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Irreducible Unitary Representations of Quantum Lorentz Group

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Abstract. A complete classification of irreducible unitary representations of a one parameter deformation $S_qL(2, C)$ $(0<q<1)$ of $SL(2, C)$ is given. It shows that in spite of a popular belief the representation theory for $S_qL(2, C)$ is not "a smooth deformation" of the one for $SL(2, C)$.

0. Introduction

A theory of quantum deformations of the classical locally compact groups still seems far from being complete. According to [16] one can distinguish a purely algebraic (Hopf-algebra or Hopf *-algebra) level, topological (C^*-algebra) level and intermediate Hilbert space (i.e. the representation theory) level. For the compact groups the algebraic and topological approaches are equivalent since the topological level is well understood and there is a natural way of passing to it from the algebraic one (see e.g. [14, 15]) via the Hilbert space level. In effect one obtains a smooth deformation of group structure and its representation theory.

This experience is a source of the popular belief that it is also the case for general locally compact groups. A class of Pontryagin duals for compact quantum groups is also well established on the C^*-algebra level [8] and seems to confirm this conviction, but for the non-compact case there is no general theory of topological quantum deformation at the moment.

The study of other examples indicates that new phenomena can occur which are not seen on the algebraic level:

– The deformation may not exist on the C^*-algebra level (cf. [16] where non-existence of comultiplication for quantum $SU(1, 1)$ group for real values of deformation parameter was proved).

– The deformation on the C^*-algebra level exists under some additional conditions (e.g. restrictions on spectra of operators involved in the theory (see the spectral condition in [16] for the case of $E(2)$ – the group of motions of the Euclidean plane)).

– The deformation exists on the C^*-algebra level but a representation theory is not similar to the undeformed one. To obtain a similarity one has to impose smoothness conditions (e.g. smooth and non-smooth finite-dimensional representations of the quantum Lorentz group in [8]).

It is clear that a study of concrete examples is very important for the proper background of the general theory. Investigating known examples one reveals a crucial role of a Hilbert space level. In the present paper we study irreducible unitary representations of the quantum Lorentz group [8]. This will be a starting point as well for the construction of Pontryagin dual (i.e. a quantum group of "characters") quantum Lorentz group on the C^*-algebra level as for the development of harmonic analysis in further investigations.

The quantum deformation $S_qL(2,C)$ $(0<q<1)$ of $S_1L(2,C)=SL(2,C)$ introduced by Podleś and Woronowicz in [8] contains the quantum deformation $S_qU(2)$ of $SU(2)$. Any unitary representation v of $S_qL(2,C)$ induces then a unitary representation v_c of $S_qU(2)$. Since irreducible (unitary) representations of $S_qU(2)$ are labeled by the integer and half-integer spin parameter $s\in S$ where

$$S=\{0,\tfrac{1}{2},1,\tfrac{3}{2},2,\tfrac{5}{2},3,\dots\}, \tag{0.1}$$

let p be a minimal spin occurring in the (unique) decomposition of v_c into a direct sum of irreducible representations. Let for any $p\in S$,

$$S_p=\{p,p+1,p+2,p+3,\dots\} \tag{0.2}$$

be a subset of S. By $\mathscr{E}_p$ we shall denote for $p\in S$ and $p\neq 0$ an ellipse in C:

$$\mathscr{E}_p=\left\{z\colon z=\frac{q}{\sqrt{1+q^2}}[(q^p+q^{-p})\cos\varphi+i(q^{-p}-q^p)\sin\varphi] \text{ for } \varphi\in[0,2\pi[\right\} \tag{0.3}$$

and for $p=0$ a closed interval

$$\mathscr{E}_0=[-\sqrt{1+q^2},\sqrt{1+q^2}]. \tag{0.4}$$

Let

$$\Sigma_q=\bigcup_{p\in S}\mathscr{E}_p. \tag{0.5}$$

Then we shall prove that for any irreducible unitary representation of $S_qL(2,C)$ the value $\mathscr{X}_0$ of the Casimir operator X for $S_qL(2,C)$ belongs to Σ_q and irreducible unitary representations can be labeled by the minimal spin p and the value $\mathscr{X}_0$ analogously as in the case of classical $SL(2,C)$. Let

$$\mathscr{L}_q=\{(p,z_p)\colon p\in S,\ z_p\in\mathscr{E}_p\},$$

then we can state our main result

Theorem 0.1. *Let $q\in]0,1[$. There is one to one correspondence between the set $\mathscr{L}_q$ and the set of unitary equivalence classes of irreducible unitary representations of quantum Lorentz group $S_qL(2,C)$.*

Moreover,

1. *there are two one-dimensional representations:*

$$\tilde{\tau}=(0,\sqrt{1+q^2}) \quad \textit{and the trivial one} \quad \tau=(0,-\sqrt{1+q^2}).$$

2. *If v is an unitary irreducible representation different from τ and $\tilde{\tau}$ with minimal spin p then it is infinite-dimensional and contains any irreducible unitary representation of $S_qU(2)$ with spin $s \in S_p$ with multiplicity one.*

To compare this result with the classical one (the case $q=1$) let us denote by $\mathscr{P}_p$ the parabola

$$\mathscr{P}_p = \{z: z = 2(t^2 - p^2 + 1) + 4ipt \text{ for } t \in]-\infty, \infty[\}$$

for $p \in S$ and $p \neq 0$ and a halfline

$$\mathscr{P}_0 = [0, \infty[$$

for $p=0$. Let

$$\Sigma_1 = \bigcup_{p \in S} \mathscr{P}_p \tag{0.6}$$

and

$$\mathscr{L}_1 = \{(p, z_p): p \in S,\ z_p \in \mathscr{P}_p\}.$$

Then the value $\mathscr{X}_0$ of the Casimir operator X (a linear combination of the Laplace operators) for $SL(2,C)$ in the unitary irreducible representation v of $SL(2,C)$ belongs to Σ_1 and the set of irreducible unitary representations (unitary equivalence classes) is labeled by $\mathscr{L}_1$ (see e.g. [5, pp. 104, 144] or [1, 2]).

Fig. 1 describes the set $\frac{\sqrt{1+q^2}}{q}\Sigma_q$ for $0<q<1$ (upper part) and Σ_1 (lower part). Let us note that

$$\left\{\frac{\sqrt{1+q^2}}{q}\mathscr{E}_p: p \in S,\ p \neq 0\right\}$$

is a family of cofocal ellipses with focuses in points $(-2,0)$ and $(2,0)$ and

$$\{\mathscr{P}_p: p \in S,\ p \neq 0\}$$

is a family of cofocal parabolas with focus in $(2,0)$

All label spaces $\mathscr{L}_q$ are homeomorphic for $0<q<1$ and not homeomorphic with $\mathscr{L}_1$. The limit $q \to 1$ is singular. It corresponds to moving one focus to infinity and then representation $\tilde{\tau}$ disappears (there are also two complementary series in the deformed case and one of them also disappears in this limit).

Let us observe that if v is an irreducible unitary representation of quantum Lorentz group then $\tilde{\tau} \otimes v$ is also an irreducible unitary representation [this corresponds to the reflection Σ_q with respect to the point $(0,0)$].

In essence it was also noticed in the case of irreducible finite-dimensional (non-unitary) representations of $S_qL(2,C)$ [8]. Since in this case the set of values of Casimir operator is discrete it was possible to divide representations into two parts – the smooth representations which are continuous deformations of representations for $q=1$ and non-smooth one. It was conjectured [8] that this exhausts all finite-dimensional irreducible representations of $S_qL(2,C)$. The affirmative answer was given in [12].

Such a division is no longer possible in the case of unitary representations and this shows that also in the case of finite-dimensional representations the non-smooth ones are also important.

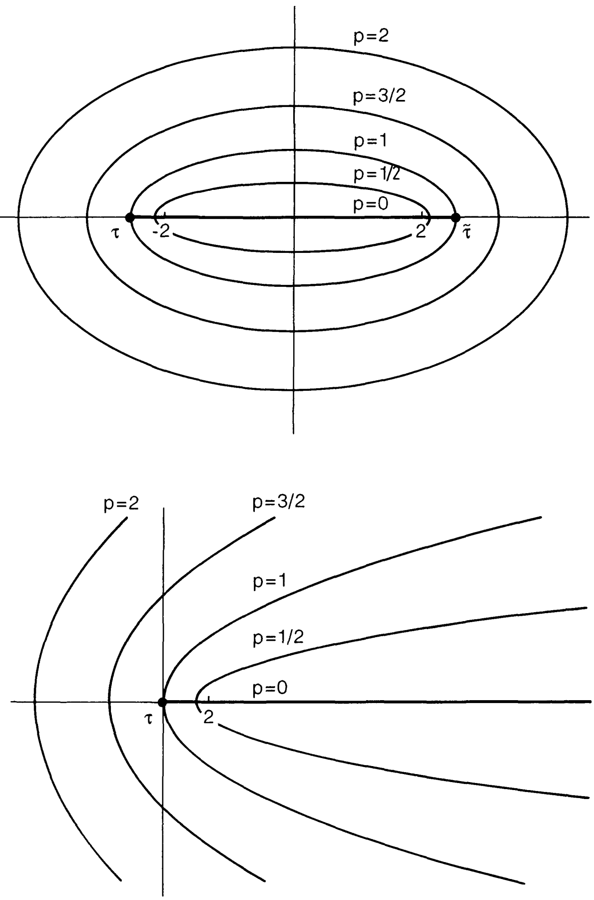


Fig. 1. Σ_q

A few remarks on notations: For any C^*-algebra B by $M(B)$ we shall denote the multiplier algebra of B. Let us note that for the algebra of compact operators $CB(H)$ on a Hilbert space H we have that $M(CB(H))=B(H)$. We shall say that $\psi\in\mathrm{Mor}(B_1,B_2)$ if ψ is *-algebra homomorphism $\psi: B_1\rightarrow M(B_2)$ such that $\psi(B_1)B_2$ is dense in B_2 (see [16, p. 402]).

We shall also use an affiliation relation in the case of "unbounded multiplier" T and denote it $T\eta B$ (cf. [16, § 1]). Then any $\psi\in\mathrm{Mor}(B_1,B_2)$ has a unique extension to *-algebra homomorphism from $M(B_1)$ to $M(B_2)$ and also to elements affiliated with B_1 [16, Theorem 1.2].

Let $\{B_n\}_{n\in N}$ be a family of finite-dimensional (unital) C^*-algebras labeled by a denumerable set N. Then

$$B=\sum_{n\in N}^{\oplus} B_n$$

will denote the (non-unital) C^*-algebra which elements are sequences $(b_n)_{n\in N}$ tending to 0 at infinity. In this case

$$M(B)=\sum_{n\in N}^{\text{bounded}} B_n\,,$$

i.e. $b \in M(B)$ if $b=(b_n)_{n\in N}$ is a uniformly bounded sequence and $b \,\eta\, B$ if $b \in \mathop{\times}\limits_{n\in N} B_n$ [16, §1, Example 5]. By $\sum_{n\in N}^{\text{finite}} B_n$ we shall denote a dense *-algebra in B of sequences $(b_n)_{n\in N}$ with finite number of elements different from 0.

In the paper we deal with compact as well as non-compact quantum groups. In any case quantum group G is a bialgebra with additional "group structure". A bialgebra is a pair $(\mathfrak{R}, \Delta)$, where $\mathfrak{R}$ is a C^*-algebra and $\Delta \in \mathrm{Mor}(\mathfrak{R}, \mathfrak{R}\otimes\mathfrak{R})$ is a coassociative comultiplication:

$$(\Delta\otimes \mathrm{id})\Delta = (\mathrm{id}\otimes\Delta)\Delta .$$

We shall say that $v \in M(CB(H)\otimes\mathfrak{R})$ is a strongly continuous representation of $G=(\mathfrak{R},\Delta)$ (acting) on the Hilbert space H if

$$(\mathrm{id}_H\otimes\Delta)v = v_{12}v_{13} . \tag{0.7}$$

[We use the leg notation: for any C^*-algebras A, B, C and $w \in M(A\otimes B)$ by w_{12} we denote the unique image of w under the canonical morphism

$$\psi_{12} \in \mathrm{Mor}(A\otimes B\otimes C),$$

where

$$\psi_{12}(a\otimes b) = a\otimes b\otimes I_C \in \mathrm{M}(A\otimes B\otimes C)$$

for any $a \in A$, $b \in B$, etc.] We shall say that the representation is unitary if element v is unitary.

Any quantum group $G=(\mathfrak{R},\Delta)$ considered in this paper is a one parameter deformation of a classical object with deformation parameter $q \in]0,1[$ and has a 2-dimensional fundamental representation

$$u = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}.$$

Then the algebra $\mathfrak{R}$ is "generated" by $\{u_{ij}, u_{ij}^*\}$ satisfying some commutation relations. The multiplication Δ is then deduced from its value on the sets of generators

$$\Delta(u_{ij}) = u_{i1}\otimes u_{1j} + u_{i2}\otimes u_{2j} .$$

There is also a counit e and coinverse (antipode) κ which can be "extended" from their values on generators:

$$e(u_{ij}) = \delta_{ij} = e(u_{ij}^*),$$

$$\kappa(u_{11}) = u_{22}, \qquad \kappa(u_{11}^*) = u_{22}^*,$$

$$\kappa(u_{22}) = u_{11}, \qquad \kappa(u_{22}^*) = u_{11}^*,$$

$$\kappa(u_{12}) = -\frac{1}{q}u_{12}, \qquad \kappa(u_{12}^*) = -qu_{12}^*,$$

$$\kappa(u_{21}) = -qu_{21}, \qquad \kappa(u_{21}^*) = -\frac{1}{q}u_{21}^* .$$

For the precise meaning of "generating" and "extension" in concrete examples we refer the reader to [8].

The paper is organized as follows. The first section is selfcontained. It embodies the idea that the Pontryagin dual quantum group (the universal "space of unitary characters" for quantum group) play the same role in the representation theory of quantum group as the Lie algebra in the Lie group theory. One has to replace derivatives by suitable functionals. This reduces the problem of study of the unitary representations to the problem (easier in general) of representations of the "algebra of functions" on the dual group.

Propositions 1.1 and 1.3 states that $S_qU(2)$ and $S_q\hat{U}(2)$ are mutually Pontryagin dual groups. This combining with the structural theorem for $S_qL(2, C)$ leads to the description of the algebra of "functions" on the dual group (Propositions 1.4 and 1.5).

To establish a useful framework for the study of representations of this algebra we develop the method of tensor operators in Sect. 2. Using this we are able to classify the unitary representations of $S_qL(2, C)$ (Propositions 3.1 and 3.2) and to realize unitary equivalence classes (Theorem 3.3) in the similar way as for the classical $SL(2, C)$.

1. Quantum Lorentz Group $S_qL(2, C)$

Recently, it was shown that the classical group $SL(2, C)$ admits many one-parameter non-equivalent deformations (see [8, 18, 19]). In this paper we focus on the first known.

For $q \in]0, 1[$ we denote by $S_qL(2, C)$ the quantum deformation of $SL(2, C)$ in the sense of Podleś and Woronowicz [8]. This deformation follows from an analog of the Iwasawa decomposition for $SL(2, C)$ and it contains the well-known quantum deformation $S_qU(2)$ of $SU(2)$ and its Pontryagin dual $S_q\hat{U}(2)$ in such a way that $S_qL(2, C)$ is an example of double-group construction applied to $S_qU(2)$. Symbolically,

$$S_qL(2, C) = S_qU(2) \bowtie S_q\hat{U}(2). \tag{1.1}$$

Since our description of unitary representations of $S_qL(2, C)$ is based upon this decomposition we recall in the first part of this section the basic results concerning $S_qU(2)$ and its representation theory. For more information we refer to [13, 8, 11, 3].

To abbreviate the notation the compact quantum group $S_qU(2)$ will be denoted by G_c, its dual $S_q\hat{U}(2)$ (quantum non-compact group) by G_d and by $G = G_c \bowtie G_d$ the resulting quantum (non-compact) Lorentz group $S_qL(2, C)$.

In the case of $G_c = (\mathfrak{R}_c, \Delta_c)$ the algebra $\mathfrak{R}_c$ of "*continuous functions*" on G_c is the C^*-algebra completion of the $*$-algebra $\mathscr{A}_c$ of "*smooth continuous functions*" on G_c, i.e. the $*$-algebra generated by two elements α_c, γ_c such that

$$\begin{gathered} \alpha_c^*\alpha_c + \gamma_c^*\gamma_c = I, \qquad \alpha_c\alpha_c^* + q^2\gamma_c^*\gamma_c = I, \\ \alpha_c\gamma_c = q\gamma_c\alpha_c, \qquad \alpha_c\gamma_c^* = q\gamma_c^*\alpha_c, \qquad \gamma_c\gamma_c^* = \gamma_c^*\gamma_c. \end{gathered} \tag{1.2}$$

To describe a group structure on G_c it is enough to define it on the set of its generators. The comultiplication $\Delta_c \in \mathrm{Mor}(\mathfrak{R}_c, \mathfrak{R}_c \otimes \mathfrak{R}_c)$ follows from the fact that

$$u^{1/2} = \begin{pmatrix} \alpha_c & -q\gamma_c^* \\ \gamma_c & \alpha_c^* \end{pmatrix} \tag{1.3}$$

is a 2-dimensional (unitary) representation of G_c, i.e.

$$\begin{aligned}\Delta_c(\alpha_c) &= \alpha_c \otimes \alpha_c - q\gamma_c^* \otimes \gamma_c\,,\\ \Delta_c(\gamma_c) &= \gamma_c \otimes \alpha_c + \alpha_c^* \otimes \gamma_c\,.\end{aligned} \tag{1.4}$$

Moreover, there exists a counit $e_c \in \mathrm{Mor}(\mathfrak{R}_c, C)$ given by

$$e_c(\alpha_c) = 1\,, \qquad e_c(\gamma_c) = 0\,. \tag{1.5}$$

It is known that unitary irreducible representations of G_c are labeled by spin parameter $s \in S$. The corresponding unitary representation u^s acts on $(2s+1)$-dimensional Hilbert space K^s and $u^s \in B(K^s) \otimes \mathscr{A}_c$. The algebra $\mathfrak{R}_d$ of "*continuous functions tending to* 0 *at infinity*" on G_d is then the C^*-algebra

$$\mathfrak{R}_d = \sum_{s \in S}^{\oplus} B(K^s)\,, \tag{1.6}$$

i.e. the algebra without unity which means that we deal with non-compact case. The algebra $\mathfrak{R}_d$ can be generated by "*unbounded continuous functions*" A_d and N_d and then we have only $A_d \,\eta\, \mathfrak{R}_d$, $N_d \,\eta\, \mathfrak{R}_d$ (cf. [16, Sect. 1, Example 5]. These operators satisfy the following relations (cf. [8, (1.35)–(1.38)]):

$$\begin{aligned}&A_d \text{ is positive selfadjoint}\,,\\ &A_d N_d = q N_d A_d\,,\\ &N_d N_d^* = N_d^* N_d + \frac{1}{1-q^2}(A_d^{-2} - A_d^2)\,.\end{aligned} \tag{1.7}$$

Let us note that N_d differs from n in [8] by the factor $(1-q^2)$. The comultiplication $\Delta_d \in \mathrm{Mor}(\mathfrak{R}_d, \mathfrak{R}_d \otimes \mathfrak{R}_d)$ on $G_d = (\mathfrak{R}_d, \Delta_d)$ is defined on generators A_d, N_d by the fact that (cf. [8, Theorem 5.1])

$$w_d = \begin{pmatrix} A_d & (1-q^2)N_d \\ 0 & A_d^{-1} \end{pmatrix} \tag{1.8}$$

is a 2-dimensional representation of G_d, i.e.

$$\begin{aligned}\Delta_d(A_d) &= A_d \otimes A_d\,,\\ \Delta_d(N_d) &= A_d \otimes N_d + N_d \otimes A_d^{-1}\,.\end{aligned} \tag{1.9}$$

Moreover, there exists a counit $e_d \in \mathrm{Mor}(\mathfrak{R}_d, C)$ defined by

$$e_d(A_d) = 1\,, \qquad e_d(N_d) = 0\,. \tag{1.10}$$

The fundamental role in the G_c-representation theory plays an unitary operator

$$U = \sum_{s \in S}^{\text{bounded}} u^s\,. \tag{1.11}$$

Clearly, $U \in M(\mathfrak{R}_d \otimes \mathfrak{R}_c)$. Since (cf. [8, (2.15) and Theorem 3.1 (3.3)])

$$(\mathrm{id}_d \otimes \Delta_c)U = U_{12} U_{13} \tag{1.12}$$

and

$$(\Delta_d \otimes \mathrm{id}_c)U^* = U_{13}^* U_{23}^* \tag{1.13}$$

it is called a bicharacter (U is a representation of G_c acting on $\mathfrak{R}_d$ and U^* is a representation of G_d acting on $\mathfrak{R}_c$). This gives a correspondence between the unitary representations of a group and the representations of an "*algebra of functions*" on the Pontryagin dual group.

Proposition 1.1. i) *Let $v_c \in M(CB(H) \otimes \mathfrak{R}_c)$ be a unitary representation of G_c acting on Hilbert space H. Then there exists the unique $\psi_d \in \mathrm{Mor}(\mathfrak{R}_d, CB(H))$ such that*

$$v_c = (\psi_d \otimes \mathrm{id}_c) U . \tag{1.14}$$

ii) *Let $\psi_d \in \mathrm{Mor}(\mathfrak{R}_d, CB(H))$. Then v_c defined by (1.14) belongs to $M(CB(H) \otimes \mathfrak{R}_c)$ and it is a unitary representation of G_c.*

Proof. The first part of the proposition follows from the more general Theorem 2.1 [8] in the special case of the C^*-algebra $CB(H)$. Since U is the unitary operator, the unitarity of v_c in (1.14) is obvious and by simple calculations using (1.12) we get that v_c is the unitary representation of G_c and ii) follows. Q.E.D.

Now we shall describe the correspondence $v_c \to \psi_d$ in a more convenient way, i.e. in terms of

$$A = \psi_d(A_d), \qquad N = \psi_d(N_d).$$

To this end let's define linear functionals φ_A, φ_R, and φ_N on the algebra $\mathscr{A}_c$ such that for any $a, b \in \mathscr{A}_c$,

$$\begin{gathered} \varphi_A(ab) = \varphi_A(a)\varphi_A(b), \qquad \varphi_R(ab) = \varphi_R(a)\varphi_R(b), \\ \varphi_N(ab) = \varphi_A(a)\varphi_N(b) + \varphi_N(a)\varphi_R(b). \end{gathered} \tag{1.15}$$

They are uniquely defined by their values on the set of generators $\{\alpha, \alpha^*, \gamma, \gamma^*\}$ of the algebra $\mathscr{A}_c$ (cf. [8, Eqs. (5.1), (5.2)] and [9]):

$$\begin{gathered} \varphi_A(\alpha_c) = q^{1/2} = \varphi_R(\alpha_c^*), \qquad \varphi_A(\alpha_c^*) = q^{-1/2} = \varphi_R(\alpha_c), \\ \varphi_N(\gamma) = -q^{-3/2}, \qquad \varphi_A(I_c) = 1 = \varphi_R(I_c), \end{gathered} \tag{1.16}$$

and the other values are 0. (Let us note that our φ_N differs by the factor $(1-q^2)$ from ξ_{12} in [8].)

Since any element $T \eta \mathfrak{R}_d = \sum_{s \in S}^{\oplus} B(K^s)$ is a sequence $(T_s)_{s \in S}$, where $T_s \in B(K^s)$ we have that

$$A_d = (A_s)_{s \in S}, \qquad N_d = (N_s)_{s \in S}. \tag{1.17}$$

It was shown (see the proof of Theorem 5.1 in [8]) that for $s \in S$,

$$(\mathrm{id}_{K^s} \otimes \varphi_A)(u^s)^* = A_s, \qquad (\mathrm{id}_{K^s} \otimes \varphi_N)(u^s)^* = N_s, \qquad (\mathrm{id}_{K^s} \otimes \varphi_R)(u^s)^* = A_s^{-1}.$$

We shall shortly write this as

$$A_d = (\mathrm{id}_d \otimes \varphi_A)U^*, \qquad N_d = (\mathrm{id}_d \otimes \varphi_N)U^*, \qquad A_d^{-1} = (\mathrm{id}_d \otimes \varphi_R)U^*. \tag{1.18}$$

A_s, N_s belong to $B(K^s)$ and satisfy relations (1.7) so they define the representation of C^* – algebra $\mathfrak{R}_d$ which corresponds to the unitary representation u^s of G_c. Moreover (cf. [8, Corollary 5.2]), there is the canonical orthonormal basis

$$\{f_m^s : m = -s, -s+1, \ldots, s-1, s\} \tag{1.19}$$

in K^s such that

$$\begin{aligned} A_s f_m^s &= q^m f_m^s, \\ N_s f_m^s &= q^{-s}\sqrt{[s-m]_q[s+m+1]_q} f_{m+1}^s, \\ N_s^* f_m^s &= q^{-s}\sqrt{[s+m]_q[s-m+1]_q} f_{m-1}^s, \end{aligned} \tag{1.20}$$

where

$$[n]_q = \frac{1-q^{2n}}{1-q^2}. \tag{1.21}$$

Let us note that the canonical basis (1.19) is determined by (1.20) up to m-independent complex number of modulus 1 and can be fixed by identification of K^s with the subspace of the q-symmetric tensors in $\bigotimes_{j=1}^{2s} K^{1/2}$. Then $f_s^s = \bigotimes_{j=1}^{2s} f_{1/2}^{1/2}$. For $u^{1/2}$ (in this case $K^{1/2} = C^2$) we have by [8, (5.8)] the correspondence

$$f_{-1/2}^{1/2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad f_{1/2}^{1/2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and then for $s=1$,

$$f_1^1 = f_{1/2}^{1/2} \otimes f_{1/2}^{1/2}, \quad f_{-1}^1 = f_{-1/2}^{1/2} \otimes f_{-1/2}^{1/2},$$

$$f_0^1 = \frac{1}{\sqrt{1+q^2}} [q f_{-1/2}^{1/2} \otimes f_{1/2}^{1/2} + f_{1/2}^{1/2} \otimes f_{-1/2}^{1/2}].$$

Now let $\psi_d \in \mathrm{Mor}(\mathfrak{R}_d, CB(H))$ and v_c be the corresponding unitary representation of G_c acting on H. Since $\mathfrak{R}_d$ is a direct sum of full matrix algebras then

$$H = \sum_{s\in S}^{\oplus} H^s = \sum_{s\in S}^{\oplus} K^s \otimes H_s \tag{1.22}$$

and $\psi_d(a_s) = a_s \otimes \mathrm{id}_{H_s} \in B(H^s)$ for any a_s belonging to the subalgebra $B(K^s)$ of $\mathfrak{R}_d$. The decomposition (1.22) corresponds to the decomposition of v_c into a direct sum of irreducible unitary representations of G_c and v_c restricted to H^s is $u^s \otimes \mathrm{id}_{H_s}$. The dimension $\dim H_s$ is a multiplicity of u^s in v_c [or equivalently $B(K^s)$ in ψ_d]. Let

$$S(v_c) := \{s \in S : H^s \neq 0\} \tag{1.23}$$

and

$$D = \sum_{s\in S(v_c)}^{\text{finite}} H^s. \tag{1.24}$$

Then $S(v_c)$ will be called *a support of* v_c (or *a support of* ψ_d) and D will be called *the natural domain for* v_c *(and* ψ_d*)* since it is a dense linear subset of H invariant under the action of v_c and an invariant essential domain for a selfadjoint operator

$$A = \psi_d(A_d) = \sum_{s\in S(v_c)} A_s \otimes \mathrm{id}_{H_s} \tag{1.25}$$

and closed operator

$$N = \psi_d(N_d) = \sum_{s\in S(v_c)} N_s \otimes \mathrm{id}_{H_s}, \tag{1.26}$$

and hence for its adjoint N^*. It is clear that A, N, and N^* satisfy (1.7) on D and completely determine ψ_d. Now let $x \in H^s$, then $v_c(x \otimes I_c) \in D \otimes_{\text{alg}} \mathscr{A}_c$ and

$$\begin{aligned} Ax &= \psi_d(A_d)x = (\psi_d \otimes \mathrm{id}_c)[(\mathrm{id}_d \otimes \varphi_A)U^*]x \\ &= (\psi_d \otimes \varphi_A)U^*(x \otimes I_c) = (\mathrm{id}_D \otimes \varphi_A)v_c^*(x \otimes I_c). \end{aligned}$$

In the same manner we can compute Nx and we get

Corollary 1.2. *Let $v_c \in M(CB(H) \otimes \mathfrak{R}_c)$ be a unitary representation of G_c acting on the Hilbert space H, $\psi_d \in \mathrm{Mor}(\mathfrak{R}_d, CB(H))$ be the corresponding representation of $\mathfrak{R}_d$ and D be the natural domain for v_c. Let $A = \psi_d(A_d)$, $N = \psi_d(N_d)$.*

Then for any $x \in D$,

$$\begin{aligned} Ax &= (\mathrm{id}_D \otimes \varphi_A)v_c^*(x \otimes I_c), \\ Nx &= (\mathrm{id}_D \otimes \varphi_N)v_c^*(x \otimes I_c). \end{aligned} \tag{1.27}$$

Remark. This shows that in the sense of operators on D we have equality

$$A = (\mathrm{id}_D \otimes \varphi_A)v_c^*, \qquad N = (\mathrm{id}_D \otimes \varphi_N)v_c^*.$$

Let us note that using unitarity of v_c and property (1.15) of φ_A, φ_N, and φ_R we can also compute that

$$(\mathrm{id}_D \otimes \varphi_A)v_c = A^{-1}, \quad (\mathrm{id}_D \otimes \varphi_R)v_c = A, \quad (\mathrm{id}_D \otimes \varphi_N)v_c = -\frac{1}{q}N \tag{1.28}$$

as for operators on D. Let $\varphi^*(a) = \overline{\varphi(a^*)}$ for any $\varphi \in \mathscr{A}_c'$ and $a \in \mathscr{A}_c$ then $\varphi^* \in \mathscr{A}_c'$. For $\varphi_{N^*} := \varphi_N^*$ we get

$$N^* = (\mathrm{id}_D \otimes \varphi_{N^*})v_c. \tag{1.29}$$

Since [cf. (1.16)] $\varphi_A^* = \varphi_R$, $\varphi_R^* = \varphi_A$ we have by (1.15) that

$$\varphi_{N^*}(ab) = \varphi_A(a)\varphi_{N^*}(b) + \varphi_{N^*}(a)\varphi_R(b), \tag{1.30}$$

and using this

$$(\mathrm{id}_D \otimes \varphi_{N^*})v_c^* = -\frac{1}{q}N^*. \tag{1.31}$$

Now we shall prove that $G_c = S_qU(2)$ is a Pontryagin dual to G_d, i.e. $\hat{G}_d = G_c$. This statement is not true for general compact quantum group and is closely related to the fact that the Haar measure h_c may not be faithful on the whole $\mathfrak{R}_c$ but this does not occur in the case of $S_qU(2)$.

Proposition 1.3. i) *Let $v_d \in M(CB(H) \otimes \mathfrak{R}_d)$ be a unitary representation of G_d acting on Hilbert space H.*

Then there exists the unique $\psi_c \in \mathrm{Mor}(\mathfrak{R}_c, CB(H))$ such that

$$v_d = \tau_d(\mathrm{id}_d \otimes \psi_c)U^*, \tag{1.32}$$

where τ_d is a flip $\tau_d: \mathfrak{R}_d \otimes B(H) \to B(H) \otimes \mathfrak{R}_d$.

ii) *Let $\psi_c \in \mathrm{Mor}(\mathfrak{R}_c, CB(H))$.*

Then v_d defined by (1.32) *belongs to $M(CB(H) \otimes \mathfrak{R}_d)$ and is a unitary representation of G_d.*

Proof. The unitarity of U^* implies that the operator v_d defined by (1.32) is unitary and simple calculation using the bicharacter property (1.13) shows that it is a representation of G_d.

Now we shall prove i). Let v_d be a unitary element of

$$M(CB(H)\otimes\mathfrak{R}_d)=M\Big(\sum_{s\in S}^{\oplus} CB(H)\otimes B(K^s)\Big)=\sum_{s\in S}^{\text{bounded}} B(H)\otimes B(K^s)$$

and $\pi_s\in\mathrm{Mor}(\mathfrak{R}_d, B(K^s))$ be the canonical projection then

$$v_d=\sum_{s\in S}^{\text{bounded}} v^s,$$

where $v^s=(\mathrm{id}_H\otimes\pi_s)v_d$. Choosing for any $s\in S$ the canonical orthonormal basis (1.19) in K^s we get the isomorphism $B(K^s)\sim M_{2s+1}(C)$. Let $\{m^s_{ij}\}$ be the matrix units for $M_{2s+1}(C)$ then $v^s\in M_{2s+1}(B(H))$ and

$$(v^s)^*=\sum_{i,j} x^s_{ij}\otimes m^s_{ij}$$

for some $x^s_{ij}\in B(H)$. Analogously, for $U=\sum_{s\in S}^{\text{bounded}} u^s$ we get

$$u^s=\sum_{i,j} m^s_{ij}\otimes a^s_{ij},$$

where $a^s_{ij}\in\mathscr{A}_c$ are matrix elements of u^s. Since the matrix elements of the irreducible unitary representations of G_c form a linear basis of the vector space $\mathscr{A}_c$ [14, Theorem 5.7(1)] then the map $\psi_c:\mathscr{A}_c\to B(H)$ defined by

$$\psi_c(a^s_{ij})=x^s_{ij} \tag{1.33}$$

is linear. We have to prove that if v_d is a representation of G_d then ψ_c is a multiplicative and $*$-preserving map. The multiplicativity of ψ_c follows from the fact that

$$(\Delta_d\otimes\mathrm{id}_c)U=U_{23}U_{13}=\sum_{s,s',i,j,i',j'} m^s_{ij}\otimes m^{s'}_{i'j'}\otimes a^{s'}_{i'j'}a^s_{ij}$$

and

$$(\mathrm{id}_H\otimes\Delta_d)v_d^*=(v_d)^*_{13}(v_d)^*_{12}=\sum_{s,s',i,j,i',j'} x^{s'}_{i'j'}x^s_{ij}\otimes m^s_{ij}\otimes m^{s'}_{i'j'}$$

so

$$\psi_c(a^{s'}_{i'j'}a^s_{ij})=x^{s'}_{i'j'}x^s_{ij}=\psi_c(a^{s'}_{i'j'})\psi_c(a^s_{ij})$$

for any $s,s'\in S$.

Now for $s=1/2$ the representation $u^{1/2}$ is given by (1.3) so

$$(v^{1/2})^*=\begin{pmatrix}\psi_c(\alpha_c) & -q\psi_c(\gamma_c^*)\\ \psi_c(\gamma_c) & \psi_c(\alpha_c^*)\end{pmatrix}\in M_2(B(H)). \tag{1.34}$$

Using the multiplicativity of ψ_c and unitarity of $u^{1/2}$ we get

$$\begin{pmatrix}\psi_c(\alpha_c) & -q\psi_c(\gamma_c^*)\\ \psi_c(\gamma_c) & \psi_c(\alpha_c^*)\end{pmatrix}\begin{pmatrix}\psi_c(\alpha_c^*) & \psi_c(\gamma_c^*)\\ -q\psi_c(\gamma_c) & \psi_c(\alpha_c)\end{pmatrix}=\begin{pmatrix}I_H & 0\\ 0 & I_H\end{pmatrix} \tag{1.35}$$

in $M_2(B(H))$. Since v_d is unitary then $(v^{1/2})^*$ in (1.34) is an unitary matrix in $M_2(B(H))$ and from (1.35) we get

$$\begin{pmatrix} \psi_c(\alpha_c^*) & \psi_c(\gamma_c^*) \\ -q\psi_c(\gamma_c) & \psi_c(\alpha_c) \end{pmatrix} = \begin{pmatrix} \psi_c(\alpha_c) & -q\psi_c(\gamma_c^*) \\ \psi_c(\gamma_c) & \psi_c(\alpha_c^*) \end{pmatrix}^{-1}$$
$$= \begin{pmatrix} \psi_c(\alpha_c) & -q\psi_c(\gamma_c^*) \\ \psi_c(\gamma_c) & \psi_c(\alpha_c^*) \end{pmatrix}^{*} = \begin{pmatrix} \psi_c(\alpha_c)^* & \psi_c(\gamma_c)^* \\ -q\psi_c(\gamma_c^*)^* & \psi_c(\alpha_c^*)^* \end{pmatrix}$$

and $\psi_c(\alpha_c^*) = \psi_c(\alpha_c)^*$, $\psi_c(\gamma_c^*) = \psi_c(\gamma_c)^*$. This means that $\psi_c : \mathscr{A}_c \to B(H)$ is the unital $*$-homomorphism. Moreover, $(v^{1/2})^*$ is the unitary matrix of the form $\begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix}$ so $a = \psi_c(\alpha_c)$, $c = \psi_c(\gamma_c)$ has to satisfy the relations (1.2) (cf. [8] remark after (1.33)) and ψ_c has the unique extension to a C^*-representation of $\mathfrak{R}_c$ in $B(H)$ [13, Theorem 1.1]. Since $\mathfrak{R}_c$ is the unital C^*-algebra this implies that $\psi_c \in \mathrm{Mor}(\mathfrak{R}_c, CB(H))$ and $v_{\hat{d}}^* = \tau_d(\mathrm{id}_d \otimes \psi_c)U$. This proves the existence of ψ_c satisfying (1.32). The uniqueness of ψ_c follows from the fact that any $\psi_c \in \mathrm{Mor}(\mathfrak{R}_c, CB(H))$ is uniquely defined by its values on $\mathscr{A}_c$ but if ψ_c satisfies (1.32) it has to satisfy (1.33) which completely determines ψ_c on $\mathscr{A}_c$. Q.E.D.

Remark. Propositions 1.1 and 1.3 show that G_c and G_d are mutually Pontryagin dual groups. Let a denote one of the letters c or d and $\hat{c} = d$, $\hat{d} = c$. Then the correspondence $v_a \to \psi_{\hat{a}}$ is natural in the sense that for any unitary representations $v_a^{(1)}, v_a^{(2)}$ of G_a acting on the Hilbert spaces H_1 and H_2, respectively and corresponding representations $\psi_{\hat{a}}^{(1)}, \psi_{\hat{a}}^{(2)}$ of $\mathfrak{R}_{\hat{a}}$ we have

$$\{t \in B(H_1, H_2) \colon (t \otimes I_a)v_a^{(1)} = v_a^{(2)}(t \otimes I_a)\}$$
$$= \{t \in B(H_1, H_2) \colon t\psi_{\hat{a}}^{(1)}(x) = \psi_{\hat{a}}^{(2)}(x)t \text{ for all } x \in \mathfrak{R}_{\hat{a}}\}$$

(cf. Remark preceding §4 in [8]). In particular, unitary representation v_a of G_a is irreducible if and only if the representation $\psi_{\hat{a}}$ of $\mathfrak{R}_{\hat{a}}$ is irreducible. Let us note also that any operator $t \in B(H_1, H_2)$ intertwines representations $v_c^{(1)}$ and $v_c^{(2)}$ if and only if it maps natural domains into itself $t : D_1 \to D_2$ and for any $y \in D_1$,

$$t\psi_d^{(1)}(A_d)y = \psi_d^{(2)}(A_d)ty\,, \qquad t\psi_d^{(1)}(N_d)y = \psi_d^{(2)}(N_d)ty\,,$$
$$t\psi_d^{(1)}(N_d^*)y = \psi_d^{(2)}(N_d^*)ty\,.$$

The last part of this section we devote to the description of unitary representations of quantum Lorentz group in terms of the representations of G_c and G_d. Let $G = (\mathfrak{R}, \Delta)$ be the quantum Lorentz group. The algebra of "*continuous functions tending to* 0 *at infinity*" on G is

$$\mathfrak{R} = \mathfrak{R}_c \otimes \mathfrak{R}_d\,.$$

If

$$\tilde{\sigma} : \mathfrak{R}_c \otimes \mathfrak{R}_d \to \mathfrak{R}_d \otimes \mathfrak{R}_c$$

is given by

$$\tilde{\sigma}(a \otimes x) = U(x \otimes a)U^*\,, \tag{1.36}$$

where U is the bicharacter (1.11) then

$$\Delta = (\mathrm{id}_c \otimes \tilde{\sigma} \otimes \mathrm{id}_d)(\Delta_c \otimes \Delta_d)\,.$$

The counit is given by $e=e_c\otimes e_d$ (cf. (4.9), (4.16), (4.17), and Theorem 4.1 in [8]).

Let $p_c=\mathrm{id}_c\otimes e_d$ and $p_d=e_c\otimes\mathrm{id}_d$ then $p_c\in\mathrm{Mor}(\mathfrak{R},\mathfrak{R}_c)$, $p_d\in\mathrm{Mor}(\mathfrak{R},\mathfrak{R}_d)$. They correspond to the embeddings $G_c\to G$, $G_d\to G$. The structural theorem for unitary representations of G is related to these embeddings.

Proposition 1.4. i) *Let $v\in M(CB(H)\otimes\mathfrak{R})$ be a unitary representation of G acting on the Hilbert space H and let*

$$v_c=(\mathrm{id}_H\otimes p_c)v\,,\qquad v_d=(\mathrm{id}_H\otimes p_d)v\,. \tag{1.37}$$

Then $v_c\in M(CB(H)\otimes\mathfrak{R}_c)$, $v_d\in M(CB(H)\otimes\mathfrak{R}_d)$ and they are the unitary representations of G_c and G_d, respectively, acting on the Hilbert space H satisfying the compatibility condition

$$(v_d)_{12}(v_c)_{13}=(\mathrm{id}_H\otimes\tilde{\sigma})(v_c)_{12}(v_d)_{13} \tag{1.38}$$

and

$$v=(v_c)_{12}(v_d)_{13}\,. \tag{1.39}$$

ii) *Let $v_c\in M(CB(H)\otimes\mathfrak{R}_c)$, $v_d\in M(CB(H)\otimes\mathfrak{R}_d)$ be the unitary representations of G_c and G_d, respectively, acting on the same Hilbert space H and satisfying the compatibility condition* (1.38).

Then v defined by (1.39) *belongs to $M(CB(H)\otimes\mathfrak{R})$ and is the unitary representation of G.*

Proof. The arguments used in the proof of Theorem 4.4 in [8] for the case of finite-dimensional representations of G are still valid. One has to check the unitarity condition only but this is obvious because p_c and p_d are morphisms. Q.E.D.

This proposition reduces the study of unitary representations of G to the study of the pairs of unitary representations of G_c and G_d, respectively, acting in the same Hilbert space and satisfying the compatibility condition (1.38). In view of Propositions 1.1 and 1.3 and Corollary 1.2 we would like to replace the unitary representations v_c, v_d by corresponding representations ψ_d, ψ_c of algebras $\mathfrak{R}_d$ and $\mathfrak{R}_c$, respectively. The main problem is to express the compatibility condition (1.38) in terms of ψ_d and ψ_c. A partial solution to this was given in [8, Proposition 4.5].

Let $\psi_c\in\mathrm{Mor}(\mathfrak{R}_c, CB(H))$ corresponds to v_d and let for $a\in\mathfrak{R}_c$,

$$a*\psi_c:=(\psi_c\otimes\mathrm{id}_c)\Delta_c(a)\,,\qquad \psi_c*a:=\tau_c(\mathrm{id}_c\otimes\psi_c)\Delta_c(a)\,,$$

where τ_c is a flip $\tau_c:\mathfrak{R}_c\otimes B(H)\to B(H)\otimes\mathfrak{R}_c$.

It was proved [8, Proposition 4.5] that v_c and v_d satisfy compatibility condition (1.38) if and only if

$$(a*\psi_c)v_c=v_c(\psi_c*a) \tag{1.40}$$

for any $a\in\{\alpha_c,\alpha_c^*,\gamma_c,\gamma_c^*\}$. To abbreviate the notation we shall denote

$$\alpha=\psi_c(\alpha_c)\,,\qquad \gamma=\psi_c(\gamma_c)\,.$$

Then $\alpha\in B(H)$, $\gamma\in B(H)$ and they satisfy relations (1.2). Now (1.40) reads

$$\begin{aligned}
(\alpha\otimes\alpha_c-q\gamma^*\otimes\gamma_c)v_c&=v_c(\alpha\otimes\alpha_c-q\gamma\otimes\gamma_c^*)\,,\\
(\alpha^*\otimes\alpha_c^*-q\gamma\otimes\gamma_c^*)v_c&=v_c(\alpha^*\otimes\alpha_c^*-q\gamma^*\otimes\gamma_c)\,,\\
(\gamma\otimes\alpha_c+\alpha^*\otimes\gamma_c)v_c&=v_c(\alpha\otimes\gamma_c+\gamma\otimes\alpha_c^*)\,,\\
(\gamma^*\otimes\alpha_c^*+\alpha\otimes\gamma_c^*)v_c&=v_c(\alpha^*\otimes\gamma_c^*+\gamma^*\otimes\alpha_c)\,.
\end{aligned} \tag{1.41}$$

We have

Proposition 1.5. *Let* $v_c \in M(CB(H), \mathfrak{R}_c)$ *and* $v_d \in M(CB(H), \mathfrak{R}_d)$ *be the unitary representations of* G_c *and* G_d *acting on the same Hilbert space* H *and let*

$$D = \sum_{s\in S(v_c)}^{\text{finite}} H^s = \sum_{s\in S(v_c)}^{\text{finite}} K^s \otimes H_s$$

be the natural domain (1.24) *for* v_c. *Let* ψ_d, ψ_c *be the corresponding representations of algebras* $\mathfrak{R}_d$ *and* $\mathfrak{R}_c$, *respectively on* H *and*

$$\begin{aligned} A &= \psi_d(A_d), \qquad N = \psi_d(N_d), \\ \alpha &= \psi_c(\alpha_c), \qquad \gamma = \psi_c(\gamma_c) \end{aligned} \tag{1.42}$$

be the generators of these representations (i.e. A and N have the form (1.25) *so they satisfy relations*

$$\left.\begin{aligned} A &= A^* > 0, \\ AN &= qNA, \\ NN^* &= N^*N + \frac{1}{1-q^2}(A^{-2} - A^2) \end{aligned}\right\} \tag{1.43}$$

on D and

$$\left.\begin{aligned} &\alpha^*\alpha + \gamma^*\gamma = I_H, \\ &\alpha\alpha^* + q^2\gamma^*\gamma = I_H, \\ &\alpha\gamma = q\gamma\alpha, \quad \gamma\gamma^* = \gamma^*\gamma, \quad \alpha\gamma^* = q\gamma^*\alpha \end{aligned}\right\} \tag{1.44}$$

on H).

Then the following conditions are equivalent:

i) v_c *and* v_d *satisfy the compatibility condition* (1.38).

ii) *The operator*

$$t = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \in M_2(B(H)) \tag{1.45}$$

is an (unitary) intertwining operator for the unitary representations

$$V_c^{(1)} = (u^{1/2})_{23}(v_c)_{13}, \qquad V_c^{(2)} = (v_c)_{13}(u^{1/2})_{23}$$

acting on $\hat{H} = H \otimes K^{1/2}$, *i.e.*

$$(t \otimes I_c)V_c^{(1)} = V_c^{(2)}(t \otimes I_c). \tag{1.46}$$

iii) $a: D \to D$ *for any* $a \in \{\alpha, \alpha^*, \gamma, \gamma^*\}$ *and the following relations*

$$\left.\begin{aligned} &A\alpha = \alpha A, \qquad A\gamma = \frac{1}{q}\gamma A, \\ &N\alpha = q\alpha N - q\gamma^* A, \qquad N\gamma = \gamma N + \frac{1}{q}(\alpha^* A - \alpha A^{-1}), \\ &N\alpha^* = \frac{1}{q}\alpha^* N + \frac{1}{q}\gamma^* A^{-1}, \qquad N\gamma^* = \gamma^* N \end{aligned}\right\} \tag{1.47}$$

are satisfied on D.

Proof. Let us identify $\hat{H}=H\otimes K^{1/2}=H\oplus H$. Then using (1.3) the condition (1.46) means that

$$\begin{pmatrix} v_c & 0 \\ 0 & v_c \end{pmatrix}\begin{pmatrix} \alpha\otimes\alpha_c - q\gamma\otimes\gamma_c^*, & -q\gamma^*\otimes\alpha_c - q\alpha^*\otimes\gamma_c^* \\ \alpha\otimes\gamma_c+\gamma\otimes\alpha_c^*, & -q\gamma^*\otimes\gamma_c+\alpha^*\otimes\alpha_c^* \end{pmatrix}$$
$$=\begin{pmatrix} \alpha\otimes\alpha_c - q\gamma^*\otimes\gamma_c, & -q\gamma^*\otimes\alpha_c^* - q\alpha\otimes\gamma_c^* \\ \alpha^*\otimes\gamma_c+\gamma\otimes\alpha_c, & -q\gamma\otimes\gamma_c^*+\alpha^*\otimes\alpha_c^* \end{pmatrix}\begin{pmatrix} v_c & 0 \\ 0 & v_c \end{pmatrix}$$

and this are exactly the relations (1.41) so i) and ii) are equivalent. Now the intertwiner property (1.46) is equivalent to the statement that t maps the natural domains $\hat{D}_1, \hat{D}_2$ for $V_c^{(1)}$ and $V_c^{(2)}$, respectively, into itself and intertwines the generators $\hat{A}_1, \hat{N}_1$ and $\hat{A}_2, \hat{N}_2$ of the corresponding representations $\psi_d^{(1)}, \psi_d^{(2)}$. Since

$$\hat{D}_1 = D\otimes K^{1/2} = D\oplus D = \hat{D}_2\,,$$

then by (1.45) $a(D)\subset D$ for any $a\in\{\alpha,\alpha^*,\gamma,\gamma^*\}$. Moreover, using (1.28) and properties (1.15) and (1.16) of functionals φ_A, φ_R, and φ_N we get

$$\hat{A}_1 = (\mathrm{id}_{\hat{D}_1}\otimes\varphi_R)V_c^{(1)} = q^{-1/2}\begin{pmatrix} A & 0 \\ 0 & qA \end{pmatrix} = (\mathrm{id}_{\hat{D}_1}\otimes\varphi_R)V_c^{(2)} = \hat{A}_2\,,$$
$$\hat{N}_1 = -q(\mathrm{id}_{\hat{D}_1}\otimes\varphi_N)V_c^{(1)} = q^{-1/2}\begin{pmatrix} qN & 0 \\ A & N \end{pmatrix},$$
$$\hat{N}_2 = -q(\mathrm{id}_{\hat{D}_2}\otimes\varphi_N)V_c^{(2)} = q^{-1/2}\begin{pmatrix} N & 0 \\ A^{-1} & qN \end{pmatrix}.$$

Now relations $t\hat{A}_1=\hat{A}_2 t$, $t\hat{N}_1=\hat{N}_2 t$ are the same as relations (1.47) so the equivalence of ii) and iii) follows. Q.E.D.

Let us note that Eqs. (1.43), (1.44), and (1.47) imply that an algebra of *continuous functions tending to* 0 *at infinity* on a dual group $\hat{G}$ of the quantum Lorentz group G is "generated" by $\{A,N,\alpha,\gamma\}$ so it contains the algebras $\mathfrak{R}_c$, $\mathfrak{R}_d$, but this is not the tensor product of them. Let us denote it symbolically by

$$\hat{\mathfrak{R}} = \mathfrak{R}_c \odot \mathfrak{R}_d\,.$$

Then one can define $\hat{\Delta}=\Delta_c\odot\Delta_d$, $\hat{e}=e_c\odot e_d$, and $\hat{\kappa}=\kappa_c\odot\kappa_d$ which will impose the Hopf $*$-algebra structure on $\hat{\mathfrak{R}}$ or equivalently it will define the Pontryagin dual group $\hat{G}$ on the Hopf $*$-algebra level. Since G_c and G_d are mutually Pontryagin dual groups one could expect that the quantum Lorentz group is selfdual, but as we see it is not the case. A Hopf algebra structure was recently studied in [10, 6].

Proposition 1.5 reduces the problem of classification of the unitary representations of quantum Lorentz group to the problem of classification of four operators $\{A,N,\alpha,\gamma\}$ satisfying on D the relations (1.43), (1.44), and (1.47). For operators A and N it is clear. To incorporate operators α and γ into this scheme we use the method of tensor operators.

2. Tensor Operators

2.1. Basic Notions and Operations

Let $v_c\in M(CB(H)\otimes\mathfrak{R}_c)$ be a unitary representation of G_c acting on the Hilbert space H, ψ_d be the corresponding representation of $\mathfrak{R}_d$ and

$$A=\psi_d(A_d)\,,\qquad N=\psi_d(N_d)$$

be the generators of this representation. Let

$$D = \sum_{s\in S(v_c)}^{\text{finite}} H^s = \sum_{s\in S(v_c)}^{\text{finite}} K^s \otimes H_s \tag{2.1}$$

be the natural domain for v_c. We shall consider an *-algebra of linear (in general unbounded) operators (cf. [4])

$$L^+(D) = \{T:\ D_T = D,\ D_{T^*} \supset D,\ T(D) \subset D,\ T^*(D) \subset D\}. \tag{2.2}$$

Clearly, any $T \in L^+(D)$ is a closeable operator on H. There is an induced action of G_c on the *-algebra $L^+(D)$

$$V_c(T) := v_c(T \otimes I_c)v_c^* \quad \text{for any} \quad T \in L^+(D) \tag{2.3}$$

since it is easy to check that

$$(\mathrm{id}_{L^+(D)} \otimes \Delta_c)V_c = (V_c \otimes \mathrm{id}_c)V_c.$$

Taking in mind (1.28), (1.15), (1.29), (1.30), and (1.31) we get for generators $\hat{A}, \hat{N}, \hat{N}^*$ of the corresponding representation of $\mathfrak{R}_d$,

$$\begin{aligned} \hat{A}(T) &= ATA^{-1}, \\ \hat{N}(T) &= NTA^{-1} - qA^{-1}TN, \\ \hat{N}^\dagger(T) &= N^*TA^{-1} - \frac{1}{q}A^{-1}TN^*. \end{aligned} \tag{2.4}$$

In what follows we shall consider s-tensor operators, i.e. sets of operators which transforms under the action of V_c according to the representation u^s of G_c. Since we will be interested only in the case of $s=0$ and $s=1$ we shall restrict our definition of s-tensor operators to this particular case.

Definition 2.1. i) An operator $X \in L^+(D)$ is called *a scalar operator* if

$$V_c(X) = v_c(X \otimes I_c)v_c^* = X \otimes I_c. \tag{2.5}$$

The set of all scalar operators will be denoted by $\mathscr{T}^0(D)$.
ii) A triplet

$$\mathbf{Y} = \begin{pmatrix} Y_1 \\ Y_0 \\ Y_{-1} \end{pmatrix}, \tag{2.6}$$

where Y_1, Y_0, Y_{-1} belong to $L^+(D)$ is called *a vector operator* if

$$V_c(Y_j)(x \otimes I_c) = v_c(Y_j \otimes I_c)v_c^*(x \otimes I_c) = \sum_{k\in\{-1,0,1\}} Y_k x \otimes u^1_{kj} \tag{2.7}$$

for $j \in \{-1, 0, 1\}$ and any $x \in D$.

The set of all vector operators will be denoted by $\mathscr{T}^1(D)$.

Using (2.4) and (1.20) we see that

1. $X \in \mathscr{T}^0(D)$ if and only if

$$AX = XA, \qquad NX = XN, \qquad N^*X = XN^* \tag{2.8}$$

on D. This implies that $\mathscr{T}^0(D)$ is a *-algebra.

2. $\mathbf{Y} \in \mathscr{T}^1(D)$ if and only if

$$\begin{gathered}
AY_1 = qY_1A, \quad AY_0 = Y_0A, \quad AY_{-1} = \frac{1}{q}Y_{-1}A, \\
NY_1 = \frac{1}{q}Y_1N, \quad N^*Y_1 = \frac{1}{q}Y_1N^* + \frac{\sqrt{1+q^2}}{q}Y_0A, \\
NY_0 = Y_0N + \frac{\sqrt{1+q^2}}{q}Y_1A, \quad N^*Y_0 = Y_0N^* + \frac{\sqrt{1+q^2}}{q}Y_{-1}A, \\
NY_{-1} = qY_{-1}N + \frac{\sqrt{1+q^2}}{q}Y_0A, \quad N^*Y_{-1} = qY_{-1}N^*
\end{gathered} \tag{2.9}$$

on D. Clearly, $\mathscr{T}^1(D)$ is a vector space and also a bimodule over $\mathscr{T}^0(D)$.

For any $\mathbf{Y} \in \mathscr{T}^1(D)$ let us define

$$\mathbf{Y}^\dagger = \begin{pmatrix} Y_1 \\ Y_0 \\ Y_{-1} \end{pmatrix}^\dagger = \begin{pmatrix} -qY_{-1}^* \\ Y_0^* \\ -\frac{1}{q}Y_1^* \end{pmatrix}. \tag{2.10}$$

Then using (2.9) it is easy to show that the map

$$\mathscr{T}^1(D) \ni \mathbf{Y} \mapsto \mathbf{Y}^\dagger \in \mathscr{T}^1(D)$$

is an antilinear involution in $\mathscr{T}^1(D)$.

Since the tensor product of two u^1 representations of G_c decomposes into a direct sum $u^0 \oplus u^1 \oplus u^2$ one can expect that there exist two bilinear maps

$$\mathscr{T}^1(D) \times \mathscr{T}^1(D) \ni (\mathbf{Y}, \mathbf{Z}) \mapsto \mathbf{Y} \bullet \mathbf{Z} \in \mathscr{T}^0(D) \tag{2.11}$$

and

$$\mathscr{T}^1(D) \times \mathscr{T}^1(D) \ni (\mathbf{Y}, \mathbf{Z}) \mapsto \mathbf{Y} \times \mathbf{Z} \in \mathscr{T}^1(D). \tag{2.12}$$

This is really the case and using (2.8), (2.9) one can check that operations

$$\begin{pmatrix} Y_1 \\ Y_0 \\ Y_{-1} \end{pmatrix} \bullet \begin{pmatrix} Z_1 \\ Z_0 \\ Z_{-1} \end{pmatrix} = -\frac{1}{q}Y_{-1}Z_1 + Y_0Z_0 - qY_1Z_{-1}, \tag{2.13}$$

$$\begin{pmatrix} Y_1 \\ Y_0 \\ Y_{-1} \end{pmatrix} \times \begin{pmatrix} Z_1 \\ Z_0 \\ Z_{-1} \end{pmatrix} = \frac{1}{\sqrt{1+q^2}} \begin{pmatrix} Y_0Z_1 - q^2Y_1Z_0 \\ q(Y_{-1}Z_1 - Y_1Z_{-1}) + (1-q^2)Y_0Z_0 \\ Y_{-1}Z_0 - q^2Y_0Z_{-1} \end{pmatrix} \tag{2.14}$$

called the scalar product and the vector product of vector operators satisfy requirements (2.11) and (2.12), respectively.

2.2. Basic Maps

Tensor operators on D are intertwiners for the appropriate actions of G_c. We use this fact to describe a structure of such operators and related operations described above.

Let $X \in \mathcal{T}^0(D)$ be a scalar operator on the natural domain D for the unitary representation v_c of G_c [i.e. D has the form (2.1)]. Then by (2.5) it is an intertwining operator for v_c on D. Since X is closeable and v_c restricted to H^s is $u^s \otimes I_{H_s}$ and u^s is an irreducible representation of G_c then

$$X = \sum_{s \in S(v_c)} I_{K^s} \otimes \Phi_s^0(X), \tag{2.15}$$

where $\Phi_s^0(X) \in B(H_s)$ is a bounded operator on H_s for any $s \in S(v_c)$. In what follows we shall identify K^s for $s \geqq 1$ as the q-symmetric part of $\bigotimes_{n=1}^{2s} K^{1/2}$ where $K^{1/2}$ is a 2-dimensional carrier Hilbert space of the fundamental representation $u^{1/2}$ [cf. (1.3)]. For $s=1$ we shall abbreviate $K^1 = K$. Let $\{f_m^s : m = -s, -s+1, \ldots, s-1, s\}$ be the canonical basis (1.19) in K^s. For $\mathbf{Y} \in \mathcal{T}^1(D)$ let us define a map

$$\begin{aligned} &\Phi(\mathbf{Y}) : K \otimes D \to D, \\ &\Phi(\mathbf{Y})(f_j^1 \otimes x) = Y_j x. \end{aligned} \tag{2.16}$$

It is clear using (2.7) that $\Phi(\mathbf{Y})$ is an intertwining operator for the G_c actions by $u^1 \otimes v_c$ on $K \otimes D$ and v_c on D. Since for any $s \in S$ the tensor product of representations u^1 and u^s has a decomposition into a direct sum $\bigoplus_{s'=|s-1|}^{s+1} u^{s'}$ we define *a proximity relations for spins* $s, s' \in S$. We shall say that s' *is near* s and denote it by $s' \sim s$ whenever $u^{s'}$ is contained in the tensor product of u^1 and u^s. This means that

$$s' \sim s \quad \text{if and only if} \quad \begin{cases} s' \in \{s-1, s, s+1\} & \text{for} \quad s \geqq 1, \\ s' \in \{1/2, 3/2\} & \text{for} \quad s = 1/2, \\ s' = 1 & \text{for} \quad s = 0. \end{cases} \tag{2.17}$$

To describe in more detail the structure of operator $\Phi(\mathbf{Y})$ and operations mentioned previously we shall fix some intertwiners for G_c actions. To this end we use technique of diagrams.

By one vertical line we denote an element of $K^{1/2}$. Since $u^{1/2} \otimes u^{1/2} = u^0 \oplus u^1$ and the corresponding decomposition is

$$K^{1/2} \otimes K^{1/2} = K^0 \oplus K^1$$

(this corresponds to the decomposition into q-antisymmetric and q-symmetric tensors) then there are two intertwiners:

$$E : K^0 = C \to K^{1/2} \otimes K^{1/2}, \qquad E' : K^{1/2} \otimes K^{1/2} \to C. \tag{2.18}$$

We shall denote them by

$$E = \bigwedge, \qquad E' = \bigvee \tag{2.19}$$

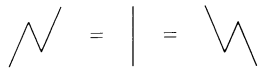
and fix as

$$E(1) = f_{-1/2}^{1/2} \otimes f_{1/2}^{1/2} - q f_{1/2}^{1/2} \otimes f_{-1/2}^{1/2}, \tag{2.20}$$

and

$$\begin{aligned} &E'(f_{-1/2}^{1/2} \otimes f_{-1/2}^{1/2}) = 0 = E'(f_{1/2}^{1/2} \otimes f_{1/2}^{1/2}), \\ &E'(f_{-1/2}^{1/2} \otimes f_{1/2}^{1/2}) = -\frac{1}{q}, \qquad E'(f_{1/2}^{1/2} \otimes f_{-1/2}^{1/2}) = 1. \end{aligned} \tag{2.21}$$

One can check that



(2.22)

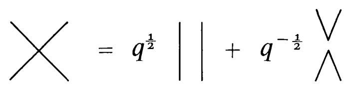
and

$$E' \cdot E = \quad$$  $$= -\frac{1+q^2}{q}\,. \tag{2.23}$$

Let us define an intertwiner

$$\sigma : K^{1/2} \otimes K^{1/2} \to K^{1/2} \otimes K^{1/2}$$

by

$\sigma =$  (2.24)

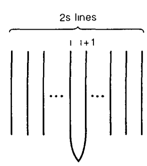
Then σ satisfies a quadratic equation

$$\sigma^2 + (q^{-3/2} - q^{1/2})\sigma - \frac{1}{q} = 0$$

and σ has two eigenvalues: $q^{1/2}$ on $K^1 = K$ and $-q^{-3/2}$ on $K^0 = C$. One can also check that σ satisfies the braid equation. Using (2.24) and (2.22) we have

(2.25)

Let $x \in \bigotimes_{n=1}^{2s} K^{1/2}$ then $x \in K^s$ if and only if



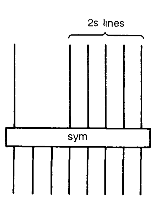
$= 0,$ (2.26)

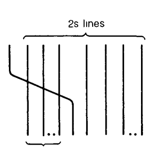
where E' is applied to any pair $(i, i+1)$ of lines.

We shall also need a symmetrization operator (intertwiners)

$$\hat{\Psi}_{(s+\frac{1}{2})s} : K^{1/2} \otimes K^s \to K^{s+\frac{1}{2}}, \qquad \Psi_{(s+1)s} : K \otimes K^s \to K^{s+1}\,.$$

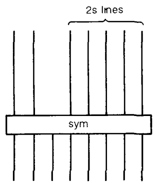
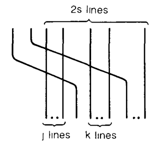
Since $K^{s+\frac{1}{2}}$ and K^{s+1} can be identified as subspaces of $K^{1/2} \otimes K^s$ and $K \otimes K^s$, respectively, we can take

$\hat{\Psi}_{(s+\frac{1}{2})s} =$ 

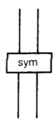
$= \frac{1}{p_{1,s}} \sum_{j=0}^{2s} q^{\frac{3}{2}j}$ 

, (2.27)

where $p_{1,s}=\sum_{j=0}^{2s} q^{2j}$ is a normalization factor,

$\Psi_{(s+1)s}=$  $=\frac{1}{p_{2,s}}\sum_{\substack{j,k=0\\ j+k\leqq 2s}} q^{\frac{3}{2}(2j+k)}$  (2.28)

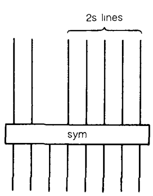
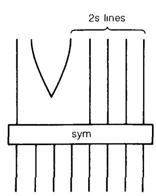
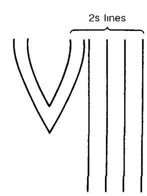
and $p_{2,s}=\sum_{j,k=0,\, j+k\leqq 2s} q^{(2j+k)}$ is a normalization factor. They are projections as can be seen using (2.26) and (2.24). In particular,

 $=$ $+\ \frac{q}{1+q^2}$  . (2.29)

Now we fix intertwiners

$$\Psi_{s's}: K\otimes K^s \rightarrow K^{s'}$$

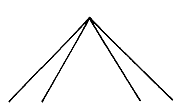
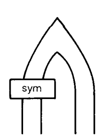
for $s'\sim s$:

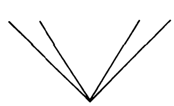
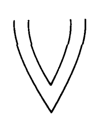
$\Psi_{(s+1)s}$  Ψ_{ss}  $\Psi_{(s-1)s}$  (2.30)

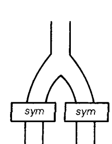
and let

$$\Theta: C\rightarrow K\otimes K, \quad \Theta^*: K\otimes K\rightarrow C, \quad T: K\rightarrow K\otimes K$$

be fixed operators given by

$C \overset{\Theta}{\downarrow} K\otimes K \ =$  $=-\frac{1}{q}$  , (2.31)

$K\otimes K \overset{\Theta^*}{\downarrow} C \ =$  $=-q$  , (2.32)

$K \overset{T}{\downarrow} K\otimes K \ =$  . (2.33)

Now for $\mathbf{Y}\in\mathscr{T}^1(D)$

$$\Phi(\mathbf{Y}): K\otimes D=\sum_{s\in S(v_c)} K\otimes K^s\otimes H_s \rightarrow \sum_{s'\in S(v_c)} K^{s'}\otimes H_{s'}$$

can be written as

$$\Phi(\mathbf{Y})=\sum_{\substack{s',s\in S(v_c)\\ s'\sim s}}\Psi_{s's}\otimes\Phi^1_{s's}(\mathbf{Y}),\tag{2.34}$$

where $\Phi^1_{s's}(\mathbf{Y})\in B(H_s,H_{s'})$ for $s,s'\in S(v_c)$, $s'\sim s$ and completely determine $\Phi(\mathbf{Y})$:

$$\begin{array}{c} K\otimes H^s \\ \Phi(\mathbf{Y})\Big\downarrow \\ H^{s'} \end{array} = \begin{array}{cc} \underbrace{K\otimes K^s}\otimes & H_s \\ \Psi_{s's}\Big\downarrow & \Big\downarrow \Phi^1_{s's}(\mathbf{Y}) \\ K^{s'}\otimes & H_{s'} \end{array} .\tag{2.35}$$

Since

$$\Theta(1)=-\frac{1}{q}f^1_{-1}\otimes f^1_1+f^1_0\otimes f^1_0-qf^1_1\otimes f^1_{-1}\tag{2.36}$$

and

$$\begin{aligned} Tf^1_1&=\frac{1}{\sqrt{1+q^2}}(f^1_0\otimes f^1_1-q^2f^1_1\otimes f^1_0),\\ Tf^1_0&=\frac{1}{\sqrt{1+q^2}}[q(f^1_{-1}\otimes f^1_1-f^1_1\otimes f^1_{-1})+(1-q^2)f^1_0\otimes f^1_0],\\ Tf^1_{-1}&=\frac{1}{\sqrt{1+q^2}}(f^1_{-1}\otimes f^1_0-q^2f^1_0\otimes f^1_{-1}) \end{aligned}\tag{2.37}$$

we get by (2.16) for $\mathbf{Y},\mathbf{Z}\in\mathscr{T}^1(D)$

$$\begin{array}{ccc} C & \otimes & D \\ \Theta\Big\downarrow & & \Big\downarrow\mathrm{id} \\ K\otimes & \underbrace{K\otimes D} & \\ \mathrm{id}\Big\downarrow & \Big\downarrow\Phi(\mathbf{Z}) & \\ \underbrace{K\ \otimes} & D & \\ & \Big\downarrow\Phi(\mathbf{Y}) & \\ & D & \end{array} = \begin{array}{c} D \\ \Big\downarrow \mathbf{Y}\bullet\mathbf{Z} \\ D \end{array}\tag{2.38}$$

and

$$\begin{array}{ccc} K & \otimes & D \\ T\Big\downarrow & & \Big\downarrow\mathrm{id} \\ K\otimes & \underbrace{K\otimes D} & \\ \mathrm{id}\Big\downarrow & \Big\downarrow\Phi(\mathbf{Z}) & \\ \underbrace{K\ \otimes} & D & \\ & \Big\downarrow\Phi(\mathbf{Y}) & \\ & D & \end{array} = \begin{array}{c} K\otimes D \\ \Big\downarrow \Phi(\mathbf{Y}\times\mathbf{Z}) \\ D \end{array} .\tag{2.39}$$

Moreover, since for $x \in D$

$$\Phi(\mathbf{Y})^* x = \sum_{j \in \{-1,0,1\}} f_j^1 \otimes Y_j^* x$$

and

$$\Theta^*(f_1^1 \otimes f_j^1) = -q\delta_{j,-1},$$
$$\Theta^*(f_0^1 \otimes f_j^1) = \delta_{j,0},$$
$$\Theta^*(f_{-1}^1 \otimes f_j^1) = -\frac{1}{q}\delta_{j,-1}$$

we have

$$\begin{array}{ccc} K & \otimes & D \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \Phi(\mathbf{Y})^*} \\ \underbrace{K \otimes K} & \otimes & D \\ {\scriptstyle \Theta^*}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ C & \otimes & D \end{array} \quad = \quad \begin{array}{c} K \otimes D \\ \downarrow{\scriptstyle \Phi(\mathbf{Y}^\dagger)} \\ D \end{array} . \tag{2.40}$$

Now we can compute $\Phi_s^0(\mathbf{Y} \bullet \mathbf{Z})$, $\Phi_{s's}^1(\mathbf{Y} \times \mathbf{Z})$, and $\Phi_{s's}^1(\mathbf{Y}^\dagger)$ in terms of $\Phi_{s's}^1(\mathbf{Y})$ and $\Phi_{s's}^1(\mathbf{Z})$.

a) *The Scalar Product of Vector Operators.* From Eqs. (2.15), (2.16), (2.13) using (2.38) and (2.36) we have

$$\begin{array}{ccc} K^s \otimes H_s \\ {\scriptstyle \mathrm{id}}\downarrow \quad \downarrow{\scriptstyle \Phi_s^0(\mathbf{Y}\bullet\mathbf{Z})} \\ K^s \otimes H_s \end{array} = \sum_{s':s'\sim s} \begin{array}{ccccc} & C & \otimes & K^s \otimes H_s & \\ & {\scriptstyle \Theta}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} & \\ K \otimes & \underbrace{K \otimes K^s} & & \otimes\ H_s & \\ {\scriptstyle \mathrm{id}}\downarrow & \downarrow{\scriptstyle \Psi_{s's}} & & \downarrow{\scriptstyle \Phi_{s's}^1(\mathbf{Z})} & \\ \underbrace{K \otimes K^{s'}} & & \otimes & H_{s'} & \\ \downarrow{\scriptstyle \Psi_{ss'}} & & & \downarrow{\scriptstyle \Phi_{ss'}^1(\mathbf{Y})} & \\ K^s & & \otimes & H_s & \end{array}$$

so

$$\Phi_s^0(\mathbf{Y} \bullet \mathbf{Z}) = \sum_{s':s'\sim s} \lambda_{ss's} \Phi_{ss'}^1(\mathbf{Y}) \Phi_{s's}^1(\mathbf{Z}), \tag{2.41}$$

where

$$\begin{array}{ccc} & C & \otimes\ K^s \\ & {\scriptstyle \Theta}\downarrow & \downarrow{\scriptstyle \mathrm{id}} \\ K \otimes & \underbrace{K \otimes K^s} & \\ {\scriptstyle \mathrm{id}}\downarrow & \downarrow{\scriptstyle \Psi_{s's}} & \\ K \otimes & K^{s'} & \\ & \downarrow{\scriptstyle \Psi_{ss'}} & \\ & K^s & \end{array} = \lambda_{ss's} \begin{array}{c} K^s \\ {\scriptstyle \mathrm{id}}\downarrow \\ K^s \end{array} . \tag{2.42}$$

The $\lambda_{ss's}$ for $s' \sim s$ in (2.42) can be computed and

$$\begin{aligned} \lambda_{s(s+1)s} &= -\frac{[2s+3]_q}{q^3[2s+1]_q}, \\ \lambda_{sss} &= \frac{[2s+2]_q}{q^2(1+q^2)[2s]_q} \quad \text{for} \quad s>0, \\ \lambda_{s(s-1)s} &= -\frac{1}{q} \qquad\qquad\qquad \text{for} \quad s \geqq 1. \end{aligned} \tag{2.43}$$

For more details see Appendix.

b) *The Vector Product of Vector Operators.* We compute $\Phi^1_{s's}(\mathbf{Y}\times\mathbf{Z})$. Using definition (2.14) and Eqs. (2.39), (2.37), (2.36) we have

$$\begin{array}{ccc} \underbrace{K \otimes K^s} \otimes H_s \\ \Psi_{s's}\Big\downarrow \qquad \Big\downarrow \Phi^1_{s's}(\mathbf{Y}\times\mathbf{Z}) \\ K^{s'} \otimes H_{s'} \end{array} = \sum_{s'':\, \substack{s''\sim s \\ s''\sim s'}} \begin{array}{c} K \otimes K^s \otimes H_s \\ T\Big\downarrow \qquad\qquad \Big\downarrow \mathrm{id} \\ K \otimes \underbrace{K \otimes K^s} \otimes H_s \\ \mathrm{id}\Big\downarrow \qquad \Big\downarrow \Psi_{s''s} \qquad \Big\downarrow \Phi^1_{s''s}(\mathbf{Z}) \\ \underbrace{K \otimes K^{s''}} \otimes H_{s''} \\ \Big\downarrow \Psi_{s's''} \qquad \Big\downarrow \Phi^1_{s's''}(\mathbf{Y}) \\ K^{s'} \otimes H_{s'} \end{array}$$

and

$$\Phi^1_{s's}(\mathbf{Y}\times\mathbf{Z}) = \sum_{s'':\, \substack{s''\sim s \\ s''\sim s'}} \varrho_{s's''s} \Phi^1_{s's''}(\mathbf{Y}) \Phi^1_{s''s}(\mathbf{Z}), \tag{2.44}$$

where

$$\begin{array}{c} K \otimes K^s \\ T\Big\downarrow \qquad \Big\downarrow \mathrm{id} \\ K \otimes \underbrace{K \otimes K^s} \\ \mathrm{id}\Big\downarrow \qquad \Big\downarrow \Psi_{s''s} \\ \underbrace{K \otimes K^{s''}} \\ \Big\downarrow \Psi_{s's''} \\ K^{s'} \end{array} = \varrho_{s's''s} \begin{array}{c} \underbrace{K \otimes K^s} \\ \Big\downarrow \Psi_{s's} \\ K^{s'} \end{array}. \tag{2.45}$$

One can compute $\varrho_{s's''s}$ for $s' \sim s''$, $s'' \sim s$, $s' \sim s$ (see Appendix):

$$\begin{aligned} &\varrho_{(s-1)(s-1)s} = \frac{q}{1+q^2} \quad \text{for} \quad s \geqq 1, \qquad \varrho_{s(s-1)s} = 1 \quad \text{for} \quad s \geqq 1, \\ &\varrho_{s(s+1)s} = -\frac{[2s]_q[2s+3]_q}{[2s+1]_q[2s+2]_q} \quad \text{for} \quad s \geqq 0, \qquad \varrho_{(s+1)(s+1)s} = -\frac{[2s+4]_q}{q(1+q^2)[2s+2]_q}, \\ &\varrho_{(s-1)ss} = -\frac{[2s+2]_q}{q(1+q^2[2s]_q} \quad \text{for} \quad s \geqq 1, \qquad \varrho_{sss} = -\frac{1+q^{(4s+2)}}{q(1+q^2)[2s]_q} \quad \text{for} \quad s>0, \\ &\varrho_{(s+1)ss} = \frac{q}{1+q^2}. \end{aligned} \tag{2.46}$$

c) *The Adjoint of Vector Operator.* From (2.40), (2.10) and since

$$\Phi(\mathbf{Y})^* = \sum_{s',s:s'\sim s} (\Psi_{ss'})^* \otimes \Phi^1_{s's}(\mathbf{Y})^*,$$

we get

$$\begin{array}{ccc} \underbrace{K\otimes K^s}\otimes H_s \\ \Psi_{s's}\downarrow \quad \downarrow \Phi^1_{s's}(\mathbf{Y}^\dagger) \\ K^{s'} \otimes H_{s'} \end{array} = \begin{array}{ccccc} K & \otimes & K^s & \otimes & H_s \\ \mathrm{id}\downarrow & & \downarrow \Psi^*_{ss'} & & \downarrow \Phi^1_{ss'}(\mathbf{Y})^* \\ \multicolumn{5}{c}{\underbrace{K\otimes K}\otimes K^{s'}\otimes H_{s'}} \\ \Theta^*\downarrow & & \downarrow \mathrm{id} & & \downarrow \mathrm{id} \\ C & \otimes & K^{s'} & \otimes & H_{s'} \end{array} \quad . \tag{2.47}$$

The computation of

$$\begin{array}{ccc} K & \otimes & K^s \\ \mathrm{id}\downarrow & & \downarrow \Psi^*_{ss'} \\ \multicolumn{3}{c}{\underbrace{K\otimes K}\otimes K^{s'}} \\ \Theta^*\downarrow & & \downarrow \mathrm{id} \\ C & \otimes & K^{s'} \end{array} = \omega_{s's} \begin{array}{c} K\otimes K^s \\ \downarrow \Psi_{s's} \\ K^{s'} \end{array} \tag{2.48}$$

gives (cf. Appendix):

$$\omega_{(s+1)s} = -\frac{1}{q}, \qquad \omega_{ss} = 1, \qquad \omega_{(s-1)s} = -q. \tag{2.49}$$

Comparing this with (2.47) one obtains:

$$\begin{aligned} \Phi^1_{(s+1)s}(\mathbf{Y}^\dagger) &= -\frac{1}{q}\Phi^1_{s(s+1)}(\mathbf{Y})^*, \\ \Phi^1_{ss}(\mathbf{Y}^\dagger) &= \Phi^1_{ss}(\mathbf{Y})^*, \\ \Phi^1_{(s-1)s}(\mathbf{Y}^\dagger) &= -q\Phi^1_{s(s-1)}(\mathbf{Y})^*. \end{aligned} \tag{2.50}$$

At the end let us remark that since $\mathscr{T}^1(D)$ is a bimodule over $\mathscr{T}^0(D)$ we have for $\mathbf{Y}\in\mathscr{T}^1(D)$ and $X\in\mathscr{T}^0(D)$ that $\mathbf{Y}\cdot X\in\mathscr{T}^1(D)$ and $X\cdot\mathbf{Y}\in\mathscr{T}^1(D)$,

$$\begin{aligned} \Phi^1_{s's}(\mathbf{Y}\cdot X) &= \Phi^1_{s's}(\mathbf{Y})\Phi^0_s(X), \\ \Phi^1_{s's}(X\cdot\mathbf{Y}) &= \Phi^0_{s'}(X)\Phi^1_{s's}(\mathbf{Y}). \end{aligned} \tag{2.51}$$

3. Irreducible Unitary Representations

This section is devoted mainly to the proof of Theorem 0.1. We describe all irreducible unitary representations of quantum Lorentz group, i.e. irreducible families of four operators $\{\alpha,\gamma,A,N\}$ satisfying relations (1.43), (1.44), and (1.47). Let us define

$$C = \frac{1}{1+q^2}\left[(1-q^2)^2 N^*N + q^2A^2 + A^{-2}\right], \tag{3.1}$$

$$\mathbf{Z}=\begin{pmatrix} Z_1 \\ Z_0 \\ Z_{-1} \end{pmatrix} = \begin{pmatrix} \frac{q(1-q^2)}{\sqrt{1+q^2}} NA \\ -\frac{1}{1+q^2}[(1-q^2)^2 N^*N + A^{-2} - A^2] \\ -\frac{1-q^2}{q\sqrt{1+q^2}} N^*A \end{pmatrix}, \tag{3.2}$$

$$X=\frac{1}{\sqrt{1+q^2}}[q(1-q^2)\gamma N-(q^2\alpha^* A+\alpha A^{-1})], \tag{3.3}$$

and

$$\mathbf{Y}=\begin{pmatrix} Y_1 \\ Y_0 \\ Y_{-1} \end{pmatrix} = \begin{pmatrix} (1-q^2)\alpha^* N+\gamma^* A^{-1} \\ \frac{1}{\sqrt{1+q^2}}[q(1-q^2)\gamma N+(\alpha^* A-\alpha A^{-1})] \\ \gamma A \end{pmatrix}. \tag{3.4}$$

Using (1.43), (1.44), and (1.47) one can check that operators C, X satisfy (2.8) and $\mathbf{Z}, \mathbf{Y}$ satisfy (2.9) so

$$C\in\mathscr{T}^0(D), \quad X\in\mathscr{T}^0(D), \quad \mathbf{Z}\in\mathscr{T}^1(D), \quad \mathbf{Y}\in\mathscr{T}^1(D).$$

The operator C is essentially selfadjoint on D and commutes with A, N, and N^* so it is a Casimir operator for $S_qU(2)$. Since the action of operators A and N on D is known, then using (1.25), (1.26), and (1.20) one can compute [cf. (2.15)] that

$$\begin{aligned} &\Phi^0_s(C)=C_s I_{H_s} && \text{for} \quad s\in S(v_c), \\ &C_s=\frac{q^{-2s}(1+q^{4s+2})}{1+q^2} && \text{for} \quad s\in S. \end{aligned} \tag{3.5}$$

Taking into account that H^s is invariant space for A, N, and N^* we see that $\Phi^1_{s's}(\mathbf{Z})$ can be $\neq 0$ only if $s'=s$ then by (2.30) and (2.34) we have

$$\begin{aligned} &\Phi^1_{s's}(\mathbf{Z})=\delta_{s',s}\mathscr{Z}_s I_{H_s} && \text{for} \quad s\in S(v_c), \\ &\mathscr{Z}_s=\frac{q(q^{-2s}-q^{2s})}{\sqrt{1+q^2}} && \text{for} \quad s\in S, \end{aligned} \tag{3.6}$$

so $\Phi(\mathbf{Z})=\sum_{s\in S(v_c)} \Psi_{ss}\otimes\mathscr{Z}_s I_{H_s}$.

Using again commutation relations (1.43), (1.44), and (1.47) one checks that X commutes with A, N, α, γ, and their adjoints so it is a Casimir operator for $S_qL(2, C)$. Moreover, A, N, α, γ and their adjoints are bounded on H^s for any $s\in S(v_c)$ and H^s is invariant subspace for X so X is a normal operator and D is an essential domain for it. Now relations (1.43), (1.44), (1.47) imply

$$X^*=X\cdot C+\mathbf{Y}\bullet\mathbf{Z}, \tag{3.7}$$

$$\mathbf{Y}^\dagger=\sqrt{1+q^2}\mathbf{Y}\times\mathbf{Z}-q^2\mathbf{Y}\cdot C, \tag{3.8}$$

$$\mathbf{Y}\bullet\mathbf{Y}=X^2-(1+q^2)I, \tag{3.9}$$

$$\mathbf{Y} \times \mathbf{Y} = \frac{1-q^2}{\sqrt{1+q^2}} X \cdot \mathbf{Y}. \tag{3.10}$$

This shows that the set of operators $\{A, N, \alpha, \gamma\}$ satisfying (1.43), (1.44), and (1.47) can be replaced by the set $\{A, N, X, Y_1, Y_0, Y_{-1}\}$ of operators satisfying (2.9) and (3.7)–(3.10). For irreducible set of operators we have an additional condition that

$$X = \mathcal{X}_0 I_D \quad \text{for some complex number} \quad \mathcal{X}_0 \in C. \tag{3.11}$$

Remark. Let us note that the C^*-algebra associated with quantum spheres of Podleś [7] can be generated equivalently by coordinates of vector operator $\mathbf{Y}$ such that

$$\mathbf{Y}^\dagger = \mathbf{Y}, \quad \mathbf{Y} \bullet \mathbf{Y} = \varrho I, \quad \mathbf{Y} \times \mathbf{Y} = \lambda \mathbf{Y}$$

for some real ϱ and λ. We see that in our case the C^*-algebra generated by coordinates of $\mathbf{Y}$ and $\mathbf{Y}^\dagger$ in the case of an irreducible unitary representation of G will correspond to "complexification" of quantum sphere.

Let p denote a minimal spin in the unitary representation v_c, i.e.

$$p = \min\{s\colon s \in S(v_c)\}, \tag{3.12}$$

then Eqs. (3.7) and (3.8) by using (2.50), (2.41), (2.44), and (3.5), (3.6) can be expressed in terms of the corresponding mappings and we get for $s \in S(v_c)$:

$$\mathcal{X}_0^* I_{H_s} = \mathcal{X}_0 C_s I_{H_s} + \lambda_{sss} \mathcal{Z}_s \Phi^1_{ss}(\mathbf{Y}), \tag{3.13}$$

$$-\frac{1}{q} \Phi^1_{s(s+1)}(\mathbf{Y})^* = [\sqrt{1+q^2}\, \varrho_{(s+1)ss} \mathcal{Z}_s - q^2 C_2] \Phi^1_{(s+1)s}(\mathbf{Y}), \tag{3.14}$$

$$\Phi^1_{ss}(\mathbf{Y})^* = [\sqrt{1+q^2}\, \varrho_{sss} \mathcal{Z}_s - q^2 C_s] \Phi^1_{ss}(\mathbf{Y}) - \mathcal{X}_0 \mathcal{Z}_s I_{H_s}, \tag{3.15}$$

$$-q \Phi^1_{s(s-1)}(\mathbf{Y})^* = [\sqrt{1+q^2}\, \varrho_{(s-1)ss} \mathcal{Z}_s - q^2 C_s] \Phi^1_{(s-1)s}(\mathbf{Y}). \tag{3.16}$$

Let us note that $\Phi^1_{(s-1)s}(\mathbf{Y}) = 0$ whenever $s < p+1$. Analogously, Eqs. (3.9) and (3.10) lead to

$$\begin{aligned} &\lambda_{s(s+1)s} \Phi^1_{s(s+1)}(\mathbf{Y}) \Phi^1_{(s+1)s}(\mathbf{Y}) + \lambda_{sss} \Phi^1_{ss}(\mathbf{Y})^2 + \lambda_{s(s-1)s} \Phi^1_{s(s-1)}(\mathbf{Y}) \Phi^1_{(s-1)s}(\mathbf{Y}) \\ &\quad = [\mathcal{X}_0^2 - (1+q^2)] I_{H_s}, \end{aligned} \tag{3.17}$$

$$\begin{aligned} &\varrho_{s(s+1)s} \Phi^1_{s(s+1)}(\mathbf{Y}) \Phi^1_{(s+1)s}(\mathbf{Y}) + \varrho_{sss} \Phi^1_{ss}(\mathbf{Y})^2 + \varrho_{s(s-1)s} \Phi^1_{s(s-1)}(\mathbf{Y}) \Phi^1_{(s-1)s}(\mathbf{Y}) \\ &\quad = \frac{1-q^2}{\sqrt{1+q^2}} \mathcal{X}_0 \Phi^1_{ss}(\mathbf{Y}), \end{aligned} \tag{3.18}$$

$$\begin{aligned} &\varrho_{(s+1)(s+1)s} \Phi^1_{(s+1)(s+1)}(\mathbf{Y}) \Phi^1_{(s+1)s}(\mathbf{Y}) + \varrho_{(s+1)ss} \Phi^1_{(s+1)s}(\mathbf{Y}) \Phi^1_{ss}(\mathbf{Y}) \\ &\quad = \frac{1-q^2}{\sqrt{1+q^2}} \mathcal{X}_0 \Phi^1_{(s+1)s}(\mathbf{Y}), \end{aligned} \tag{3.19}$$

$$\begin{aligned} &\varrho_{(s-1)(s-1)s} \Phi^1_{(s-1)(s-1)}(\mathbf{Y}) \Phi^1_{(s-1)s}(\mathbf{Y}) + \varrho_{(s-1)ss} \Phi^1_{(s-1)s}(\mathbf{Y}) \Phi^1_{ss}(\mathbf{Y}) \\ &\quad = \frac{1-q^2}{\sqrt{1+q^2}} \mathcal{X}_0 \Phi^1_{(s-1)s}(\mathbf{Y}). \end{aligned} \tag{3.20}$$

At first let us observe that for an irreducible representation v by (3.14)–(3.16) the support (1.23) of corresponding v_c is contained in the set S_p [cf. (0.2)].

Moreover, since $\lambda_{sss}\mathscr{X}_s \neq 0$ for $s>0$ by (2.43) and (3.6) then (3.13) determines $\Phi^1_{ss}(\mathrm{Y})$. For $s=0$ (it can occur only if $p=0$) there is no λ_{000} [the map $\Phi^1_{00}(\mathrm{Y})$ does not exist since 0 is not near 0] and (3.13) is an additional condition for $\mathscr{X}_0$. Since by (3.5) $C_0=1$ we see that if $p=0$ then $\mathscr{X}_0=\bar{\mathscr{X}}_0$, i.e. $\mathscr{X}_0$ has to be real. In what follows we shall consider separately two cases: $p>0$ and $p=0$. Since the decomposition of v_c into direct sum of irreducible representations is unique it is clear that the minimal spin p is invariant of unitary equivalence. We shall denote also $S(v)=S(v_c)$ and call it *the support of v*.

Proposition 3.1. i) *Let v be an unitary irreducible representation of quantum Lorentz group G with minimal spin $p>0$ and let $\mathscr{X}_0$ be the value of Casimir operator X for v. Then $\mathscr{X}_0 \in \mathscr{E}_p$.*

ii) *Let $\mathscr{X}_0 \in \mathscr{E}_p$ for some $p \in S$ and $p>0$.*

Then there exists a unique (up to unitary equivalence) irreducible unitary representation v of G with minimal spin p and for which the value of Casimir operator X is $\mathscr{X}_0$.

Moreover $S(v)=S_p$ and for any $s \in S(v)\dim H_s=1$.

Proof. From (3.13) we get for $s \geqq p$

$$\Phi^1_{ss}(\mathrm{Y})=\mathscr{Y}_{ss}I_{H_s} \quad \text{with} \quad \mathscr{Y}_{ss}=\frac{\bar{\mathscr{X}}_0-C_s\mathscr{X}_0}{\lambda_{sss}\mathscr{X}_s}. \tag{3.21}$$

Now for $s=p$ from (3.17) and (3.18) we get the minimal equations

$$\lambda_{p(p+1)p}\Phi^1_{p(p+1)}(\mathrm{Y})\Phi^1_{(p+1)p}(\mathrm{Y})+\lambda_{ppp}\Phi^1_{pp}(\mathrm{Y})^2=[\mathscr{X}_0^2-(1+q^2)]I_{H_p}, \tag{3.22}$$

$$\varrho_{p(p+1)p}\Phi^1_{p(p+1)}(\mathrm{Y})\Phi^1_{(p+1)p}(\mathrm{Y})+\varrho_{ppp}\Phi^1_{pp}(\mathrm{Y})^2=\frac{1-q^2}{\sqrt{1+q^2}}\mathscr{X}_0\Phi^1_{pp}(\mathrm{Y}). \tag{3.23}$$

Eliminating $\Phi^1_{p(p+1)}(\mathrm{Y})\Phi^1_{(p+1)p}(\mathrm{Y})$ from (3.22), (3.23) and substituting $\mathscr{Y}_{pp}$ from (3.21) and using (3.6) and (2.43) we get the additional condition on $\mathscr{X}_0$:

$$q^{-2p}(1+q^{4p})|\mathscr{X}_0|^2-(\mathscr{X}_0^2+\bar{\mathscr{X}}_0^2)=\frac{q^{2-4p}(1-q^{4p})^2}{1+q^2}$$

which implies that $\mathscr{X}_0 \in \mathscr{E}_p$ and this proves i).
Now let $\mathscr{X}_0 \in \mathscr{E}_p$ for some $p>0$, i.e. [cf. (0.3)]

$$\mathscr{X}_0=\frac{q}{\sqrt{1+q^2}}[(q^p+q^{-p})\cos\varphi+i(q^{-p}-q^p)\sin\varphi] \tag{3.24}$$

for some $\varphi \in [0,2\pi[$. To prove the existence of v one has to show that Eqs. (3.13)–(3.20) have an unique solution under the condition that v has to be irreducible. From (3.22) using (3.24), (3.6), and (2.43) we get

$$\mathscr{Y}_{ss}=-q\left[\frac{q^{-s}-q^s}{q^{-(s+1)}+q^{s+1}}(q^p+q^{-p})\cos\varphi+i\frac{q^{-s}+q^s}{q^{-(s+1)}-q^{s+1}}(q^{-p}-q^p)\sin\varphi\right]. \tag{3.25}$$

Then Eq. (3.15) is fulfilled identically. From (3.14) we get

$$\Phi^1_{s(s+1)}(\mathrm{Y})^*=q^{2s+3}\Phi^1_{(s+1)s}(\mathrm{Y}) \tag{3.26}$$

and (3.16) imply that for $s>p$,

$$q^{2s+1}\Phi^1_{s(s-1)}(\mathrm{Y})^*=\Phi^1_{(s-1)s}(\mathrm{Y}) \tag{3.27}$$

which is equivalent (3.26) by passing to adjoint and replacing $s \to s+1$. Taking this into account let for $s \geqq p$,

$$R_s^+ = \Phi^1_{s(s+1)}(\mathbf{Y})\Phi^1_{(s+1)s}(\mathbf{Y}) = q^{2s+3}|\Phi^1_{(s+1)s}(\mathbf{Y})|^2, \tag{3.28}$$

and for $s > p$

$$R_s^- = \Phi^1_{s(s-1)}(\mathbf{Y})\Phi^1_{(s-1)s}(\mathbf{Y}) = q^{-(2s+1)}|\Phi^1_{(s-1)s}(\mathbf{Y})|^2, \tag{3.29}$$

then $R_s^+ \geqq 0$ and $R_s^- \geqq 0$. On the other hand since $\Phi^1_{ss}(\mathbf{Y})$ is known then (3.17), (3.18) is the set of linear equations for R_s^+, R_s^- and we get

$$|\Phi^1_{(s+1)s}(\mathbf{Y})|^2 = q^2 \frac{[s+p+1]_q[s-p+1]_q}{[2s+2]_q[2s+3]_q} \times [(q^{s+1}+q^{-(s+1)})^2 - 4\cos^2\varphi] I_{H_s}. \tag{3.30}$$

$$|\Phi^1_{(s-1)s}(\mathbf{Y})|^2 = q^{4s+4} \frac{[s+p]_q[s-p]_q}{[2s]_q[2s+1]_q} [(q^s+q^{-s})^2 - 4\cos^2\varphi] I_{H_s}. \tag{3.31}$$

The minimal value of (q^s+q^{-s}) is 2 for $s=0$ so $|\Phi^1_{(s+1)s}(\mathbf{Y})|^2 > 0$ for $s \geqq p$ and $|\Phi^1_{(s-1)s}(\mathbf{Y})|^2 > 0$ for $s > p$. We denote

$$\mathscr{Y}_{(s+1)s} = q\sqrt{\frac{[s+p+1]_q[s-p+1]_q}{[2s+2]_q[2s+3]_q}[(q^{s+1}+q^{-(s+1)})^2 - 4\cos^2\varphi]}, \tag{3.32}$$

$$\mathscr{Y}_{(s-1)s} = q^{2s+2}\sqrt{\frac{[s+p]_q[s-p]_q}{[2s]_q[2s+1]_q}[(q^s+q^{-s})^2 - 4\cos^2\varphi]}. \tag{3.33}$$

Let now $e_p \neq 0$ be a normalized vector in H_p. Then e_p is an eigenvector for $\Phi^1_{pp}(\mathbf{Y})$ with the eigenvalue $\mathscr{Y}_{pp}$. Using (3.30) we see that $\Phi^1_{(p+1)p}(\mathbf{Y})e_p \neq 0$ and normalizing it we get the unit vector e_{p+1} such that

$$\Phi^1_{(p+1)p}(\mathbf{Y})e_p = \mathscr{Y}_{(p+1)p}e_{p+1}.$$

Now Eq. (3.19) shows that e_{p+1} is an eigenvector for $\Phi^1_{(p+1)(p+1)}(\mathbf{Y})$ with the eigenvalue $\mathscr{Y}_{(p+1)(p+1)}$. Moreover by (3.28)

$$\Phi^1_{p(p+1)}(\mathbf{Y})e_{p+1} = q^{2p+3}\mathscr{Y}_{(p+1)p}e_p.$$

In the same manner we can define e_{p+2} and by induction we see that starting from a unit vector $e_p \in H_p$ we get for $s \in S_p$ a unit vector $e_s \in H_s$ such that

$$\Phi^1_{ss}(\mathbf{Y})e_s = \mathscr{Y}_{ss}e_s,$$

$$\Phi^1_{(s+1)s}(\mathbf{Y})e_s = \mathscr{Y}_{(s+1)s}e_{s+1}, \qquad \Phi^1_{(s-1)s}(\mathbf{Y})e_s = \mathscr{Y}_{(s-1)s}e_{s-1}.$$

Let H'_s be a vector space spaned by e_s. Clearly $\dim H'_s = 1$ and

$$D' = \sum_{s \in S_p} K^s \otimes H'_s \subset D$$

is invariant under the action of A, N, N^*, Y_j, Y_j^* $(j=-1,0,1)$ so by irreducibility $D' = D$ since D' is the natural domain for the action of v on the invariant subspace $H = \sum^{\oplus}_{s \in S_p} K^s \otimes H'_s$ and $H'_s = H_s$ for all $s \in S_p$. This ends the proof of ii). Q.E.D.

For minimal spin $p=0$ we have

Proposition 3.2. i) *Let v be an unitary irreducible representation of quantum Lorentz group G with minimal spin $p=0$ and let $\mathscr{X}_0$ be the value of Casimir operator X for v. Then $\mathscr{X}_0 \in \mathscr{E}_0$.*
ii) *Let $\mathscr{X}_0 \in \mathscr{E}_0$.*

Then there exists a unique (up to unitary equivalence) irreducible unitary representation v of G with minimal spin $p=0$ and for which the value of Casimir operator X is $\mathscr{X}_0$.
Moreover
1. *If $\mathscr{X}_0 = \pm\sqrt{1+q^2}$ then v is a 1-dimensional representation.*
2. *If $\mathscr{X}_0 \in]-\sqrt{1+q^2}, \sqrt{1+q^2}[$ then $S(v)=S_0$ and for any $s\in S(v)$ $\dim H_s=1$.*

Proof. If v is an irreducible unitary representation of G with minimal spin $p=0$ then we know that $\mathscr{X}_0$ is real. Using the fact that [cf. (2.43)] $\lambda_{010} = -\frac{[3]_q}{q^3}$ Eq. (3.17) gives

$$\Phi^1_{01}(\mathbf{Y})\Phi^1_{10}(\mathbf{Y}) = \frac{q^3}{[3]_q}[(1+q^2)-\mathscr{X}_0^2]. \tag{3.34}$$

Now from (3.14) and (3.5), (3.6) for $s=0$ we get as before [cf. (3.26)] that $\Phi^1_{01}(\mathbf{Y}) = q^3\Phi^1_{10}(\mathbf{Y})$ and

$$|\Phi^1_{10}(\mathbf{Y})|^2 = \frac{1}{q^3}[(1+q^2)-\mathscr{X}_0^2]I_{H_0} \tag{3.35}$$

has to be $\geqq 0$ so $\mathscr{X}_0 \in \mathscr{E}_0$. This proves i).

Let $\mathscr{X}_0 \in \mathscr{E}_0$. Then we can write

$$\mathscr{X}_0 = \sqrt{1+q^2}\cos\varphi \quad \text{for some} \quad \varphi\in[0,2\pi[. \tag{3.36}$$

Let us assume that $\mathscr{X}_0 = \pm\sqrt{1+q^2}$. Then by (3.34), (3.35) $\Phi^1_{10}(\mathbf{Y})=0$ and $\Phi^1_{01}(\mathbf{Y})=0$ so $H^0 = K^0 \otimes H_0 = H_0$ is invariant under the action of Y_j, Y_j^* $(j=-1,0,1)$ and $Y_j=0$ on H^0. Since by (1.25), (1.26), (1.20) $A=1$, $N=0=N^*$ on H^0 and by (3.4) $Y_{-1}=\gamma A=\gamma=0$ we have by (1.44) that $\alpha=\pm 1$. From (3.3) we get

$$\begin{aligned} \{A,N,\alpha,\gamma\} &= \{1,0,1,0\} \quad \text{for} \quad \mathscr{X}_0 = -\sqrt{1+q^2}, \\ \{A,N,\alpha,\gamma\} &= \{1,0,-1,0\} \quad \text{for} \quad \mathscr{X}_0 = \sqrt{1+q^2}. \end{aligned} \tag{3.37}$$

Since in any case $\{A,N,\alpha,\gamma\}$ is the set of commuting operators it is clear that $\dim H^0 = 1$.
We assume now that $|\mathscr{X}_0| < \sqrt{1+q^2}$. Then we can argue as in the case $p>0$. For $s>0$ we get by (3.13) and (3.36),

$$\Phi^1_{ss}(\mathbf{Y}) = \mathscr{Y}_{ss}I_{H_s} \quad \text{with} \quad \mathscr{Y}_{ss} = -(1+q^2)\frac{q^{-s}-q^s}{q^{-(s+1)}+q^{s+1}}\cos\varphi. \tag{3.38}$$

Using this we get for $s\in S(v)$:

$$\begin{aligned} |\Phi^1_{(s+1)s}(\mathbf{Y})|^2 = q^2\frac{[s+1]_q^2}{[2s+2]_q[2s+3]_q}[(q^{s+1}+q^{-(s+1)})^2 \\ -(q+q^{-1})^2\cos^2\varphi]I_{H_s}, \end{aligned} \tag{3.39}$$

$$\begin{aligned} |\Phi^1_{(s-1)s}(\mathbf{Y})|^2 = q^{4s+4}\frac{[s]_q^2}{[2s]_q[2s+1]_q}[(q^s+q^{-s})^2 \\ -(q+q^{-1})^2\cos^2\varphi]I_{H_s}. \end{aligned} \tag{3.40}$$

Since in this case $\cos^2\varphi<1$ we see that $\Phi^1_{(s+1)s}(\mathbf{Y})\neq 0$ for $s\in S_0$ and for $s>0$ also $\Phi^1_{(s-1)s}(\mathbf{Y})\neq 0$. In the same manner as before we can construct a set of normalized vectors $\{e_s : e_s\in H_s \text{ for } s\in S_0\}$ such that

$$\Phi^1_{ss}(\mathbf{Y})e_s=\mathcal{Y}_{ss}e_s,$$

$$\Phi^1_{(s+1)s}(\mathbf{Y})e_s=\mathcal{Y}_{(s+1)s}e_{s+1}, \qquad \Phi^1_{(s-1)s}(\mathbf{Y})e_s=\mathcal{Y}_{(s-1)s}e_{s-1},$$

where $\mathcal{Y}_{ss}$ is given by (3.37) and

$$\mathcal{Y}_{(s+1)s}=q\sqrt{\frac{[s+1]_q^2}{[2s+2]_q[2s+3]_q}[(q^{s+1}+q^{-(s+1)})^2-(q+q^{-1})^2\cos^2\varphi]}, \tag{3.41}$$

$$\mathcal{Y}_{(s-1)s}=q^{2s+2}\sqrt{\frac{[s]_q^2}{[2s]_q[2s+1]_q}[(q^s+q^{-s})^2-(q+q^{-1})^2\cos^2\varphi]}, \tag{3.42}$$

and we conclude that $S(v)=S_0$ and $\dim H_s=1$ for any $s\in S_0$. This ends the proof of ii). Q.E.D.

The proof of Theorem 0.1 is now a straightforward corollary.

Remark. Let v be an unitary representation of G and X be the corresponding Casimir operator. Then X is normal operator and it is clear that SpX is an invariant of unitary equivalence. Let $\Sigma_q=\bigcup_{p\in S}\mathcal{E}_p$ then above propositions show that $SpX\subset\Sigma_q$. Moreover SpX and multiplicities does not completely determine the representation since for $\mathcal{X}_0=\pm\sqrt{1+q^2}$ and $\mathcal{X}_0=\pm\frac{1}{\sqrt{1+q^2}}(q^{3/2}+q^{1/2})$ there are two types of nonequivalent irreducible unitary representations: the minimal spin $p\in\{0,1\}$ or $p\in\{0,1|2\}$ respectively. It is also clear that the minimal spin p is also invariant of unitary equivalence so for irreducible representations it is not a function of $\mathcal{X}_0$.

The last part of this section we devote to the description of the irreducible unitary representations of quantum Lorentz group in terms of operators $\{A,N,\alpha,\gamma\}$. Let us note that if v is an irreducible unitary representation of the quantum Lorentz group G with minimal spin $p\in S$ acting on the Hilbert space $H=\sum^{\oplus}_{s\in S(v)}H^s$ then for any $s\in S(v)=S_p$ the space H^s is canonically isomorphic to K^s (remember that $\dim H_s=1$):

$$K^s\ni x\mapsto x\otimes e_s\in K^s\otimes H_s=H^s.$$

We shall identify elements of H^s with elements of K^s by this isomorphism. Using this we have

Theorem 3.3. i) *Let v be an unitary irreducible representation of the quantum Lorentz group G with minimal spin $p\in S$ and Casimir operator value $\mathcal{X}_0\in\mathcal{E}_p$ (i.e.*

$$\mathcal{X}_0=\begin{cases}\frac{q}{\sqrt{1+q^2}}[(q^p+q^{-p})\cos\varphi+i(q^{-p}-q^p)\sin\varphi] & \text{for}\quad p>0\\ \sqrt{1+q^2}\cos\varphi & \text{for}\quad p=0\end{cases} \tag{3.43}$$

for some $\varphi\in[0,2\pi[)$ acting on the Hilbert space $H=\sum^{\oplus}_{s\in S(v)}H^s$.

Then there exists for any $s\in S_p$ an orthonormal basis

$$\{f^s_m : m=-s,-s+1,\dots,s-1,s\}$$

(cf. (1.19)*) in* H^s *such that*

$$A f_m^s = q^m f_m^s,$$

$$N f_m^s = q^{-s}\sqrt{[s-m]_q[s+m+1]_q} f_{m+1}^s, \tag{3.44}$$

$$N^* f_m^s = q^{-s}\sqrt{[s+m]_q[s-m+1]_q} f_{m-1}^s,$$

$$\begin{aligned}\alpha f_m^s = {} & q^{-s}\sqrt{[s-m]_q[s+m]_q}\, a_{s-1} f_m^{s-1} \\ & - \left[\frac{q^m+q^{-m}}{1-q^2} q^{s+1} b_s + i\frac{q^{-m}-q^m}{1-q^2} q^{s+1} c_s\right] f_m^s \\ & - q^{s+1}\sqrt{[s-m+1]_q[s+m+1]_q}\, a_s f_m^{s+1},\end{aligned} \tag{3.45}$$

$$\begin{aligned}\alpha^* f_m^s = {} & -q^s\sqrt{[s-m]_q[s+m]_q}\, a_{s-1} f_m^{s-1} \\ & - \left[\frac{q^m+q^{-m}}{1-q^2} q^{s+1} b_s - i\frac{q^{-m}-q^m}{1-q^2} q^{s+1} c_s\right] f_m^s \\ & + q^{-s-1}\sqrt{[s-m+1]_q[s+m+1]_q}\, a_s f_m^{s+1},\end{aligned} \tag{3.46}$$

$$\begin{aligned}\gamma f_m^s = {} & q^{-m}\sqrt{[s+m]_q[s+m-1]_q}\, a_{s-1} f_{m-1}^{s-1} \\ & + \sqrt{[s+m]_q[s-m+1]_q}\,(b_s+ic_s) f_{m-1}^s \\ & + q^{m-1}\sqrt{[s-m+1]_q[s-m+2]_q}\, a_s f_{m-1}^{s+1},\end{aligned} \tag{3.47}$$

$$\begin{aligned}\gamma^* f_m^s = {} & q^m\sqrt{[s-m]_q[s-m-1]_q}\, a_{s-1} f_{m+1}^{s-1} \\ & + \sqrt{[s-m]_q[s+m+1]_q}\,(b_s-ic_s) f_{m+1}^s \\ & + q^{-m-1}\sqrt{[s+m+1]_q[s+m+2]_q}\, a_s f_{m+1}^{s+1},\end{aligned} \tag{3.48}$$

where

$$a_s = \begin{cases} \dfrac{q^{2(s+1)}}{[2s+2]_q}\sqrt{\dfrac{[s+p+1]_q[s-p+1]_q}{[2s+1]_q[2s+3]_q}[(q^{s+1}+q^{-s-1})^2-4\cos^2\varphi]} \\ \qquad\qquad\qquad \text{for} \quad p>0, \\ \dfrac{q^{2(s+1)}}{[2s+2]_q}\sqrt{\dfrac{[s+1]_q^2}{[2s+1]_q[2s+3]_q}[(q^{s+1}+q^{-s-1})^2-(q+q^{-1})^2\cos^2\varphi]} \\ \qquad\qquad\qquad \text{for} \quad p=0 \end{cases} \tag{3.49}$$

$$b_s+ic_s = \begin{cases} \dfrac{(1-q^2)q^s}{(1+q^{2s})(1+q^{2(s+1)})}(q^p+q^{-p})\cos\varphi \\ \quad + i\dfrac{(1-q^2)q^s}{(1-q^{2s})(1-q^{2(s+1)})}(q^{-p}-q^p)\sin\varphi \quad \text{for} \quad p>0 \\ \dfrac{(1-q^2)q^s}{(1+q^{2s})(1+q^{2(s+1)})}(q+q^{-1})\cos\varphi \qquad \text{for} \quad p=0. \end{cases} \tag{3.50}$$

ii) *Let* $p\in S$ *and* $\varphi\in[0,2\pi[$ *be fixed. Then operators* $\{A,N,\alpha,\gamma\}$ *defined by* (3.44)–(3.48) *satisfy the commutation relations* (1.43), (1.44), (1.47) *on* $D=\sum_{s\in S_p} K^s$ *and* *they describe an irreducible unitary representation v of G with minimal spin p and Casimir operator value* (3.43).

Proof. Let $\{f_m^s : m=-s,-s+1,\ldots,s\}$ be the orthonormal basis (1.19) in $K^s \sim H^s$ then (3.44) is clear by (1.20) and (1.25), (1.26). To compute (3.46) and (3.47) let us remark that from (3.4) and (3.3) we get

$$\gamma = Y_{-1}A^{-1}, \quad \alpha^* = \frac{1}{\sqrt{1+q^2}}(Y_0 - X)A^{-1} \tag{3.51}$$

so

$$\gamma f_m^s = q^{-m} Y_{-1} f_m^s, \tag{3.52}$$

$$\alpha^* f_m^s = \frac{q^{-m}}{\sqrt{1+q^2}}(Y_0 f_m^s - \mathscr{X}_0 f_m^s), \tag{3.53}$$

and it is enough to compute $Y_{-1}f_m^s$ and $Y_0 f_m^s$. By the definitions (2.16) and (2.34) we know that for $j \in \{-1,0,1\}$,

$$Y_j f_m^s = \Phi(\mathrm{Y})(f_j^1 \otimes f_m^s) = \Psi_{(s-1)s}(f_j^1 \otimes f_m^s)\mathscr{Y}_{(s-1)s} + \Psi_{ss}(f_j^1 \otimes f_m^s)\mathscr{Y}_{ss} + \Psi_{(s+1)s}(f_j^1 \otimes f_m^s)\mathscr{Y}_{(s+1)s} \tag{3.54}$$

and for $j=-1,0$ we have to compute $\Psi_{s's}(f_j^1 \otimes f_m^s)$ for $s' \sim s$. At first using (2.30), (2.28), (2.27), and (2.21) we can compute

$$\Psi_{(s-1)s}(f_{-1}^1 \otimes f_s^s) = \frac{1}{q^2} f_{s-1}^{s-1},$$

$$\Psi_{ss}(f_{-1}^1 \otimes f_s^s) = -\frac{q^{2s-2}}{\sqrt{[2s]_q}} f_{s-1}^s, \tag{3.55}$$

$$\Psi_{(s+1)s}(f_{-1}^1 \otimes f_s^s) = \frac{q^{4s}\sqrt{1+q^2}}{\sqrt{[2s+1]_q[2s+2]_q}} f_{s-1}^{s+1}.$$

Now we can use the fact that $\Psi_{s's}$ is an intertwining operator for $u^1 \otimes u^s$ and $u^{s'}$,

$$u^{s'}(\Psi_{s's} \otimes I_c) = (\Psi_{s's} \otimes I_c)(u^1 \otimes u^s). \tag{3.56}$$

Applying φ_{N^*} to both sides of (3.56) and using (1.29), (1.30), (1.28) we get

$$N_{s'}^* \Psi_{s's} = \Psi_{s's}(N_1^* \otimes A_s + A_1^{-1} \otimes N_s^*)$$

and since $N_1^* f_{-1}^1 = 0$ then we get a recurrence formula

$$N_{s'}^* \Psi_{s's}(f_{-1}^1 \otimes f_m^s) = q \Psi_{s's}(f_{-1}^1 \otimes N_s^* f_m^s). \tag{3.57}$$

Starting with $m=s$ by (3.55) and (1.20) we obtain from (3.57):

$$\Psi_{(s-1)s}(f_{-1}^1 \otimes f_m^s) = \frac{1}{q^2}\sqrt{\frac{[s+m-1]_q[s+m]_q}{[2s-1]_q[2s]_q}} f_{m-1}^{s-1},$$

$$\Psi_{ss}(f_{-1}^1 \otimes f_m^s) = -q^{s+m-2}\frac{\sqrt{[s+m]_q[s-m+1]_q}}{[2s]_q} f_{m-1}^s, \tag{3.58}$$

$$\Psi_{(s+1)s}(f_{-1}^1 \otimes f_m^s) = q^{2(s+m)}\sqrt{\frac{[s-m+1]_q[s-m+2]_q}{[2s+1]_q[2s+2]_q}} f_{m-1}^{s+1}.$$

Combining this with (3.54), (3.52) and using (3.25), (3.32), (3.33) or (3.38), (3.41), (3.42) we arrive at (3.47).

Now we are ready to compute also $\Psi_{s's}(f_0^1 \otimes f_m^s)$. Applying φ_N to both sides of (3.56) and using (1.28), (1.15) we get

$$N_{s'}\Psi_{s's} = \Psi_{s's}(N_1 \otimes A_s + A_1^{-1} \otimes N_s).$$

By (1.20) and (3.58) this gives

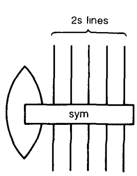
$$\begin{aligned}
\Psi_{(s-1)s}(f_0^1 \otimes f_m^s) &= -\sqrt{1+q^2}\, q^{s+m-2} \sqrt{\frac{[s+m]_q[s-m]_q}{[2s-1]_q[2s]_q}}\, f_m^{s-1}, \\
\Psi_{ss}(f_0^1 \otimes f_m^s) &= \frac{1-q^{2(s+m)}(1+q^2)+q^{4s+2}}{q(1-q^{4s})\sqrt{1+q^2}}\, f_m^s, \qquad (3.59)\\
\Psi_{(s+1)s}(f_0^1 \otimes f_m^s) &= \sqrt{1+q^2}\, q^{s+m} \sqrt{\frac{[s-m+1]_q[s+m+1]_q}{[2s+1]_q[2s+2]_q}}\, f_m^{s+1}.
\end{aligned}$$

This by (3.54), (3.53) and (3.32), (3.33), (3.25) or (3.38), (3.41), (3.42) leads to (3.46). Taking adjoints in (3.47) and (3.46) we get (3.48) and (3.45). This proves i). One can check by computations that the operators $\{A, N, \alpha, \gamma\}$ defined by (3.44)–(3.48) satisfy the commutation relations (1.43), (1.44), (1.47) on $D = \sum_{s \in S_p} K^s$. It is clear that the minimal spin is p and using (3.3) we get (3.43). Using Proposition 1.5 and Remark after the proof of Proposition 1.3 we end the proof of ii). Q.E.D.

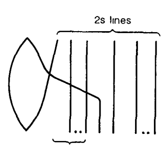
A. Appendix

In this section we give examples of calculations of coefficients listed in Sect. 2.

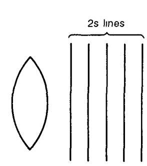
a) *The Scalar Product of Vector Operators.* By using the properties of maps previously defined we compute for example $\lambda_{s(s+1)s}$ and this means that we have to compute diagram (2.42) for $s' = s+1$. Using the definition (2.27) and the fact that $p_{1,s} = [2s+1]_q$ we shall compute at first a simpler diagram



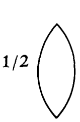
$$= \frac{1}{[2s+1]_q} \sum_{j=0}^{2s} q^{\frac{3}{2}j}$$



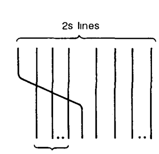
$$= \frac{1}{[2s+1]_q} \Bigg\{$$



$$+ \sum_{j=1}^{2s} q^{\frac{3}{2}j} \Big[q^{1/2}$$

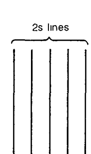


$$+ q^{-1/2} \Big]$$

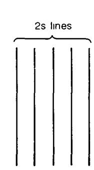


$$\Bigg\}$$

$$= -\frac{1}{[2s+1]_q}\left(\frac{1}{q} + q + q\sum_{j=0}^{2s-1} q^{2j}\right)$$



$$= Q_{2s}$$



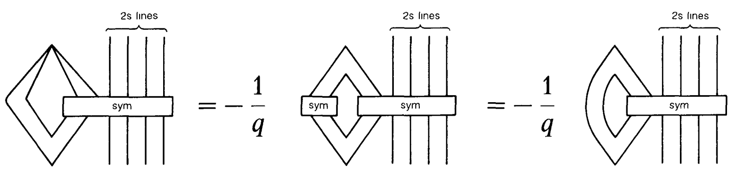
,

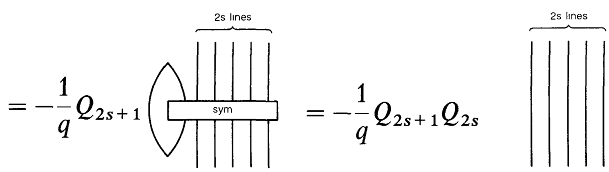
where

$$Q_{2s} = -\frac{[2s+2]_q}{q[2s+1]_q}. \tag{A.1}$$

In the second row we have used (2.27), (2.23) and since σ has the eigenvalue $q^{1/2}$ on symmetric tensors we got the last equality.

Using this we have from (2.30) and (2.31)





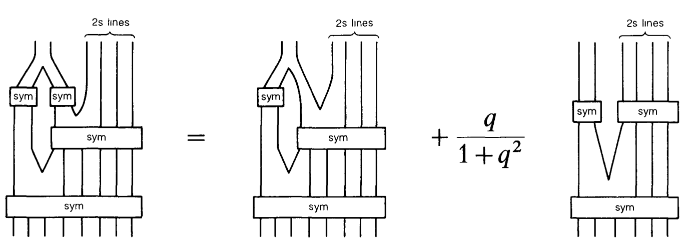
where we used (2.29) to omit the first symmetrization and

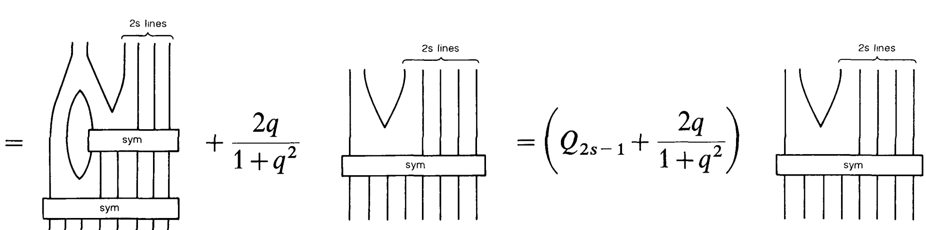
$$\lambda_{s(s+1)s} = -\frac{1}{q} Q_{2s+1} Q_{2s} = -\frac{[2s+3]_q}{q^3[2s+1]_q}.$$

The rest values can be obtained in the same manner.

b) *The Vector Product of Vector Operators.* We compute diagram (2.45) in the case ϱ_{sss}.

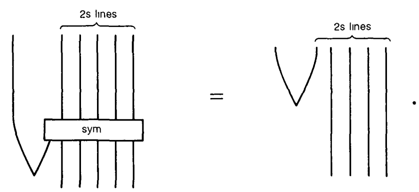
Using (2.29), property (2.22) and again (2.29) we get





(A.2)

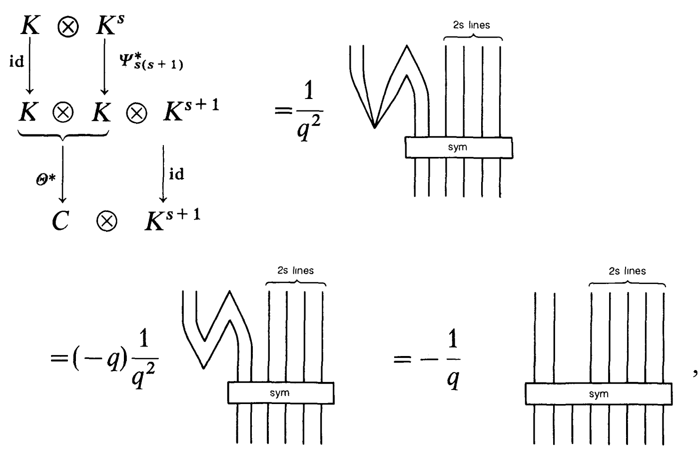
where in the second equality we used the fact that on the symmetric tensors



From (A.2) and (A.1) we obtain then

$$\varrho_{sss} = \left(Q_{2s-1} + \frac{2q}{1+q^2}\right) = -\frac{1+q^{4s+2}}{q(1+q^2)[2s]_q}.$$

c) *The Adjoint of Vector Operator.* The computation of (2.48) is simple. We use the fact that the diagram for $\Psi^*_{ss'}$ comes from the diagram for $\Psi_{s's}$ by reflecting it in a horizontal line. By (2.20), (2.21) we have $(E')^* = -\frac{1}{q}E$. This implies that symmetrization is unchanged by this operation since it uses a selfadjoint σ [cf. (2.24)]. Taking this into account we get for example for $\omega_{(s+1)s}$ by (2.30) and (2.32),



where we used property (2.22). From this by (2.30) $\omega_{(s+1)s} = -\frac{1}{q}$.

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