strings, to which the relation between modular invariance and cancellation of chiral anomalies of [8] (easily generalized to higher level, see [9]) would apply. Hence any such meromorphic theory would manifest itself in the gravitational anomaly (and the gauge anomaly if there are gauge fields) of the field theory. Since only the two gauge groups $E_8 \times E_8$ and $SO(32)$ were found to be allowed we know that no such theory can exist.

This argument does not apply to the meromorphic $c = 24$ theories. Indeed, several theories are already known that cannot be described by free bosons on Niemeier lattices. The first example is the "monster module" [10], a meromorphic $c = 24$ theory without any spin-1 operators, and which therefore lacks the 24 bosonic operators ∂X that are present in any lattice theory. This theory can be obtained by applying an orbifold $\mathbf{Z}_2$-twist $X \to -X$ to one of the 24 Niemeier lattice CFTs [11, 12]. Such a twist does not directly yield a meromorphic CFT, but a theory with four primary fields E, O, σ_O and σ_E, with spins $0, 1, \frac{3}{2}$ and 2 respectively. The corresponding states form respectively the odd and even states of the original Hilbert space, and the odd and even states of the twisted Hilbert space. The fusion rules of these primary fields may be found in [13]. In particular one has $O^2 = \sigma_{\mathrm{O}}^2 = \sigma_{\mathrm{E}}^2 = E$, showing that all of them are simple currents. Hence if they have integral spin they can be put into the chiral algebra. Putting O into the chiral algebra projects out the twist fields σ and τ, and gives us back the original lattice theory, with character $\chi = \chi_E + \chi_O$. The more interesting possibility is to put σ_{E} into the chiral algebra. This projects out J and σ, and gives a new, meromorphic CFT whose character is $X_E + \chi_{\sigma_{\mathrm{E}}}$. The

[1] This terminology is in fact used in a broader sense in [6]. Throughout this paper we use the adjective "meromorphic" to indicate a "one loop modular invariant meromorphic," or, in the terminology of [6], "bosonic self-dual meromorphic" theory

number of spin-1 operators in this new theory is easy to compute [11], and is given by $\mathcal{N}_{\text{twisted}} = \frac{1}{2}\mathcal{N}_{\text{lattice}} - 12$, where $\mathcal{N}$ denotes the number of spin-1 operators in the chiral algebra.

This twisting procedure can be applied to any of the 24 Niemeier lattices. Although it always yields a different theory, that theory may itself be another lattice theory. In [6] it was shown in a very elegant way, using the theory of codes, that in 15 cases one obtains something new. One of those 15 is the monster module, obtained from the Leech lattice CFT, which has $\mathcal{N} = 24$. Thus we know now altogether 39 meromorphic $c = 24$ theories.

The existence of one more theory can be inferred [14] from the existence of a ten-dimensional (non-supersymmetric) heterotic string theory with Kac-Moody algebra $E_{8,2}$ [15] (here and in the following $X_{m,n}$ denotes an untwisted Kac-Moody algebra of type X, rank m and level n). In [16] a mapping from ten-dimensional heterotic strings constructed by means of the covariant lattice construction (or free complex fermions) to a subset of the Niemeier lattices was described. This produced an easy classification of all ten-dimensional heterotic strings with a rank-16 gauge group. However, the same map takes *any* ten-dimensional heterotic string to some meromorphic $c = 24$ theory. The $E_{8,2}$ theory does not have a rank-16 gauge group and cannot be described by a lattice, but it is still mapped to some $c = 24$ theory. This theory has 384 spin-1 currents forming a $B_{8,1}E_{8,2}$ Kac-Moody algebra, which does not correspond to any twisted or untwisted Niemeier theory. Hence it must be a new item on the growing list of meromorphic $c = 24$ theories.

The last example known to us before starting this work was found more or less accidentally, as a result of a computer search for integer spin extensions of Kac-Moody algebras [17, 14]. A modular invariant of $F_{4,6}$ emerged that was neither a simple current invariant nor a conformal embedding. Although this new theory is not a meromorphic theory, it turned out that its six characters could be combined with the six characters of $A_{2,2}$, so that a meromorphic theory was formed with $\mathcal{N} = 60$. A twisted Niemeier theory exists with the same number of spin-1 currents, but that theory has a Kac-Moody symmetry $(C_{2,2})^6$, and is thus clearly different. This brings the total so far to 41.

The goal of this paper is to complete the list of meromorphic $c = 24$ theories. This goal will indeed be achieved, but under three additional assumptions. First of all our methods require that $\mathcal{N} \neq 0$. It has been conjectured that there is just one theory with $\mathcal{N} = 0$; for counting purposes this will be assumed to be true in the following. Secondly, we will assume that the chiral algebra is generated by a finite number of currents. This is indeed true for all unitary rational conformal field theories we know, and it might be possible to derive this rather than assume it. Finally, we will not really construct conformal field theories, but modular invariant combinations of Kac-Moody characters. It will be shown in the next section that if $\mathcal{N} \neq 0$, then the chiral algebra contains a spin-1 algebra with a total central charge 24. Hence the partition function of any such theory must be a modular invariant combination of Kac-Moody characters, and these combinations will be enumerated completely. There are 69, not including the Leech lattice and the monster module. Barring the unlikely possibility of having two or more distinct conformal field theories per combination (which in any case must all have the same representation content in each excitation level), this limits the set of possible distinct $c = 24$ theories to 71. It remains to be proved that a conformal field theory corresponding to all of these 71 partition functions actually exists. In particular, one would like to write down the operator product algebra of the set of the higher spin fields appearing in the partition function. This is simply an example of the familiar problem of writing down operator products for non-diagonal theories. Methods to address this problem exist and have been applied to various examples, but they are rather laborious, and will not be pursued here. The existence of non-diagonal modular invariant partition functions requires a large number of conditions to be satisfied, and it is difficult to believe that this would be a mere coincidence without having the significance it strongly suggests. For this reason we conjecture that a meromorphic $c = 24$ theory exists for any of the new partition functions.

Explicit constructions exist for the twisted and untwisted Niemeier theories as well as the $B_{8,1}E_{8,2}$ theory (which can be built out of real fermions), but for the second example of [14] only the modular invariant character combination is known.

There are several motivations for attempting to classify the $c = 24$ meromorphic conformal field theories. Originally our interest in this problem was related to the aforementioned relation between this classification and that of ten-dimensional heterotic strings. Indeed, from a list of the $c = 24$ theories one can obtain a list of all $d = 10$ heterotic strings by simply looking for all possible embeddings of $D_{8,1}$. Several $d = 10$ heterotic strings have been constructed by various methods [15, 18, 19], (in particular orbifolding, fermionic and lattice constructions), but none of these methods has any claim to completeness. In a recent paper we have proved completeness for $d = 10$ heterotic strings from a partial classification of the meromorphic $c = 24$ theories [20]. This work made it clear that with some more effort a complete classification of the latter should be possible. Although at present our main motivation for completing the classification is just curiosity, there are many interesting facts related to $c = 24$ that suggest these theories might have their rôle to play (for example in connection with the bosonic string or the intriguing even self-dual Lorentzian lattice $\Gamma_{25,1}$, or in connection with the Monster group). In fact, we hoped that the complete list might reveal an underlying structure that was not apparent from the partial list, but this hope has not been fulfilled so far.

Finally, the new solutions will provide us with interesting information about a class of Kac-Moody modular invariants that is still not understood at all. There are three known methods for constructing in a systematic way extensions of the chiral algebra of Kac-Moody algebras: simple currents, conformal embeddings and rank-level duality. Simple currents [17] yield generalizations of the D-invariants of $SU(2)$.[2] Conformal embeddings $H \subset G$ [27] imply extensions of the chiral algebra of H to that of G [28]. They can be recognized by the presence of spin-1 currents in the extension. Sometimes this method can be applied to embeddings $H_1 \times H_2 \subset G$ to infer the existence of invariants of H_2 from those of H_1. Rank-level duality [29–32] can be viewed as a special case of this, and implies relations between the modular group representations and invariants of the pairs $SU(n)$ level $k \leftrightarrow SU(k)$ level n, $C_{n,k} \leftrightarrow C_{k,n}$ and $SO(n)$ level $k \leftrightarrow SO(k)$ level n.

Unfortunately this is not sufficient to obtain all extensions of Kac-Moody algebras. Only one genuine exception (which cannot be obtained by any combination of these methods) was known so far, namely the $F_{4,6}$ invariant of [14] (note that we are not considering fusion rule automorphisms here). The list of meromorphic $c = 24$ theories yields several additional examples.

[2] In the special case of Kac-Moody algebras, most of these invariants can be obtained by orbifolding with respect to certain extended Dynkin diagram automorphisms, which form a group isomorphic to the center of the Lie algebra [21–24]. The only exception [25, 26] is $E_{8,2}$, which has a simple current that can yield extensions of the chiral algebra in certain tensor products

The methods we used can be summarized as follows. The starting point of the classification is the fact, mentioned above, that the $c = 8$ and $c = 16$ theories can be classified using anomaly cancellation in superstring theory. For $c = 24$ we do not use chiral anomalies of some string theory, but consider directly the same trace-identities that imply anomaly cancellation in superstring theory [8], and which contain in fact far more information. This will be explained in the next section (some of the results have already appeared in [20]). The crucial observation is that the spin-1 currents of any $c = 24$ theory must form a Kac-Moody algebra with $c = 24$, or a product of 24 $U(1)$'s. (Note that this property does not hold for $c > 24$; a trivial counter-example is the monster module tensored with $E_{8,1}$. The trouble is that there could be non-trivial counter-examples as well.) Furthermore all Kac-Moody algebras appearing in a given theory must have the same ratio g/k (where g is the dual Coxeter number and k the level), and g/k can be computed from $\mathcal{N}$, the total number of spin-1 currents. There are only 221 solutions to these three conditions.

This is a very small number in comparison with the number of ways of writing 24 as a sum of central charges of Kac-Moody algebras (not to mention rational $U(1)$'s). Since this simple argument gets us so close to the final answer, it is worthwhile to try to continue. From here on, further reduction of the number of solutions is considerably harder, though.

The next step is to use higher trace identities to rule out accidental solutions. This is a fairly laborious task, but one is finally left with 69 Kac-Moody combinations for which one or more candidates for the second excited level (with 196884 elements) exist that satisfy all trace identities. Now we consider directly the modular invariance conditions for the remaining candidates. This looks hopeless at first, since the total number of primary fields can be huge (for example 5^{12}), and often the number of integer spin fields is much too large as well. Here simple currents come to the rescue. In many cases, we can conclude from the already known results at the second level that certain simple currents of spin 2 are present in the chiral algebra. This implies that some primary fields are projected out, and the remaining ones are organized into orbits. Each independent simple current of order N reduces the number of primary fields by a factor of N^2 (ignoring fixed points). This makes it possible to find the solution. Indeed, for all of the 69 combinations for which a second-level solution exists we find precisely one modular invariant partition function (up to outer automorphisms).

2. Trace Identities

In this section we construct the character-valued partition functions for (unitary, modular invariant) meromorphic conformal field theories with central charge $8n$, $n \in \mathbf{Z}$, and derive the resulting trace identities for $c = 24$.

Consider a unitary meromorphic theory with spin-1 operators, i.e. $\mathcal{N} \neq 0$. Then these operators must have the following product [33, 6]:

$$J^a(z)J^b(w) = \frac{k\delta^{ab}}{(z-w)^2} + \frac{1}{z-w}\, if^{abc}J^c(w) + \text{finite terms}\,.$$

If for some label a all f^{abc} vanish, then J^a generates a $U(1)$ factor. Otherwise the coefficients f^{abc} must be structure constants of a semi-simple Lie algebra, and the current algebra is then a Kac-Moody algebra. The central charge of the Sugawara energy momentum tensor of this algebra is smaller than or equal to the total central charge. The states in the theory are transforming in certain representations of the zero-mode algebra, and this allows us to write down a "character-valued" partition function containing this information:

$$P(q,\vec{F}) = \mathrm{Tr}\, e^{\vec{F}\cdot\vec{J}_0} q^{L_0-c/24}, \tag{2.1}$$

where $\vec{F}\cdot\vec{J} \equiv \sum_a F^a J^a$, and F^a is a set of real coefficients. (In anomaly applications F^a would be a Yang-Mills two-form, but this will not be needed here.)

In general, the Kac-Moody algebra generated by the spin-1 current consists of several simple factors, and the partition function can be expressed in terms of the characters $\chi^\ell_{i_\ell}$ of the ℓ^{th} factor and an unknown function without spin-1 contributions:

$$P(q,\vec{F}_1,\ldots,\vec{F}_L) = \sum_{i_1,\ldots,i_L} \chi^1_{i_1}(q,\vec{F}_1)\ldots\chi^L_{i_L}(q,\vec{F}_L)\chi_{i_1,\ldots,i_L}(q)\,.$$

Now we wish to make use of the modular transformation properties of the theory. For $c=8n$ it transforms with $S=1$ and $T=e^{-2\pi i n/3}$. It is convenient to multiply P with $\eta(q)^{8n}$ to remove the phase in the T transformation. Then the function $\widehat{P}(q,0,\ldots,0)=(\eta(q))^{8n}P(q,0,\ldots,0)$ transforms as a modular function of weight $4n$. Furthermore we know the transformation properties of the Kac-Moody characters [34]

$$\begin{aligned} \tau\to\tau+1: &\quad \chi_i(\tau+1,\vec{F}) = e^{2\pi i(h_i-c/24)}\chi_i(\tau\vec{F})\,,\\ \tau\to-\frac{1}{\tau}: &\quad \chi_i\Big(-\frac{1}{\tau},\frac{\vec{F}}{\tau}\Big) = e^{-\frac{i}{8\pi\tau}\frac{k}{g}\mathrm{Tr}_{\mathrm{adj}}F^2} S_{ij}\chi_j(\tau,\vec{F})\,, \end{aligned} \tag{2.2}$$

where

$$\chi_i(\tau,\vec{F}) = \mathrm{Tr}_i\, e^{\vec{F}\cdot\vec{J}_0} e^{2\pi i\tau(L_0-c/24)}, \tag{2.3}$$

with the trace evaluated over the positive norm states of the representation "i." In (2.2) g is the dual Coxeter number of the Kac-Moody algebra, and we have traded q for τ, with $q=e^{2\pi i\tau}$. The trace in (2.2) is evaluated in the adjoint representation,[3] except for $U(1)$ factors, where one can use any non-trivial representation, provided that k/g is replaced by some normalization N.[4] This normalization turns out to be irrelevant for our purposes.

Using (2.2) and the fact that the $\widehat{P}$ must be a modular function for $\vec{F}=0$, we can derive how it must transform when $\vec{F}\neq 0$. (This is precisely the same argument as was used in [8] to derive the transformation properties of the chiral partition function of heterotic strings, from which one can derive the Green-Schwarz factorization of the anomaly.) One finds

$$\widehat{P}\left(\frac{a\tau+b}{c\tau+d},\frac{\vec{F}}{c\tau+d}\right) = \exp\left[\frac{-ic}{8\pi(c\tau+d)}\mathscr{F}^2\right](c\tau+d)^{4n}\widehat{P}(\tau,\vec{F})\,, \tag{2.4}$$

where we have defined

$$\mathscr{F}^2 = \sum_\ell \frac{k_\ell}{g_\ell}\mathrm{Tr}_{\mathrm{adj}} F_\ell^2\,,$$

with the appropriate modifications for $U(1)$'s as explained above.

[3] Conventions: J_0^a is Hermitian, $f_{abc}f_{abe}=2g\delta_{ce}$. A factor $i/2\pi$ in the usual definition of Chern characters has been absorbed in F

[4] The natural choice is the charge-1 representation, where "charge" is defined to be the eigenvalue of the spin-1 operator "∂X", i.e. the "lattice momentum". In that case $N=1$

To analyse the consequences of these transformation properties we need the Eisenstein functions, for convenience normalized as follows:

$$E_2(q) = 1 - 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n},$$
$$E_4(q) = 1 + 240\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n},$$
$$E_6(q) = 1 - 504\sum_{n=1}^{\infty}\frac{n^5q^n}{1-q^n}.$$

The last two are entire modular functions of weight 4 and 6 respectively, whereas E_2 has an anomalous term in its modular transformation

$$E_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2E_2(\tau) - \frac{6i}{\pi}c(c\tau+d).$$

The anomalous term in the E_2 transformation can be used to cancel the exponential prefactor in (2.4). Indeed, if we define

$$\widetilde{P}(q,\vec{F}) = e^{-1/48E_2(q)\mathscr{F}^2}\widehat{P}(q,\vec{F})$$

we find

$$\widetilde{P}\left(\frac{a\tau+b}{c\tau+d},\frac{\vec{F}}{c\tau+d}\right) = (c\tau+d)^{4n}\widetilde{P}(\tau,\vec{F}). \tag{2.5}$$

Expanding $\widetilde{P}$ in powers of F one finds that the expansion coefficients of terms of order m must be modular functions of weight $4n+m$. To proceed, we need to know that they are regular in the upper half-plane.

The Kac-Moody characters $\chi_\imath(\tau,\vec{F})$ are explicitly known and regular by inspection. However, the functions $\chi_{i_1,\ldots,i_L}$ are characters of an unknown conformal field theory. It can be shown [35] that these characters are regular if the chiral algebra of this unknown theory is generated by a finite number of currents, essentially because this limits the growth of the number of states with increasing level. The unknown conformal field theory is in any case unitary and rational (since the modular group closes on the finite set of characters $\chi_{i_1,\ldots,i_L}$), and all known theories of this type have a finitely generated chiral algebra. It might be possible to prove that this is true in general, but at present the best we can do is assume it. The multiplication with η^{8n} removes the singularity at $q=0$, so that the coefficient functions are entire modular functions of weight $4n+m$ on the upper half-plane *including* $\tau = i\infty$. Basic theorems on modular functions can then be invoked to show that all coefficient functions must be polynomials in E_4 and E_6.

We define the functions $\mathscr{E}_n$ as polynomials in E_4 and E_6 with total weight n. These functions have one or more free parameters. For example $\mathscr{E}_8 = \alpha(E_4)^2$, $\mathscr{E}_{10} = \alpha E_4E_6$, $\mathscr{E}_{12} = \alpha(E_4)^3 + \beta(E_6)^2$, $\mathscr{E}_{14} = \alpha(E_4)^2E_6$, etc. Clearly $\mathscr{E}_{12k+l}$ depends on $k+1$ parameters for $l = 0, 4, 6, 8$ and 10, and k for $l = 2$. An important linear combination of the two weight 12 functions is

$$\Delta(q) = \frac{1}{1728}[(E_4)^3 - (E_6)^2] = (\eta(q))^{24}.$$

The characters out of which P was built are traces over exponentials of the representation matrices of each excitation level. This yields traces of arbitrary order and over different representations. Traces over different representations can always be expressed in terms of traces over some fixed representation (called the reference representation in the following). Furthermore all traces can be expressed in terms of a number (equal to the rank) of basic traces $\mathrm{Tr}\,F^s$, where s is equal to the order of one of the fundamental Casimir operators of the Lie algebra (these are equal to the "exponents" of the Lie algebra plus 1). The reference representation must be chosen so that for all s these basic traces are non-trivial and cannot be expressed in terms of lower-order traces (unfortunately, this often excludes the adjoint representation). In the following all traces will be over the reference representation unless a different one is explicitly indicated.

Thus we arrive at the following expression for the character-valued partition function

$$P(q, F_1, \ldots, F_L) = e^{\frac{1}{48} E_2(q) \mathscr{F}^2} (\eta(q))^{-8n} \sum_{m=0}^{\infty} \sum_i \mathscr{E}_{4n+m}(i) \mathscr{T}_i^m \,. \tag{2.6}$$

Here $\mathscr{T}_i^m$ denotes a trace of total order m, and i labels the various traces of that order. Such a trace has the general form

$$\mathscr{T}_i^m = \prod_{\ell=1}^{L} [\mathrm{Tr}(F_\ell)^{s(\ell,i)}]^{m(\ell,i)} \,.$$

with $\sum_\ell s(\ell, i) m(\ell, i) = m$, and $s(\ell, i) - 1$ is one of the exponents of the $(\ell)^{\mathrm{th}}$ Lie algebra. Each such trace can have a different coefficient function, as indicated by the argument (i) of $\mathscr{E}$.

Since the ground state is a singlet representation of the theory, it does not contribute to any of the higher traces. This fixes some of the parameters in the coefficient functions $\mathscr{E}$. In the absence of the exponential "anomaly" factor this would simply mean that all coefficient functions for $m > 0$ must start with q^1 rather than q^0. However, because of the extra factor this is not true if $\mathscr{T}_i^m$ is a product of second-order traces. In that case the coefficient function must cancel the traces generated by the exponential factor multiplying the $m = 0$ terms in the sum. We can take the required terms out of the coefficient functions $\mathscr{E}$ by rewriting the partition function in the following way:

$$\begin{aligned} P(q, F) = {} & \exp\left(\frac{1}{48} E_2(q) \mathscr{F}^2\right) \eta(q)^{-8n} \\ & \times \left\{ \mathscr{E}_{4n}(0) + (E_4(q))^n \left[\cosh\left(\frac{1}{48}\sqrt{E_4(q)} \mathscr{F}^2\right) - 1 \right] \right. \\ & - (E_4(q))^{n-3/2} E_6(q) \left[\sinh\left(\frac{1}{48}\sqrt{E_4(q)} \mathscr{F}^2\right) \right] \\ & \left. + \sum_{m=2}^{\infty} \sum_i \Delta\mathscr{E}_{4n+m-12}(i) \mathscr{T}_i^m \right\}, \end{aligned} \tag{2.7}$$

where $\mathscr{E}_{4n}(0)$ has a leading term equal to 1. Note that the cosh and sinh terms, when expanded in F, produce coefficient functions that are polynomials in E_4 and E_6 of the correct weight. We can take out a factor Δ from the remaining coefficient functions, because we know that they must be proportional to q. This leaves $\mathscr{E}_{4n+m}/\Delta$, which is an entire modular function of weight $4n + m - 12$ (since Δ has no zeroes). The functions $\mathscr{E}_l$ exist only for $l = 0$ and $l \geq 4$, l even. For all other values that occur in the sum they must be interpreted as 0.

Now consider the first excited level. Expanding (2.7) to second order in F one gets

$$\mathscr{N} + \left(15 - \frac{31}{6}n + \frac{\mathscr{N}}{48}\right)\mathscr{F}^2 + \sum_\ell \alpha_\ell \,\mathrm{Tr}_{\mathrm{adj}}\, F_\ell^2 \,. \tag{2.8}$$

Here α_ℓ is the leading coefficient of $\mathscr{E}_{4n-10}(\ell)$ (times a factor for the conversion from reference to adjoint representation). This term vanishes if $n \leq 3$. Since by construction the first excited level (the spin-1 currents) consists entirely of adjoint representation of the Kac-Moody algebras, the result should be equal to the Chern-character $\mathrm{Tr}\, e^{F.\Lambda}$, where Λ is the adjoint representation matrix. Upon expansion this yields, for non-Abelian algebras

$$\sum_\ell \left(\dim_\ell + \frac{1}{2}\,\mathrm{Tr}_{\mathrm{adj}}\, F_\ell^2\right) . \tag{2.9}$$

For $U(1)$ factors there is no F^2 contribution in (2.9), and any non-trivial representation can be used for the other traces. Comparing (2.8) and (2.9) we get, for non-Abelian algebras

$$\begin{gathered}\sum_\ell \dim_\ell = \mathscr{N} \quad \text{and} \\ \left(30 - \frac{31}{3}n + \frac{\mathscr{N}}{24}\right)\frac{k_\ell}{g_\ell} + \alpha_\ell = 1 \,.\end{gathered} \tag{2.10}$$

For $n > 3$ (i.e $c \geq 32$) the second equation simply determines the coefficients α_ℓ, and one does not learn anything about the possible Kac-Moody algebras. However, for $n \leq 3$ these coefficients are absent, and we get

$$\frac{g_\ell}{k_\ell} = 30 - \frac{31}{3}n + \frac{\mathscr{N}}{24} \,. \tag{2.11}$$

which is independent of ℓ. For $U(1)$ factors the right-hand side of the second equation in (2.10) is zero instead of one, and k_ℓ/g_ℓ is replaced by the non-vanishing normalization constant N_ℓ. Hence in this case we find (if $n \leq 3$)

$$\mathscr{N} = 248n - 720 \,. \tag{2.12}$$

This makes sense only if $n = 3$. Then one finds that $\mathscr{N} = 24$, and substituting this into (2.11) we conclude that any non-Abelian factor that might still be present must have vanishing dual Coxeter number. Since this is not possible, all 24 spin-1 currents must generate $U(1)$'s. This saturates the central charge, and hence the entire theory can be written in terms of free bosons with momenta on a Niemeier lattice. The only such lattice with 24 spin-1 currents is the Leech lattice. Therefore this is the only meromorphic $c = 24$ theory in which Abelian factors appear.

Hence we may ignore $U(1)$'s from here on, and focus on non-Abelian factors. One can find the solutions of (2.12) by determining for each $\mathcal{N}$ the allowed Kac-Moody algebras, and then trying to combine them in such a way that the total adjoint dimension is $\mathcal{N}$. In addition the total Kac-Moody central charge must be less than $8n$. For $n = 1, 2$ we get only four solutions: $E_{8,1}$ for $n = 1$, and $(E_{8,1})^2$, $D_{16,1}$, and $B_{8,1}$ for $n = 2$. However, for $n = 1$ and 2 the number of spin-1 currents is not a free parameter: it must be equal to $248n$, which eliminates the fourth solution.

It is instructive to compute the total Kac-Moody central charge:

$$c_{\rm tot} = \sum_{\ell} \frac{k_\ell \dim_\ell}{k_\ell + g_\ell} = 24 \frac{\mathcal{N}}{248(3-n) + \mathcal{N}},$$

which is valid only if $n \leq 3$. For $n = 3$ we see that the result is always equal to 24, which implies that the Kac-Moody system "covers" the entire theory, and that the unknown part of the theory defined above is necessarily trivial. Our results so far can be summarized as follows:

Theorem[5]. *Let $\mathcal{C}$ be a modular invariant meromorphic $c = 24$ theory whose chiral algebra is finitely generated and contains $\mathcal{N}$ spin-1 currents, with $\mathcal{N} \neq 0$. Then either $\mathcal{N} = 24$, and $\mathcal{C}$ is the conformal field theory of the Leech lattice, or $\mathcal{N} > 24$, and the spin-1 currents form a Kac-Moody algebra with total central charge 24. The values of g/k for each simple factor of this algebra are equal to one another, and given by $\mathcal{N}/24 - 1$.*

This is all that can be learned from the trace identities at the first level. The identities for higher-order traces involve (for $n \geq 3$) always unknown coefficients analogous to α_ℓ above. These coefficients can be determined and then used to compute traces over the second excitation level.

To write down these higher-order trace identities we first need some definitions. The indices $J_{m_1,\ldots,m_r}(R)$ of a representation R of a simple Lie algebra are defined as

$$\mathrm{Tr}_R F^m = \sum J_{m_1,\ldots,m_r}(R) \prod_{i=1}^{r} \mathrm{Tr}(F^{s_i})^{m_i},$$

where the traces on the right-hand side are over the reference representation, and $\sum_i m_i s_i = m$. Here r is the rank of the Lie algebra, and the sum is over all combinations of basic traces with the correct total order m. Note that with this definition the indices depend on the reference representation. For our purposes it will be sufficient to consider the coefficients $J_{m,0,\ldots,0}$, i.e. the coefficient of $(\mathrm{Tr}\, F^2)^m$. In a tensor product of L Kac-Moody algebras we will denote the coefficient of $(\mathrm{Tr}(F_1)^2)^{n_1} \times \ldots \times (\mathrm{Tr}(F_L)^2)^{n_L}$ for a representation $R = (R, \ldots, R_\ell)$ as $K_R(n_1, \ldots, n_L)$. Thus

$$K_R(n_1, \ldots, n_L) = \prod_{\ell=1}^{L} J_{n_\ell,0,\ldots}(R_\ell).$$

[5] For the special case of simple laced, level-1 Kac-Moody algebras (yielding even self-dual lattices) this result has been proved by Venkov [36], who also observed that all the solutions to these conditions correspond precisely to the Niemeier lattices

The second-level trace identities can now be derived from (2.7). After a rather lengthy computation we get

$$\sum_R K_R(n_1,\ldots,n_L) = \left[\prod_{\ell=1,n_\ell\neq 0}^{L} \frac{(2n_\ell-1)!}{2^{n_\ell-1}(n_\ell-1)!}\left(\frac{k_\ell}{2N_\ell}\right)^{n_\ell}\right] \times\left[C_P - \sum_{\ell=1}^{L}\sum_{k=1}^{n_\ell}\frac{2^{k+1}n_\ell!}{(n_\ell-k)!(P+k-1)!B_{2k}}\left(\frac{2N_\ell}{k_\ell}\right)^k C_{k,\ell}\right], \tag{2.13}$$

which is valid if the total order, $P = \sum_\ell n_\ell$, is smaller than or equal to 5. The sum on the left-hand side is over all representations appearing at the second excitation level. The identity is valid for any (non-trivial) choice of reference representation. The dependence on this choice enters via the exponential "anomaly" factor in (2.7), and manifests itself through the normalization constants N_ℓ. They are defined by the quadratic trace of the reference representation matrices Λ_ℓ in the ℓ^{th} group

$$\mathrm{Tr}\,\Lambda_\ell^a\Lambda_\ell^b = 2N_\ell\delta^{ab}\,.$$

If one chooses the adjoint representation one must set $N_\ell = g_\ell$ (the adjoint is a valid choice as long as only quadratic traces appear). The coefficients $C_{k,\ell}$ are the indices of the adjoint representation in the ℓ^{th} factor, e.g. $C_{k,1} = K_{\text{adj}}(k,0,\ldots,0)$, with respect to the reference representation. The coefficients C_L in (2.13) are respectively equal to 196884, 32760, 5040, 720, 96, and 12 for $P = 0,1,2,3,4,$ and 5, where P is the total order of the trace, $P = \sum_\ell n_\ell$. Finally, B_{2k} are the Bernoulli numbers.

There is a trace identity of order P whenever the function $\mathscr{E}_{2P}$ has one (or fewer) parameters. Hence one expects also an identity for $P = 7$. This one is more subtle, since one has to cancel the undetermined parameter of $\mathscr{E}_{12}$ by subtracting traces of order 12. The result is

$$12K(n_1,\ldots,n_L) - \sum_{\ell=1,n_\ell\neq 0}^{L}\frac{k_\ell}{2N_\ell}n_\ell(2n_\ell-1)K(n_1,\ldots,n_{\ell-1},n_\ell-1,n_{\ell+1},\ldots,n_\ell)$$
$$= \left[\prod_{\ell=1,n_\ell\neq 0}^{L}\frac{(2n_\ell-1)!}{2^{n_\ell-1}(n_\ell-1)!}\left(\frac{k_\ell}{2N_\ell}\right)^{n_\ell}\right] \times\left[-8+\sum_{\ell=1}^{L}\sum_{k=1}^{n_\ell}\frac{2^{k+1}n_\ell!(k-6)(k+5)}{(n_\ell-k)!(k+6)!B_{2k}}\left(\frac{2N_\ell}{k_\ell}\right)^l C_{l,\ell}\right]. \tag{2.14}$$

Once the correct linear combination for the left-hand side has been determined the expression on the right-hand side can be derived from (2.13), which is still valid for $P = 6$ and $P = 7$, except that the coefficients depend on an undetermined parameter α. Parametrizing $\mathscr{E}_{12}$ in a certain way one gets for example $C_6 = \frac{9}{4} - \alpha$, $C_7 = \frac{31}{48} - \frac{7}{12}\alpha$. The parameter α cancels if one combines the seventh- and sixth-order traces as indicated above.[6]

[6] There is in fact a separate free parameter for each distinct subtrace of order 12. The precise from of the left-hand side of (2.14) is obtained by requiring the cancellation of all these parameters

As already mentioned, these identities hold independent of the choice of reference representation. For example, the lowest-order trace identity reads

$$\sum_R K_R(1,0,\ldots,0) = \frac{k_1}{2N_1}\left[32760 - 24\left(\frac{2N_1}{k_1}\right)C_{1,1}\right].$$

If the reference representation is the adjoint representation, then $C_{1,1} = 1$ and $N_1 = g_1$ and the right hand-side becomes $\frac{k_1}{2g_1}[32808 - 2\mathcal{N}]$; if for example the first Kac-Moody factor is of type A_n and we choose the vector representation, then $N = \frac{1}{2}$ and $C_{1,1} = 2g_1$. Now the right-hand side is larger than before by a factor $2g_1$, but the same is true for all indices on the left-hand side.

For higher-order traces this independence is less manifest. The indices J can be computed by means of the symmetric invariant tensors of the Lie algebra [37]. There is one such tensor for each exponent, and they are uniquely defined, up to normalization, if one requires that their contraction with all lower-order tensors should vanish. For example to fourth order one has (ignoring odd traces):

$$\begin{aligned}\mathrm{Tr}\,\Lambda^a\Lambda^b &= I_2(R)g^{ab}\,,\\ \mathrm{Tr}\,\Lambda^{(a}\Lambda^b\Lambda^c\Lambda^{d)} &= I_4(R)g^{abcd} + I_{2,2}(R)g^{(ab}g^{cd)}\,,\end{aligned} \tag{2.15}$$

where Λ^a is a representation matrix of an irreducible representation R, and the round brackets denote symmetrization with weight 1. The second-rank invariant tensor g^{ab} can be chosen equal to δ^{ab} by a suitable basis choice. In this basis tensor g^{abcd} is traceless, and is fully determined by fixing the value of $I_4(R)$ for one representation. A general expression is known for $I_{2,2}$:

$$I_{2,2}(R) = \frac{3I_2(R)^2}{D+2}\left[\frac{D}{\dim(R)} - \frac{1}{6}\frac{I_2(\text{adj})}{I_2(R)}\right],$$

where D is the dimension of the adjoint representation. The indices $I_2, I_{2,2}$ and I_4 are closely related to the indices $J_{1,0,\ldots,0}$, $J_{2,0,\ldots,0}$, and $J_{0,1,0,\ldots,0}$ (or $J_{0,0,1,0,\ldots,0}$ if there is a trace of order 3, which, for obvious notational purposes, we will from now on assume that there is not) defined above, but they are not quite the same. Note in particular that $I_{2,2}$ does not depend on a choice of a reference representation.

To compute the indices J one can choose some reference representation, contract all indices in (2.15) with vectors F_a, and then solve for F_2 and $g^{abcd}F_aF_bF_cF_d$ in terms of the indices of the reference representation. Then one can express the traces over all other representations in terms of those of the reference representation, and read off the indices J. This yields

$$\begin{aligned}J_{1,0,\ldots,0}(R) &= \frac{I_2(R)}{I_2(\text{ref})}\,,\\ J_{0,1,0,\ldots,0}(R) &= \frac{I_4(R)}{I_4(\text{ref})}\,,\\ J_{2,0,\ldots,0}(R) &= I_{2,2}(R)\left(\frac{I_2(R)}{I_2(\text{ref})}\right)^2 - I_{2,2}(R)(\text{ref})\frac{I_4(R)}{I_4(\text{ref})}\,.\end{aligned} \tag{2.16}$$

The dependence on the reference representation is partly through the normalizations $I_2(\text{ref})$, $I_4(\text{ref})$ and $(I_2(\text{ref}))^2$, which cancel as explained above. The main complication is that the last formula contains an extra term. However, this term is proportional to $I_4(R)$ and contains no other dependence on R. Since (2.13) must hold independently of the choice of reference representation, both terms must satisfy separate trace identities. From this we may conclude that the terms proportional to I_4 (appearing on both sides of (2.13) because K_R as well as $C_{2,\ell}$ are modified) must satisfy the trace identity for $J_{0,1,0\dots,0}$. This can easily be checked explicitly. Furthermore the term proportional to $I_{2,2}$ must satisfy the $J_{2,0,\dots,0}$ trace identity even when the second term in (2.16) is omitted.

This gives us one method for computing the left-hand side of (2.13) for $P = 2$. If the third index vanishes (which is true for all Lie algebras except A_n, $n \geq 2$) there is also a formula for $I_{2,2,2}$, which may be used instead of (but is not equal to) $J_{3,0,\dots,0}$ for analogous reasons:

$$I_{2,2,2}(R) = \frac{15 I_2(R)^3}{(D+2)(D+4)}\left[\left(\frac{D}{\dim(R)}\right)^2 - \frac{1}{2}\frac{I_2(\text{adj})}{I_2(R)}\frac{D}{\dim(R)} + \frac{1}{12}\left(\frac{I_2(\text{adj})}{I_2(R)}\right)^2\right].$$

This allows us to use trace identities for $P \leq 3$.

A method for computing the indices J directly is to use Chern characters. For example in A_n Lie algebras the Chern characters of the anti-symmetric tensor representations can be expressed easily in terms of the Chern character of the vector representation. Suppose the Chern characters are known for some set of representations $\mathscr{S}$. Now tensor each element of $\mathscr{S}$ with one of the antisymmetric tensor representations. If only one new representation appears in the product, one can compute its Chern character and enlarge the set $\mathscr{S}$ (Chern characters are multiplied for tensor products and added for direct sums). It is easy to see that for A_n this procedure will yield all representations. For other algebras we are already able to go to sixth-order in F ($P = 3$), which turns out to be sufficient.

In these computations one has to take into account the vanishing relations due to the non-existence of certain fundamental traces. For A_n these are the traces of order larger than the rank plus one. To remove them one starts by expanding the Chern character of the vector representation up to the required order, and then on substitutes the vanishing relations. This yields a polynomial involving only fundamental traces, whose coefficients are the indices J of the vector representation (the natural choice for the reference representation). To obtain the indices of all other representations is then a matter of straightforward multiplication and addition of polynomials. The vanishing relations, as well as a more detailed account of this method, may be found in [8].

Our strategy is now to compute the right-hand side of (2.13), and then try to match it with the traces of some set of irreducible representations on the left-hand side. This set of representations consists of the descendants of the ground state and the spin-1 states, plus a choice of the spin-2 primary fields of a given Kac-Moody combination. Equation (2.13) yields diophantine equations for the multiplicities of these primary fields, which must be positive integers. If these equations do not have a solution, there cannot exist a modular invariant partition function or the combination under consideration.

The main purpose of this method is to rule out "fake" solutions to the first level conditions. These conditions (summarized in the theorem above) yield 221 solutions. Especially for small groups such as $SU(2)$, accidental solutions should certainly be anticipated. One may hope to eliminate them by means of second-level trace identities. The main advantage of using the trace identities instead of directly checking the conditions for modular invariance is of course that we only have to deal with spin-2 currents, whereas in the latter case all integer spin currents need to be included. However, in many cases the number of spin-2 currents is still much too large. For example, one of the 221 combinations is $(A_{1,16})^9$. This has $(17)^9$ primary fields, of which 581820 have spin 2. The number of equations we have at our disposal to determine all these variables is "only" 8437. Clearly this is still untractable. Note that since only A_1 Kac-Moody algebras appear, whose only fundamental trace is the quadratic one, there is no chance to get more equations than we already have.

The large number of primary fields in this (and many other examples) turns out to be due to large combinatorial factors arising from permutations of identical Kac-Moody factors. The solution to this problem is to sum over all permutations of the orders $n_1, \ldots, n_L$ of the equations within identical Kac-Moody factors. In the present example, this reduces the number of equations to just 34, but it also reduces the number of variables, since fields that are permutations of each other become indistinguishable from the point of view of the symmetrized equations. Note that the equations are already insensitive to the difference between (complex) conjugation and $SO(8)$ triality. Conjugation forms, together with permutation of identical factors and triality, the group of outer automorphisms of the Dynkin diagram of the Lie algebra. By symmetrizing the equations we are thus identifying all representations related to one another by outer automorphisms. In the example, the number of variables is reduced to 62. This would still be too much for real variables, but since they are positive integers the situation improves drastically. In this case we find that no positive integer solution exists. Of course, if solutions do exist, we still have to disentangle the symmetrization.

For all 221 combinations this computation is now manageable. The maximal number of variables that occurs is 288. Typically, the number of equations is roughly the same or much larger than the number of variables. A computer was used to solve these equations, but no limit was imposed on the size of the integer coefficients (of course there is an absolute maximum, namely 196884). We are finally left with 69 combinations for which there are solutions to all equations considered, including of course the 39 known cases.

3. Modular Invariance

Since "accidental" solutions to all trace identities are highly improbable, we expect modular invariant partition functions to exist in all 69 cases. Therefore it is not worthwhile to consider trace identities for spin-3 currents. Although, given the representation content of the second level, the spin-3 currents have to satisfy even more equations (traces of order up to 26 can be used), the computations become forbiddingly complicated, and would anyhow not settle the existence of modular invariants definitively.

In principle, the conditions for modular invariance are much simpler than the trace identities. The partition function has the form

$$\mathscr{P}(\tau) = \sum m_i \chi_i(\tau)\,, \tag{3.1}$$

where χ_i is a combination of Kac-Moody characters for the combination of fields labelled by i. Invariance under T implies that i must have integer spin, and invariance under S that the positive integers m_i must be an eigenvector of S with eigenvalue 1. The obvious strategy for solving this is to enumerate all integer spin fields, and then solving the set of linear equations $\sum_i S_{fi} m_i = 0$, where f is a fractional spin field. The number of fractional spin fields is much larger than that of integer spin ones, so that there is no lack of equations. (In all cases considered, the solutions m_i turn out to satisfy also the remaining equations, $\sum_i S_{ji} m_i = m_j$, where j has integral spin.)

The problem with this approach is again the large number of variables that occur in certain cases due to permutations. Now symmetrization does not help, since this does not determine m_i completely and does not even settle the existence of a solution, only its non-existence. The solution to this problem is to make use of simple currents.

Simple currents are primary fields J whose fusion rules with any primary field yield just one field. Obviously this organizes the set of fields into orbits, and it also assigns charges to all fields. One would expect the presence of a simple current in the chiral algebra to greatly reduce the amount of work needed to determine the rest of the algebra, since this effectively reduces the number of primary fields by $\frac{1}{N^2}$, where N is the order of the current. One factor of N is due to the fact that fractionally charged fields are projected out, and the second one is due to combining N fields into a single one, a primary field of an algebra which has been extended by J (this counting argument is modified if the current has fixed points). The idea is now to consider the spin-2 content of the theory, previously determined from the trace identities, and check whether any of those fields are simple currents. This knowledge can then be used to simplify the search for modular invariants.

First, some of the previous intuitive statements have to be made more precise. Consider thus a partition function built out of characters of some CFT, as in (3.1), and suppose that one of those characters corresponds to a simple current J. Closure of the chiral algebra implies that J, acting on any other current in the algebra, must yield another such current. This immediately rules out fields with fractional charge with respect to J, since in that case J changes the conformal weight by a fractional amount, leading to a violation of T-invariance. Now we prove

Theorem. *Suppose that a simple current J appears in a modular invariant of the form* (3.1) *with multiplicity $m_J > 0$. Then m_i is constant on the orbits of J.*

Proof. Define $m'_i = m_{Ji}$, where Ji denotes the field obtained from i by the action of J. In the presence of simple currents, the matrix S satisfies [38, 39]

$$S_{j,Ji} = e^{2\pi i Q(j)},$$

where $Q(j)$ is the charge of the field j. Modular invariance implies

$$m_i = \sum_j S_{ij} m_j \,.$$

The summation index j is equal to Jk for some other field k, uniquely determined by j. Thus we get

$$m_i = \sum_k S_{i,Jk} m_{Jk} = e^{2\pi i Q(i)} \sum_k S_{i,k} m'_k \,.$$

Now we make use of the fact that J appears in the algebra. This implies that all other fields in the algebra must have integer charge, i.e. either $m_i = 0$ or $Q(i) \in \mathbf{Z}$. Hence the phase in the foregoing equation may be omitted, and we get

$$m'_k = \sum_i S^{-1}_{k,i} m_i = m_k \,,$$

because of modular invariance of m. Hence m is invariant under J-shifts, i.e. m must be constant on the J-orbits.

An immediate consequence is that the multiplicity of J itself must be equal to that of the identity, i.e. equal to one. (The generalization of this result to arbitrary modular invariants $\chi_i M_{ij} \chi_j^*$ is the statement that $M_{ij} = M_{J_L i j} = M_{i, J_R j}$, if J_L (J_R) is a current in the left (right) chiral algebra. This was proved in a somewhat different way in [40], but only in the case where the chiral algebra exists only out of simple currents, which is not true here.)

A similar result applies to charge conjugation. Since $S^2 = C$, the charge conjugation matrix, one gets immediately $Cm = S^2 m = m$, so that the coefficients are invariant under simultaneous charge conjugation in all Kac-Moody factors.

The knowledge that each simple current can appear only once is usually enough to reconstruct the set of spin-2 simple currents from the symmetrized multiplicities. In a few complicated cases it was helpful (though probably not necessary) to determine the spin-2 multiplicities with less than maximal symmetrization. Having determined a set of simple-current orbits that must appear in the chiral algebra, we eliminate all fractional charge fields, and rewrite the equations for m_i as follows :

$$\sum_{i_0} S_{f,i_0} N_{i_0} m_{i_0} = 0 \,,$$

where f is an integral charge, fractional spin fixed, and the sum is over all orbits of integral charge, integral spin fields. Each such orbit is represented by one field i_0, and N_{i_0} is the number of fields on an orbit. It should be emphasized that S is the original Kac-Moody modular transformation matrix, and *not* the matrix of the theory with a chiral algebra extended by simple currents. The latter is in general not easy to determine because of fixed points.

An illustrative example is $(A_{2,3})^6$. This combination has 10^6 primary fields and 6819 spin-2 currents, which are permutations of only 9 distinct primary field combinations. The unique solution to the trace identities for the spin-2 fields is

$$30 \times [(3,0)^2(0,0)^4] + 15 \times [(1,1)^4(0,0)^2] + 30 \times [(0,2)^2(0,1)^4] + 12 \times [(1,2)(0,1)^5] \,,$$

where (n, m) are A_2 Dynkin labels, and the square brackets denote a representative from a set of fields identified under charge conjugation and permutation. The result tells us to select 30 elements of the set of 60 spin-2 simple currents (i.e. 15 permutations of $[(3,0)^2(0,0)^4]$ times a factor 4 from charge conjugation). Requiring locality of the currents with respect to each other, and using the fact that each of them can appear only once, one easily determines the solution, which is unique up to conjugation in each $A_{2,3}$ factor. It consists of all permutations of $[(3,0)^2(0,0)^4]$ plus their conjugates. The simple currents generate a $(\mathbf{Z}_3)^5$ subgroup of the center, and the "naïve" estimate of the number of integral charge orbits is thus $10^6/3^{10} \approx 17$. Because of fixed points the actual number is somewhat larger (53), including 9 integral spin orbits. Solving the equations for these 19 variables is easy, and one finds that all occur with multiplicity one, except the fixed point field $(1,1)^6$, which occurs with multiplicity 6.

In many cases we find that after taking into account the simple currents, all remaining orbits occur in the algebra with multiplicity 1. There are several exceptions where some orbits do not appear, and a few with multiplicities larger than 1. This happens only for orbits that are fixed points of the simple currents, and can be interpreted as follows. If one were to extend the original Kac-Moody algebras with the simple currents, the new diagonal invariant has multiplicities N_0/N_f on its diagonal, where N_0 is the length of the identity orbit, and N_f the length of the fixed-point orbit. The wellknown interpretation is that the extended theory contains N_0/N_f fields corresponding to this term in the modular invariant. In the example discussed above there are thus 243 fields in the "intermediate" theory with representation $(1, 1)^6$. When the algebra is extended further, six of those fields appear in the chiral algebra. From the point of view of the intermediate theory, these are however distinct fields (or at least that is a logical possibility; to check this, one has to construct the matrix S of the intermediate theory by resolving the fixed points, which is not an easy task). All occurrences of higher multiplicities in the 69 solutions are consistent with this interpretation.

In some cases there are no simple currents with spin 2 (for example $C_{4,10}$ has a simple current of spin 10). Luckily, in those cases it turned out to be possible to determine the m_i's completely without reducing the set of primary fields by means of simple currents.

For all 69 solutions of the trace identities we wish to prove existence as well as uniqueness (up to equivalence) of the meromorphic modular invariant. No further work is needed for one third of the solutions, for which both of these features follow from the work of Niemeier. For the 14 theories corresponding to $\mathbf{Z}_2$ Niemeier lattices existence has been proved in [12], but this does not rule out the existence of other partition functions for the same Kac-Moody combination.

Uniqueness holds in general only up to the outer automorphisms described above. Furthermore, we often find additional solutions with spin-1 currents in the chiral algebra.[7] Their presence implies an enlargement of the Kac-Moody algebra of the theory, with the original theory conformally embedded in the new one. Clearly these invariants should not be counted as separate theories, since they will be encountered again when the enlarged Kac-Moody algebra is studied directly. Note that conformal embeddings will never appear as solutions to the trace identities, because they were derived under the assumption that there are no additional spin-1 currents.

Apart from outer automorphisms and conformal embeddings, we have found exactly one modular invariant partition function of the form (3.1) for each of the 69 combinations.

4. Results

The complete results are listed in the table. Columns 1–3 are self-explanatory. Column 4 lists the simple current orbits that appear in the chiral algebra. Since the simple currents form an Abelian group under addition, it is sufficient to list a set of generators of this group. In all cases except D_n, the simple currents generate a Z_M group, and the elements J^m of this group are labelled by m. In algebras of type $A_{n,k}$, J is chosen to be the field with Dynkin labels $(k, 0, \ldots, 0)$; in $E_{6,k}$ the one with Dynkin labels $(k, 0, 0, 0, 0, 0)$; B_n, C_n, E_7, and $E_{8,2}$ have only one non-trivial simple current, and $G_2, F_4, E_{8,k}$, $k \neq 2$ have none. Finally the D_n simple currents are denoted as v, s, or c if their Dynkin labels are k times those of the vector, spinor or conjugate

[7] In particular all combinations with $\mathcal{N} = 48$ have al least one meromorphic modular invariant of this type, since one can easily show that they can all be embedded conformally in the D_{24} Niemeier CFT

spinor representation. The simple currents appearing in the case of simply laced level-1 algebras (i.e. even self-dual lattices) were taken from [41], where they were called *glue vectors*. We have adopted this terminology also for the other combinations in the table.

Column 5 contains the orbits and the multiplicity with which each orbit appears. Note that sometimes the choice of orbit representative may hide symmetries and other relevant features.

Some notation has been introduced to deal with permutations. By $\{A_1; A_2; \ldots; A_K\}$ we mean all permutations of the entries separated by semicolons; $\{\ \}_E$ means even permutations only, and [ ] means all cyclic permutations; $(\ )^n$ means of course that the entry between round brackets is to be repeated n times. In principle, if there is more than one Kac-Moody factor, their simple currents and representations are separated by commas and enclosed in round brackets. However, these symbols are omitted when no confusion is possible.

Finally, the last column indicates where a certain conformal field theory or modular invariant partition function has appeared before. (Although the twisted Niemeier theories have been listed in [12], this paper does not contain the partition functions.)

A few cases require a separate discussion:

No. 0 This theory, the "monster module," can of course be described in terms of a $\mathbf{Z}_2$-twisted Leech lattice. Apparently no explicit expression is known giving its modular invariant partition functions in terms of simpler theories, although one would expect this to be possible.

No. 1 The Leech lattice [42] can be described as a modular invariant partition function of 24 copies of "D_1," putting it on a more or less equal footing with the other Niemeier lattices. An example of a set of simple currents yielding the Leech lattice can be found in [43], although a simpler presentation might be possible. In principle one can obtain various representations of the Leech lattice from the trace identities. We have investigated this by solving the equations for just one $U(1)$ factor, allowing different radii for the $U(1)$. The trace identities are $((2N+1)!! = 1 \times 3 \times 5 \times \ldots \times 2N+1)$

$$\sum_{q=0}^{n} M_q \left(\frac{q}{\sqrt{2n}}\right)^{2P} = C_P(2P-1)!! \quad (P = 0, \ldots, 5)$$

and

$$\sum_{q=0}^{n} M_q \left[12\left(\frac{q}{\sqrt{2n}}\right)^{14} - 91\left(\frac{q}{\sqrt{2n}}\right)^{12}\right] = -8 \times 13!!\,,$$

where C_P are the coefficients appearing in the trace identities (2.13), and n defines the radius, in such a way that one obtains a rational $U(1)$ with $2n$ primary fields with charges $q/\sqrt{2n}$, $q = -n+1, \ldots, n$. Note that opposite charges give the same contribution to the traces, so that the sums can be reduced to half this range. (One can reduce them even further by requiring that the conformal weights do not exceed 2, i.e. $q \leq 2\sqrt{2n}$.) For $n = 1$ these identities are not valid because they do not take into account the charge $\pm\sqrt{2}$ states (the $SU(2)$ roots) appearing at the first level; likewise, for $n = 2$ there are charge ± 2 ($q = 4$) descendants at the second level, which must be included in the sum on the left-hand side, with multiplicity M_4 equal to 2 (i.e. 1 for each charge). For larger n the first such descendant appears at level n, and does not affect the argument. One may try to solve the equations to obtain the unknown multiplicities M_q of the primary fields. For $n \leq 10$ we always found one or more solutions (since the number of contributing fields increases with n it does not make much sense to consider larger values). For example, for $n = 2$ the equations are satisfied with $M_0 = 93474$, $M_1 = 94208$, and $M_2 = 9200$. This can be inverted to obtain a somewhat strange closed formula for the coefficients C_P, $P \geq 1$:

$$C_P(2P-1)!! = 92(100 + 2^{10-2P}) + 2^{2P+1}.$$

The fact that solutions exist for larger values of n indicates that there are many other ways to write the Leech lattice as a product of rational $U(1)$'s. Constructing some of these by considering traces over more than one $U(1)$ factors might be feasible; but proving that all solutions are in fact the same, up to rotations, is certainly impossible with these methods alone. Fortunately, this is already known by other arguments.
No. 2 This was one of the most difficult cases, owing to the presence of a large number of fixed-point fields. The solution is, with the set of simple currents listed in the table:

$$(0)^{12} + (1)^{11}3 + \{(2;)^9(0;)^3\} + 12 \times (2)^{12} + 132 \quad \text{chosen from} \quad \{2^6 0^6\},$$

i.e. all 220 permutations of the spin-3 fields $(2)^9(0)^3$ appear once, but only $\frac{1}{7}$ of the permutations of $(2)^6(0)^6$. Using identities symmetrized on subsets one can show that all 132 must be different, but the hard problem is to select them. The answer can be characterized as follows. Take the set of 11 vectors

$$1[0;0;1;0;0;0;1;1;1;0;1],$$

and generate all 113 vectors obtained by adding them modulo 3. One then obtains 729 different vectors, each appearing with a multiplicity 243. Now replace the non-vanishing entries of all vectors (which are either 1 or 2) by the $SU(2)$ fixed-point field (2), and divide by 2 the multiplicities of all these fields, except the identity. In this way one obtains 12 copies of $(2)^{12}$, all 220 permutations of $(2)^9(0)^3$, and 132 different permutations of $(2)^6(0)^6$ ($2 \times (132 + 220 + 12) + 1 = 729$). This is the solution, up to permutations of the $SU(2)$ factors. The description given here was obtained by working out the $\mathbf{Z}_2$ twist of the $(A_2)^{12}$ Niemeier lattice.[8] However, this was only used to obtain a presentable description of the 132 spin-2 fields. Uniqueness was proved by solving the modular invariance conditions, as in all other cases.
No. 5 As in the previous case, the only complication is the determination of combinations of $SU(2)$ fixed points. The answer is

$$(0)^{16} + 8 \times (1)^{16} + J\mathscr{S},$$

where $\mathscr{S}$ is the set of 30 vectors

$$n_0(1)^{16} + n_1(0)^8(1)^8 + n_2((0)^4(1)^4)^2 + n_3((0)^2(1)^2)^4 + n_4(01)^8 \bmod 2,$$

with n_i defined modulo 2, with at least one of the n_i, $1 \geq i \geq 4$ equal to 1. As indicated, the vector entries (which of course are $SU(2)$ Dynkin labels) are added modulo 2. The combination $(0)^8(1)^8$ has spin $\frac{3}{2}$. A spin-2 field is obtained by acting once with the $SU(2)$ simple current J on one of the identity components. The simple

[8] The spin-2 field $(1)^{11}3$, 12-fold degenerate because of the simple current action on it, plays the rôle of one of the twist fields, referred to as σ_E in the introduction

currents in the chiral algebra imply that it does not matter in which factor one performs this action.

No. 12 The notation in column 5 uses a double cyclic permutation, which may be confusing. It means to perform all cyclic permutations of the six representations, combined with all permutations of the last two pains of representations, with all the distinct combinations that are obtained counted once. The total number of representations obtained this way is 30.

The other modular invariants do not require further explanation.

An interesting by-product of these results is a list of some new invariants of simple Kac-Moody algebras. They can be read off from the table by looking for cases where non-trivial primary fields appear in combination with the identity of all other simple Kac-Moody algebras (if any) in the theory. In this way we find modular invariants (with extension of the chiral algebra with spins 2 and higher) for $D_{4,12}$, $D_{5,8}$, $A_{7,4}$, $D_{5,4}$, $D_{6,5}$, $C_{5,3}$, $A_{8,3}$, $D_{7,3}$, $C_{7,2}$, $A_{9,2}$, $C_{10,1}$, $C_{4,10}$, $A_{5,6}$, $A_{6,7}$, $D_{9,2}$, $B_{12,2}$, $F_{4,6}$, $E_{6,4}$, and $E_{7,3}$. These invariants correspond neither to conformal embeddings, nor are they simple current invariants. However, the first 11 are presumably[9] related to conformal embeddings by rank-level duality, The invariants to which they are related are respectively those of $D_{6,8}$, $D_{4,10}$, $A_{3,8} \equiv D_{3,8}$, $D_{2,10} \equiv (A_{1,10})^2$, $B_{2,12}$, $C_{3,5}$, $A_{2,9}$, $B_{1,14} \equiv A_{1,28}$, $C_{2,7} \equiv B_{2,7}$, $A_{1,10}$ and $C_{1,10} \equiv A_{1,10}$. Of the remaining ones, $C_{4,10}$ is dual to $C_{10,4}$, which cannot be conformally embedded in any Kac-Moody algebra. Hence we anticipate the existence of a higher spin extension of $C_{10,4}$. The algebras $A_{5,6}$ and $A_{6,7}$ are "self-dual," and $D_{9,2}$ and $B_{12,2}$ are formally related to $SO(2)$, but since this is Abelian it is difficult to give meaning to the level. Finally, the three exceptional algebras are not dual to anything. Some of these Kac-Moody invariants have already been obtained or conjectured on the basis of rank-level duality, and the $F_{4,6}$ invariant appeared first in [17].

5. Discussion

In this paper we have shown that all meromorphic $c = 24$ conformal field theories with finitely generated chiral algebras containing at least one spin-1 current have a partition function which can be written entirely as a modular invariant combination of Kac-Moody character. Furthermore we have enumerated all 69 such partition functions, 30 of which were not yet known. The actual construction of the conformal field theories corresponding to these new partition functions remains to be done. If there is exactly one CFT per modular invariant, and only one theory without spin-1 currents, then the total number of $c = 24$ meromorphic CFT's is 71 (an interesting fact, though perhaps a meaningless coincidence is that 71 is precisely the largest prime in the order of the monster group).

We hope that these theories have an interesting rôle to play in physics or mathematics, but this remains to be elucidated. The list itself could have revealed some underlying structure, but if it exists it must be rather subtle. In the spirit of generalizing from lattices to conformal field theories there are several questions that suggest themselves. For example, for self-dual lattices a formula exists [44] for the

[9] This has not been checked, but is conjectured here on the basis of the existence of a rank-level dual conformal embedding, using the duality relations $C_{n,k} \leftrightarrow C_{k,n}$, $SO(n)_k \leftrightarrow SO(k)_n$, and $SU(n)_k \leftrightarrow SU(k)_n$

sum of the inverse orders of the lattice automorphism groups (this is known as the Minkowsky-Siegel formula). If there is a generalization of this formula to conformal field theory one could use it to prove completeness of our list, and at the same time prove uniqueness of the monster module. A second interesting fact about the Niemeier lattices is that they can all be embedded into the unique Lorentzian self-dual lattice $\Gamma_{25,1}$, and are "orthogonal" to certain lightlike vectors on this lattice. It would be very interesting to see if this fact has a generalization to conformal field theory.

The last two points are pure speculation, but in any case the list has enabled us to make some modest progress in two other classification problems, namely string theory in 10 dimensions, and modular invariants of Kac-Moody algebras. The main lesson learned about the latter classification problem is that we still know essentially nothing about it. Several new invariants were found that could not have been anticipated with any known method. One might hope that this exhausts the list of exceptional extensions of the chiral algebra of simple Kac-Moody algebras, but there is not really any good reason to believe that. The situation is much worse for semi-simple Kac-Moody algebras. A very large number of exceptional invariants for such algebras can be read off from the table. It would be very strange indeed if no new exceptional invariants appear on a list of $c = 32$ (or larger) conformal field theories, which undoubtedly will never by enumerated.

Indeed, the present classification has made it clear once more that something changes drastically beyond $c = 24$ (this can also be seen in other ways, e.g. from the Minkowsky-Siegel formula, or from the properties of $\Gamma_{8n+1,1}$ Lorentzian self-dual lattices). Even though the number of meromorphic conformal field theories is too large to allow a complete listing (already the number of lattices is much larger than 8×10^7), one would at least like to have a finite algorithm that can produce the list in principle. Even for the subclass that has a $c = 32$ spin-1 algebra the methods we used for $c = 24$ do not yield a finite algorithm, since one has to allow a priori $U(1)$ factors with arbitrary radii. These difficulties are closely related to the unsolved problem of arriving at a practical classification of rational conformal field theories.

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Table 1. Modular invariant partition functions of the 69 theories with a non-Abelian spin-1 algebra. The monster module and the Leech lattice are included for completeness as Nos. 0 and 1. The table is explained in Sect. 4

No.	$\mathcal{N}$	Spin-1 algebra	Glue	Orbits	Ref.
0	0	–			[10]
1	24	$U(1)^{24}$		(0)	[42]
2	36	$(A_{1,4})^{12}$	$1[1;(0;)^{10}]$	See text	[12]
3	36	$D_{4,12}A_{2,6}$	$(0,1)+(s,0)+(v,0)$	$(0000+0006+0060+0066+0400+3033,00)$ $+(0204+0240+0300+0244+1411+2122,03)$ $+(0044+0600+1213+1231+1233+2022,11)$ $+(0004+0040+0048+0320+0302+0324$ $+1033+1035+1053+3\times 2222,22)$	
4	36	$C_{4,10}$	1	$0000+0024+0040+0044+00,10,0+0260+0321$ $+0323+0500+0800+1051+1430+1431+2\times 2222$ $+2242+3031+4140$	
5	48	$(A_{1,2})^{16}$	$11[11;(00;)^6]$ $+1010[1010;(0000;)^2]$ $+(1000)^4$	See text	[12]
6	48	$(A_{2,3})^6$	$1[1;(0;)^4]$	$(00)^6+\{(11;)^4(00;)^2\}+(01)^5(12)+(10)^5(21)+6\times(1,1)^6$	
7	48	$(A_{3,4})^3A_{1,2}$	$[1;0;0]1$	$((000)^3+(012)^3,0)+(\{002;010;111\},1)+4\times((111)^3,1)$ $+([000;020;020],2)+([012;020;020],2)$	
8	48	$A_{5,6}C_{2,3}A_{1,2}$	$(1,0,1)+(0,1,1)$	$(00000+02020,00,0)+(00003+00211,30,1)$ $+(00200+02020,20,2)+(00130+03100,11,1)$ $+(00022,01,0)+(00030,00,2)+(01102,10,1)$ $+(01121,20,0)+(01210,01,2)+2\times(11111,11,1)$	
9	48	$(A_{4,5})^2$	$(1,0)+(0,1)$	$(0000,0000)+(0102,0102)+4\times(1111,1111)$ $+[0021;0110]+[1111;0013]$	
10	48	$D_{5,8}A_{1,2}$	$(s,0)$	$(00000+00222+03011+10111,0)+(00113+00131$ $+2\times 11111,1)+(00044+00200+01022+01211,2)$	
11	48	$A_{6,7}$	1	$000000+001301+103100+002030+010122+3\times 111111$	

Table 1 (continued)

No.	$\mathcal{N}$	Spin-1 algebra	Glue	Orbits	Ref.
12	60	$(C_{2,2})^6$	$1[1;(0;)^5]$	$(00)^6+[00;(20;)^5]+[00;(01;)^5]+[11;(10;)^5]$ $+\{(01;)^3(20;)^3\}+[(00;)^2[(01;)^2;(20;)^2]]$	[12]
13	60	$D_{4,4}(A_{2,2})^4$	$(0,1,1,1,0)$ $+(0,2,1,0,1)$ $+(v,0,0,0,0)$ $+(s,0,0,0,0)$	$(0000,(00)^4)+(0100,(11)^4)$ $+(0200,[11;(00;)^3])+(1011,[00;(11;)^3])$ $+(0002,00,11,11,00)+(0002,11,00,00,11)$ $+(0020,00,00,11,11)+(0020,11,11,00,00)$ $+(0022,00,11,00,11)+(0022,11,00,11,00)$	
14	60	$F_{4,6}A_{2,2}$	–	$(0000+0004+0030+1100,00)$ $+(0003+0006+0021+2010,11)$ $+(0101+1012,10+01)+(0102+2000,02+20)$	[14]
15	72	$(A_{1,1})^{24}$	$1[(0;)^5 1;0;1;(0;0;1;1;)^2$ $0;1;0;(1;)^4]$	(0)	[7]
16	72	$(A_{3,2})^4(A_{1,1})^4$	$(1100)^2+(1010)^2$ $+(1001)^2$ $+(2,(0)^3,(1)^4)$	$((000)^4+[000;(101;)^3],(0)^4)$ $+((001)^3,011,0,1,0,0)$	[12]
17	72	$A_{5,3}D_{4,3}(A_{1,1})^3$	$(0,s,0,1,1)+(0,v,1,1,0)$ $+(1,0,1,1,1)$	$((00000,0000)+(00111,1011)+(01002,0100),(0)^3)$ $+((01010,0002+0020+2000),(0)^3)$	
18	72	$A_{7,4}(A_{1,1})^3$	$(1,1,0,0)$	$(0000000+0001101,(0)^3)$ $+(0000202+0010100,0,1,1)$ $+(0010011,0,0,1)+(0010011,0,1,0)$	
19	72	$D_{5,4}C_{3,2}(A_{1,1})^2$	$(0,1,1,1)+(s,1,0,0)$	$((00000,000)+(00200,000)+(01011,020),0,0)$ $+((00011,010)+(00022,020)+(00200,010),1,1)$ $+((00100,110)+(00111,001)+(10011,011),0,1)$	
20	72	$D_{6,5}(A_{1,1})^2$	$(s,0,1)+(c,1,0)$	$(000000+010002+010020,0,0)$ $+(100111+002000+200100,0,0)$	
21	72	$C_{5,3}G_{2,2}A_{1,1}$	$(1,0,1)$	$(00000+00020+03000,00,0)$ $+(00002+02000+10110,02,0)$ $+(10001+00030+02010,10,0)$ $+(00200+01101+20010,01,0)$	

Table 1 (continued)

No.	$\mathcal{N}$	Spin-1 algebra	Glue	Orbits	Ref.
22	84	$C_{4,2}(A_{4,2})^2$	$(1,0,0)+(0,1,2)$	$(0000,(0000)^2)+(0020,(0110)^2)+(0100,(1001)^2)$ $+(0001,[0000;0110])+(0200,[0000;1001])$ $+(1010,[0110;1001])$	
23	84	$(B_{3,2})^4$	$1[1;0;0]$	$(000)^4+(001)^3(101)+\{000;010;100;002\}_E$ $+[100;(002;)^3]+[002;(010;)^3]+[010;.(100;)^3]$	[12]
24	96	$(A_{2,1})^{12}$	$2[1;1;2;(1;)^3(2;)^3 1;2]$	(0)	[7]
25	96	$(D_{4,2})^2(C_{2,1})^4$	$([s;0],1,1,0,0)$ $+([v,0],0,1,1,0)$ $+(0,0,1,1,1,1)$	$((0000)^2,(00)^4)+([0000;\dot{0}100],(10)^4)$ $+((0001)^2,00,01,10,10)+((0001)^2,10,10,00,01)$ $+((0010)^2,00,10,01,10)+((0010)^2,10,00,10,01)$ $+((0011)^2,00,10,10,01)+((0011)^2,10,00,01,10)$ $+((0100)^2,(00)^3,01)$	[12]
26	96	$(A_{5,2})^2C_{2,1}(A_{2,1})^2$	$(1,0,1,1,1)$+$(0,1,1,1,2)$	$((00000)^2,(00)^3)+((01010)^2,01,(00)^2)$ $+([00001;10010],10,01,00)$	
27	96	$A_{8,3}(A_{2,1})^2$	$(1,1,1)$	$(00000000,(00)^2)+(00010101,[01;10])+(00100011,(00)^2)$	
28	96	$E_{6,4}C_{2,1}A_{2,1}$	$(1,0,1)$	$(000000+100011,00,00)+(110000+000110,10,00)$ $+(200020+001000,01,00)$	
29	108	$(B_{4,2})^3$	$1[1;0]$	$(0000)^3+(0001,0001,1001)+(0010,0010,0010)$ $+[0000;(0010;)^2]+[0100;(0002;)^2]$ $+[000\dot{2};(1000;)^2]+[1000;(0100;)^2]$	[12]
30	120	$(A_{3,1})^8$	$3[2;0;0;1;0;1;1]$	(0)	[7]
31	120	$(D_{5,2})^2(A_{3,1})^2$	$(0,s,1,1)$ $+(s,0,3,1)$	$((00000)^2+[00011;01000],(000)^2)$ $+((00001)^2,010,001)$	[12]
32	120	$E_{6,3}(G_{2,1})^3$	$(1,0,0,0)$	$(000000,(00)^3)+(000001,(01)^3)$ $+(001000,[00;00;01])+(100010,[00;01;01])$	
33	120	$A_{7,2}(C_{3,1})^2A_{3,1}$	$(1,0,1,1)$ $+(0,1,1,2)$	$(0000000,(000)^2,000)+(0010100,(010)^2,000)$ $+(0100010,[000;010],000)$	
34	120	$D_{7,3}A_{3,1}G_{2,1}$	$(s,1,0)$	$(0000000+1000100,000,00)+(0000011+1010000,000,01)$	
35	120	$C_{7,2}A_{3,1}$	$(1,2)$	$(0000000+0000200+1000001+0101000,000)$ $+(0010010,001+100)$	

Table 1 (continued)

No.	$\mathcal{N}$	Spin-1 algebra	Glue	Orbits	Ref.
36	132	$A_{8,2}F_{4,2}$	$(3,0)$	$(00000000,0000)+(00011000,1000)+(10000001,0002)$ $+(00100100,0010)+(01000010,0001)$	
37	144	$(A_{4,1})^6$	$1[0;1;4;4;1]$	(0)	[7]
38	144	$(C_{4,1})^4$	$1[1;0;0]$	$(0000)^4+[0000;(0100;)^3]+((0010)^3,1000)$	[12]
39	144	$D_{6,2}C_{4,1}(B_{3,1})^2$	$(s,0,0,1)$ $+(0,1,1,1)$ $+(v,0,1,1)$	$(000000,0000,000,000)+(000100,0100,000,000)$ $+(000011,0100,001,001)+(001000,0000,001,001)$ $+(000001,0010,000,001)+(000010,0010,001,000)$	[12]
40	144	$A_{9,2}A_{4,1}B_{3,1}$	$(1,2,1)$	$(000000000+001000100,0000,000)+(001001000,0001,001)$	
41	156	$(B_{6,2})^2$	$(1,1)$	$(000000,000000)+(000001,100001)$ $+[000010;100000]+[000100;000002]+[001000;010000]$	[12]
42	168	$(D_{4,1})^6$	$(s)^6+0[0;v;c;c;v]$	(0)	[7]
43	168	$(A_{5,1})^4D_{4,1}$	$2[0;2;4]0$ $+(3,3,0,0,s)$ $+(3,0,3,0,v)$ $+(3,0,0,3,c)$	(0)	[7]
44	168	$E_{6,2}C_{5,1}A_{5,1}$	$(1,1,1)$	$((000000,00000)+(000001,00010)+(100010,01000),00000)$	
45	168	$E_{7,3}A_{5,1}$	$(1,3)$	$(0000000+0000011,00000)+(0000100,00010+01000)$	
46	192	$(A_{6,1})^4$	$1[2;1;6]$	(0)	[7]
47	192	$D_{8,2}(B_{4,1})^2$	$(s,0,0)+(v,1,1)$	$(00000000,(0000)^2)+(00000100,(0001)^2)$ $+(00000001,[1000;0001])+(00010000,0000,1000)$	[12]
48	192	$(C_{6,1})^2B_{4,1}$	$(1,0,1)+(0,1,1)$	$((000000)^2,0000)+((000100)^2,1000)$ $+([000010;001000],0001)$	
49	216	$(A_{7,1})^2(D_{5,1})^2$	$(1,1,s,v)+(1,7,v,s)$	(0)	[7]
50	216	$D_{9,2}A_{7,1}$	$(s,2)$	$(000000000+000001000,0000000)+(000000001,0000100)$	[12]

Table 1 (continued)

No.	$\mathcal{N}$	Spin-1 algebra	Glue	Orbits	Ref.
51	240	$(A_{8,1})^3$	$[1;1;4]$	(0)	[7]
52	240	$C_{8,1}(F_{4,1})^2$	$(1,0,0)$	$(00000000, (0000)^2) + (00000100, (0001)^2)$ $+ (00010000, [0000; 0001])$	
53	240	$E_{7,2}B_{5,1}F_{4,1}$	$(1,1,0)$	$(0000000, 00000, 0000) + (0000001, 00001, 0000)$ $+ (0000010, 00001, 0001) + (0000100, 00000, 0001)$	
54	264	$(D_{6,1})^4$	$\{0; s; v; c\}_E$	(0)	[7]
55	264	$(A_{9,1})^2 D_{6,1}$	$(2,4,0)$ $+ (5,0,s) + (0,5,c)$	(0)	[7]
56	288	$C_{10,1}B_{6,1}$	$(1,1)$	$(0000000000 + 0000010000, 000000) + (0000001000, 000001)$	
57	300	$B_{12,2}$	–	$000000000000 + 100000000001$ $+ 000000000100 + 000010000000$	[12]
58	312	$(E_{6,1})^4$	$1[0;1;2]$	(0)	[7]
59	312	$A_{11,1}D_{7,1}E_{6,1}$	$(1,s,1)$	(0)	[7]
60	336	$(A_{12,1})^2$	$(1,5)$	(0)	[7]
61	360	$(D_{8,1})^3$	$[s;v;v]$	(0)	[7]
62	384	$E_{8,2}B_{6,1}$	$(1,1)$	$(00000000, 00000000) + (10000000, 00000001)$	[14]
63	408	$A_{15,1}D_{9,1}$	$(2,s)$	(0)	[7]
64	456	$D_{10,1}(E_{7,1})^2$	$(s,1,0) + (c,0,1)$	(0)	[7]
65	456	$A_{17,1}E_{7,1}$	$(3,1)$	(0)	[7]
66	552	$(D_{12,1})^2$	$[s;v]$	(0)	[7]
67	624	$A_{24,1}$	(5)	(0)	[7]
68	744	$(E_{8,1})^3$	—	(0)	[7]
69	744	$D_{16,1}E_{8,1}$	$(s,0)$	(0)	[7]
70	1128	$D_{24,1}$	(s)	(0)	[7]