Commun. Math. Phys. 152, 539–564 (1993)

Communications in
Mathematical
Physics

Geometry and Integrability of Topological-Antitopological Fusion

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Received May 22, 1992

Abstract. Integrability of equations of topological-antitopological fusion (being proposed by Cecotti and Vafa) describing the ground state metric on a given 2D topological field theory (TFT) model, is proved. For massive TFT models these equations are reduced to a universal form (being independent on the given TFT model) by gauge transformations. For massive perturbations of topological conformal field theory models the separatrix solutions of the equations bounded at infinity are found by the isomonodromy deformations method. Also it is shown that the ground state metric together with some part of the underlined TFT structure can be parametrized by pluriharmonic maps of the coupling space to the symmetric space of real positive definite quadratic forms.

Introduction

The idea of topological field theories (TFT) as solvable models without local, propagating degrees of freedom was proposed in [1]. In [1–4] it was shown that topological correlators (at tree level) in a 2D TFT model are holomorphic functions on moduli of the TFT model obeying an overdetermined system of nonlinear PDE (the equations of associativity of primary operator algebra). Integrability of these equations was proved in [5].

The problem of calculation of the ground state metric of a family of TFT was studied in a general situation (for both massless and massive theories) in [6]. In this paper a system of PDE for the ground state metric (being a Hermitian metric on the moduli space of TFT) was derived. The topological and "antitopological" (i.e. complex conjugate) correlators serve as coefficients of these PDE. This general construction of calculating of ground state metric was called in [6] a *topological-antitopological fusion.* The equation of the same form arises for the metric on moduli space of Calabi-Yau varieties [7, 8]. The Hermitian metric on the moduli

★ On leave of absence from Dept. Mech. & Math., Moscow State University, 119899, Moscow space in this case is the same as the Zamolodchikov metric [9] of the underlying $N = 2$ superconformal field theories. In this case the metric is Kähler having special properties. The geometry of the moduli space with this metric is called in [7, 8] *special geometry*. Thus general solutions of the equations of [6] can be called also *generalized special geometry*. Also in [6, 10, 11] a number of particular integrable reductions of the main equations was found. It was shown that under some symmetry assumption the equations of topological-antitopological fusion can be reduced to affine Toda equations (particularly, to Euclidean sinh-Gordon) and to some other integrable systems of the soliton theory. For massive perturbations of topological conformal field theory (TCFT) many particular reductions of the main equations can be solved via the Painlevé transcendents of the third kind. More complicated reduction of the equations of topological-antitopological fusion was investigated numerically in [12].

The present paper can be considered as a continuation of the investigations having been started in [5] of the rôle of integrable systems in the TFT. The main aim of this paper is to prove integrability of the equations of topological-antitopological fusion in the general case. This integrability immediately follows from the zero-curvature representation of these equations depending on a spectral parameter being obtained in Sect. 1.

In Sect. 2 it was proved for massive TFT models that the equations of topological-antitopological fusion can be reduced to a universal integrable PDE system with constant coefficients (i.e. not depending on the given TFT model). For models with two primaries this system coincides with the Euclidean sinh-Gordon. For massive perturbations of TCFT the ground state metric can be found from the equations (generalizing the Painlevé III) of isomonodromy deformations of a linear operator with rational coefficients (again this operator is universal, i.e. it does not depend on the concrete TCFT model). The separatrix solutions of these equations satisfying at infinity the boundary conditions of [6] are found in Sect. 2 using Riemann boundary value problem machinery.

A nice geometrical reformulation of the equations of topological-antitopological fusion is given in Sect. 3. It is shown that any solution of these equations determines a pluriharmonic map (i.e. harmonic along complex directions) of the moduli space of the TFT model to the symmetric space of real positive definite quadratic forms (in fact a loop in the space of pluriharmonic maps). Conversely, any such a map determines a family of topological-antitopological fusion structures on the moduli space together with the underlying TFT structure. Functional parameters of the family can be described explicitly in differential geometric terms. This relation to the theory of harmonic maps probably can be useful to describe possible topological-antitopological fusion structures "in large" (i.e. using appropriate information about topology of the moduli space of the TFT model).

1. Zero-Curvature Representation for the Equations of Topological-Antitopological Fusion

Let $\mathbf{M}$ be a complex manifold of (complex) dimension n with a nondegenerate holomorphic complex quadratic form

$$\eta = \eta_{ab}(z)\,dz^a dz^b\,, \qquad \det(\eta_{ab}) \neq 0 \tag{1.1}$$

and a Hermitian positive definite form

$$g = g_{\bar{a}b}(z,\bar{z})\,d\bar{z}^a dz^b\,, \qquad \overline{g_{\bar{b}a}} = g_{\bar{a}b}\,, \qquad \det(g_{\bar{a}b}) \neq 0 \tag{1.2}$$

(the bar means complex conjugations).

Definition 1.1. The pair η, g is called *compatible* if there exists a complex connection $D = (\Gamma^c_{ab}, \Gamma^{\bar{c}}_{\bar{a}\bar{b}} = \overline{\Gamma^c_{ab}})$, where for any complex vector field $X = X^a\partial_a$

$$D_c X^a = \partial_c X^a + \Gamma^a_{cb}X^b\,, \tag{1.3a}$$
$$D_{\bar{c}} X^a = \bar{\partial}_c X^a\,, \tag{1.3b}$$
$$D_{\bar{c}} = \bar{D}_c\,, \tag{1.3c}$$

such that

$$D_c\eta_{ab} \equiv \partial_c\eta_{ab} - \Gamma^d_{ca}\eta_{db} - \Gamma^d_{cb}\eta_{db} = 0\,, \tag{1.4a}$$
$$D_c g_{\bar{a}b} \equiv \partial_c g_{\bar{a}b} - \Gamma^d_{cb} g_{\bar{a}d} = 0\,. \tag{1.4b}$$

Note that the equations $D_{\bar{c}}\overline{\eta_{ab}} = 0$, $D_{\bar{c}} g_{\bar{a}b} = 0$ follow from (1.4).

It is clear that the connection D for a compatible pair η, g is determined uniquely. The matrices $\Gamma_c = (\Gamma^b_{ca})$ have the form

$$\Gamma_c = g^{-1}\partial_c g\,, \qquad g = (g_{\bar{a}b})\,. \tag{1.5}$$

It immediately follows from the definition that the tensor $M = (M^b_{\bar{a}})$,

$$M^b_{\bar{a}} = g_{\bar{a}c}\eta^{cb}\,, \qquad (\eta^{cb}) = (\eta_{cb})^{-1} \tag{1.6}$$

obeys the equations

$$M\bar{M} = \text{const. } 1\,. \tag{1.7}$$

The compatible pair η, g is called *normalized* if

$$M\bar{M} = 1\,. \tag{1.8}$$

On a complex manifold with a normalized compatible pair η, g there is a canonical $\mathbf{C}$-linear isomorphism between the spaces of complex tensors of the type $\begin{pmatrix} p & p' \\ q & q' \end{pmatrix}$ (i.e., tensors with the components of the form $T^{i_1\ldots i_p\bar{k}_1\ldots\bar{k}_{p'}}_{j_1\ldots j_q\bar{l}_1\ldots\bar{l}_{q'}}$ – the notations as in [13]) for given $p + p' + q + q'$. In other words, the operations of raising and lowering of indices via η_{ab}, $\overline{\eta_{ab}}$, and $g_{\bar{a}b}$ commute. The parallel transport being specified by the connection Γ respects this isomorphism. Also an anticomplex involution τ acting on the (complexified) tangent space $T\mathbf{M} \otimes \mathbf{C} = T^{1,0}\mathbf{M} \oplus T^{0,1}\mathbf{M}$ is defined as follows:

$$\tau(X^a\partial_a + X^{\bar{a}}\bar{\partial}_a) = M^b_{\bar{a}}\overline{X^a}\partial_b + \overline{M^b_{\bar{a}}}\,\overline{X^{\bar{a}}}\bar{\partial}_b\,, \tag{1.9a}$$
$$\tau(T^{1,0}\mathbf{M}) = T^{1,0}\mathbf{M}\,, \qquad \tau(T^{0,1}\mathbf{M}) = T^{0,1}\mathbf{M}\,, \tag{1.9b}$$
$$\tau^2 = 1\,, \qquad \tau(\lambda x) = \bar{\lambda}\tau(x) \quad \text{for} \quad \lambda \in \mathbf{C}\,. \tag{1.9c}$$

The operator τ commutes with the standard complex conjugation

$$T^{1,0} \to T^{0,1}\,, \qquad X \mapsto \bar{X}\,, \tag{1.10a}$$
$$\tau(\bar{X}) = \overline{\tau(X)}\,. \tag{1.10b}$$

The complex inner product

$$\langle X, Y\rangle = \eta_{ab}X^aY^b \tag{1.11}$$

and the Hermitian scalar product

$$(X, Y) = g_{\bar{a}b}\overline{X^a}Y^b \tag{1.12}$$

are related by the equation

$$(X, Y) = \langle \tau(X), Y\rangle . \tag{1.13}$$

The operator τ is antiorthogonal with respect to the inner product $\langle\, ,\rangle$:

$$\langle \tau(X), \tau(Y)\rangle = \overline{\langle X, Y\rangle} . \tag{1.14}$$

It is covariantly constant with respect to the complex connection D:

$$D_c M^b_{\bar{a}} \equiv \partial_c M^b_{\bar{a}} + \Gamma^a_{cd} M^d_{\bar{b}} = 0 . \tag{1.15}$$

Also one should have the condition of positive definiteness

$$\langle \tau(X), X\rangle > 0 \quad \text{for} \quad X \neq 0 . \tag{1.16}$$

Proposition 1.1. *All compatible normalized pairs η, g with fixed η are in* 1–1 *correspondence with anticomplex involutions of the form* (1-9)–(1-16).

The proof is straightforward.

An anticomplex involution τ with the above properties also will be called *compatible* with the complex metric η.

The group of holomorphic automorphisms $A = (A^b_a(z))$ of $T^{1,0}\mathbf{M}$ acts on normalized compatible pairs as follows:

$$\eta \mapsto A^T \eta A , \qquad g \mapsto A^\dagger g A , \qquad M \mapsto A^{-1} M \bar{A} , \tag{1.17a}$$

$$\Gamma_a \mapsto A^{-1}\Gamma_a A + A^{-1}\partial_a A . \tag{1.17b}$$

The connection D for the compatible pair η, g is not symmetric. If $\hat{\Gamma} = (\hat{\Gamma}^c_{ab})$ is the Levi-Civita connection for the metric η (i.e. $\hat{\Gamma}^c_{ab} = \hat{\Gamma}^c_{ba}$, $\hat{D}_c \eta_{ab} \equiv \partial_c \eta_{ab} - \hat{\Gamma}^d_{ca}\eta_{db} - \hat{\Gamma}^d_{cb}\eta_{ad} = 0$) then the difference

$$T^c_{ab} = \Gamma^c_{ab} - \hat{\Gamma}^c_{ab} \tag{1.18a}$$

is a $\binom{1}{2}$ tensor. It obeys the symmetry

$$T^c_{ab}\eta_{cd} + T^c_{ad}\eta_{cb} = 0 . \tag{1.18b}$$

If one of the metric η or g is flat then vanishing of the tensor T^c_{ab} (in fact, vanishing of the skew-symmetric part $T^c_{[ab]} = \Gamma^c_{[ab]}$) is equivalent to simultaneous reducibility via holomorphic change of coordinates of the pair η, g to a constant form. Note that the holomorphic part of the Riemann curvature tensor of the connection Γ^c_{ab} vanishes:

$$[D_a, D_b] \equiv 0 . \tag{1.19}$$

Remark. For any anticomplex involution τ in a n-dimensional complex space T there exists a n-dimensional τ-invariant real subspace $V \subset T$ such that T is isomorphic to the complexification of V

$$\tau|_V = 1 , \qquad T = V \oplus iV . \tag{1.20}$$

Indeed, let

$$V_\pm = \left(\frac{1 \pm \tau}{2}\right) T .$$

One has

$$T = V_+ \oplus V_- , \qquad \tau|_{V_\pm} = \pm 1 .$$

Because of the antilinearity (1.9c) we obtain

$$iV_+ \subset V_- , \qquad iV_- \subset V_+ .$$

Hence $V_- = iV_+$. Putting $V = V_+$ we obtain (1.20). If a basis of the space T is chosen in V then the operator τ in this basis is represented by the unity matrix. In other words, any matrix $M = (M_{\bar{a}}^b)$ satisfying (1.8) can be represented in the form

$$M = \Phi \bar{\Phi}^{-1} \tag{1.21a}$$

for some complex matrix Φ. The matrices of the tensors η, g in such a basis coincide

$$G = \Phi^T \eta \Phi = \Phi^\dagger g \Phi , \tag{1.21b}$$

so G is a real symmetric matrix. Hence the only algebraic invariant of a normalized pair η, g is the signature of the Hermitian metric g. For the positive definite Hermitian metric g matrices of all the tensors M, η, g can be reduced simultaneously to the unity matrix (in one point of the manifold $\mathbf{M}$). Globally on a quasi-Fröbenius manifold $\mathbf{M}$ the distribution of the kernels

$$V = \ker(1 - \tau) \subset T^{1,0}\mathbf{M} \tag{1.22a}$$

determines a real n-dimensional bundle $\mathscr{V}$ over $\mathbf{M}$ such that

$$\mathscr{V} \subset T^{1,0} . \tag{1.22b}$$

One has

$$T^{1,0} = \mathscr{V} \otimes \mathbf{C} . \tag{1.22c}$$

This bundle uniquely determines the antiinvolution τ (i.e. the tensor $M_{\bar{a}}^b$). The tensors η, g specify a positive definite quadratic form on $\mathscr{V}$. In other words, they specify a section G [see (1.21b)] of the associated bundle $Q(\mathscr{V})$ of positive definite quadratic forms on the bundle $\mathscr{V}$.

To write the equations of topological-antitopological fusion (or, equivalently, the generalised equations of special geometry) we need to introduce the notion of Fröbenius manifold (see [5]). I recall that a commutative associative algebra A with a unity is called *Fröbenius* if there is a nondegenerate invariant inner product $\langle\, ,\rangle$ on A:

$$\langle ab, c\rangle = \langle a, bc\rangle . \tag{1.23}$$

$\mathbf{M}$ is called a (complex) quasi-Fröbenius manifold if a structure of Fröbenius algebra over the ring $\mathscr{F}(\mathbf{M})$ of holomorphic functions on $\mathbf{M}$ is fixed on the space Vect($\mathbf{M}$) of holomorphic vector-fields. It is assumed that the invariant inner product on Vect($\mathbf{M}$) is specified by a nondegenerate holomorphic quadratic form η [see (1.1)]. In local complex coordinates the multiplication law and the inner product should read

$$(X \cdot Y)^c(z) = X^a(z) Y^b(z) c_{ab}^c(z) , \tag{1.24a}$$

$$\langle X, Y\rangle = \eta_{ab}(z) X^a(z) Y^b(z) , \tag{1.24b}$$

where c_{ab}^c, η_{ab} are holomorphic tensors on $\mathbf{M}$. These satisfy the equations

$$c_{ba}^c = c_{ab}^c , \tag{1.25a}$$

$$c_{ab}^s c_{sc}^d = c_{as}^d c_{bc}^s , \tag{1.25b}$$

$$c_{abc} \equiv c_{ab}^s \eta_{sc} = c_{acb} . \tag{1.25c}$$

If $e = (e^a)$ is the unity (holomorphic) vector field then

$$e^a c_{ab}^c = \delta_b^c \tag{1.26}$$

(the Kronecker delta).

A quasi-Fröbenius $\mathbf{M}$ is called a *Fröbenius manifold* (see [5]) if the curvature of the connection

$$\tilde{\nabla}_X^{(\lambda)} Y = \nabla_X Y + \lambda X \cdot Y \tag{1.27}$$

vanishes identically in the spectral parameter λ. Here ∇ is the Levi-Civita connection for η. The complex metric η on a Fröbenius manifold $\mathbf{M}$ is flat. That means that in appropriate local coordinates t^α, $\alpha = 1, \ldots, n, \eta$ has a constant form

$$\eta = \eta_{\alpha\beta} dt^\alpha dt^\beta \,, \qquad \eta_{\alpha\beta} = \text{const.} \tag{1.28}$$

The structure constants $c_{\alpha\beta\gamma}(t)$ in the flat coordinates can be represented in the form

$$c_{\alpha\beta\gamma}(t) = \partial_\alpha \partial_\beta \partial_\gamma F(t) \tag{1.29}$$

for some function $F(t)$, $\partial_\alpha = \partial/\partial t^\alpha$. In a topological field theory (TFT) with n primary fields $\phi_1, \ldots, \phi_n$ the tensors $\eta_{\alpha\beta}$ and $c_{\alpha\beta\gamma}$ are the (tree-level) double and triple correlators of the primaries respectively. The coordinates $t^1, \ldots, t^n$ are the coupling constants of the perturbed TFT model, where the Lagrangian should be perturbed as $\mathscr{L} \mapsto \mathscr{L} - \sum t^\alpha \int \phi_\alpha$ (see [1–4] for details). And $F(t)$ coincides with the primary free energy (at tree-level).

The associativity conditions (1.25b) read as a system of nonlinear PDE for the function $F(t)$. This system of PDE was called in [5] the Witten-Dijkgraaf-E. Verlinde-H. Verlinde (WDVV) system. Flatness of the connection (1.27) gives the zero-curvature representation ("Lax pair") depending on the spectral parameter λ for WDVV equations.

Particularly, a (quasi-) Fröbenius manifold $\mathbf{M}$ is called *massive* if the Fröbenius algebra $(c_{\alpha\beta}^\gamma, \eta_{\alpha\beta})$ is semisimple (i.e. has no nilpotents) for any t. Local structure of massive Fröbenius manifolds can be described using an appropriate version of the inverse spectral transform (see [5] and Sect. 2 below) for WDVV.

A quasi-Fröbenius structure on $\mathbf{M}$ is called *integrable* if local coordinates $u^1, \ldots, u^n$ exist such that the structure tensor $c = (c_{ij}^k)$ in these coordinates does not depend on u. Particularly, any massive Fröbenius manifold is integrable [5]. In other words, *canonical* local coordinates $u^1, \ldots, u^n$ exist on a massive Fröbenius manifold $\mathbf{M}$ such that the law of the multiplication (1.24a) of the corresponding basic vector fields $\partial_i = \partial/\partial u^i$ has the form

$$\partial_i \cdot \partial_j = \delta_{ij} \partial_i \,. \tag{1.30}$$

These coordinates are determined uniquely up to permutations and shifts.

Let us come back to the arbitrary quasi-Fröbenius manifold $(c_{ab}^c(z), \eta_{ab}(z))$. I am going to define (following [6]) special geometry structure on a given quasi-Fröbenius manifold. If $\mathbf{M}$ is a Fröbenius manifold (i.e. a TFT model) then these special geometry structures on $\mathbf{M}$ are also called topological-antitopological fusions of the given TFT model [6].

Let us denote by C_a the operators

$$C_a = (c_{ab}^c(z)) \,. \tag{1.31}$$

Definition 1.2 (see [6]). A compatible pair η, g (or η, M) on a quasi-Fröbenius manifold $\mathbf{M}$ determines a special geometry (or topological-antitopological fusion) structure on it if

$$D_a C_b = D_b C_a \,, \tag{1.32a}$$

$$[D_a, D_{\bar{b}}] = -[C_a, C_{\bar{b}}] \,, \tag{1.32b}$$

where

$$C_{\bar{b}} = M \overline{C_b} \bar{M} \,. \tag{1.32c}$$

The normalized Hermitian metric

$$\tilde{g}_{\bar{a}b} d\bar{z}^a dz^b = \frac{g_{\bar{a}b} d\bar{z}^a dz^b}{g_{\bar{a}b} \bar{e}^a e^b} \,,$$

where e^a is the unity vector field, is called generalized Zamolodchikov metric on $\mathbf{M}$.

Remark. We have seen above that a compatible normalized pair η, g (or η, M) on a manifold $\mathbf{M}$ can be encoded by a pair $(\mathscr{V}, G)$, where $\mathscr{V}$ is a real n-dimensional subbundle in $T^{1,0}\mathbf{M}$ and G is a section of the associated bundle $Q(\mathscr{V})$ of positive definite quadratic forms on $\mathscr{V}$. It will be shown in Sect. 3 that the equations of special geometry have a nice geometric reformulation in terms of the pair $(\mathscr{V}, G)$: $\mathscr{V}$ is a flat bundle (i.e. it admits a connection of zero curvature) and G is a pluriharmonic section of $Q(\mathscr{V})$.

The equations of special geometry (together with the equations of compatibility of η and M) can be written as the following overdetermined system of equations for the matrix-valued function M (with Γ of the form (1.15):

$$\bar{\partial}_b(\partial_a M \cdot \bar{M}) = C_a M \bar{C}_b \bar{M} - M \bar{C}_b \bar{M} C_a \,. \tag{1.33}$$

It turns out that this system imposes a constraint for the quasi-Fröbenius manifold.

Proposition 1.2. *If eigenvalues of some $C_a = (c_{ab}^c(z))$ are simple and a special geometry structure on* $\mathbf{M}$ *exists then the quasi-Fröbenius structure $(c_{ab}^c(z))$ is integrable.*

The proof of this proposition will be given in Sect. 3.

Let us obtain now a "zero-curvature representation" (depending on a spectral parameter) of Eq. (1.32).

Proposition 1.3. *The equations*

$$\partial_a \xi = \lambda C_\alpha \xi - \Gamma_a \xi \,, \tag{1.34a}$$

$$\bar{\partial}_a \xi = \lambda^{-1} M \overline{C_a} \bar{M} \xi \,, \tag{1.34b}$$

where the matrix coefficients

$$C_a = (c_{ab}^c(z)) \,, \qquad \Gamma_a = (\Gamma_{ab}^c) \,, \qquad M = M_{\bar{a}}^b \tag{1.35}$$

obey the conditions

$$M \bar{M} = 1 \,, \tag{1.36a}$$

$$M^T \eta M = \bar{\eta} \,, \tag{1.36b}$$

$$\partial_k \eta = \eta \Gamma_k + \Gamma_k^T \eta \,, \tag{1.36c}$$

$$\eta C_a = C_a^T \eta \,, \tag{1.36d}$$

for a holomorphic symmetric nondegenerate $\eta = (\eta_{ab})$ are compatible identically in the spectral parameter λ iff *the pair η, M is compatible with Γ_{ab}^c as the corresponding* *connection* (1.4) *determining a special geometry structure on the quasi-Fröbenius manifold.*

Note that Eq. (1.36c) provides invariance of the compatibility conditions $[\partial_a, \partial_b] = [\bar\partial_a, \bar\partial_b] = [\partial_a, \bar\partial_b] = 0$ with the symmetry (1.36d) and with (1.36a, b).

Remark. Compatibility of the linear equations (1.34) can be interpreted as vanishing of the curvature of the λ-dependent "connection",

$$\tilde D_X^{(\lambda)} Y = D_X Y - \lambda X \cdot Y \,, \tag{1.37a}$$

$$\tilde D_{\bar X}^{(\lambda)} Y = D_{\bar X} Y - \lambda^{-1}\tau(X \cdot \tau(Y)) \tag{1.37b}$$

[cf. (1.27) above]. For $|\lambda| = 1$ the operation $\tilde D^{(\lambda)}$ determines a connection on $\mathbf{M}$ (i.e. it respects the complex conjugation).

Proof. The compatibility $\partial_k \partial_l = \partial_l \partial_k$ implies (1.32a) and the commutativity and associativity of the algebra $(c_{ab}^c(z))$. The compatibility $\partial_k \bar\partial_l = \bar\partial_l \partial_k$ is equivalent to the equations

$$\bar\partial_l \Gamma_k = [C_k, M\bar C_l \bar M] \tag{1.38}$$

and

$$\partial_k (M\bar C_l \bar M) = [M\bar C_l \bar M, \Gamma_k] \,. \tag{1.39}$$

It is enough to show that $\partial_k M + \Gamma_k M = 0$. Using $M\bar M = 1$ (1.39) can be rewritten as

$$[\bar M(\partial_k M\bar M + \Gamma_k) M, \bar C_l] = 0 \,. \tag{1.40}$$

Let us prove that the operator $\bar M(\partial_k M \bar M + \Gamma_k) M$ is $\bar\eta$-skew-symmetric. Equivalently, $\partial_k M\bar M + \Gamma_k$ should be η-skew-symmetric. Indeed,

$$0 = \partial_k (M^T \eta M) = \partial_k M^T \eta M + M^T(\eta \Gamma_k + \Gamma_k^T \eta) M + M^T \eta \partial_k M \,.$$

Multiplying by $M^\dagger$ and by $\bar M$ from the l.h.s and r.h.s. respectively one obtains the η-skew-symmetry

$$\eta(\partial_k M\bar M + \Gamma_k) + (\partial_k M\bar M + \Gamma_k)^T \eta = 0 \,.$$

Lemma. *Let $\mathcal{A}$ be a commutative associative algebra with a unity and $A: \mathcal{A} \to \mathcal{A}$ a linear operator such that*

$$A(ab) = aA(b) = bA(a) \,. \tag{1.41}$$

Then A is the operator of multiplication $A(a) = \alpha a$, $\alpha \in \mathcal{A}$.

Proof. Put $\alpha = A(1)$.

Corollary. *If A is a linear operator on a Fröbenius algebra satisfying* (1.41) *and being skew-symmetric* w.r.t. $\langle\, , \rangle$ *then* $A = 0$.

From the corollary it follows that

$$D_a M = 0$$

for the connection Γ_{ab}^c. And $D_a \eta = 0$ by (1.36c). Equation (1.38) coincides with (1.32b). The proposition is proved.

As a consequence of the proposition we obtain

Theorem 1. *Equation* (1.32) *of topological-antitopological fusion are integrable.* [1]

Remark. The system (1.32) provides no restriction (additional to (1.23), (1.36c)) for the invariant inner product η. Indeed, let us consider, for example, the quasi-Fröbenius structure with the operators C_a of the form

$$c_{ab}^c = \delta_a^c \delta_{ab} \tag{1.42}$$

(in the given coordinates). Then the invariant inner product η should be diagonal in these coordinates. Any diagonal holomorphic matrix η' determines another invariant inner product. The gauge transformation (1.17) with

$$A = \sqrt{\eta' \eta^{-1}} \tag{1.43}$$

transforms the special geometry structure for c_{ab}^c, η to a special geometry structure for c_{ab}^c, η'.

Example. Let us consider special geometry structures on the "trivial" Fröbenius manifold c_{ij}^k = const., η_{ij} = const. In the nonnilpotent case one can consider the direct sum of n copies of the 1-dimensional Fröbenius algebra,

$$c_{ij}^k = \delta_i^k \delta_{ij}, \qquad \eta_{ij} = \delta_{ij}. \tag{1.44}$$

The first part (1.32a) of the equations reads

$$[\Gamma_i, C_j] = [\Gamma_j, C_i], \tag{1.45a}$$

$$\Gamma_i^T = -\Gamma_i. \tag{1.45b}$$

These can be solved in the form

$$\Gamma_i = [q, C_i], \qquad q^T = q, \tag{1.46}$$

where the off-diagonal symmetric matrix $q = (q_{ij})$ is determined uniquely. The system (1.32b) reads

$$\partial_i M = [C_i, q] M, \tag{1.47a}$$

$$\bar{\partial}_i q = M \bar{C}_b \bar{M} - \mathrm{diag}(M \bar{C}_b \bar{M}), \tag{1.47b}$$

where "diag" means the diagonal part of the matrix $M\bar{C}_b\bar{M}$. The matrix q also satisfies the equations

$$[\partial_i q, C_j] - [\partial_j q, C_i] + [[q, C_i], [q, C_j]] = 0. \tag{1.48}$$

This follows from (1.32a). The matrix $M = (m_{\bar{i}j}))$ satisfies the constraints

$$M^T M = 1, \qquad M = M^\dagger. \tag{1.49}$$

[1] In [6, 10, 11] many integrable reductions of (1.32) were found. Nevertheless the "Lax pair" of [6] (coinciding with (1.34) for $\lambda = \pm 1$) provides no possibility to solve the system in the general case using an appropriate version of the inverse spectral transform

Also it should be positive definite. In the coordinate form the system (1.47), (1.48) reads

$$\partial_k q_{ij} = q_{ik} q_{kj}\,, \qquad i, j, k \text{ distinct}\,, \tag{1.50a}$$

$$\sum_k \partial_k q_{ij} = 0\,, \qquad i \neq j\,, \tag{1.50b}$$

$$\bar\partial_k q_{ij} = m_{\bar k i} m_{\bar k j}\,, \qquad i \neq j\,, \tag{1.50c}$$

$$\partial_k m_{\bar i j} = m_{\bar i k} q_{kj}\,, \qquad k \neq j\,, \tag{1.50d}$$

$$\sum_k \partial_k m_{\bar i j} = 0 \tag{1.50e}$$

(one should add also the complex conjugate equations).

In the first nontrivial case $n = 2$ positive definite matrices M of the form (1.49) can be represented in the form

$$M = \begin{pmatrix} \cosh \frac{\alpha}{2} & -i \sinh \frac{\alpha}{2} \\ i \sinh \frac{\alpha}{2} & \cosh \frac{\alpha}{2} \end{pmatrix} \tag{1.51}$$

for real α. All the functions q_{ij}, $m_{\bar i j}$ depend on the difference

$$z = z^1 - z^2\,,$$

From (1.50d) it immediately follows that

$$q_{12} = q_{21} = -\frac{i}{2}\,\partial\alpha\,,$$

where $\partial = \partial/\partial z$. From (1.50c) one obtains the Euclidean sinh-Gordon equation for α

$$\partial\bar\partial\alpha = \sinh\alpha\,. \tag{1.52}$$

Recently this equation (with the opposite sign) proved to be important in the theory of surfaces of constant mean curvature [14–16].

In Sect. 2 it will be shown that all special geometries on massive Fröbenius manifolds locally can be described by the system (1.50).

Remark. Equations of compatibility of rational operator pencils of the form

$$\partial_x \xi = \lambda C \xi - \gamma \xi\,, \tag{1.53a}$$

$$\partial_y \xi = \lambda^{-1} \tilde C \xi \tag{1.53b}$$

with $n \times n$ matrix coefficients C, $\tilde C$, Γ were studied for $n > 2$ in the paper [17] as a multicomponent generalization of the Sine-Gordon eq. (for $n = 2$ (1.53) gives [18] the Sine-Gordon equation in the light-cone variables x, y). In [19] it was shown that these equations are gauge equivalent to the integrable [20] equations of principal chiral field. The idea of this gauge equivalence is important for the constructions of Sect. 3. "Multidimensionalization" of the system (1.53) by adding dependence on higher times (i.e. isospectral deformations of (1.53)) was considered recently in [21] without investigating of reality constraints of the type (1.36).

2. Topological-Antitopological Fusions of a Massive TFT

Let us consider in more detail topological-antitopological fusions of a given massive TFT. Equations (1.32) can be represented as the compatibility conditions of the linear system

$$\partial_\alpha \xi = (\lambda C_\alpha - [C_\alpha, V])\xi\,, \tag{2.1a}$$

$$\bar{\partial}_\alpha \xi = \lambda^{-1}(M\bar{C}_\alpha \bar{M})\xi\,, \tag{2.1b}$$

$\partial_\alpha = \partial/\partial t^\alpha$, $\bar{\partial}_\alpha = \partial/\partial \bar{t}^\alpha$, the matrix $V = (V_\alpha^\beta(t,\bar{t}))$ satisfies

$$\eta V = V^T \eta\,. \tag{2.2}$$

As it was proved in [5], a canonical coordinate system $u^1(t), \ldots, u^n(t)$ locally exists on a massive Fröbenius manifold such that in these coordinates

$$c_{ij}^k = \delta_i^k \delta_{ij}\,, \tag{2.3a}$$

$$\eta_{ij} = h_i^2(u)\delta_{ij} \tag{2.3b}$$

for some analytic functions $h_i(u)$. The *rotation coefficients*

$$\gamma_{ij}(u) = \frac{\partial_j h_i(u)}{h_j(u)}\,, \qquad i \neq j\,, \tag{2.4}$$

$\partial_i = \partial/\partial u^i$, are symmetric in i, j. They satisfy the following integrable system of PDE,

$$\partial_k \gamma_{ij} = \gamma_{ik}\gamma_{kj}\,, \qquad i, j, k \text{ are distinct,} \tag{2.5a}$$

$$\sum_k \partial_k \gamma_{ij} = 0\,, \tag{2.5b}$$

$$\gamma_{ji} = \gamma_{ij}\,. \tag{2.5c}$$

The zero curvature representation of the system (2.5) has the form [22],

$$\partial_k \psi = (\lambda E_k - [E_k, \Gamma])\psi\,, \qquad k = 1, \ldots, n\,, \tag{2.6a}$$

$$(E_k)_{ij} = \delta_{ij}\delta_{jk}\,, \qquad \Gamma = (\gamma_{ij})\,. \tag{2.6b}$$

Any solution $\gamma_{ij}(u)$ of the system (2.5) determines a TFT as follows. Let $\psi_{i\alpha}(u)$, $\alpha = 1, \ldots, n$, be a basis of solutions of the linear system (2.6) for $\lambda = 0$. Then

$$\eta_{ii}(u) = \psi_{i1}^2(u)\,, \tag{2.7a}$$

$$\eta_{\alpha\beta} = \sum_i \psi_{i\alpha}(u)\psi_{i\beta}(u)\,, \tag{2.7b}$$

$$\frac{\partial t_\alpha}{\partial u^i} = \psi_{i1}(u)\psi_{i\alpha}(u)\,, \qquad t_\alpha = \eta_{\alpha\beta}t^\beta\,, \tag{2.7c}$$

$$c_{\alpha\beta\gamma}(u) = \sum_i \frac{\psi_{i\alpha}\psi_{i\beta}\psi_{i\gamma}}{\psi_{i1}}\,. \tag{2.7d}$$

Let us do a gauge transformation of the linear problem (2.1) of the form

$$\varphi_i(u, \bar{u}, \lambda) = \psi_{i\alpha}(u)\xi^\alpha(t, \bar{t}, \lambda)\,, \qquad u = u(t) \tag{2.8a}$$

or, equivalently,

$$\xi^\alpha(t,\bar t,\lambda) = \eta^{\alpha\beta}\sum_i \psi_{i\beta}(u)\varphi_i(u,\bar u,\lambda)\,. \tag{2.8b}$$

The functions $\psi_{i\alpha}(u)$ were determined above.

Proposition 2.1. *After the substitution* (2.8) *the system* (2.1) *transforms to the following gauge equivalent system:*

$$\partial_k\varphi = (\lambda E_k - [E_k, q])\varphi\,, \tag{2.9a}$$

$$\bar\partial_k\varphi = \lambda^{-1} m E_k \bar m \varphi\,, \tag{2.9b}$$

$k = 1, \ldots, n$, *where the symmetric off-diagonal matrix* $q = (q_{ij})$ *has the form*

$$q_{ij} = \gamma_{ij} - v_{ij}\,, \tag{2.10a}$$

where

$$V_\alpha^\beta = \sum_{i,j} \psi_{i\alpha}\psi_j^\beta v_{ij}\,, \tag{2.10b}$$

$$M_{\bar a}^b = \sum_{i,j} \bar\psi_{i\alpha} m_{\bar i j}\psi_j^\beta\,. \tag{2.10c}$$

The matrix m is Hermitian, positive definite and orthogonal.

The proof is straightforward.
As a consequence we obtain

Theorem 2. *Equations* (1.32) *of special geometry on a massive Fröbenius manifold are gauge equivalent to Eq.* (1.50).

Note that the system (1.50) is universal, i.e. it does not depend on the concrete TFT model. The gauge transformation (1.50) $\rightarrow$ (1.32) is determined only by the TFT model. So for massive models the WDVV equations for the correlators and the equations of topological-antitopological fusion for the ground state metric can be decoupled.

Let us consider the trivial solution of the system (1.50):

$$q = 0\,, \qquad m = 1\,. \tag{2.11}$$

This gives the following special geometry structure:

$$ds^2 = \sum |\eta_{ii}|\, d\bar u^i du^i = \sum_i \bar\psi_{i\alpha}\psi_{i\beta} d\bar t^\alpha dt^\beta\,. \tag{2.12}$$

The curvature of the corresponding connection D [see (1.4)] for the trivial solution vanishes identically.

Solutions of the system (1.50) close to the trivial one can be found form the linearized system. The leading approximation for the matrix m has the form

$$m_{\bar i j} \cong \delta_{ij} + i\alpha_{ij}(u^i - u^j, \bar u^i - \bar u^j)\,, \tag{2.13a}$$

where

$$\alpha_{ji}(u,\bar u) = -\,\alpha_{ij}(u,\bar u) \tag{2.13b}$$

are real solutions to the linearized sinh-Gordon

$$\partial\bar\partial\alpha_{ij} = \alpha_{ij}\,. \tag{2.13c}$$

Let us consider massive perturbations of topological conformal field theory (TCFT, see [3]). They are described by a massive Fröbenius manifold with a one-parameter group of conformal automorphisms. In the flat coordinates this group acts as scaling transformations

$$t^\alpha \mapsto c^{1-q_\alpha} t^\alpha \,, \tag{2.14a}$$

where q_α are the charges of the primary fields, $q_1 = 0$,

$$\eta_{\alpha\beta} \mapsto c^{q_\alpha + q_\beta - d} \eta_{\alpha\beta} \,. \tag{2.14b}$$

where d is the *dimension* of the model (i.e. $\eta_{\alpha\beta} = 0$ if $q_\alpha + q_\beta \neq d$),

$$c_{\alpha\beta\gamma} \mapsto c^{q_\alpha + q_\beta + q_\gamma - d} c_{\alpha\beta\gamma} \,. \tag{2.14c}$$

In the conformal point $t = 0$ the primary correlators form a graded Fröbenius algebra with q_α as the weights of the generators. In the canonical coordinates u^i the group acts in the standard way

$$u^i \mapsto c u^i \,, \tag{2.15a}$$

$$\gamma_{ij} \mapsto c^{-1} \gamma_{ij} \,. \tag{2.15b}$$

The similarity reduction (2.15) of the system (2.5) describes the isomonodromy deformations of the linear ODE system with rational coefficients

$$\lambda \partial_\lambda \psi = (\lambda U - [U, \Gamma]) \psi \,, \tag{2.16a}$$

$$U = \mathrm{diag}(u^1, \ldots, u^n) \,, \tag{2.16b}$$

$$\Gamma = (\gamma_{ij}(u)) \,. \tag{2.16c}$$

The integration of the similarity reduction of (2.5), (2.15) was given in [5]. Particularly, all the scaling dimensions q_α, d are calculated as the monodromy indices of (2.16) in the point $\lambda = 0$. For $n = 3$ this similarity reduction can be reduced to a particular case of the Painlevé-VI equation; for $n > 3$ this is an appropriate high order generalisation of the Painlevé-VI.

Let us consider now topological-antitopological fusions of a massive perturbation of TCFT. It is natural to assume a special geometry structure on a scaling invariant Fröbenius manifold to be covariant with respect to the group (2.14), $c = \exp i\phi$. In the massive case the similarity reduction of the system (1.50) is obtained by adding the equations

$$\sum_{k=1}^n (u^k \partial_k - \bar{u}^k \bar{\partial}_k) q_{ij} = - q_{ij} \,, \tag{2.17a}$$

$$\sum_{k=1}^n (u^k \partial_k - \bar{u}^k \bar{\partial}_k) m_{\bar{i}j} = 0 \,. \tag{2.17b}$$

The system (1.50), (2.17) can be reduced to a system of ODE of the order $n(n-1)$. For $n = 2$ this is equivalent to the similarity reduction of the sinh-Gordon equation (1.52) (i.e. to a particular case of the Painlevé-III equation [23]. For $n > 2$ the system (1.50), (2.17) can be considered as a high-order generalisation of the Painlevé-III.

Lemma. *The similarity reduction* (2.17) *of the system* (1.50) *is equivalent to the compatibility conditions of the linear problem* (2.9) *and of the linear differential equation in* λ

$$\lambda \partial_\lambda \varphi = [\lambda U - [U, q] - \lambda^{-1} m \bar{U} \bar{m}] \varphi \,. \tag{2.18}$$

Here U has the form (2.16b), $q = (q_{ij})$.

Proof. Equation (2.18) is nothing but the similarity equation for the function $\varphi = \varphi(u, \bar{u}, \lambda)$ of the form

$$[\sum(u^k \partial_k - \bar{u}^k \bar{\partial}_k) - \lambda \partial_\lambda]\varphi = 0 . \tag{2.19}$$

Compatibility of (2.19) with (2.9) is equivalent to the similarity equations (2.17). Lemma is proved.

Corollary. *The system* (1.50), (2.17) *gives the isomonodromy deformations of the linear operator* (2.18).

Isomonodromy deformations of linear operators of the form (2.18) (for $n > 2$ without any constraints of the type (1.36)) were described in [24]. Generic self-similar solutions of (1.50), (2.17) can be found easily using an appropriate Riemann boundary value problem (see [23]). Here we study more thoroughly the massive case. Semi-classical arguments of [6] for topological-antitopological fusion of Landau-Ginsburg models give rise to specification of $n(n-1)/2$-dimensional subfamily of separatrix solutions of (1.50), (2.17). As it follows from the formulae of Appendix B of [6] for such models the special geometry structure should trivialize for $|u| \to \infty$:

$$q \to 0 , \qquad m \to 1 \quad \text{for} \quad |u| \to \infty . \tag{2.20}$$

Asymptotics of these solutions for large u can be found by solving the similarity reduction of the Helmholtz equation (2.13c). This gives

$$m_{\bar{i}j} \cong \delta_{ij} + i\mu_{ij} K_0(2|u^i - u^j|) , \tag{2.21a}$$

where

$$\mu_{ij} = -\mu_{ij} \tag{2.21b}$$

are some real constants, $K_0(x)$ is the Bessel function (i.e. the solution of the modified Bessel equation

$$y'' + \frac{1}{x} y' - y = 0 \tag{2.22}$$

vanishing for $x \to \infty$). Let us construct these separatrix solutions.

To do it we formulate the following Riemann boundary value problem: to find $n \times n$ matrix-valued functions $\Psi_+(u, \bar{u}, \lambda)$ and $\Psi_-(u, \bar{u}, \lambda)$ analytic in λ in the half-planes $\operatorname{Re}\lambda > 0$ and $\operatorname{Re}\lambda < 0$ resp. satisfying the following boundary conditions on the imaginary axis (here $\varrho > 0$):

$$\Psi_-(u, \bar{u}, i\varrho) = \Psi_+(u, \bar{u}, i\varrho) S , \tag{2.23a}$$

$$\Psi_-(u, \bar{u}, -i\varrho) = \Psi_+(u, \bar{u}, -i\varrho) S^T , \tag{2.23b}$$

where $S = (s_{pq})$ is a complex $n \times n$ matrix satisfying the following conditions:

$$S\bar{S} = 1 , \tag{2.24a}$$

$$s_{pp} = 1 , \qquad s_{pq} = 0 \quad \text{for} \quad \pi \le \alpha_{pq} < 2\pi , \tag{2.24b}$$

where

$$\alpha_{pq} = \arg(u^p - u^q) , \qquad 0 \le \alpha_{pq} < 2\pi . \tag{2.24c}$$

The functions $\Psi_\pm$ should obey the following normalization condition:

$$\Psi_\pm(u, \bar{u}, \lambda) \exp[-\lambda U - \lambda^{-1} \bar{U}] \to 1 \quad \text{for} \quad \lambda \to \infty . \tag{2.25}$$

Proposition 2.2. *For sufficiently large* $|u|$ *there exists a unique solution of the above Riemann* b.v.p. *The matrix*

$$M = \lim_{\lambda \to 0} \Psi_{\pm}(u, \bar{u}, \lambda) \exp[-\lambda U - \lambda^{-1} \bar{U}] \tag{2.26}$$

satisfies Eqs. (1.33) *(for the Fröbenius manifold* (1.44)*) being a regular function for these large* $|u|$, *and* $M \to 1$ *for* $u \to \infty$.

Note [2] that the solution M depends exactly on $n(n-1)/2$ parameters (i.e. on the Stokes matrix S). For generic self-similar solutions of (2.9) the eigenfunction has a jump also on a unit circle.

Proof. Let

$$\Psi_{\pm}(u, \bar{u}, \lambda) \exp[-\lambda U - \lambda^{-1} \bar{U}] = \tilde{\Psi}_{\pm}(u, \bar{u}, \lambda) .$$

For the functions $\tilde{\Psi}_{\pm}(u, \bar{u}, \lambda)$ the Riemann b.v.p. (2.23) reads

$$\tilde{\Psi}_{-}(u, \bar{u}, \pm i\varrho) = \tilde{\Psi}_{+}(u, \bar{u}, \pm i\varrho) \tilde{S}^{\pm}(u, \bar{u}, \pm i\varrho) ,$$

where the matrices $\tilde{S}^{\pm}(u, \bar{u}, \lambda) = (\tilde{s}^{\pm}_{pq}(u, \bar{u}, \lambda))$ have the form

$$\tilde{s}^{+}_{pq}(u, \bar{u}, \lambda) = \exp[\lambda(u^p - u^q) + \lambda^{-1}(\bar{u}^p - \bar{u}^q)]\, s_{pq} ,$$
$$\tilde{s}^{-}_{pq}(u, \bar{u}, \lambda) = \exp[\lambda(u^p - u^q) + \lambda^{-1}(\bar{u}^p - \bar{u}^q)]\, s_{qp} .$$

These matrices tend exponentially to 1 when $|u| \to \infty$ because of (2.24). The b.v.p. (2.23) is equivalent to the singular integral equation for the function $\tilde{\Psi}_{+}$

$$\tilde{\Psi}_{+}(u, \bar{u}, \lambda) = 1 - \frac{1}{2\pi i} \left[\int_0^{i\infty} \frac{\tilde{\Psi}_{+}(u, \bar{u}, \zeta)\, [1 - \tilde{S}^{+}(u, \bar{u}, \zeta)]}{\zeta - \lambda + 0} d\zeta \right.$$
$$\left. + \int_{-i\infty}^{0} \frac{\tilde{\Psi}_{+}(u, \bar{u}, \zeta)\, [1 - \tilde{S}^{-}(u, \bar{u}, \zeta)]}{\zeta - \lambda + 0} d\zeta \right] . \tag{2.27}$$

For $|u| \to \infty$ the kernel of this equation vanishes exponentially. This proves existence and uniqueness of the solution.

The piecewise analytic function $\Psi = (\Psi_{+}, \Psi_{-})$ satisfies the following identities:

$$\Psi(\lambda) \Psi^T(-\lambda) = 1 , \tag{2.28}$$
$$\overline{\Psi(\bar{\lambda}^{-1})} = \bar{M} \Psi(\lambda) , \tag{2.29}$$

where M is defined in (2.26). This gives

$$M^T M = 1 , \qquad M^{\dagger} = M . \tag{2.30}$$

The logarithmic derivatives

$$\partial_k \Psi \cdot \Psi^{-1} , \qquad \bar{\partial}_k \Psi \cdot \Psi^{-1} , \qquad \lambda \partial_\lambda \Psi \cdot \Psi^{-1}$$

are analytic in $\lambda \in \mathbf{C}$, $\lambda \neq 0$, $\lambda \neq \infty$. Investigation of the analytic properties of these gives rise to infer that they are rational functions in λ. This immediately proves that

[2] I acknowledge A. Its for explaining me how one can specify the separatrix solutions of sinh-Gordon in the framework of the isomonodromy deformations approach

Ψ is an eigenfunction of the linear problems (2.9) and (2.18), where M has the form (2.26), and the matrix V is determined by the formula

$$V(u, \bar{u}) = \lim_{\lambda \to \infty} \lambda[1 - \Psi(u, \bar{u}, \lambda) \exp(-\lambda U - \lambda^{-1} \bar{U}] . \tag{2.31}$$

I recall (see Proposition 1.3 above) that compatibility of (2.9), (2.18) implies that M satisfies Eqs. (1.50), (2.17). The proposition is proved.

Calculation of the parameters μ_{ij} in the asymptotics (2.21) via the Stokes matrix S will be given in the next publication. The problem of global behaviour of the solutions of (1.50), (2.17) looks more complicated (note that the Riemann b.v.p. (2.23) should be redefined when some of the arguments of $u^p - u^q$ pass through the real axis). To do it in the first nontrivial case $N = 2$ one should use the connection formulae of [25].

3. Topological-Antitopological Fusions and Pluriharmonic Maps

It is wellknown that harmonic functions are the solutions $G = G(u, \bar{u})$ of the equation

$$\partial \bar{\partial} G = 0 .$$

Pluriharmonic functions (or vector-functions) $G(z, \bar{z})$, $z = (z^1, \ldots, z^n)$, are defined as solutions of the overdetermined system

$$\partial_k \bar{\partial}_l G = 0 , \qquad k, l = 1, \ldots, n .$$

Equivalently, the restriction of G onto any holomorphic curve $z^k = z^k(u)$, $k = 1, \ldots, n$ should be holomorphic.

Also for any complex manifold $\mathbf{M}$ and a real Riemannian manifold Q the class of pluriharmonic maps

$$G : \mathbf{M} \to Q$$

is well-defined. Particularly, pluriharmonic maps of a compact complex manifold $\mathbf{M}$ to compact Lie groups Q were studied recently [26, 27]. They proved to have many nice features of harmonic maps of Riemann surfaces to a compact Lie group (see [15, 28, 29]).

In this section it will be shown that any solution of Eqs. (1.32) of topological-antitopological fusion on a quasi-Fröbenius manifold $\mathbf{M}$ locally determines a pluriharmonic map of $\mathbf{M}$ to the symmetric space $Q = Gl(n)/O(n)$ of $n \times n$ positive definite quadratic forms (in fact, even a loop in the space of pluriharmonic maps $\mathbf{M} \to Q$). Conversely, it will be shown that, under some additional assumptions, a pluriharmonic map $\mathbf{M} \to Q$ determines a family of quasi-Fröbenius structures together with a special geometry structure on given $\mathbf{M}$. Globally instead of pluriharmonic maps $G : \mathbf{M} \to Q$ one has to consider pluriharmonic sections $G : \mathbf{M} \to Q(\mathscr{V})$ of the bundle of positive definite quadratic forms on a real n-dimensional flat subbundle $\mathscr{V} \subset T^{1,0}\mathbf{M}$ (see Sect. 1 above).

Let us fix a solution of the system (1.32). Let $\Phi = \Phi_\varphi(z, \bar{z})$ be the fundamental matrix of the system (1.34) for $\lambda = \exp i\varphi$ for some real φ being normalized by the condition

$$\bar{\Phi} = \bar{M} \Phi . \tag{3.1}$$

It is easy to see that such a normalization is compatible with (1.34). More than that, it can be done simultaneously for any φ so Φ_φ is a periodic function of φ. Equivalently, the matrix M is factorized as

$$M = \Phi\bar{\Phi}^{-1} . \tag{3.2}$$

The condition $M = M^\dagger$ is equivalent to reality of the symmetric matrix

$$G = \Phi^T \eta \Phi . \tag{3.3}$$

Starting from this point we will consider only special geometries with positive definite Hermitian products $g = (g_{\bar{a}b})$. Then

$$G = \Phi^\dagger g \Phi \tag{3.4}$$

is a matrix of a real positive definite quadratic form. We obtain therefore (locally in $\mathbf{M}$) a map (depending on the parameter φ)

$$G = G_\varphi(z, \bar{z}) : \mathbf{M} \to Q , \tag{3.5}$$

where $Q = Gl(n)/O(n)$ is the symmetric space of real positive definite quadratic forms. It will be proved below that this map is pluriharmonic.

Let us analyze now the global properties of the construction.

The matrix Φ normalized as in (3.1) is determined uniquely up to the transformations

$$\Phi \mapsto \Phi S \tag{3.6}$$

for an arbitrary real nondegenerate matrix S. We obtain therefore an isomorphism between the holomorphic tangent bundle $T^{1,0}\mathbf{M}$ and the complexification of some n-dimensional *real* bundle $\mathscr{V} \subset T^{1,0}\mathbf{M}$ on $\mathbf{M}$,

$$\Phi : \mathscr{V} \otimes \mathbf{C} \to T^{1,0}\mathbf{M} . \tag{3.7}$$

Indeed, the columns of the matrix $\Phi = (\Phi_I^a)$ under holomorphic changes of coordinates transform as holomorphic tangent vectors. In the intersection of two coordinate charts (z^a) and $(z^{a'})$ the matrices (Φ_I^a) and $(\Phi_{I'}^{a'})$ are related by the transformation (3.6) with a constant real S. This gives the construction of the real bundle $\mathscr{V}$. This isomorphism transforms the antiinvolution τ (with the matrix M) to the identity map on $\mathscr{V}$, and the complex and Hermitian quadratic forms $\Phi^*\eta$, Φ^*g coincide on $\mathscr{V}$ (i.e., they have the same real symmetric matrix G). We obtain that globally the formula (3.3) determines a section of the bundle $Q(\mathscr{V})$ of positive definite quadratic forms on the real n-dimensional subbundle $\mathscr{V} = \ker(\tau - 1) \subset T^{1,0}\mathbf{M}$. From the construction of the bundle $\mathscr{V}$ it immediately follows

Proposition 3.1. *For any solution Eqs.* (1.32) *of topological-antitopological fusion the bundle* $\mathscr{V} = \ker(\tau - 1) \subset T^{1,0}\mathbf{M}$ *is flat (*i.e. *it admits a connection of zero curvature).*

Particularly, on a simply-connected $\mathbf{M}$ the bundle $\mathscr{V}$ is trivial. So G is a map of $\mathbf{M}$ to the symmetric space Q. For non-simply-connected $\mathbf{M}$ G is an automorphic map with respect to some linear representation

$$\pi_1(\mathbf{M}) \to Gl(n) \tag{3.8}$$

(twisted pluriharmonic map [31]).

Let us come back to pluriharmonic maps. The group $Gl(n)$ acts transitively on Q as follows:

$$G \mapsto S^T G S , \qquad S \in Gl(n) . \tag{3.9}$$

The stationary subgroup is isomorphic to $O(n)$. The invariant metric on Q has the form

$$\operatorname{tr}(G^{-1}dGG^{-1}dG). \tag{3.10}$$

A map $G = G(z, \bar{z})\colon \mathbf{M} \to Q$ is called *pluriharmonic* if the functional

$$\frac{i}{4}\int \operatorname{tr}(G^{-1}\partial G G^{-1}\bar{\partial} G)\,du d\bar{u} \tag{3.11}$$

has extremum for any holomorphic curve

$$\begin{aligned} z^k = z^k(u)\,, \qquad k = 1, \dots, n\,, \\ \partial = \partial/\partial u\,, \qquad \bar{\partial} = \partial/\partial\bar{u}\,. \end{aligned} \tag{3.12}$$

Particularly, representing

$$G = \exp \alpha\tilde{G}$$

where $\alpha = \dfrac{1}{n}\log\det G$, $\det\tilde{G} = 1$, one obtains

$$\operatorname{tr}(G^{-1}\partial G G^{-1}\bar{\partial}G) = |\partial\alpha|^2 + \operatorname{tr}(\tilde{G}^{-1}\partial\tilde{G}\tilde{G}^{-1}\bar{\partial}\tilde{G})\,.$$

Hence $\log\det G$ is a pluriharmonic function (cf. [6]). The matrix $\tilde{G}$ determines a pluriharmonic map to the irreducible symmetric space $\tilde{Q} = SL(n)/SO(n)$.

The function $G(z, \bar{z})$ should obey an overdetermined system of equations. This system can be rewritten in a simple form using matrix-valued currents

$$A_k = G^{-1}\partial_k G\,, \qquad A_{\bar{k}} = G^{-1}\bar{\partial}_k G = \bar{A}_k\,, \tag{3.13}$$

$$\partial_k A_l - \partial_l A_k = [A_l, A_k]\,, \tag{3.14a}$$

$$\bar{\partial}_l A_k = \tfrac{1}{2}\,[A_k, A_{\bar{l}}]\,. \tag{3.14b}$$

This system will be investigated thoroughly below. Pluriharmonicity for sections of $Q(\mathscr{V})$ is determined in a similar way.

The main claim of this section is

Theorem 3. *Let $\Phi = \Phi(z, \bar{z})$ be the fundamental matrix of solutions of the system* (1.34) *for $\lambda = \exp i\varphi$ normalized by the condition* (1.3). *Then $G = \Phi^T\eta\Phi$ is a pluriharmonic section of the bundle $Q(\mathscr{V})$. Conversely, let $\mathscr{V}$ be any flat real n-dimensional bundle on* $\mathbf{M}$ *satisfying* (1.22b, c) *and G a pluriharmonic section of $Q(\mathscr{V})$. If some of the operators $A_k = G^{-1}\partial_k G$ are semisimple* (i.e. *it has a simple spectrum) then the pluriharmonic map $G\colon \mathbf{M} \to Q$ (being defined locally) induces on* $\mathbf{M}$ *a family of integrable massive quasi-Fröbenius structures together with special geometry structures. All these quasi-Fröbenius structures have the same canonical coordinates* (1.30) *with an arbitrary diagonal in these coordinates holomorphic tensor η as the invariant inner product. Fixation of this tensor η specifies uniquely the quasi-Fröbenius and special geometry structure on* $\mathbf{M}$.

Particularly, if the diagonal tensor η satisfies in the canonical coordinates the system (2.5) (the so-called Egoroff metric) then the above construction gives all the Fröbenius manifold structures on $\mathbf{M}$ with marked atlas of canonical coordinates together with the topological-antitopological fusions of them.

Proof. Let

$$\xi = \Phi\chi\,. \tag{3.15}$$

After this gauge transform one obtains from (1.34)

$$\partial_k \chi = -\frac{A_k}{\zeta+1}\chi, \tag{3.16a}$$

$$\bar\partial_k \chi = \frac{A_{\bar k}}{\zeta-1}\chi, \tag{3.16b}$$

where

$$A_k = 2\Phi^{-1} C_k \Phi, \tag{3.17}$$

$$A_{\bar k} = \bar A_k, \tag{3.18}$$

$$\lambda = \frac{\zeta - 1}{\zeta + 1}. \tag{3.19}$$

It is easy to see that

$$A_k = G^{-1}\partial_k G \tag{3.20}$$

for $G = \Phi^T \eta \Phi$. Compatibility of (3.16) (identically in the spectral parameter ζ) reads

$$[A_k, A_l] = \partial_l A_k - \partial_k A_l = 0, \tag{3.21a}$$

$$\bar\partial_l A_k = \tfrac{1}{2}[A_k, A_{\bar l}]. \tag{3.21b}$$

Hence the function $G(z, \bar z)$ is pluriharmonic.

To prove the converse statement one needs first to prove

Lemma. *Any solution of* (3.14) *satisfies also* (3.21).

For the case of pluriharmonic maps to a compact Lie group such a statement was proved in [27].

Proof. Compatibility conditions of (3.14) imply

$$[A_{\bar k}, [A_p, A_q]] = 0 \tag{3.22}$$

for any p, q, k. Let us prove that $[A_p, A_q] = 0$ follows from these equations. Let us consider G as a Euclidean structure in a n-dimensional real vector space V. The operators A_k, $A_{\bar k}$ act in $V \otimes \mathbf{C}$. Let us introduce a positive Hermitian inner product in $V \otimes \mathbf{C}$ by the formula

$$(v, w) = G_{IJ}\bar v^I w^J \tag{3.23}$$

(I recall that this is $\Phi^* g$ – see (3.4)). One has

$$A_p^\dagger = A_{\bar p}. \tag{3.24}$$

Let us order anyhow pairs $\alpha = (p, q)$, $p < q$, and let

$$\mathscr{A}_\alpha = [A_p, A_q], \qquad \alpha = (p, q). \tag{3.25}$$

It follows from (3.22), (3.24) that

$$[\mathscr{A}_\alpha, \mathscr{A}_\beta^\dagger] = 0. \tag{3.26}$$

Particularly, the operators $\mathscr{A}_\alpha$ are normal. Let us reduce the operator $\mathscr{A}_1$ to a diagonal form in an orthonormal basis:

$$\mathscr{A}_1 = \mathrm{diag}(\lambda_1^{(1)}, \ldots, \lambda_n^{(1)}),$$

$$\mathscr{A}_1^\dagger = \mathrm{diag}(\bar\lambda_1^{(1)}, \ldots, \bar\lambda_n^{(1)}),$$

If $\mathscr{A}_2 = (a^{(2)}_{IJ})$ then the equality $[\mathscr{A}_2, \mathscr{A}_1^\dagger] = 0$ reads

$$(\bar\lambda^{(1)}_I - \bar\lambda^{(1)}_J) a^{(2)}_{IJ} = 0 .$$

Therefore

$$[\mathscr{A}_1, \mathscr{A}_2] = 0 .$$

Let us diagonalize these two operators simultaneously. By induction we can reduce simultaneously to a diagonal form in an orthonormal basis all the operators $[A_p, A_q]$.

Let $V' \subset V \otimes \mathbf{C}$ be a maximal subspace where all the operators $[A_p, A_q]$ (and, therefore, $[A_{\bar p}, A_{\bar q}]$) are scalars. Because of (3.22) V' is invariant with respect to all A_k. Then $[A_p, A_q]|_{V'} = 0$ since the trace of a commutator vanishes. Hence all the eigenvalues of all $[A_p, A_q]$ vanish. Again using their normality we obtain $[A_p, A_q] = 0$. The lemma is proved.

Note that the above semisimplicity assumption has not been used in this proof.

As it has been proved pluriharmonicity of a map $\mathbf{M} \to Q$ is equivalent to compatibility of the system (3.16). Hence Eqs. (3.14) of pluriharmonic maps to the symmetric space Q are integrable [3].

Now I'll try to explain the geometric idea of the final part of the proof before proceeding to the calculations. For simplicity let me consider the problem locally (so all the bundles will be trivial).

Let V be an n-dimensional real vector space. The pull-back of the Levi-Civita connection on Q (for the invariant metric (3.10)) determines a complex connection ∇_k, $\nabla_{\bar k}$ on the trivial bundle $\mathbf{M} \times V \otimes \mathbf{C}$ (the formula (3.37) below). The operator

$$d''_G = d\bar z^k \nabla_{\bar k} \tag{3.27}$$

satisfies $d''^2_G = 0$ and, therefore, it determines on $\mathbf{M} \times V \otimes \mathbf{C}$ a structure of the holomorphic vector bundle $\mathscr{V} \times \mathbf{C}$. The commuting operators A_k act on $\mathscr{V} \otimes \mathbf{C}$. They appear to be holomorphic sections of the bundle

$$E = T^{1,0}_* \mathbf{M} \otimes \mathrm{End}(\mathscr{V} \otimes \mathbf{C}) . \tag{3.28}$$

The commutativity (3.14a) can be rewritten in the form

$$A \wedge A = 0 , \tag{3.29}$$

where

$$A = A_k dz^k \tag{3.30}$$

is the section of E. Also the matrix-valued 1-form A is closed. Such a pair (E, A) was called a Higgs bundle in [31]. It is very important that in our case there is a Euclidean scalar product on $\mathbf{M} \times V$ (being specified by the matrix G). It proves to be holomorphic with respect to d''_G. So it determines a holomorphic nondegenerate quadratic form $\langle\, ,\rangle$ on $\mathscr{V} \times \mathbf{C}$ being invariant for the operators A_k. We obtain therefore a holomorphic family of commuting operators being symmetric with respect to a holomorphic inner product $\langle\, ,\rangle$. This looks so similar to the deformation of Fröbenius algebras! To complete the construction one needs to identify $\mathscr{V} \otimes \mathbf{C}$ and $T^{1,0}\mathbf{M}$ (i.e. to construct an isomorphism Φ – see (3.7)). Here the semisimplicity assumption is

[3] General solution of these equations can be obtained using the ideas of [30]

essential. We construct locally a basis of holomorphic sections $v_1, \ldots, v_n$ of $\mathscr{V} \times \mathbf{C}$ and such a coordinate system $u^1, \ldots, u^n$ on $\mathbf{M}$ that

$$Av_k = du^k v_k \tag{3.31}$$

(no summation over the repeated indices here!). The isomorphism we need is constructed then as follows

$$v_k \mapsto \partial/\partial u^k \,. \tag{3.32}$$

Let us proceed now to the detailed proof.

Let $\omega = \omega_k dz^k$ be a 1-form such that

$$A_k v = \omega_k v \tag{3.33}$$

for some common eigenvector $v = v^I(z, \bar{z})$. The genericity assumption provides existence of n linearly independent forms $\omega^1, \ldots, \omega^n$ being the "weights" (3.33) of the commutative algebra $A_1, \ldots, A_n$. Let $v_1, \ldots, v_n$ be the corresponding eigenvectors. Let

$$\langle v, w\rangle = G_{IJ} v^I w^J \tag{3.34}$$

be a symmetric nondegenerate inner product in V (and in $V \otimes \mathbf{C}$). The operators $A_1, \ldots, A_n$ are symmetric with respect to this inner product. The standard consequence of this fact is that

$$\langle v_I, v_J\rangle = 0 \quad \text{for} \quad I \neq J \,, \tag{3.35}$$

$$\langle v_I, v_I\rangle \neq 0 \,. \tag{3.36}$$

Let us prove that all the weights ω are holomorphic forms on $\mathbf{M}$. Indeed,

$$0 = \bar{\partial}_l (A_k v - \omega_k v) = \tfrac{1}{2}\, [A_k, A_{\bar{l}}] v + (A_k - \omega_k) \bar{\partial}_l v - \bar{\partial}_l \omega_k v \,.$$

Multiplying by v one obtains

$$\bar{\partial}_l \omega_k \langle v, v\rangle = \tfrac{1}{2}\, \langle v, [A_k, A_{\bar{l}}] v\rangle = 0 \,.$$

The next step: to prove that the weights are closed forms. The proof:

$$\partial_l (A_k v) - \partial_k (A_l v) = (\partial_l \omega_k - \partial_k \omega_l) v + \omega_k \partial_l v - \omega_l \partial_k v \,.$$

After scalar multiplication by v one obtains

$$(\partial_l \omega_k - \partial_k \omega_l)\, \langle v, v\rangle = 0 \,.$$

It is convenient to introduce the connection in the trivial bundle $\mathbf{M} \times V$,

$$\nabla_k v = \partial_k v + \tfrac{1}{2}\, A_k v \,, \qquad \nabla_{\bar{k}} v = \bar{\partial}_k v + \tfrac{1}{2}\, A_{\bar{k}} v \,, \tag{3.37}$$

(the pull-back of the Levi-Civita connection on Q). It is compatible with the scalar product (3.34):

$$\partial_k \langle v, w\rangle = \langle \nabla_k v, w\rangle + \langle v, \nabla_k w\rangle \,,$$
$$\bar{\partial}_k \langle v, w\rangle = \langle \nabla_{\bar{k}} v, w\rangle + \langle v, \nabla_{\bar{k}} w\rangle$$

for any vectors v, w.

Let us prove that the eigenvectors can be normalized in such a way that

$$\nabla_{\bar{k}} v = 0 \tag{3.38}$$

(i.e. that they are holomorphic sections of $\mathscr{V}\otimes\mathbf{C}$ w.r.t. (3.37)). Indeed, if v, v' are two eigenvectors with the eigenvalues ω, ω', then from

$$0=\bar\partial_l(A_kv-\omega_kv)=\tfrac{1}{2}\,[A_k,\bar A_l]v+(A_k-\omega_k)\bar\partial_lv$$

after multiplication by v' one obtains

$$(\omega_k'-\omega_k)\left\langle v',\bar\partial_lv+\tfrac{1}{2}\,\bar A_lv\right\rangle=0\,.$$

Hence

$$\nabla_{\bar l}v=f_{\bar l}v$$

for some functions $f_{\bar l}$. It is easy to se that the 1-form $f_{\bar l}d\bar z^l$ is closed. After renormalisation of v we obtain (3.38).

The norms $\langle v,v\rangle$ of the normalized eigenvectors are holomorphic functions:

$$\bar\partial_l\langle v,v\rangle=2\langle\nabla_{\bar l}v,v\rangle=0\,.$$

Let $v_1,\ldots,v_n$ be any normalized basis of the eigenvectors (3.33). Let

$$\eta_{ij}=\langle v_i,v_j\rangle \tag{3.39}$$

(a diagonal holomorphic matrix). Then we put

$$\Phi=\hat v^{-1}\,, \tag{3.40}$$

where the coordinates of the eigenvectors $v_1,\ldots,v_n$ are written as the columns of the matrix $\hat v$. One has

$$G=\Phi^T\eta\Phi\,, \tag{3.41a}$$

$$A_k=2\Phi^{-1}C_k\Phi \tag{3.41b}$$

where

$$2C_k=\operatorname{diag}(\omega_k^1,\ldots,\omega_k^n)\,. \tag{3.42}$$

To define the connection Γ let us represent the covariant derivatives ∇_kv_i as linear combinations of the basis vectors

$$\nabla_kv_i=\Gamma_{ki}^jv_j\,. \tag{3.43}$$

The connection Γ_{ki}^j is compatible with η_{ij} as it follows from the definition. It is easy to see that after the gauge transformation

$$\chi=\hat v\xi\,,$$

and the substitution (3.41)–(3.43) the linear problem (3.16) transforms to (1.34), where

$$M=\Phi\bar\Phi^{-1}\,.$$

We are to prove only that the operators C_k determine a closed algebra with a unity. Indeed, since the forms ω^i are closed, they have the form

$$\omega^i=2du^i \tag{3.44}$$

for some functions u^i. In the coordinates $u^1,\ldots,u^n$ the operators C_i become the matrix unities

$$(C_i)_p^q=\delta_i^q\delta_{ip}\,. \tag{3.45}$$

This completes the proof of the theorem.

Note that Proposition 1.2 follows from (3.45).

Appendix. WDVV Equations for Massive TFT Models and Correlators of the Impenetrable Bose Gas

After the main body of this paper was written I was informed by Cecotti that he also obtained (in a joint work with Vafa) the universal form (1.50), (2.17) of the equations of topological-antitopological fusion in TCFT for the particular case of topological minimal models. In the derivation [32] Cecotti and Vafa also used the canonical coordinates (2.3) coinciding [5] for the case of minimal models with critical values of Landau-Ginsburg superpotential. Moreover, they observed [32] that these equations coincide [4] with the equations [33] for multipoint correlators in 2D Ising model.

In this appendix I will show that the equations [34] for multipoint correlators in the impenetrable Bose gas (as functions of distances) are in close relation with WDVV equations (written in the canonical coordinates (2.3)) for a massive TFT model with even number of primaries.

I recall that the Hamiltonian of the one-dimensional non-relativistic Bose gas [35] has the form

$$H = \int_{-\infty}^{+\infty} (\partial_z \psi^+ \partial_z \psi + \psi^+ \psi^+ \psi \psi - h \psi^+ \psi)\, dz\,, \tag{A.1}$$

where $\psi(z)$, $\psi^+(z)$ are canonical Bose fields, $[\psi(z), \psi^+(z')] = \delta(z - z')$, h is a chemical potential. For the case of impenetrable bosons the coupling constant is $c = +\infty$. The mean value of an operator $\mathscr{O}$ at temperature $T > 0$ is defined in a standard way

$$\langle \mathscr{O} \rangle_T = \mathrm{tr}(\mathscr{O} \exp(-H/T))/\mathrm{tr}(\exp(-H/T))\,. \tag{A.2}$$

Multi-point correlators are defined as the mean values of the product of the operators $\psi^+(z_i^+)$, $\psi(z_j^-)$ for different real z_i, z_j:

$$\langle \psi^+(z_1^+) \dots \psi^+(z_N^+) \psi(z_1^-) \dots \psi(z_N^-) \rangle_T = T^{N/2} G_N(x^+, x^-, t) \tag{A.3}$$

for some function G_N of the variables

$$\begin{gathered} x^\pm = (x_1^\pm, \dots, x_N^\pm)\,, \qquad x_i^\pm = z_i \sqrt{T}\,, \qquad i = 1, \dots, N\,, \\ t = h/T\,. \end{gathered} \tag{A.4}$$

This function can be represented in the form [34]

$$\begin{aligned} G_N(x^+, x^-, t) &= (1/4)^N (-1)^{[(N+1)/2]} \prod_{j<k} \mathrm{sign}(x_k^+ - x_j^+)\, \mathrm{sign}(x_k^- - x_j^-) \\ &\quad \times \det_N (V_{lm}(x, t, \kappa)) \Delta(x, t, \kappa))\big|_{\kappa = \frac{2}{\pi}}\,. \end{aligned} \tag{A.5}$$

Here $\det_N (V_{lm})$ means determinant of the $N \times N$ minor of the form $1 \le l \le N$, $N + 1 \le m \le 2N$. The real functions $V_{lm}(x, t, \kappa)$ and $\Delta(x, t, \kappa)$ are determined in terms of the linear integral operator

$$(Kf)(\lambda) = \int_{-\infty}^{\infty} \sum_{m=1}^{2N} (-1)^m \frac{e_m^+(\lambda) e_m^-(\mu)}{2i(\lambda - \mu)} f(\mu)\, d\mu\,, \tag{A.6}$$

[4] I acknowledge S. Cecotti for his explanation of the intrinsic physical reasons of this coincidence

where

$$e_m^{\pm}(\lambda) = \sqrt{\theta(\lambda)}\exp(\pm i\lambda x_m)\,, \tag{A.7}$$

$\theta(\lambda)$ is the Fermi weight

$$\theta(\lambda) = [1 + \exp(\lambda^2 - t)]^{-1} \tag{A.8}$$

(dependence of $\theta(\lambda)$ on t and of $e_m^{\pm}(\lambda)$ on x, t is suppressed in the formulae). Namely,

$$V_{lm}(x, t, \kappa) = (-1)^l \kappa \int_{-\infty}^{\infty} e_l^-(\mu) f_m^+(\mu)\, d\mu\,, \tag{A.9}$$

where the functions $f_m^{\pm}(\mu)$ (depending also on x, t, κ) are determined by the formulae

$$f_m^+ - \kappa K f_m^+ = e_m^+\,, \tag{A.10a}$$

$$f_m^- - \kappa K^\dagger f_m^- = e_m^-\,. \tag{A.10b}$$

The function $\Delta(x, t, \kappa)$,

$$\Delta(x, t, \kappa) = \det(1 - \kappa K)\,. \tag{A.11}$$

The functions V_{lm} obey the following integrable system of PDE [34]:

$$\partial_k V_{lm} = \tfrac{1}{2} V_{lk} V_{km}\,, \qquad k \neq m, l\,, \tag{A.12a}$$

$$\sum_{k=1}^{2N} \partial_k V_{lm} = 0\,, \tag{A.12b}$$

$$V_{lm} = (-1)^{l+m} V_{lm}\,, \tag{A.12c}$$

$$\sum V_{mm} = 0 \tag{A.12d}$$

as functions of the $2N$-component vector $x = (x^+, x^-) = (x_1, \ldots, x_{2N})$, $\partial_k = \partial/\partial x_k$. Equations containing the "time" derivative are [ibid]

$$\begin{aligned}\partial_t(\partial_l - \partial_m) V_{lm} &+ (x_l - x_m) V_{lm} + (\partial_t V_{mm} - \partial_t V_{ll}) V_{lm} \\ &+ \tfrac{1}{2} \sum_{p \neq l,m} (V_{lp} \partial_t V_{pm} - V_{pm} \partial_t V_{lp}) = 0\,.\end{aligned} \tag{A.13}$$

Dependence of the determinant (A.11) on x, t is specified by the equations

$$\partial_m \log \Delta = -\tfrac{1}{2} V_{mm}\,, \tag{A.14a}$$

$$\partial_t \log \Delta = -\tfrac{1}{2} \sum_{m=1}^{2N} x_m \partial_t V_{mm} + \tfrac{1}{4} \sum (\partial_t V_{lm})(\partial_t V_{ml})\,. \tag{A.14b}$$

After substitution

$$V_{lm} = (-1)^l \gamma_{lm}\,, \tag{A.15a}$$

$$x_l = 2(-1)^l u^l\,, \tag{A.15b}$$

one immediately obtains that the t-independent part (A.12) of the system for the off-diagonal part of the matrix V coincides with the system (2.5) for $n = 2N$ (i.e. with WDVV eqs. in the canonical coordinates u^l). The functions V_{mm} and $\log \Delta$ are determined by the off-diagonal terms V_{lm} from Eqs. (A.12) and (A.14).

We obtain that multipoint correlators of the impenetrable Bose-gas determine a two-parameter family of massive TFT models (depending on t and κ) with even number of primaries. The zero-temperature limit of these correlators can be expressed via Painlevé transcendents of the fifth kind (for $N = 1$) and their high-order generalisations [36]. It is interesting that this limit does not coincide with TCFT, where all the correlators are given in terms of the Painlevé-VI transcendents and their high-order generalisations.

Acknowledgements. I am acknowledged to S. Cecotti for explanation of some important points of the paper [6]. I wish to thank A. Its for help in using the isomonodromy deformations method. I am grateful to E. Witten for bringing my attention to the papers [31]; this was useful for giving Sect. 3 its final shape.

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Communicated by N.Yu. Reshetikhin