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Classical A_n-W-Geometry

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Abstract. By analyzing the *extrinsic* geometry of two dimensional surfaces chirally embedded in $\mathscr{C}P^n$ (the $\mathscr{C}P^n$ W-surface [1]), we give exact treatments in various aspects of the classical W-geometry in the conformal gauge: First, the basis of tangent and normal vectors are defined at regular points of the surface, such that their infinitesimal displacements are given by connections which coincide with the vector potentials of the (conformal) A_n-Toda Lax pair. Since the latter is known to be intrinsically related with the W symmetries, this gives the geometrical meaning of the A_n W-Algebra. Second, W-surfaces are put in one-to-one correspondence with solutions of the conformally-reduced WZNW model, which is such that the Toda fields give the Cartan part in the Gauss decomposition of its solutions. Third, the additional variables of the Toda hierarchy are used as coordinates of $\mathscr{C}P^n$. This allows us to show that W-transformations may be extended as particular diffeomorphisms of this target-space. Higher-dimensional generalizations of the WZNW equations are derived and related with the Zakharov-Shabat equations of the Toda hierarchy. Fourth, singular points are studied from a global viewpoint, using our earlier observation [1] that W-surfaces may be regarded as instantons. The global indices of the W-geometry, which are written in terms of the Toda fields, are shown to be the instanton numbers for associated mappings of W-surfaces into the Grassmannians. The relation with the singularities of W-surface is derived by combining the Toda equations with the Gauss-Bonnet theorem.

1. Introduction

The geometry behind W-algebras [2,3] has been an important open problem of solvable quantum field theories ([4–13]). It is supposed to give generalizations of the

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★★ Present address: Dept. of Physics, Univ. of Tokyo, Hongo 7-3-1, Nunkyo-ku, Tokyo, Japan two-dimensional gravity by including higher-spin gauge fields. It is also important for understanding the relationship between solvable models in two dimensions and higher. Two-dimensional gravity in the conformal gauge is notoriously equivalent to the Liouville theory which is the conformal Toda theory associated with the Lie algebra A_1. A step to generalize this situation was made in [2] where it was shown that the Toda theory associated with any given simple Lie algebra gives two Noether realizations of the corresponding W-algebra. Thus, if the above A_1-Liouville scheme is repeated for the other Lie algebras, there should exist generalizations of two-dimensional gravity (called W-gravities) which are invariant by generalized diffeomorphisms, and coincide with the conformal Toda theories when a particular local coordinate frame is used.

Until now, however, this conjecture could not be proven. One of the main difficulties is that for W gravities the Virasoro algebra is replaced by W-algebras which are non-linear. Many studies start by using a *linearized* W-symmetry, and try to include the non-linear effects by perturbation. Although this strategy is quite popular, it becomes clear that one cannot get the final answer this way in a closed form. Since those approaches are approximate by nature, they cannot yield the exact geometrical structure of W algebras. In the same way, as for Einstein's general relativity, understanding this geometry is certainly the key. Our strategy is to unravel the geometry behind the Toda theories that include the full non-linearity of W-algebras from the start.

In this paper, we give the details of our previous proposal [1] that one can regard the W-geometry as the extrinsic geometry of particular two dimensional surfaces (W-surface) embedded in higher dimensional Kähler manifolds. (We will restrict ourselves to the simplest particular situation, i.e. our target space is $\mathscr{C}P^n$ which corresponds to the A_n-type W-geometry.) Instead of introducing higher-spin gauge generators, our approach makes use of the *extrinsic* curvatures of the embedded surface, and relates it with the Toda dynamics mentioned above. The main virtue of our approach is that it is very simple to begin with. A W surface is characterized by the specific *chiral structure* of its embedding which we will call chiral for short. More explicitly, a Kähler manifold has a natural description by pairs of conjugate coordinates, and a W surface is such that, basically, the first (resp. second) is an analytic (resp. anti-analytic) function of the surface parameters. Since the case of real surface coordinates will be considered simultaneously, we shall use most of the time the words chiral and anti-chiral instead of holomorphic (analytic) and anti-holomorphic (anti-analytic). A related fact is that our analytic embedding functions are not necessarily complex conjugate of the anti-analytic ones. We need to pick this choice in order to make the analytic and anti-analytic W-algebras independently act on the embedding functions. Thus we have to deal with two dimensional surfaces and not with curves.

The idea of getting solvable models from the embedding of two dimensional surfaces into higher dimensional spaces has a long history. For example, it is a classical fact that the sine-Gordon equation can be obtained from embedding into a three dimensional flat space. In [14], Saveliev made a general analysis of surface-embeddings into spaces equipped with Lie algebra structures, and connected them with Toda theories. However, he considered the embedding of general surfaces, and thus his picture is much more complicated [1]. On the other hand, a proposal was already made by Sotokov and Stanishkov [4], along lines similar to ours. They realize W_3-geometry by using the extrinsic geometry of a two dimensional surface in the three

[1] Thus he could not deal with A_n as we do here

dimensional affine space A_3. The difference between our work and theirs is that while they use a light-cone gauge fixing, we choose the equivalent of the conformal gauge. As usual, this latter gauge choice is more practical than the light-cone approach. In their work, the restriction of the embedding function is rather complicated and it is increasingly difficult to generalize it to higher rank cases. On the contrary, in the conformal gauge, we make use of the intimate relationship [2] between conformal Toda field theories[2] and W-symmetries. The restriction to the embedding functions is particularly simple, i.e. they should be chiral, and it is presumably universal for any kind of W-algebras. The simplicity of our gauge choice makes it easy to approach problems which have been difficult to solve previously.

Another novelty of our approach is that we need to combine the chiral and antichiral sectors, using a Kähler structure. This feature is in contrast with most of the current discussions [5–13], apart from [14 and 4] mentioned above. It is crucial in our opinion, since both chiral and anti-chiral components appear in the conformal theories we are considering, although they certainly have holomorphic-antiholomorphic decompositions.

In the first three sections we consider regular points of W surfaces and our viewpoint is strictly local. We construct a set of orthonormalized tangents and normals, study their infinitesimal displacements, and construct a well suited local reference frame of the target space in a neighborhood of the W surface. In the last section we consider singular points instead. Then it is more significant to discuss global properties, that is, what are the possible singularities (ramification indices) of a W surface with a given genus.

Since this article contains many different points developed one after the other, we present it as a succession of theorems, propositions, and so on, for clarity[3] Yet the language should be familiar to physicists of the field. The body of the paper is divided into three main sections after the present introduction which is called Sect. 1.

The *main section 2* deals with the regular points of a W-surface Σ where the Taylor expansions of the coordinates of the surface generate linearly independent vectors. *The mere introduction of the extrinsic geometry is not enough to understand* W-*geometry. We need to organize them such that we do not have redundant degree of freedoms.* This program is carried out by a string analogue of Frenet-Serret relations. It is shown that the Lay pair of A_n Toda theory appears naturally and that their compatibility condition, that is the Gauss-Codazzi equations [15], is equivalent to Toda equations. Next, we show that there is a complete equivalence between conformally reduced WZNW models [17] and W-surfaces in $\mathscr{C}P^n$. The generalized Frenet-Serret formula derived in Sect. 2.2 is connected with the Gauss decomposition of the solution of the WZNW equations.

The *main section 3* deals with the interpretation of W-symmetry as target-space diffeomorphism. First, the present geometrical interpretation makes a frequent use of determinants, so that it is natural to introduce fermionic operators. We use this free-fermion approach [18] to introduce the additional coordinates of the KP hierarchy in $\mathscr{C}P^n$-W-surfaces. They give particular parametrizations of the target-manifold. (We call them W-parametrizations). They allow us to extend the W-transformations to the target space, obtaining a special class of diffeomorphisms. We next study the W-parametrizations from the viewpoint of the Toda hierarchy. The link between Riemannian geometry and the latter is established by showing that the integrability

[2] We only deal with conformal Toda theories, in the present article. They will simply be called Toda theories

[3] Contrary to the equations, they are not numbered by section

conditions for W-parametrizations coincide with the Zakharov-Shabat equations. Next, it is noted that W-parametrization gives rise to a higher dimensional analogue of the solvable WZNW equations for their Christoffel symbols.

Concerning the *main section 4*, we first reformulate our approach in terms of the intrinsic geometry of the family of associated surfaces in the Grassmannians $G_{n+1,k+1}$, $k = 1, \ldots, n$. This is needed to study singular points and global aspects of W-surfaces following the general scheme of our letter [1]. The aim is to establish the generalization of the Gauss-Bonnet theorem to the W-surfaces discussed above. The instanton number associated with each mapping can be regarded as the global invariant of W-geometry. We relate them with the singularity indices of the W-surface.

Before beginning our journey through the W-geometrical landscape, we note that it is quite attractive an idea that it comes out from the geometry of embeddings. Indeed, 2D conformal systems are notoriously related to string theories, and the present viewpoint goes in the same direction. It is compatible with the fact that W-strings [23], if they exist, may come out spontaneously when on looks for the true vacuum of the much-wanted string-theory of Nature.

2. Local Structure of the Embedding at Regular Points

2.1. The Gauss-Codazzi–Frenet-Serret Equations

In order to get correctly the W-geometry in the conformal gauge, we need to carefully define the target space and the restriction of the embedding function. Compared to the lightcone gauge approach [4], our definition of the restriction becomes quite simple (See Definition 2 in the following.) In this section we only deal with trivial target-manifolds. This will be used later on to deal with the complex projective spaces.

Definition 1. ***$\mathscr{C}^n$ target manifolds.*** They will be taken to be Riemannian manifolds with $2n$ real dimensions, noted $\mathscr{C}^n$, whose points represented by boldface letters X, have components $X^{\underline{A}}$, $\underline{A} = 1, \ldots, 2n$. It is assumed that
1) there exists a preferred class of coordinates X^A, $\bar{X}^{\bar{A}}$, $1 \leq A, \bar{A} \leq n$, such that the line-element takes the form $ds^2 = 2\Sigma_A dX^A d\bar{X}^A$,
2) there exists a conjugation-operation noted with a star such that:

$$(X^{\underline{A}})^* = \sum_{\underline{B}=1}^{2n} \widetilde{C}^{\underline{A}}_{\underline{B}} X^{\underline{B}}, \tag{2.1}$$

which leaves the line-element invariant.

This is very close to the standard definition of $\mathscr{C}^n$, but, contrary to the common practice[4], we do not assume that X^A and $\bar{X}^{\bar{B}}$ are complex coordinates such that $(X^A)^* = \bar{X}^A$. Our past knowledge of string theory shows that one must be more flexible. For instance, if we think of a string in constant Minkowski-metric, the components that involve the time-direction, say X^0 and $\bar{X}^0$, are real. This will be shown, on an explicit example at the end of the section. We shall call chiral (resp. anti-chiral) components, the set $\{X^A, 1 \leq A \leq n\}$ (resp. $\{\bar{X}^{\bar{A}}, 1 \leq \bar{A} \leq n\}$).

Our basic strategy is to study embeddings of two-dimensional surfaces Σ with chiral parametrizations. This chirality is defined with respect to surface parameters

[4] Thus we use the symbol $\mathscr{C}^n$ instead of the usual C^n

noted z and $\bar{z}$. On the surface, we shall make use of a trivial two-dimensional complex structure of the usual type. However, taking z to be a standard complex variable is not the only choice. It corresponds to a Euclidean parametrization where the real coordinates are $x_1 = (z+\bar{z})/2$, and $x_2 = (z-\bar{z})/2i$. Another possibility is to be working with real surface-parameters. Then, as it is well known, the square-root of -1 is represented by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ acting on the two-component vector $\begin{pmatrix} \bar{z} \\ z \end{pmatrix}$. The "real," and "imaginary parts" are now $x_0 = (z+\bar{z})/2$, $x_1 = (z-\bar{z})/2$. This parametrization is of the Minkowski type, where x_0 is a time-like parameter.

Next, a function is called chiral if it only depends upon one of the two coordinates z or $\bar{z}$ (if z is a complex variable this means of course analytic or anti-analytic). A basic object of the present W-geometry is specified by

Definition 2. W-***Surfaces.*** A $\mathscr{C}^n$ W-surface is a two-dimensional manifold Σ with a chiral embedding into $\mathscr{C}^n$. A chiral embedding is defined by equations of the form

$$X^A = f^A(z), \;\; A = 1, \ldots, n, \quad \bar{X}^{\bar{A}} = \bar{f}^{\bar{A}}(\bar{z}), \;\; \bar{A} = 1, \ldots, n. \tag{2.2}$$

We shall also use the words chiral surfaces with the same meaning as W surfaces[5]. We do not assume any general link between the conjugation in $\mathscr{C}^n$ and in the surface-parameter-space. Thus f^A and $\bar{f}^{\bar{A}}$ are independent functions. For string applications this is needed, basically, since the two chiral components may be associated with the right- and left-moving modes which are independent if the string is closed. We shall give an example of this fact, at the end of the section, by considering the case of free bosonic strings.

Our first result is that Toda field-equations naturally arise from the Gauss-Codazzi equations of the chiral embedding of W-surfaces. These equations are integrability conditions for derivatives of the tangents and of the normals to the surface. The latter are introduced by extending Frenet-Serret formulae as follows. At each point of the surface, one considers the Taylor expansion of f^A and $\bar{f}^{\bar{A}}$ up to the n-th order, and introduce the corresponding matrix of inner products:

$$g_{\bar{\jmath} i} = g_{i \bar{\jmath}} \equiv \sum_{A\bar{B}} \delta_{A\bar{B}} \partial^{(i)} f^A(z) \bar{\partial}^{(\bar{\jmath})} \bar{f}^{\bar{B}}(\bar{z}), \quad 1 \le i, \bar{\jmath} \le n. \tag{2.3}$$

∂ and $\bar{\partial}$ are shorthands for $\partial/\partial z$ and $\partial/\partial\bar{z}$ respectively. $\partial^{(i)}$ stands for $(\partial)^i$. Later on we shall exhibit a particular parametrization of $\mathscr{C}^n$, called W-parametrization, where the vectors $\partial^{(i)} f^A(z)$, $i > 1$, and $\bar{\partial}^{(\bar{\jmath})} \bar{f}^{\bar{B}}(\bar{z})$ $\bar{\jmath} > 1$ will become tangent vectors, so that the covariance properties of the present discussion will become more transparent. At this moment, we are concerned with generic regular points of Σ, where the Taylor expansions of f^A and $\bar{f}^{\bar{A}}$ give linearly independent vectors. Then $f^{(a)}$, and $\bar{f}^{(a)}$, $a = 1, \ldots, n$, (upper indices in between parenthesis denote derivatives) span the following

[5] The emphasis is on their extrinsic geometry

Definition 3. ***Moving Frame.*** Consider the vectors e_a, and $\bar{e}_a$, $a = 1, \ldots, n$, with components

$$e_a^A = \frac{1}{\sqrt{\Delta_a \Delta_{a-1}}} \begin{vmatrix} g_{1\bar{1}} & \cdots & g_{a\bar{1}} \\ \vdots & & \vdots \\ g_{1\overline{a-1}} & \cdots & g_{a\overline{a-1}} \\ f^{(1)A} & \ldots & f^{(a)A} \end{vmatrix}, \quad e_a^{\bar{A}} = 0, \tag{2.4}$$

$$\bar{e}_a^A = 0, \quad \bar{e}_a^{\bar{A}} = \frac{1}{\sqrt{\Delta_a \Delta_{a-1}}} \begin{vmatrix} g_{\bar{1}1} & \cdots & g_{\bar{a}1} \\ \vdots & & \vdots \\ g_{\bar{1}a-1} & \cdots & g_{\bar{a}a-1} \\ \bar{f}^{(1)\bar{A}} & \ldots & \bar{f}^{(a)\bar{A}} \end{vmatrix}, \tag{2.5}$$

Δ_a is the determinant

$$\Delta_a \equiv \begin{vmatrix} g_{1\bar{1}} & \cdots & g_{a\bar{1}} \\ \vdots & & \vdots \\ g_{1\bar{a}} & \cdots & g_{a\bar{a}} \end{vmatrix}. \tag{2.6}$$

Denote by (x, y) the inner product $\sum_A (x^A \bar{y}^A + y^A \bar{x}^A)$. One has the

Proposition 1. *The moving frame defined above is orthonormal, that is*

$$(e_a, e_b) = (\bar{e}_a, \bar{e}_b) = 0, \quad (e_a, \bar{e}_b) = \delta_{a,b}. \tag{2.7}$$

Proof. These last relations immediately follow from the fact that

$$(e_a, \bar{f}^{(b)}) = 0, \quad \text{and} \quad (\bar{e}_a, f^{(b)}) = 0 \quad \text{for} \quad a > b \tag{2.8}$$

together with the definition Eq. (2.6) of Δ_a. Q.E.D.

For the following it is important to note that, according to Eq. (2.8), Eqs. (2.4), (2.5) take the form

$$e_a = \sum_{b \le a} C_{ab}(z, \bar{z}) \sqrt{\frac{\Delta_{a-1}}{\Delta_a}} f^{(b)}(z), \quad \text{with} \quad C_{aa} = 1, \tag{2.9}$$

$$\bar{e}_a = \sum_{b \le a} A_{ba}(z, \bar{z}) \sqrt{\frac{\Delta_{a-1}}{\Delta_a}} \bar{f}^{(b)}(\bar{z}), \quad \text{with} \quad A_{aa} = 1. \tag{2.10}$$

This equation is also valid for $a = 1$ if we define Δ_0 to be equal to one, as we shall do. The vectors e_1 and $\bar{e}_1$ are tangents to the surface, while the other vectors are clearly normals. Thus the Gauss-Codazzi equations may be derived by studying their derivatives along Σ. The main result of the present section is the

Theorem 1. ***Generalized Frenet-Serret Formulae.*** *The derivative of the moving frame is given by*

$$
\begin{aligned}
\partial e_a =& \frac{1}{2}\partial \ln\left(\frac{\Delta_a}{\Delta_{a-1}}\right) e_a + \sqrt{\frac{\Delta_{a-1}\Delta_{a+1}}{\Delta_a^2}}\, e_{a+1}, \quad a \le n-1\,, \\
\partial e_n =& \frac{1}{2}\partial \ln\left(\frac{\Delta_n}{\Delta_{n-1}}\right) e_n\,, \\
\bar\partial e_a =& -\frac{1}{2}\bar\partial \ln\left(\frac{\Delta_a}{\Delta_{a-1}}\right) e_a - \sqrt{\frac{\Delta_{a-2}\Delta_a}{\Delta_{a-1}^2}}\, e_{a-1}, \quad 2 \le a\,, \\
\bar\partial e_1 =& -\frac{1}{2}\bar\partial \ln(\Delta_1) e_1\,,
\end{aligned}
\tag{2.11}
$$

with similar equations for $\bar e$.

Proof. It is easy to see that these derivatives may be written as

$$
\begin{aligned}
\partial e_a =& \sum_b R_{ab} e_b\,, \quad \bar\partial e_a = \sum_b S_{ab} e_b\,; \\
\bar\partial \bar e_a =& \sum_b \bar R_{ab} \bar e_b\,, \quad \partial \bar e_a = \sum_b \bar S_{ab} \bar e_b\,.
\end{aligned}
\tag{2.12}
$$

According to Eq. (2.7), one has

$$
R_{ab} + \bar S_{ba} = \bar R_{ab} + S_{ba} = 0\,. \tag{2.13}
$$

Since $\partial \bar f^{(a)} = \bar\partial f^{(a)} = 0$, it follows from Eqs. (2.8), (2.9), (2.10) that S_{ab} and $\bar S_{ab}$ vanish for $a < b$. Moreover, it is easy to verify that ∂e_a and $\bar\partial \bar e_a$ may be respectively expanded in terms of $f^{(b)}$ and $\bar f^{(b)}$ with $b \le a+1$ only. Thus R_{ab} and $\bar R_{ab}$ vanish for $b > a+1$. Combining with Eq. (2.13), one sees that the only non-vanishing elements are $R_{aa+1} \equiv \kappa_a$, $\bar R_{aa+1} \equiv \bar\kappa_a$, for $a \le n-1$, and $R_{aa} \equiv \sigma_a$, $\bar R_{aa} \equiv \bar\sigma_a$, for $a \le n$. Equation (2.12) becomes

$$
\partial e_a = \kappa_a e_{a+1} + \sigma_a e_a\,, \quad \bar\partial e_a = -\bar\sigma_a e_a - \bar\kappa_{a-1} e_{a-1}\,, \tag{2.14}
$$

$$
\bar\partial \bar e_a = \bar\kappa_a \bar e_{a+1} + \bar\sigma_a \bar e_a\,, \quad \partial \bar e_a = -\sigma_a \bar e_a - \kappa_{a-1} \bar e_{a-1}\,. \tag{2.15}
$$

Making use of Eqs. (2.8), (2.9), (2.10), one next easily obtains

$$
\sigma_a = -(\partial \bar e_a, e_a) = \frac{1}{2}\partial \ln\left(\frac{\Delta_a}{\Delta_{a-1}}\right), \quad \bar\sigma_a = -(\bar\partial e_a, \bar e_a) = \frac{1}{2}\bar\partial \ln\left(\frac{\Delta_a}{\Delta_{a-1}}\right) \tag{2.16}
$$

$$
\kappa_a = (\partial e_a, \bar e_{a+1}) = \sqrt{\frac{\Delta_{a-1}\Delta_{a+1}}{\Delta_a^2}}\,, \qquad \bar\kappa_a = (\bar\partial \bar e_a, e_{a+1}) = \kappa_a\,, \tag{2.17}
$$

and the proof is completed by using the fact that the e's and $\bar e$'s form a complete basis. Q.E.D.

This theorem generalizes, for W-surfaces, the Frenet-Serret formulae which are standard for curves. Equations (2.11) have a form which is closely related to the A_{n-1} Toda equations, if we define the Toda-like fields by

$$
\phi_a \equiv -\ln(\Delta_a)\,, \quad \text{for} \quad a = 1, \ldots, n\,. \tag{2.18}
$$

This may be neatly expressed as follows. As is well known, the Lie algebra A_{n-1} may be explicitly realized with n fermionic operators $\mathfrak{b}_i$, which satisfy $[\mathfrak{b}_i, \mathfrak{b}_j^+]_+ = \delta_{i,j}$. One writes

$$h_i = \mathfrak{b}_i^+\mathfrak{b}_i - \mathfrak{b}_{i+1}^+\mathfrak{b}_{i+1}\,, \quad E_i = \mathfrak{b}_i^+\mathfrak{b}_{i+1}\,, \quad E_{-i} = E_i^+\,, \quad i = 1, \ldots, n-1\,, \tag{2.19}$$

in the Chevalley basis where h_i generate the Cartan subalgebra, and $E_{\pm i}$ are associated with the standard set of simple roots. The basic difference in our case is that, contrary to the A_{n-1}-Toda case, ϕ_n is not zero, and we need another "Cartan" generator

$$h_n = \mathfrak{b}_n^+\mathfrak{b}_n \tag{2.20}$$

which is realized by the fermionic operators but does not belong to A_{n-1}. Altogether, the generators we have introduced satisfy the commutation relations

$$[h_i, E_{\pm j}] = \pm K_{ji}^{gl(n)} E_{\pm j}\,, \quad [E_j, E_{-k}] = \delta_{j,k} h_j\,, \tag{2.21}$$

where i goes from 1 to n, while j and k run from 1 to $n-1$. Concerning Lie algebras, we denote by $gl(n)$ the Lie algebra of $n \times n$ matrices (Lie algebra of the linear group). The matrix $K_{ij}^{gl(n)}$ may be regarded as the $n \times n$ Cartan matrix of $gl(n) \sim A_{n-1} \oplus gl(1)$. For i and j between 1 and $n-1$, in coincides with the Cartan matrix of A_{n-1}, and

$$K_{jn}^{gl(n)} = -\delta_{j,n-1}\,. \tag{2.22}$$

The reality conditions are most simply discussed with Minkowski surface parameters (z and $\bar{z}$ real). Then one is using the most non-compact real form of the Lie algebras we encounter. In particular, the Lie group associated with $gl(1)$ is the multiplication by real positive numbers. This should be understood from now on. Let us go back to Eq. (2.11). Together with the anti-chiral parts, it takes the form

$$\begin{aligned} \partial e_a &= \sum_b \omega_{za}^b e_b\,, \quad \bar{\partial} e_a = \sum_b \omega_{\bar{z}a}^b e_b\,, \\ \bar{\partial} \bar{e}_a &= \sum_b \bar{\omega}_{\bar{z}a}^b \bar{e}_b\,, \quad \partial \bar{e}_a = \sum_b \bar{\omega}_{za}^b \bar{e}_b\,. \end{aligned} \tag{2.23}$$

It follows trivially from Eq. (2.7) that $\omega_{za}^b + \bar{\omega}_{zb}^a = 0$ and $\omega_{\bar{z}a}^b + \bar{\omega}_{\bar{z}b}^a = 0$. The generators (2.19), (2.20) commute with the number operator $N = \sum_{i=1}^n \mathfrak{b}_i^+\mathfrak{b}_i$. The subspace with $N = 1$ has dimension n. We identify it with the space span by the e_a and write

$$\omega_{za}^b = <0|\mathfrak{b}_b \omega_z \mathfrak{b}_a^+|0>\,, \quad \omega_{\bar{z}a}^b = <0|\mathfrak{b}_b \omega_{\bar{z}} \mathfrak{b}_a^+|0>\,, \tag{2.24}$$

where $|0>$ is the vacuum state of the oscillators $\mathfrak{b}_i$. Using the formulae just given, one sees that Eq. (2.11) is equivalent to

$$\begin{aligned} \omega_z &= -\frac{1}{2}\sum_{i=1}^n h_i \partial\phi_i + \sum_{i=1}^{n-1} \exp\left(\sum_{j=1}^n K_{ij}^{gl(n)}\phi_j/2\right) E_{-i}\,, \\ \omega_{\bar{z}} &= \frac{1}{2}\sum_{i=1}^n h_i \bar{\partial}\phi_i - \sum_{i=1}^{n-1} \exp\left(\sum_{j=1}^n K_{ij}^{gl(n)}\phi_j/2\right) E_i\,. \end{aligned} \tag{2.25}$$

Remarkably, one sees that the right member is just the Toda Lax-pair [16]. Toda equations are equivalent to the zero-curvature condition on ω. Let us remember that, in the language of Riemannian geometry [15], the associated second fundamental form is given by the projection of the derivatives of the tangent vectors onto the normals. Its non-vanishing components are

$$(\partial^2 f, \bar{e}_a) \equiv \Omega^a_{zz} = \delta_{a,2}\sqrt{\frac{\Delta_2}{\Delta_1}}\,, \quad (\bar{\partial}^2 \bar{f}, e_a) \equiv \bar{\Omega}^a_{\bar{z}\bar{z}} = \delta_{a,2}\sqrt{\frac{\Delta_2}{\Delta_1}}\,, \tag{2.26}$$

where $a \neq 1$. Similarly, the third fundamental form is given by the projection of the derivatives of the normals:

$$\begin{aligned}(\partial e_a, \bar{e}_b) &= \omega^b_{za}\,, \quad (\bar{\partial} e_a, \bar{e}_b) = \omega^b_{\bar{z}a}\,,\\ (\partial \bar{e}_a, e_b) &= \bar{\omega}^b_{za}\,, \quad (\bar{\partial} \bar{e}_a, e_b) = \bar{\omega}^b_{\bar{z}a}\,,\end{aligned} \tag{2.27}$$

where $a, b \geq 2$. Going to the second derivatives, we next derive the

Theorem 2. ***Gauss-Codazzi Equations.*** *The integrability conditions of the Frenet-Serret equations Eqs.* (2.11) *coincide with the Toda equations associated with $gl(n)$:*

$$\begin{aligned}0 = [\partial, \bar{\partial}]e_a &= \sum_b F^b_{z\bar{z}a} e_b \equiv \sum_b < 0|\flat_b F_{z\bar{z}} \flat^+_a|0 > e_b\,,\\ F_{z\bar{z}} &= \sum_{i=1}^{n} h_i \partial\bar{\partial}\phi_i + \sum_{i=1}^{n-1} h_i \exp\left(\sum_{j=1}^{n} K^{gl(n)}_{ij}\phi_j\right)\end{aligned} \tag{2.28}$$

Proof. Straightforward computations using Eqs. (2.21), (2.22), and (2.24), (2.25). Q.E.D.

It is instructive to directly compare the above formulae with the explicit solution of [2]. Equation (2.3) shows that $g_{11} \equiv \exp(-\phi_1)$ has chiral components are $\chi^A \equiv f^{(1)A}$ and $\bar{\chi}^A \equiv \bar{f}^{(1)A}$. A simple calculation starting from Eqs. (2.6), and (2.18) shows that

$$e^{-\phi_k} = \sum_{i_1<\cdots<i_k} \begin{vmatrix} \chi^{i_1} & \dots & \chi^{i_k}\\ \chi^{(1)i_1} & \dots & \chi^{(1)i_k}\\ \vdots & \dots & \vdots\\ \chi^{(k-1)i_1} & \dots & \chi^{(k-1)i_k}\end{vmatrix}\,, \begin{vmatrix} \bar{\chi}^{i_1} & \dots & \bar{\chi}^{i_k}\\ \bar{\chi}^{(1)i_1} & \dots & \bar{\chi}^{(1)i_k}\\ \vdots & \dots & \vdots\\ \bar{\chi}^{(k-1)i_1} & \dots & \bar{\chi}^{(k-1)i_k}\end{vmatrix}\,. \tag{2.29}$$

This exactly coincides with the explicit form of the A_{n-1}-Toda solutions of [2]. The only difference is that the right member of the above is not equal to one for the $k = n$, so that ϕ_n does not vanish. However, in the present case, this right member factorizes into the product of a single function of z times another function of $\bar{z}$, so that ϕ_n is a solution of $\partial\bar{\partial}\phi_n = 0$. These explicit formulae of course confirm our previous calculations, that is, Eqs. (2.28). The removal of the additional field, might simply be done by imposing that the Wronskians of the functions χ^A and of the functions $\bar{\chi}^A$ be equal to one. At the present stage, this would be an artificial condition without geometrical significance, since these functions are the first derivatives of the embedding functions. We will see that this additional $gl(1)$ factor will be removed naturally in the $\mathscr{C}P^n$ case. This will be the subject of the coming section. Beside this $gl(1)$ factor, the present situation has another unwanted feature. The induced metric on Σ is $g_{11} = \exp(-\phi_1)$, while for the Liouville theory, say, it is $\exp(2\phi_1)$! This disaster will be repaired in Appendix A.1, explicitly, by deriving Proposition 10.

The Example of Free Bosonic String. In this subsection, we show the necessity of our more general conjugation (see Eq. (2.1)) on a simple stringy example. Let Y^α,

$\alpha = 1, \ldots, 2n$, be the space-time coordinates of the string which are, of course, real. As we shall see, it is essential to work with the Minkowski signature, *without performing the Wick rotation.* The target-space metric, $\tilde{\eta}_{\alpha\beta} = \pm\delta_{\alpha,\beta}$ is taken to be constant and diagonal. For the sake of the coming argument, we shall allow for several time-like directions (this possibility cannot be ruled out a priori for W-strings [23]). Thus we take the target-space metric to be $\tilde{\eta}_{\alpha\beta} = -\delta_{\alpha,\beta}$, for $\alpha = 1, \ldots, s$, $\tilde{\eta}_{\alpha\beta} = \delta_{\alpha,\beta}$, for $\alpha = s+1, \ldots, 2_n$. s is the number of "time" axis. This metric will become off-diagonal, as required by the definition 1, if one defines

$$\left.\begin{aligned} X^\alpha &\equiv (Y^\alpha + Y^{n+\alpha})/\sqrt{2} \\ \bar{X}^\alpha &\equiv (-Y^\alpha + Y^{n+\alpha})/\sqrt{2} \end{aligned}\right\}, \quad \alpha = 1, \ldots, s,$$
$$\left.\begin{aligned} X^\alpha &\equiv (Y^\alpha + iY^{n+\alpha})/\sqrt{2} \\ \bar{X}^\alpha &\equiv (Y^\alpha - iY^{n+\alpha})/\sqrt{2} \end{aligned}\right\}, \quad \alpha = s+1, \ldots, n. \tag{2.30}$$

In this way the inner product is indeed $(X_1, X_2) = \sum_A (X_1^A \bar{X}_2^A + X_2^A \bar{X}_1^A)$. Obviously, one has $(X^A)^* = X^A$, for $A \leq s$, and $(X^A)^* = \bar{X}^A$, for $A > s$. This is an example of Eq. (2.1), which is not the usual conjugation of $\mathscr{C}^n$. For a free string, the surface swept by the string-positions is given by

$$Y^\alpha = q^\alpha + p^\alpha \ln(z) + \tilde{p}^\alpha \ln(\bar{z}) + i \sum_{r\neq 0} \left[\frac{a_r^\alpha}{r} z^{-r} + \frac{\tilde{a}_r^\alpha}{r} \bar{z}^{-r} \right], \tag{2.31}$$

where a_r^α (resp. $\tilde{a}_r^\alpha$) are the right-moving (resp. left-moving) oscillator-modes which satisfy $(a_r^\alpha)^* = a_{-r}^\alpha$ (resp. $(\tilde{a}_r^\alpha)^* = \tilde{a}_{-r}^\alpha$). q^α (resp. $(p^\alpha + \tilde{p}^\alpha)/2$) is the center-of-mass position (resp. total momentum) of the string which must be real. $(p^\alpha - \tilde{p}^\alpha)/2i$ is the winding-number which is an integer. Equation (2.31) will describe a W-surface, if the embedding functions computed from Eqs. (2.30) satisfy the Cauchy-Riemann relations $\partial \bar{f}^{\bar{A}} = \bar{\partial} f^A = 0$. One gets

$$\begin{array}{lllll} p^\alpha = p^{n+\alpha}, & \tilde{p}^\alpha = -\tilde{p}^{n+\alpha}, & a_r^\alpha = a_r^{n+\alpha}, & \tilde{a}_r^\alpha = -\tilde{a}_r^{n+\alpha}, & \alpha \leq s, \\ p^\alpha = ip^{n+\alpha}, & \tilde{p}^\alpha = -i\tilde{p}^{n+\alpha}, & a_r^\alpha = ia_r^{n+\alpha}, & \tilde{a}_r^\alpha = -i\tilde{a}_r^{n+\alpha}, & \alpha > s. \end{array} \tag{2.32}$$

Clearly, the reality-condition forces us to take $f^A = \bar{f}^A = 0$, for $A > s$. The number of components of the W-string-surface is only equal to $2s$. This is why the Minkowski metric was essential for the present example. For $A, \bar{A} = 1, \ldots, s$, the embedding functions are

$$\begin{aligned} f^A(z) &= \{(q^A + q^{A+n})/2 + p^A \ln(z) + i \sum_{r\neq 0} a_r^A z^{-r}/r\}\sqrt{2}, \\ \bar{f}^{\bar{A}}(\bar{z}) &= \{(q^A - q^{A+n})/2 + \tilde{p}^A \ln(\bar{z}) + i \sum_{r\neq 0} \tilde{a}_r^{\bar{A}} \bar{z}^{-r}/r\}\sqrt{2}. \end{aligned} \tag{2.33}$$

For Euclidean world-sheet-parametrizations where z is a complex number, they satisfy the conditions $(f^A(z))^* = f^A(\bar{z})$, and $(\bar{f}^A(\bar{z}))^* = \bar{f}^A(z)$, in contrast with the conditions of real analyticity which are usually assumed (in particular for algebraic curves). Physically, this reflects the fact that, for closed strings, left and right modes are not correlated. On the contrary, if we consider an open string with parameters running, say, in the upper half-plane, the boundary condition is that $\{z\partial - \bar{z}\bar{\partial}\}Y^\alpha = 0$, for $z = \bar{z}$. As a result one has $(f^A(z))^* = \bar{f}^A(\bar{z})$, and one recovers the standard mathematical situation of real analytic functions. A similar situation occurs for Liouville theory with boundaries [25, 26].

2.2. $\mathscr{C}P^n$ W-*Surface and* A_n *Toda Lax Pair*

Definition 4. ***$\mathscr{C}P^n$ target space.*** The complex projective space $\mathscr{C}P^n$ is defined[6] to be the quotient of the space $\mathscr{C}^{n+1}$ of Definition 1, by the equivalence relation

$$X \sim Y\,, \quad \text{if} \quad X^A = Y^A \varrho(Y)\,, \quad \text{and} \quad \bar{X}^{\bar{A}} = \bar{Y}^{\bar{A}} \bar{\varrho}(\bar{Y})\,, \tag{2.34}$$

where ϱ and $\bar{\varrho}$ are arbitrary chiral functions.

It will be convenient to denote the $n+1$ homogeneous coordinates by X^A, $\bar{X}^{\bar{A}}$ with $A, \bar{A} = 0, 1, 2, \ldots, n$. In our definition of $\mathscr{C}P^n$, ϱ and $\bar{\varrho}$ are independent, since we do not impose any general reality condition on X^A, and $\bar{X}^{\bar{A}}$. The metric which is invariant under the rescaling Eq. (2.34) is the Fubini-Study metric [22],

$$G_{A\bar{A}} = \left(\delta_{A\bar{A}}\left(\sum_{B=0}^{n} X^B \bar{X}^B\right) - X^{\bar{A}} \bar{X}^A\right) \Bigg/ \left(\sum_{B=0}^{n} X^B \bar{X}^B\right)^2, \tag{2.35}$$

whose Kähler potential is given by $\mathscr{K} = \ln \sum_{A=0}^{n} X^A \bar{X}^{\bar{A}}$. We note that the variation Eq. (2.34) shifts this potential by $\ln \varrho + \ln \bar{\varrho}$. The metric is invariant if ϱ (resp. $\bar{\varrho}$) is only function of X^A (resp. $\bar{X}^{\bar{A}}$), as required by the above definition. This will be called a local rescaling. At this point, there are two ways to parametrize $\mathscr{C}P^n$. On the one hand, it is customary to use the Fubini-Study metric and to impose the condition $X^0 = \bar{X}^0 = 1.$[7]. This procedure is developed in Appendix A.1. It gives an example of the treatment of the Gauss-Codazzi equations with non-trivial target metric. On the other hand, and for the present purpose, it is more convenient to proceed as follows. First, instead of using the curved Fubini-Study metric Eq. (2.35), we use the *flat* metric of $\mathscr{C}^{n+1}$,

$$G_{A\bar{A}} = \delta_{A\bar{A}}\,, \quad (A, \bar{A} = 0, 1, \ldots, n)\,. \tag{2.36}$$

We shall work with the $2(n+1)$ homogeneous coordinates, without fixing the local scale (its choice is to be made only at the end). This is done by keeping the $2(n+1)$ embedding-functions, and making our discussion covariant under the $gl(1)$ local-rescaling symmetry,

$$f^A(z) \to \varrho(z) f^A(z)\,, \quad \bar{f}^{\bar{A}}(\bar{z}) \to \bar{\varrho}(\bar{z}) \bar{f}^{\bar{A}}(\bar{z})\,. \tag{2.37}$$

For this, one constructs the moving frame starting from the *zero*$^{\text{th}}$ order derivative of the embedding functions f and $\bar{f}$. The appropriate choice of the local scale will turn out to depend upon the W-surface considered.

---

[6] Here again, as for $\mathscr{C}^{n+1}$, we use different letters, that is $\mathscr{C}P^n$ instead of CP^n, to emphasize that our definition is somewhat non-standard

[7] Clearly, this approach can be only used locally, i.e. as long as the embedding function f^0 and $\bar{f}^0$ are nonvanishing. Since we use higher order derivative, our approach is not covariant under the target space reparametrization except for the special case (W-parametrization) we shall develop later

Except for theses modifications, the construction of the moving frame is completely parallel to the one of the previous chapter.

Definition 5. ***Toda fields.*** Introduce the matrix of inner products

$$\eta_{r\bar{s}} = \sum_{A=0}^{n} f^{(r)A}(z)\bar{f}^{(\bar{s})A}(\bar{z}), \quad 0 \le r, \bar{s} \le n. \tag{2.38}$$

The Toda fields Φ_ℓ, ($\ell = 1, \ldots, n+1$) are given by

$$\Phi_\ell(z,\bar{z}) = -\ln \tau_\ell(z,\bar{z}), \quad \tau_\ell(z,\bar{z}) \equiv \begin{vmatrix} \eta_{0\bar{0}} & \cdots & \eta_{\ell-1\bar{0}} \\ \vdots & & \vdots \\ \eta_{0\overline{\ell-1}} & \cdots & \eta_{\ell-1\overline{\ell-1}} \end{vmatrix}. \tag{2.39}$$

Define also $\tau_0 \equiv 1$ and $\Phi_0 \equiv 0$.

Definition 6. ***$\mathscr{C}P^n$ Moving Frame.*** The following vectors are orthonormal:

$$\tilde{e}_\ell = \frac{1}{(\tau_\ell \tau_{\ell+1})^{1/2}}\tilde{v}_\ell, \qquad \tilde{\bar{e}}_\ell = \frac{1}{(\tau_\ell \tau_{\ell+1})^{1/2}}\tilde{\bar{v}}_\ell, \tag{2.40}$$

$$\tilde{v}_\ell = \begin{vmatrix} \eta_{0\bar{0}} & \cdots & \eta_{\ell\bar{0}} \\ \vdots & & \vdots \\ \eta_{0\overline{\ell-1}} & \cdots & \eta_{\ell\overline{\ell-1}} \\ f & \cdots & f^{(\ell)} \end{vmatrix} \qquad \tilde{\bar{v}}_\ell = \begin{vmatrix} \eta_{\bar{0}0} & \cdots & \eta_{\bar{\ell}0} \\ \vdots & & \vdots \\ \eta_{\bar{0}\ell-1} & \cdots & \eta_{\bar{\ell}\ell-1} \\ \bar{f} & \cdots & \bar{f}^{(\ell)} \end{vmatrix}. \tag{2.41}$$

In the last equation, which is similar to Eqs. (2.4) and (2.5), the determinants are to be computed for each components of the last lines, and only non-vanishing components are written. The vectors $\tilde{v}$ are introduced for later convenience. They satisfy

$$(\tilde{v}_\ell, \tilde{\bar{v}}_{\ell'}) = \tau_\ell \tau_{\ell+1}\delta_{\ell\ell'}. \tag{2.42}$$

Proposition 2. ***Frenet-Serret Formulae for $\mathscr{C}P^n$.*** *The above vectors satisfy (ℓ runs from* 0 *to* n, *with* $\tilde{e}_{-1} = \tilde{e}_{n+1} \equiv 0$*)*

$$\begin{aligned} \partial\tilde{e}_\ell &= \frac{1}{2}\,\partial(\Phi_\ell - \Phi_{\ell+1})\tilde{e}_\ell + e^{(1/2)(2\Phi_{\ell+1}-\Phi_\ell-\Phi_{\ell+2})}\tilde{e}_{\ell+1}, \\ \bar{\partial}\tilde{e}_\ell &= -\frac{1}{2}\,\bar{\partial}(\Phi_\ell - \Phi_{\ell+1})\tilde{e}_\ell - e^{(1/2)(2\Phi_\ell-\Phi_{\ell-1}-\Phi_{\ell+1})}\tilde{e}_{\ell-1}. \end{aligned} \tag{2.43}$$

Proof. Calculations similar to the ones of the previous section. Q.E.D.

The next important point is the

Theorem 3. ***Covariance under Local Rescaling.*** *Under the transformation Eq. (2.37), the moving frame transforms covariantly:*

$$\begin{aligned} &\tau_\ell \to \varrho^\ell \bar{\varrho}^\ell \tau_\ell, && \Phi_\ell \to \Phi_\ell - \ell\ln\varrho - \ell\ln\bar{\varrho}, \\ &\tilde{v}_\ell \to \varrho^{\ell+1}\bar{\varrho}^\ell \tilde{v}_\ell, && \tilde{\bar{v}}_\ell \to \bar{\varrho}^{\ell+1}\varrho^\ell \tilde{\bar{v}}_\ell, \\ &\tilde{e}_\ell \to (\varrho/\bar{\varrho})^{1/2}\tilde{e}_\ell, && \tilde{\bar{e}}_\ell \to (\bar{\varrho}/\varrho)^{1/2}\tilde{\bar{e}}_\ell. \end{aligned} \tag{2.44}$$

Proof. Trivial computations show that

$$
\begin{aligned}
f^{(a)} &\to \sum_{b=0}^{a} \left\{ \binom{a}{b} \partial^{(a-b)} \varrho \right\} f^{(b)} \equiv \sum_{b=0}^{a} \Lambda_{ab} f^{(b)} , \quad \Lambda_{aa} = \varrho , \\
\bar{f}^{(a)} &\to \sum_{b=0}^{a} \left\{ \binom{a}{b} \bar{\partial}^{(a-b)} \bar{\varrho} \right\} \bar{f}^{(b)} \equiv \sum_{b=0}^{a} \bar{\Lambda}_{ba} f^{(b)} , \quad \bar{\Lambda}_{aa} = \bar{\varrho} .
\end{aligned}
\tag{2.45}
$$

The transformation of the matrix $\eta_{r\bar{s}}$ Eq. (2.38) is $\eta_{r\bar{s}} \to \sum_{j\bar{k}} \Lambda_{rj} \eta_{j\bar{k}} \bar{\Lambda}_{\bar{k}\bar{s}}$. It follows that the tau-, v-, and $\bar{v}$-functions are multiplied by sub-determinants of Λ and $\bar{\Lambda}$. Since Λ (resp. $\bar{\Lambda}$) is lower (resp. upper) triangular, these sub-determinants are equal to the products of their diagonal elements, and the result follows. Q.E.D.

Thus the above moving frame may be called homogeneous.

Corollary 1. *The Frenet-Serret formula Eq.* (2.43) *are invariant under local rescaling.*

Theorem 4. ***Toda equations.*** *There exists a choice of local rescaling such that the Gauss-Codazzi equations coincide with the A_n Toda equations.*

Proof. The compatibility conditions of the Frenet-Serret equations give

$$
\partial\bar{\partial}\Phi_\ell = -\exp(2\Phi_\ell - \Phi_{\ell-1} - \Phi_{\ell+1}) \quad (\ell = 1, \ldots, n) , \tag{2.46}
$$

$$
\partial\bar{\partial}\Phi_{n+1} = 0 . \tag{2.47}
$$

The first equation is precisely the A_n Toda equation. The second equation implies that τ_{n+1} is the product of two chiral functions. For later use we introduce two functions $U_0(z)$, and $\bar{U}_0(\bar{z})$, which are such that

$$
\tau_{n+1} = U_0(z)\bar{U}_0(\bar{z}) . \tag{2.48}
$$

Given the $2(n+1)$ embedding functions, we apply the local rescaling Eq. (2.37) with

$$
\varrho = (U_0(z))^{-1/(n+1)}, \quad \bar{\varrho} = (\bar{U}_0(\bar{z}))^{-1/(n+1)}, \tag{2.49}
$$

which precisely puts $\tau_{n+1} = 1$, killing the unwanted degree of freedom. Q.E.D.

Finally we get the A_n Toda equation *exactly*. By looking at the explicit solution, similar to Eq. (2.29), one sees that $U_0(z)$ (resp. $\bar{U}_0(\bar{z})$) is the Wronskian of the embedding functions f (resp. $\bar{f}$). These do not vanish at regular points, so that this choice of parametrization is really possible. We shall call it the Wronskian scale-choice. In the forthcoming it is however, more convenient to work in a scale-invariant way, without making this particular choice.

If one compares with the previous section, one sees that, in the present scheme, the distinction between intrinsic and extrinsic geometries is not so clear anymore, however we shall soon show that this separation is not invariant by W transformations.

2.3. Connection with the WZNW *Model*

2.3.1. Preamble. It has been shown [17] that there is a deep connection between Toda equations and the so-called conformally reduced WZNW equations. In this section we show how the latter, which contain more degrees of freedom than the former, are directly related to the present W-geometry. For completeness, we first recall the

Definition 7. ***Conformally reduced*** A_n**-WZNW** ***Model.*** Let z and $\bar{z}$ be Minkowski surface-parameters, let $\theta(z,\bar{z})$ be a $(n+1)\times(n+1)$ real matrix of determinant one, and

$$\mathscr{J}\equiv\theta^{-1}\partial\theta\,,\quad \bar{\mathscr{J}}\equiv(\bar{\partial}\theta)\theta^{-1}. \tag{2.50}$$

The conformally reduced WZNW equations are

$$\bar{\partial}\mathscr{J}=\partial\bar{\mathscr{J}}=0\,, \tag{2.51}$$

$$\mathrm{tr}(\mathscr{J}E_{-\alpha})=\mu_\alpha\,,\quad \mathrm{tr}(\bar{\mathscr{J}}E_{\alpha})=\bar{\mu}_\alpha\,, \tag{2.52}$$

α runs over a set of positive roots. The parameters $\mu_\alpha=\bar{\mu}_\alpha=-1$ if α is simple[8], and vanish otherwise.

Of course there is a similar definition for complex z obtained by Wick's rotation. We shall actually need the following generalization:

Definition 8. ***Conformally reduced*** $gl(n+1)$**-WZNW** ***Model.*** Same as above, but the determinant of θ is arbitrary.

This generalization incorporates an additional $gl(1)$ gauge degree of freedom. As it is well known the general solution of Eqs. (2.51) is $\theta=\theta_L(\bar{z})\theta_R(z)$, where θ_L and θ_R are arbitrary chiral matrices. Then conditions Eqs. (2.52) lead to solutions of the

Definition 8. ***Drinfeld-Sokolov Equations.*** They are of the form [20]

$$\begin{gathered}\partial\varUpsilon_r(z)-\sum_{s=0}^{n}\mathscr{D}_{rs}(z)\varUpsilon_s(z)=0\,,\\ \mathscr{D}_{rs}=0\,,\quad\text{for}\quad s-r>1\,,\quad \mathscr{D}_{rr+1}=1\,.\end{gathered} \tag{2.53}$$

If $\mathrm{tr}(\mathscr{D})=0$ this (DS) equation is associated with A_n. Otherwise it is associated with $gl(n+1)$.

It is easy to see that, writing $\varUpsilon_r^{(\ell)}=\theta_R^{-1}(z)_{r\ell}$, and $\bar{\varUpsilon}_r^{(\ell)}=\theta_L^{-1}(\bar{z})_{\ell r}$, give $2n$ solutions of the Drinfeld-Sokolov equations just defined.

2.3.2. WZNW Dynamics for $\mathscr{C}P^n$*-W-surfaces.* Our starting point is Eq. (2.38):

$$\eta_{\imath\bar{\jmath}}=\sum_{A=0}^{n}f^{(i)A}(z)\bar{f}^{(\bar{\jmath})A}(\bar{z})\,, \tag{2.54}$$

It is quite clear from the start that the above matrix is of the form $\eta=\eta_R(z)\eta_L(\bar{z})$, and thus satisfies equations of the WZNW type. More precisely, we have the

Theorem 5. ***Conformally reduced*** **WZNW** ***Solutions form*** **W** ***Surfaces.*** *The matrix* $\theta\equiv\eta^{-1}$ *is a solution of the conformally reduced* $gl(n+1)$ WZNW *equations introduced by Definition 8.*

[8] There may be arbitrary constants for primitive roots, but the present restricted choice is sufficient for our purpose

Proof. The currents are given by

$$\partial \eta_{i\bar{\jmath}} = -\mathscr{J}_{ik}\eta_{k\bar{\jmath}}\,, \quad \bar{\partial} \eta_{i\bar{\jmath}} = -\eta_{i\bar{k}}\bar{\mathscr{J}}_{\bar{k}\bar{\jmath}}\,. \tag{2.55}$$

They obviously satisfy

$$\bar{\partial}\mathscr{J}_{ik} = \partial \bar{\mathscr{J}}_{\bar{\imath}\bar{k}} = 0\,. \tag{2.56}$$

Moreover, since by construction, $\partial f^{(i)A} = f^{(i+1)A}$ and $\bar{\partial}\bar{f}^{(i)A} = \bar{f}^{(i+1)A}$, one has, according to Eq. (2.54), $\mathscr{J}_{ij} = -\delta_{j,i+1}$, and $\bar{\mathscr{J}}_{ji} = -\delta_{j,i+1}$ for $i \le n-1$, and this completes the derivation. Q.E.D.

Concerning $\mathscr{J}_{nj}$, it is well known that the $n+1$ functions F^A are automatically solutions of the differential equation ($A = 0, \ldots, n$)

$$\begin{vmatrix} f^0 & \ldots & f^n & f^A \\ \vdots & & \vdots & \vdots \\ f^{(n+1)0} & \ldots & f^{(n+1)n+1} & f^{(n+1)A} \end{vmatrix} \equiv \left\{ \sum_{k=0}^{n+1} U_{n+1-k}\partial^{(k)} \right\} f^A = 0\,, \tag{2.57}$$

which allows us to write

$$\partial^{(n+1)} f^A = \sum_{k=0}^{n} \frac{U_{n-k}}{U_0} f^{(k)A} \equiv \sum_{k=0}^{n} \lambda_k f^{(k)A}, \tag{2.58}$$

since U_0, which is equal to the Wronskian of the $n+1$ functions f^A, does not vanish at regular generic points. Thus one sees that $\mathscr{J}_{nj} = \lambda_j$, and $\bar{\mathscr{J}}_{jn} = \bar{\lambda}_j$, where $\bar{\lambda}_j$ is related to the differential equation satisfied by the functions f. $\mathscr{J}$ and $\bar{\mathscr{J}}$ are finally given by

$$\bar{\mathscr{J}} = -\bar{I} - \bar{\lambda}\,, \qquad \mathscr{J} = -I - \lambda\,, \tag{2.59}$$

$$\bar{I} = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & 0 & 1 & 0 \end{bmatrix}, \qquad I = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \ldots & & 0 & 1 \\ 0 & \ldots & & 0 & 0 \end{bmatrix}, \tag{2.60}$$

$$\bar{\lambda} = \begin{bmatrix} 0 & 0 & 0 & \ldots & \bar{\lambda}_0 \\ 0 & 0 & 0 & \ldots & \bar{\lambda}_1 \\ \vdots & & \ddots & & \vdots \\ 0 & & \ldots & 0 & \bar{\lambda}_{n-1} \\ 0 & 0 & \ldots & 0 & \bar{\lambda}_n \end{bmatrix}, \qquad \lambda = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & \ldots & 0 & 0 \\ \lambda_0 & \lambda_1 & \ldots & \lambda_{n-1} & \lambda_n \end{bmatrix}. \tag{2.61}$$

The current $\mathscr{J}$ and $\bar{\mathscr{J}}$ may be expressed in terms of the generators introduced in Sect. 2.1 [see Eqs. (2.19), (2.20)] except that, here, i runs from zero to n, and we are dealing with $gl(n+1)$. In this section we are always in the sector $N \equiv \sum_{i=0}^{n} \mathfrak{b}_i^+\mathfrak{b}_i = 1$, that is, in the defining representation, where h_i and $E_{\pm j}$ are $(n+1)\times(n+1)$ matrices. We keep the same notation as in Sect. 2.1, for the generators since there may be no confusion.

Corollary 2. ***Embedding Functions as Solutions of DS Equations.*** *For given A and $\bar{A}$, the set of derivatives of the embedding functions $\{f^{(j)A}, j = 0, \ldots, n\}$, and*

$\{\bar{f}^{(\bar{\jmath})\bar{A}}, \bar{\jmath} = 0,\dots,n\}$, *are solutions of the* $gl(n+1)$ DS *equations introduced in Definition 9.*

Proof. It immediately follows from Eqs. (2.55), and (2.56) that we have

$$\partial f^{(\jmath)A} + \sum_k \mathscr{J}_{\jmath k} f^{(k)A} = 0\,, \quad \bar\partial \bar f^{(\bar\jmath)\bar A} + \sum_{\bar k} \bar{\mathscr{J}}_{\bar k \bar\jmath} f^{(\bar k)\bar A} = 0 \tag{2.62}$$

which coincides with the Drinfeld-Sokolov equations associated with $gl(n+1)$ gauge introduced by Definition 9. Q.E.D.

One should note, however, that the currents associated with the embedding functions are of a more restricted type, since there are many more vanishing matrix elements in Eqs. (2.60), and (2.61) than required by Definition 9. In this connection, Eq. (2.57), and its anti-chiral counterpart show that U_ℓ and $\bar U_\ell$ should be regarded as W-potentials. The fact that U_0 and $\bar U_0$ are not constant, of course, reflects the existence of the additional $gl(1)$ degree of freedom. Accordingly, the current satisfy the $gl(n+1)$ DS equations, and not the one related with A_n.

Next we give the geometrical interpretation of the degrees of freedom that appear in the conformally reduced WZNW dynamics, and not in the Toda equation. This results from the

Theorem 6. ***Gauss Decomposition from Moving Frame.*** *The moving-frame equations*

$$\tilde e_a = \sum_{b\le a} C_{ab}(z,\bar z)\sqrt{\frac{\tau_a}{\tau_{a+1}}}\, f^{(b)}(z)\,, \quad \text{with} \quad C_{aa}=1\,, \tag{2.63}$$

$$\tilde{\bar e}_a = \sum_{b\le a} A_{ba}(z,\bar z)\sqrt{\frac{\tau_a}{\tau_{a+1}}}\, \bar f^{(b)}(\bar z)\,, \quad \text{with} \quad A_{aa}=1 \tag{2.64}$$

are such that the matrix $\theta = \eta^{-1}$ has the Gauss decomposition

$$\theta_{rs} = \sum_{a=0}^{n}\sum_{b=0}^{n} A_{ra}B_{ab}C_{bs}\,. \tag{2.65}$$

Denote by $\mathscr{N}_+^\cdot$ (resp. $\mathscr{N}_-^\cdot$) *the sub-groups generated by the step operators associated with positive (resp. negative) roots, and by* $\mathscr{D}_0$ *the group generated by the Cartan generators* including h_n. *Then*

$$A \in \mathscr{N}_+^\cdot\,, \quad B \in \mathscr{D}_0\,, \quad C \in \mathscr{N}_-^\cdot\,. \tag{2.66}$$

Proof. Equations (2.63), and (2.64) are slight modifications of Eqs. (2.9) and (2.10). Denote by $\varphi_\pm$ the set of positive (resp. negative) roots of A_n. The matrices C_{ab} (resp. A_{ab}) vanish unless $a \ge b$ (resp. $a \le b$), and their diagonal matrix elements are equal to 1. Thus we may write

$$C = \exp\Bigl(\sum_{a\in\varphi_+} y^\alpha E_{-\alpha}\Bigr)\,, \quad A = \exp\Bigl(\sum_{a\in\varphi_+} x^\alpha E_{\alpha}\Bigr)\,. \tag{2.67}$$

Equations (2.63), (2.64) give

$$f^{(a)}(z) = \sum_b C^{-1}_{ab}(z,\bar z)\sqrt{\frac{\tau_{b+1}}{\tau_b}}\,\tilde e_b\,, \quad \bar f^{(a)}(\bar z) = \sum_b A^{-1}_{ba}(z,\bar z)\sqrt{\frac{\tau_{b+1}}{\tau_b}}\,\tilde{\bar e}_b\,. \tag{2.68}$$

Inserting this into Eq. 2.54, one indeed verifies the decomposition Eq. (2.65), if one lets

$$B_{ab} = \frac{\tau_a}{\tau_{a+1}}\,\delta_{a,b} = e^{\Phi_{a+1}-\Phi_a}\delta_{a,b}\,. \tag{2.69}$$

This completes the proof. Q.E.D.

In terms of the Cartan generators, B may be written as

$$B = \exp\left(\sum_{i=0}^{n}\Phi_{i+1}h_i\right). \tag{2.70}$$

In the Toda theory, the only remaining degrees of freedom are the Toda fields Φ_i. Thus we see that the geometrical interpretation of the matrices A and C is that they specify the embedding.

Corollary 3. ***Geometrical Interpretation of the* WZNW *Equations.*** *The* WZNW *equations for A, B, and C [17], that is*

$$(\bar\partial C)C^{-1} = -B^{-1}\bar I B\,, \quad A^{-1}\partial A = -BIB^{-1}, \tag{2.71}$$

are direct consequences of the chirality conditions of the W *surface (Cauchy-Riemann equations):* $\partial\bar f^A = \bar\partial f^A = 0$.

Proof. Consider the last equation as an example. Equation (2.68) gives

$$\sum_b\left((\bar\partial C^{-1}_{ab})e_b\sqrt{\frac{\tau_{b+1}}{\tau_b}}\right) + \sum_b\left(C^{-1}_{ab}\bar\partial\left(e_b\sqrt{\frac{\tau_{b+1}}{\tau_b}}\right)\right) = 0\,. \tag{2.72}$$

It follows from Eqs. (2.43) that

$$\bar\partial\left(e_b\sqrt{\frac{\tau_{b+1}}{\tau_b}}\right) = \prod_j e^{K^{gl(n+1)}_{b-1j}\Phi_j}\left(e_{b-1}\sqrt{\frac{\tau_b}{\tau_{b-1}}}\right), \tag{2.73}$$

and the previous equations gives $-(\bar\partial C)C^{-1} = \sum_{j=1}^{n} E_{-j}\exp\left(\sum_i K^{gl(n+1)}_{j-1i}\Phi_i\right)$ which is indeed equal to $B^{-1}\bar I B$, according to Eqs. (2.21). Treating similarly, the equation for $\bar f$, we get

$$(\bar\partial C)C^{-1} = -B^{-1}\bar I B\,, \quad A^{-1}\partial A = -BIB^{-1}, \tag{2.74}$$

which coincide [17] with the WZNW equations for A, B and C. Q.E.D.

Our next topic is concerned with the additional field Φ_{n+1}, and the associated $gl(1)$ gauge invariance. We shall prove the

Proposition 3. $gl(1)$ ***Invariance of the*** $gl(n+1)$ WZNW ***equations.*** *Given two arbitrary functions* $\varrho(z)$ *and* $\bar\varrho(\bar z)$, *the* $gl(n+1)$ WZNW *equations are invariant under the transformation*

$$A \to \bar\varrho\bar\Lambda^{-1}A\,, \quad C \to \varrho C\Lambda^{-1}, \quad B \to B/\varrho\bar\varrho\,, \tag{2.75}$$

where Λ *and* $\bar\Lambda$ *are given by Eq.* (2.45), *that is*

$$\Lambda_{ab} = \binom{a}{b}\partial^{(a-b)}\varrho\,, \quad \bar\Lambda_{ba} = \binom{a}{b}\bar\partial^{(a-b)}\bar\varrho\,. \tag{2.76}$$

Proof. Clearly $(\bar{\partial}C)C^{-1}$ and $A^{-1}\partial A$ are invariant. This is also trivially true for the right-hand sides. Q.E.D. As a consequence we have the

Proposition 4. WZNW ***Gauge Equivalence.*** *The solutions of the* $gl(n+1)$-WZNW *equations are gauge-equivalent to the ones of the* A_n-WZNW *equations.*

Proof. Since θ is a solution of the WZNW equations its determinant is a product of chiral functions. Thus it may be set equal to one by a $gl(1)$ transformation of the type introduced by Proposition 3. Q.E.D.

Concerning the DS equations, the following is useful

Lemma 1. ***Basic Properties of the Transformations*** Λ.
1) *For an infinitesimal transformation* $\varrho = 1+\varepsilon$, $\Lambda = 1 + s$. *One has*

$$< b|(\partial s + [s, I])|a >= \begin{cases} 0 & , \text{ if } \ b < n \\ \begin{pmatrix} n+1 \\ a \end{pmatrix} \partial^{n+1-a}\varepsilon\,, & \text{if } \ b = n\,. \end{cases} \tag{2.77}$$

2) *The form (Eqs.* (2.59), (2.60), (2.61)*) of the current* $\mathscr{J}$ *is gauge invariant.*

Proof. 1) The derivation uses easy calculations bases on the standard recursion relation for binomial coefficients: $\begin{pmatrix} b+1 \\ a \end{pmatrix} - \begin{pmatrix} b \\ a-1 \end{pmatrix} = \begin{pmatrix} b \\ a \end{pmatrix}$.
2) It is sufficient to work with infinitesimal transformations. Then

$$\mathscr{J} \to \mathscr{J} + \delta\mathscr{J} \equiv \mathscr{J} + \partial s + [s, I] + [s, \lambda]\,. \tag{2.78}$$

In the terms of the fermionic modes $\mathfrak{b}_j$, we may write $\lambda = \sum_b \mathfrak{b}_n^+ \mathfrak{b}_b \lambda_b$, and

$$[s, \lambda] = -\sum_{a=0}^{n-1} \mathfrak{b}_n^+ \mathfrak{b}_a \sum_{b\geq a} \begin{pmatrix} b \\ a \end{pmatrix} \lambda_b \partial^{b-a}\varepsilon\,. \tag{2.79}$$

$$\delta\mathscr{J} = \sum_{a=0}^{n-1} \mathfrak{b}_n^+ \mathfrak{b}_a \left[\begin{pmatrix} n+1 \\ a \end{pmatrix} \partial^{n+1-a}\varepsilon - \sum_{b\geq a} \begin{pmatrix} b \\ a \end{pmatrix} \lambda_b \partial^{b-a}\varepsilon \right]. \tag{2.80}$$

Thus the form of $\mathscr{J}$ is indeed preserved with

$$\delta\lambda_a = \begin{pmatrix} n+1 \\ a \end{pmatrix} \partial^{n+1-a}\varepsilon - \sum_{b\geq a} \begin{pmatrix} b \\ a \end{pmatrix} \lambda_b \partial^{b-a}\varepsilon\,. \tag{2.81}$$

This completes the proof. Q.E.D.

This lemma leads to the

Proposition 5. DS ***Gauge Equivalence.*** *The solutions of the* $gl(n+1)$-DS *equations are gauge equivalent to those of the* A_n-DS *equations.*

Proof. This must be true since the corresponding WZNW are gauge equivalent. Indeed it has been shown in [20] that a general DS current $\mathscr{D}$ of Definition 9 is gauge equivalent to a current $\mathscr{J}$ of the form Eq. (2.59), (2.60), (2.61). According to the last lemma, one may thus perform a $gl(1)$ transformation such that $\lambda_n \to 0$, and h_n decouples. Q.E.D.

The $gl(1)$ invariance is directly connected with the rescaling of $\mathscr{C}P^n$ W surfaces as shown by the

Proposition 6. ***Rescaling.*** *The $gl(1)$ gauge transformations Eqs.* (2.75) *corresponds to the rescaling Eq.* (2.37) *of the embedding functions*

$$f^A(z) \to \varrho(z) f^A(z)\,, \quad \bar f^{\bar A}(\bar z) \to \bar\varrho(\bar z) \bar f^{\bar A}(\bar z)\,. \tag{2.82}$$

Proof. Combining Eqs. (2.44) with (2.68), one verifies Eqs. (2.75) for A and C. Substituting Eqs. (2.44) into Eq. (2.70), gives $B \to B\exp(-\ln(\varrho\bar\varrho)\sum_{i=0}^{n}(i+1)h_i)$. Making use of the explicit expressions Eqs. (2.19), (2.20), one verifies that $\sum_{i=0}^{n}(i+1)h_i$ is equal to the identity operator, and this completes the proof. Q.E.D. Of course the additional generator h_n is instrumental in the proof. Finally we arrive at the

Theorem 7. ***Equivalence between*** **WZNW** ***Solutions and*** $\mathscr{C}P^n$ **W-*Surfaces.*** *There exists a one-to-one correspondence between the solutions of the conformally reduced A_n* WZNW *and the* W*-surfaces in $\mathscr{C}P^n$.*

Proof. 1) First it is clear from the previous discussions that there exists a unique solution of the conformally reduced WZNW equations associated with a given W-surface in $\mathscr{C}P^n$. Indeed a W surface in $\mathscr{C}P^n$ is described by the homogeneous formalism displayed in Sect. 2.2, and there is a one-to-one correspondence between the local rescaling of the homogeneous description of $\mathscr{C}P^n$ and the $gl(1)$-gauge of the $gl(n+1)$ WZNW and DS equations.
2) The proof of the converse goes as follows. Let θ be a solution of the A_n-WZNW-model: $\partial\theta = \theta K$, $\bar\partial\theta = \bar K\theta$, such that, for $\alpha \in \varphi_+$, $-\operatorname{tr}(KE_{-\alpha})$, and $-\operatorname{tr}(\bar K E_\alpha)$ are equal to 1, if α is simple, and 0 otherwise. For the solution

$$\theta = \theta_L(\bar z)\theta_R(z)\,, \quad K = \theta_R^{-1}\partial\theta_R\,, \quad \bar K = (\bar\partial\theta_L)\theta_L^{-1}\,, \tag{2.83}$$

the conditions on K and $\bar K$ are left invariant by the gauge transformations $\theta_L \to \alpha_L\theta_L$, and $\theta_R \to \theta_R\alpha_R$ such that α_L (resp. α_R) belong to subgroups $\mathscr{N}_+$ (resp. $\mathscr{N}_-$). One may verify [20] that there exist gauge transformations such that the gauge transformed current $\mathscr{J} = \alpha_R^{-1}\partial\alpha_R + \alpha_R^{-1}K\alpha_R$, and $\bar{\mathscr{J}} = (\bar\partial\alpha_L)\alpha_L^{-1} + \alpha_L\bar K\alpha_L^{-1}$, take the form Eqs. (2.59), (2.60), (2.61). Of course, since K and $\bar K$ belong to the Lie algebra A_n, λ_n and $\bar\lambda_n$ are found to vanish. Letting $\eta = \theta^{-1}$, one sees that we have $\eta = \eta_R(z)\eta_L(\bar z)$ with

$$\begin{aligned} \partial\eta_{R\,kl} &= \eta_{R\,k+1\,l}\,, \quad \text{for } k \le n-1\,, \quad \partial\eta_{R\,nl} = \sum_{b=0}^{n-1}\lambda_b\eta_{R\,bl}\,, \\ \bar\partial\eta_{R\,kl} &= \eta_{R\,kl+1}\,, \quad \text{for } l \le n-1\,, \quad \bar\partial\eta_{R\,kn} = \sum_{b=0}^{n-1}\bar\lambda_b\eta_{R\,kb}\,, \end{aligned} \tag{2.84}$$

as consequences of the Drinfeld-Sokolov equations. Thus we let

$$\eta_{R\,0A} = \tilde f^A\,, \quad \eta_{L\,A0} = \tilde{\bar f}^A. \tag{2.85}$$

Equations (2.84) are satisfied iff

$$\partial^{n+1}\tilde f = \sum_{b=0}^{n-1}\lambda_b\partial^{(b)}\tilde f\,, \quad \bar\partial^{n+1}\tilde{\bar f} = \sum_{b=0}^{n-1}\bar\lambda_b\partial^{(b)}\tilde{\bar f}\,, \tag{2.86}$$

which means the Wronskians of $\tilde{f}$, and of $\tilde{\bar{f}}$ are constant. These conditions are easily removed by performing the rescalings $\tilde{f} \to f = \varrho\tilde{f}$, $\tilde{\bar{f}} \to \bar{\varrho}\bar{f}$, which are such that the new Wronskians are equal to ϱ^{n+1}, and $\bar{\varrho}^{n+1}$, respectively. The f and $\bar{f}$ are coordinates of a W-surface with local rescaling invariance. Writing the Gauss decomposition $\eta = C^{-1}B^{-1}A^{-1}$ gives back the Frenet-Serret formula, and this establishes the complete correspondence between $\mathscr{C}P^n$-W-surfaces and the conformally reduced A_n-WZNW-dynamics. Q.E.D.

3. KP Coordinates and W-Geometry

3.1. Free-fermion description of chiral embedding

In the construction of the moving frame, we have seen that the determinants of the embedding functions play a central rôle. This fact leads us to suspect that some kind of fermionic structure underlies the geometry of W-surfaces. In the description of the Toda theory, it is known [18] that the free-fermions neatly describe their solution-space as they do for the KP hierarchy. In our situation, the embedding is connected to the A_n Toda theory, and the corresponding fermion theory becomes non-relativistic in contrast with the KP case. The present free fermions are identical to those which appear in the matrix-model. Although the present main section is devoted to the case of regular points, this subsections deals with the more general situation which we will encounter in the next main section.

Let us summarize our free-fermion conventions following[9] [18].

$$[\psi_n, \psi_m]_+ = [\psi_n^+ \psi_m^+]_+ = 0\,,$$
$$[\psi_n, \psi_m^+]_+ = \delta_{n,m}\,, \quad (n, m = 0, 1, \ldots) \tag{3.1}$$
$$\psi_n|\emptyset\rangle = 0 \quad \langle\emptyset|\psi_n^+ = 0 \quad \forall n\,. \tag{3.2}$$

We use the semi-infinite indices $n = 0, 1, 2, \ldots, \infty$ for the fermion-operators. The vacuum states $|\emptyset\rangle$ and $\langle\emptyset|$ correspond to the no-particle states. The n-particle ground state is created from them in the standard way:

$$|n\rangle = \psi_{n-1}^+ \psi_{n-2}^+ \cdots \psi_0^+ |\emptyset\rangle\,, \quad \langle n| = \langle\emptyset|\psi_0\psi_1 \cdots \psi_{n-1}\,. \tag{3.3}$$

The current operators,

$$J_n = \sum_{s=0}^{\infty} \psi_{n+s}^+ \psi_s\,, \quad \bar{J}_n = \sum_{s=0}^{\infty} \psi_s^+ \psi_{n+s}\,, \tag{3.4}$$

will be taken as Hamiltonians as one does for the KP hierarchy [24]. (W-parameters) already mentioned in Sect. 2.1. The rôle of these fermions may be understood as follows. Take the case where z is a complex variable. Then the embedding functions f^A are analytic, and each of them is entirely determined by its Taylor expansion around a single point of its analyticity domain. Its behaviour at any other point of its Riemann surface is fixed by analytic continuation. The following free-fermion formalism realizes this continuation automatically. Consider the Taylor expansions at the point z:

$$f^A(z + x) = \sum_{s=0}^{\infty} f^{(s)A}(z)\frac{x^s}{s!}\,, \quad \bar{f}^{\bar{A}}(\bar{z} + \bar{x}) = \sum_{s=0}^{\infty} \bar{f}^{(s)A}(\bar{z})\frac{\bar{x}^s}{s!}\,. \tag{3.5}$$

[9] We actually make a slight modifications by interchanging ψ and ψ^+ to follow the usual convention

To these developments, we associate the free-fermion operators,

$$\psi_{fA(z)} = \sum_{s=0}^{\infty} f^{(s)A}(z)\psi_s\,, \quad \psi^+_{\bar f\bar A(\bar z)} = \sum_{s=0}^{\infty} \bar f^{(s)\bar A}(\bar z)\psi^+_s\,. \tag{3.6}$$

The basic property of these operators are

Proposition 7. ***Fermionic Representation of Chiral Functions.*** 1) *Any change of the Taylor-expansion point z, $\bar z$ can be absorbed by the action of the Hamiltonians J_1, and $\bar J_1$. In particular, one has*

$$\psi_{fA(z)} = e^{-J_1 z}\psi_{fA(0)}e^{J_1 z}, \quad \psi^+_{\bar f\bar A(\bar z)} = e^{\bar J_1\bar z}\psi^+_{\bar f\bar A(0)}e^{-\bar J_1\bar z}. \tag{3.7}$$

2) *The embedding functions are represented by the fermion expectation-values*

$$f^A(z) = \langle\emptyset|\psi_{fA(z_0)}e^{J_1(z-z_0)}|1\rangle\,, \quad \bar f^{\bar A}(\bar z) = \langle 1|e^{\bar J_1(\bar z-\bar z_0)}\psi^+_{\bar f\bar A(\bar z_0)}|\emptyset\rangle\,. \tag{3.8}$$

Proof. 1) is a consequence of the identities, $e^{-J_1 z}\psi_s e^{J_1 z} = \sum_{t=0}^{s} z^t\psi_{s-t}/t!$ and of their anti-chiral counterparts.
2) comes from the relations $\langle\emptyset|\psi_s J_1^t|\emptyset\rangle = \delta_{s,t}$. Q.E.D.

Due to 1), it is equivalent to work with $\psi_{fA(z_0)}$, and $\psi^+_{\bar f\bar A(\bar z_0)}$ at any fixed z_0 and $\bar z_0$. Hence we put $z_0 = \bar z_0 = 0$ in the following and write $\psi_{fA(0)}$ and $\psi^+_{\bar f\bar A(0)}$ as ψ_{fA} and $\psi^+_{\bar f\bar A}$ for simplicity. 2) implies that we can translate the chiral embedding into $\mathscr{C}P^n$ in the free fermion language. The basic object of this approach is the

Definition 10. ***Embedding Operator.*** It is an operator in the fermionic Fock space defined by

$$\begin{aligned}\mathscr{G}(z,\bar z) &= \sum_{a=0}^{n+1}\mathscr{G}(z,\bar z)_a\\ &\equiv \sum_{a=0}^{n+1}\ \sum_{0\le A_1<\cdots<A_a\le n} \psi^+_{\bar f\bar A_1(\bar z)}\cdots\psi^+_{\bar f\bar A_a(\bar z)}|\emptyset\rangle\langle\emptyset|\psi_{fA_a(z)}\cdots\psi_{fA_1(z)}\,.\end{aligned} \tag{3.9}$$

It clearly follows from Eq. (3.7) that

$$\mathscr{G}(z,\bar z) = e^{\bar J_1\bar z}\mathscr{G}(0,0)e^{J_1 z}. \tag{3.10}$$

$\mathscr{G}$ is a sort of density matrix of the embedding functions. It has a natural restriction to the Fock space generated by the operators ψ_{fA} and $\psi^+_{\bar f\bar A}$ acting on the no-particle state, where it becomes a finite matrix. Thus a may be regarded as specifying a representation of $gl(n)$ (note that $\mathscr{G}_a = 0$ if $a > n+1$). This is the analogue of the $gl(\infty)$ matrix which appears in the Toda hierarchy. The main difference between these two is that the rank of $\mathscr{G}$ in Eq. (3.9) is finite ($= n+1$), i.e. it is degenerate. If we take the limit $n\to\infty$, it coincides with the matrix of the Toda hierarchy.

The readers might be curious about the relationship between these fermions and the operators $\flat_j$ introduced in Sect. 2.1. They obviously act on different indices of $f^{(s)A}$. $\flat$-fermions act on A and ψ-fermions act on s. There is a complicated relation between the two, which is connected with the uniformization of the Drinfeld-Sokolov equation. We shall give explicit forms of these transformations in the proof of the Hirota equation, for example, see Eqs. (3.19), (3.20). This connection is simple in the limit when $n \to \infty$ since, then, we can choose the basis, $f^{(s)A} \propto \delta_{A,s}$.

In order to get the relationship between $\mathscr{G}$ and the embedding considered in Sect. 2, we need the following

Theorem 8. ***Tau-functions.*** *One has*

$$\tau_a = \langle a|\mathscr{G}(z,\bar{z})|a\rangle \,, \tag{3.11}$$

that is, the tau-functions associated with $\mathscr{G}$ *coincide with the functions* τ_a *defined by Eq.* (2.39).

Proof. It is easy to verify that, from Eq. (3.8),

$$\eta_{j\bar{k}} = \langle\emptyset|\psi_{\bar{k}}\mathscr{G}(z,\bar{z})\psi_j^+|\emptyset\rangle \,. \tag{3.12}$$

The theorem follows by computing the determinants of Eq. (2.39), by means of Wick's theorem. Q.E.D.

These are non-chiral versions of the tau-functions of the KP hierarchy.

Next the basic tool of the fermionic approach is the

Theorem 9. ***Hirota Equation.*** *The embedding operator* $\mathscr{G}$ *satisfies*

$$\sum_{s=0}^{\infty}\psi_s^+\mathscr{G}(z,\bar{z})\bigotimes\psi_s\mathscr{G}(z,\bar{z}) = \sum_{s=0}^{\infty}\mathscr{G}(z,\bar{z})\psi_s^+\bigotimes\mathscr{G}(z,\bar{z})\psi_s \,. \tag{3.13}$$

Proof. The Hirota equations have been discussed in many places. Usually (see, e.g. [27]), however, they only give a proof of the Hirota equations of the KP *hierarchy*,

$$\sum_{s=0}^{\infty}\psi_s^+\mathscr{G}(z,\bar{z})|0\rangle\bigotimes\psi_s\mathscr{G}(z,\bar{z})|0\rangle = 0 \,,$$

which is a little different from ours. We believe that it would be pedagogically useful to give the proof for the Toda hierarchy here. [10]

First we remark that z and $\bar{z}$ can be set equal to zero in order to prove this theorem. Indeed, the explicit form of the embedding operator $\mathscr{G}$ Eqs. (3.9), and (3.10) implies that the z and $\bar{z}$ dependence can be eliminated by a suitable re-definition of the Taylor-expansion point. Since our proof is valid for any ψ_f and ψ_f^+, and is carried out at a level of formal series, it automatically includes this modification, even if we deal with $\mathscr{G}(0,0)$ as we shall do. The derivative is carried out step by step. The simplest situation is studied first, before being gradually generalized.

Situation 1. *Assume that the embedding functions are given by* ($A, \bar{A} = 0, \ldots, n$),

$$f^A(z) = z^A/A! \,, \quad \bar{f}^{\bar{A}}(\bar{z}) = \bar{z}^{\bar{A}}/\bar{A}! \,. \tag{3.14}$$

[10] The rest of this subsection is solely devoted to the proof of this theorem. Those who do not bother about its proof may simple skip it

Proof. In this case $\psi_{fA} = \psi_A$ and $\psi^+_{\bar{f}\bar{A}} = \psi^+_{\bar{A}}$. By a direct computation, we can easily confirm that

$$\sum_{s=0}^{\infty} \psi_s^+ \mathscr{G}(0,0)_a \bigotimes \psi_s \mathscr{G}(0,0)_b = \sum_{s=0}^{\infty} \mathscr{G}(0,0)_{a+1} \psi_s^+ \bigotimes \mathscr{G}(0,0)_{b-1} \psi_s \quad (3.15)$$

$$= \sum_{0\le i \le n} \psi_i^+ \mathscr{G}(0,0)_a \bigotimes \mathscr{G}(0,0)_{b-1} \psi_i \, . \quad (3.16)$$

Q.E.D.

We recall that, in this simplest case, the first $n+1$ fermion-operators ψ_{fA} and $\psi^+_{\bar{f}\bar{A}}$ coincide with the $\flat$ fermions, as already mentioned.

Situation 2. *The case when the fermionic representation of the embedding functions have the following form,*

$$\psi_{fA} = \psi_A + \sum_{s=n+1}^{\infty} f^{(s)A} \psi_s \, , \quad \psi^+_{\bar{f}\bar{A}} = \psi^+_{\bar{A}} + \sum_{s=n+1}^{\infty} \bar{f}^{(s)\bar{A}} \psi_s^+ \, , \qquad (3.17)$$

This is the canonical form of the embedding functions at the regular points.

Proof. The idea is to make a Bogoliubov transformation of the free-fermion basis such that the problem reduces to Situation 1. In doing so, we need to keep the orthonormality properties of the free-fermion basis. We give the explicit form of such transformation. Introduce[11]

$$\begin{aligned}
\Psi_\ell^{(0)} &= \begin{cases} \psi_\ell + \sum\limits_{s=n+1}^{\infty} f^{(s)\ell} \psi_s = \psi_{f\ell} & : \ell = 0,1,\ldots,n \\ \psi_\ell & : \ell > n \end{cases} , \\
\Psi_\ell^{(0)*} &= \begin{cases} \psi_\ell^+ & : \ell = 0,1,\ldots,n \\ \psi_\ell^+ - \sum\limits_{s=0}^{n} f^{(\ell)s} \psi_s^+ & : \ell > n \end{cases} ,
\end{aligned} \qquad (3.18)$$

$$\begin{aligned}
\Psi_\ell^{(\infty)*} &= \begin{cases} \psi_\ell^+ + \sum\limits_{s=n+1}^{\infty} \bar{f}^{(s)\ell} \psi_s^+ = \psi^+_{\bar{f}\ell} & : \ell = 0,1,\ldots,n \\ \psi_\ell^+ & : \ell > n \end{cases} , \\
\Psi_\ell^{(\infty)} &= \begin{cases} \psi_\ell & : \ell = 0,1,\ldots,n \\ \psi_\ell - \sum\limits_{s=0}^{n} \bar{f}^{(\ell)s} \psi_s & : \ell > n \end{cases} .
\end{aligned} \qquad (3.19)$$

We remark that we need separate the Bogoliubov transformations for the chiral and anti-chiral embedding functions. They are distinguished by (0) and (∞). These new fermion-basis satisfy the standard anti-commutation relations,

$$\begin{aligned}
&[\Psi_\ell^{(0)}, \Psi_{\ell'}^{(0)*}]_+ = \delta_{\ell\ell'} \, , \quad [\Psi_\ell^{(0)}, \Psi_{\ell'}^{(0)}]_+ = 0 \, , \quad [\Psi_\ell^{(0)*}, \Psi_{\ell'}^{(0)*}]_+ = 0 \, , \\
&[\Psi_\ell^{(\infty)}, \Psi_{\ell'}^{(\infty)*}]_+ = \delta_{\ell\ell'} \, , \quad [\Psi_\ell^{(\infty)}, \Psi_{\ell'}^{(\infty)}]_+ = 0 \, , \quad [\Psi_\ell^{(\infty)*}, \Psi_{\ell'}^{(\infty)*}]_+ = 0 \, .
\end{aligned} \qquad (3.20)$$

[11] Here, * does not mean the complex conjugation

Furthermore, they keep the bilinear combinations

$$\sum_{i=0}^{\infty} \psi_i \bigotimes \psi_i^{+} = \sum_{i=0}^{\infty} \Psi_i^{(0)} \bigotimes \Psi_i^{(0)*} = \sum_{i=0}^{\infty} \Psi_i^{(\infty)} \bigotimes \Psi_i^{(\infty)*}. \tag{3.21}$$

Due to these identities, the LHS and RHS of the Hirota equation can be rewritten as

$$\begin{aligned} &\sum_{i=0}^{\infty} \psi_i^{+} \mathscr{G}(0,0)_a \bigotimes \psi_i \mathscr{G}(0,0)_b = \sum_{i=0}^{\infty} \Psi_i^{(\infty)*} \mathscr{G}(0,0)_a \bigotimes \Psi_i^{(\infty)} \mathscr{G}(0,0)_b\,, \\ &\sum_{i=0}^{\infty} \mathscr{G}(0,0)_{a+1} \psi_i^{+} \bigotimes \mathscr{G}(0,0)_{b-1} \psi_i = \sum_{i=0}^{\infty} \mathscr{G}(0,0)_{a+1} \Psi_i^{(0)*} \bigotimes \mathscr{G}(0,0)_{b-1} \Psi_i^{(\infty)}\,. \end{aligned} \tag{3.22}$$

In terms of the new basis, the $\mathscr{G}(0,0)$ matrix of Eq. (3.9) becomes

$$\mathscr{G}(0,0)_a = \sum_{0 \le A_1 < \cdots < A_a \le n} \Psi_{\bar{A}_a}^{(\infty)*} \cdots \Psi_{\bar{A}_1}^{(\infty)*} |\emptyset\rangle\langle\emptyset| \Psi_{A_1}^{(0)} \cdots \Psi_{A_a}^{(0)}\,. \tag{3.23}$$

It is now clear that the problem reduces to Situation 1. Q.E.D.

Situation 3. *Define the following sets of non-negative integers,*

$$\begin{aligned} \Xi_0 &= \{0, 1+\beta_1, 2+\beta_1+\beta_2, \ldots, .n+\beta_1+\ldots+\beta_n\}\,, \\ \Xi_\infty &= \{0, 1+\bar{\beta}_1, 2+\bar{\beta}_1+\bar{\beta}_2, \ldots, .n+\bar{\beta}_1+\ldots+\bar{\beta}_n\}\,, \end{aligned} \tag{3.24}$$

where β_i and $\bar{\beta}_i$ $(i = 1, 2, \ldots, n)$ are two sets of non-negative integers. Define also $\Xi_0^{(-)}$ and $\Xi_\infty^{(-)}$ as the set of non-negative integers that do not belong to Ξ_0 and Ξ_∞, respectively. We also introduce two mappings σ and $\bar{\sigma}$ from the set $\{0, 1, 2, \ldots, n\}$ to Ξ_0 and Ξ_∞, respectively:

$$\sigma(0) = \bar{\sigma}(0) = 0\,, \quad \sigma(\ell) = \ell + \sum_{j=1}^{\ell} \beta_j\,, \quad \bar{\sigma}(\ell) = \ell + \sum_{j=1}^{\ell} \bar{\beta}_j\,, \tag{3.25}$$

for $i = 1, \ldots, n$. With these notations, we define the embedding functions[12] *in this situation by,*

$$f^A(z) = \frac{z^{\sigma(A)}}{\sigma(A)!} + \sum_{s \in \Xi_0^{(-)}} \frac{f^{(s)A}}{s!} z^s, \quad \bar{f}^{\bar{A}}(\bar{z}) = \frac{\bar{z}^{\bar{\sigma}(\bar{A})}}{\bar{\sigma}(\bar{A})!} + \sum_{s \in \Xi_0^{(-)}} \frac{\bar{f}^{(s)\bar{A}}}{s!} \bar{z}^s. \tag{3.26}$$

[12] We remark that, in this case, the embedding becomes singular at $z = 0$. The geometrical property of such embeddings will be the main topics of Sect. 4. The integers β, $\bar{\beta}$ are called ramification indices

Proof. This problem reduces to Situation 2 by the mappings σ and $\bar\sigma$. More explicitly, we modify the definition of the Bogoliubov-transformed fermions by,

$$\Psi_\ell^{(0)} = \begin{cases} \psi_\ell + \sum\limits_{s\in\Xi_0^{(-)}} f^{(s)\sigma^{-1}\ell}\psi_s = \psi_{f\sigma^{-1}(\ell)} & \ell\in\Xi_0 \\ \psi_\ell & \ell\in\Xi_0^{(-)} \end{cases},$$
$$\Psi_\ell^{(0)*} = \begin{cases} \psi_\ell^+ & \ell\in\Xi_0 \\ \psi_\ell^+ - \sum\limits_{s=0}^{n} f^{(\ell)s}\psi_{\sigma(s)}^+ & \ell\in\Xi_0^{(-)} \end{cases}, \tag{3.27}$$

$$\Psi_\ell^{(\infty)*} = \begin{cases} \psi_\ell^+ + \sum\limits_{s\in\Xi_0^{(-)}} \bar f^{(s)\bar\sigma^{-1}(\ell)}\psi_s^+ = \psi^+_{\bar f\bar\sigma^1(\ell)} & \ell\in\Xi_\infty \\ \psi_\ell^+ & \ell\in\Xi_\infty^{(-)} \end{cases},$$
$$\Psi_\ell^{(\infty)} = \begin{cases} \psi_\ell & \ell\in\Xi_\infty \\ \psi_\ell - \sum\limits_{s\in\Xi_0^{(-)}} \bar f^{(\ell)s}\psi_{\bar\sigma(s)} & \ell\in\Xi_\infty^{(-)} \end{cases}. \tag{3.28}$$

These fermions are orthonormal, and the embedding operator is given by

$$\mathscr{G}(0,0)_a = \sum_{0\le A_1<\cdots<A_a\le n} \Psi^{(\infty)*}_{\bar\sigma(\bar A_a)}\cdots\Psi^{(\infty)*}_{\bar\sigma(\bar A_1)}|\emptyset\rangle\langle\emptyset|\Psi^{(0)}_{\sigma(A_1)}\cdots\Psi^{(0)}_{\sigma(A_a)}\,. \tag{3.29}$$

In terms of this basis, both sides of the Hirota equation produce the following term,

$$\sum_{0\le i\le n} \Psi^{(\infty)*}_{\bar\sigma(i)}\mathscr{G}(0,0)_a \bigotimes \mathscr{G}(0,0)_{b-1}\Psi^{(0)}_{\sigma(i)}\,. \tag{3.30}$$

Q.E.D.

Situation 4. *The proof of the Theorem,* i.e. *the general situation.*

Proof. By means of $(n+1)\times(n+1)$ constant matrices, $S^{(0)}$ and $S^{(\infty)}$ $\tilde f^A = (S^{(0)-1})^A_B f^B$ and $\tilde{\bar f}^{\bar A} = (S^{(\infty)-1})^{\bar A}_B \bar f^{\bar B}$, we can return to the previous normal form Eqs. (3.26) (more about this in Sect. 4.3). Denote by $\widetilde\Psi$ and $\widetilde\Psi^*$ the fermion basis in terms of the normal form. Introduce Ψ and Ψ^* which correspond to the original embedding functions by[13]

$$\Psi_\ell = \sum_{\ell'=0}^{n} S_{\ell\ell'}\widetilde\Psi_{\ell'}\,, \quad \Psi_\ell^* = \sum_{\ell'=0}^{n} \widetilde\Psi_{\ell'}(S^{-1})_{\ell'\ell}\,. \tag{3.31}$$

for $\ell\in\Xi$ and $\Psi_\ell = \widetilde\Psi_\ell$, $\Psi_\ell^* = \widetilde\Psi_\ell^*$ for $\ell\in\Xi^{(-)}$. It is easy to check that the new Ψ- and Ψ^*-basis have exactly the same properties as the normal basis which we introduced in the previous situation. Q.E.D.

[13] We omit the superscripts (0) and (∞) because there are discussed in complete parallel

3.2 W-*Parametrization from* KP *Coordinates*

Definition 11. W-***Transformations.*** A general infinitesimal W-transformation is a change of embedding functions which takes the form

$$\delta_W f^A(z) = \sum_{j=0}^{n} w^j(z)\partial^{(j)} f^A(z)\,, \quad \delta_W \bar{f}^{\bar{A}}(\bar{z}) = \sum_{j=0}^{n} \bar{w}^j(\bar{z})\bar{\partial}^{(j)} \bar{f}^{\bar{A}}(\bar{z})\,, \tag{3.32}$$

where $w^j(z)$, and $\bar{w}^j(\bar{z})$ are arbitrary functions of one variable.

This is standard definition. The purpose of this section is to introduce a special class of parametrizations for the target-manifold which is such that these W-transformations of Σ may be extended as special types of diffeomorphisms of $\mathscr{C}P^n$.

Definition 12. W-***Parametrizations of*** $\mathscr{C}^{n+1}$**.** Given a W-surface embedded into a $\mathscr{C}^{n+1}$, the associated W-parameters of the target space are $n+1$ variables $z^{(0)}$, $z^{(1)} = z$, $z^{(2)}, \ldots, z^{(n)}$, noted $[z]$, and $\bar{z}^{(0)}, \bar{z}^{(1)} = \bar{z}, \bar{z}^{(2)} = \bar{z}^{(n)}$, noted $[\bar{z}]$. The change of coordinates from X^A, $\bar{X}^{\bar{A}}$ to $[z]$, $[\bar{z}]$ is defined by

$$X^A = f^A([z])\,, \bar{X}^{\bar{A}} = \bar{f}^{\bar{A}}([\bar{z}])\,, \tag{3.33}$$

where $f^A([z])$, and $\bar{f}^{\bar{A}}([\bar{z}])$, are the solutions of the equations

$$\frac{\partial}{\partial z^{(\ell)}} f^A([z]) = \frac{\partial^\ell}{\partial z^\ell} f^A([z])\,, \quad \frac{\bar{\partial}}{\partial \bar{z}^{(\ell)}} \bar{f}^{\bar{A}}([\bar{z}]) = \frac{\bar{\partial}^\ell}{\partial \bar{z}^\ell} \bar{f}^{\bar{A}}([\bar{z}]) \tag{3.34}$$

with the initial conditions $f^A([z]) = f^A(z)$ for $z^{(0)}, z^{(2)}, \ldots, z^{(n)} = 0$, and $\bar{f}^{\bar{A}}([\bar{z}]) = \bar{f}^{\bar{A}}(\bar{z})$ for $\bar{z}^{(0)}, \bar{z}^{(2)}, \ldots, \bar{z}^{(n)} = 0$.

These coordinates coincide with the higher variables of the KP hierarchy. Indeed, their definition is most natural in the free-fermion language, where it is easy to see that

$$f^A([z]) = \left\langle \emptyset | \psi_{f^A} e^{\sum_0^n J_s z^{(s)}} | 1 \right\rangle, \quad \bar{f}^{\bar{A}}([\bar{z}]) = \left\langle 1 | e^{\sum_0^n \bar{J}_t \bar{z}^{(t)}} \psi^+_{\bar{f}^{\bar{A}}} | \emptyset \right\rangle. \tag{3.35}$$

The dependence in $[z]$ and $[\bar{z}]$ is dictated by the action of the higher currents J, $\bar{J}$, defined by Eq. (3.4), that is, $J_1 z \to \sum_{i=0}^{n} J_i z^{(i)}$, $\bar{J}_1 \bar{z} \to \sum_{i=0}^{n} \bar{J}_i \bar{z}^{(i)}$ in Eq. (2.39). Thus we shall call them KP-*coordinates*. Equations (3.34) are propagation equations. They define the $z^{(k)}$ and $\bar{z}^{(k)}$ variables as long as no singularity develops, that is, in a neighborhood of the W-surface in $\mathscr{C}P^n$. In agreement with the above, the embedding operator, and tau-functions are re-defined by modification of the Hamiltonian in Eqs. (3.8), and (3.10), that is:

Definition 13. ***Generalized tau-functions, and embedding operator.***

$$\begin{aligned} \mathscr{G}([z],[\bar{z}]) \equiv& e^{\sum_0^n \bar{J}_t \bar{z}^{(t)}} \mathscr{G}(0,0) e^{\sum_0^n J_s z^{(s)}}\,, & (3.36)\\ \tau_\ell([z],[\bar{z}]) =& \langle \ell | \mathscr{G}([z],[\bar{z}]) | \ell \rangle\,. & (3.37) \end{aligned}$$

From the physicist's viewpoint, it is illuminating to realize that these tau-functions play the rôle of partition functions since the relevant fermionic matrix elements are obtained by taking derivatives of them with respect to $[z]$ and $[\bar{z}]$. This is contained in the so-called bosonization rules which may be derived, using a method which is an adaptation of the proof of the relativistic fermion [24] to the present non-relativistic ones:

Theorem 10. ***Bosonization Rules.***

$$\begin{aligned}
\langle \ell+1|\mathscr{G}([z],[\bar{z}])\psi^{+}_{\ell+s}|\ell\rangle &= \chi^{\mathrm{Sch}}_s([\partial])\tau_{\ell+1}([z],[\bar{z}])\,,\\
\langle \ell|\mathscr{G}([z],[\bar{z}])\psi_{\ell-s}|\ell+1\rangle &= \chi^{\mathrm{Sch}}_s(-[\partial])\tau_{\ell+1}([z],[\bar{z}])\,,\\
\langle \ell+1|\psi^{+}_{l-s}\mathscr{G}([z],[\bar{z}])|\ell\rangle &= \chi^{\mathrm{Sch}}_s([\bar{\partial}])\tau_{\ell}([z],[\bar{z}])\,,\\
\langle \ell|\psi_{l+s}\mathscr{G}([z],[\bar{z}])|\ell+1\rangle &= \chi^{\mathrm{Sch}}_s(-[\bar{\partial}])\tau_{\ell}([z],[\bar{z}])\,,
\end{aligned} \tag{3.38}$$

where the differential operators are given by Schur's polynomials,

$$\chi^{\mathrm{Sch}}_s([\partial]) = \sum_{i_1+2i_2+\ldots+si_s=s}\left(\prod_{\alpha=1}^{s}\frac{1}{i_\alpha!}\left(\frac{1}{\alpha}\frac{\partial}{\partial z^{(\alpha)}}\right)^{i_\alpha}\right). \tag{3.39}$$

For example, one has

$$\begin{aligned}
&\chi^{\mathrm{Sch}}_0([\partial]) = 1\,, \quad \chi^{\mathrm{Sch}}_1([\partial]) = \frac{\partial}{\partial z^{(1)}}\,,\\
&\chi^{\mathrm{Sch}}_2([\partial]) = \frac{1}{2}\left(\frac{\partial}{\partial z^{(2)}} + \left(\frac{\partial}{\partial z^{(1)}}\right)^2\right).
\end{aligned} \tag{3.40}$$

Going back to our main line, we note that Eqs. (3.35) give the extension to $\mathscr{C}^{n+1}$ of the η matrix (2.54):

$$\eta_{i\bar{j}}([z],[\bar{z}]) = \sum_{A=0}^{n}\partial_i f^A([z])\bar{\partial}_{\bar{j}}\bar{f}^A([z])\,, \quad \partial_i \equiv \frac{\partial}{\partial z^{(i)}}\,, \quad \bar{\partial}_{\bar{j}} \equiv \frac{\bar{\partial}}{\partial \bar{z}^{(j)}}\,. \tag{3.41}$$

Now, only first-order derivatives appear. As a matter of fact this expression coincides with the *true* Riemannian metric with respect to the KP coordinates. We call the corresponding frame, span by $\partial_s f$, $\bar{\partial}_s \bar{f}$, the W-*frame*. In terms of the W-frame and modified η, the moving frame[14] is given by,

[14] In this context the name "moving frame" is somehow misleading. They are actually **the local Lorenz frame** of the target space

Definition 14. ***Moving Frame with the* KP *Coordinates.*** The following set of vectors is orthonormal (η is given by Eq. (3.41))

$$\begin{aligned}
\tilde{e}_\ell([z],[\bar{z}]) &= \frac{1}{\sqrt{\tau_\ell([z],[\bar{z}])\,\tau_{\ell+1}([z],[\bar{z}])}}\,\tilde{v}([z],[\bar{z}])\,,\\
\tilde{v}_\ell([z],[\bar{z}]) &= \begin{vmatrix} \eta_{0\bar{0}} & \cdots & \eta_{\ell\bar{0}} \\ \vdots & & \vdots \\ \eta_{0\overline{\ell-1}} & \cdots & \eta_{\ell\overline{\ell-1}} \\ \partial_0 f([z]) & \cdots & \partial_\ell f([z]) \end{vmatrix}\,,\\
\tilde{\bar{e}}_\ell([z],[\bar{z}]) &= \frac{1}{\sqrt{\tau_\ell([z],[\bar{z}])\,\tau_{\ell+1}([z],[\bar{z}])}}\,\tilde{\bar{v}}([z],[\bar{z}])\,,\\
\tilde{\bar{v}}_\ell([z],[\bar{z}]) &= \begin{vmatrix} \eta_{\bar{0}0} & \cdots & \eta_{\bar{\ell}0} \\ \vdots & & \vdots \\ \eta_{\ell-1\bar{0}} & \cdots & \eta_{\bar{\ell}\ell-1} \\ \bar{\partial}_0 \bar{f}([\bar{z}]) & \cdots & \bar{\partial}_\ell \bar{f}([\bar{z}]) \end{vmatrix}\,.
\end{aligned} \tag{3.42}$$

Each W-parametrization depends upon the W-surface considered. The latter is obviously recovered by letting $z^{(k)} = \bar{z}^{(k)} = 0$, for $k = 2, \ldots, n$. This link between the target-space parametrization and the W-surface allows us to relate its intrinsic and extrinsic geometries. This is a key step in this whole scheme.

Next we show that the W-transformations may be extended to $\mathscr{C}^{n+1}$ by requiring that the differential equations Eq. (3.34) be left invariant. This will give a special calls of diffeomorphisms of $\mathscr{C}^{n+1}$. We shall only consider the holomorphic sector explicitly. The calculation in the anti-holomorphic sector is completely analogous. First we consider the limit $n \to \infty$. It is known that the W-transformations become linear (so-called $w_{1+\infty}$ transformations) and the coordinate transformations are simplified. We work at the level of formal series, without considering convergence problems. The desired result follows from the

Lemma 2. ***Invariance of the Differential Equations.*** *Given an arbitrary function $\varepsilon(z)$, define the functions $\varepsilon^{(s)}([z])$, $s = 0, \ldots, \infty$, from the generating function*

$$e^{-H(\zeta,[z])}\varepsilon(\partial_\zeta)e^{H(\zeta[z])} = \sum_{s=0}^{\infty} \zeta^s \varepsilon^{(s)}([z])\,, \quad H(\zeta[z]) \equiv \sum_{s=0}^{\infty} \zeta^s z^{(s)}. \tag{3.43}$$

1) *To first order, and for any given positive integer ℓ, the differential Eq.* (3.34) *are left invariant by the change of* W*-parameters*

$$\delta_\varepsilon^{(\ell)} z^{(r)} = \varepsilon^{(r-\ell)}([z])\,, \;\; r \geq \ell\,, \quad \delta_\varepsilon^{(\ell)} z^{(r)} = 0\,, \;\; r < \ell\,. \tag{3.44}$$

2) *Conversely, any first-order reparametrization of* $[z]$ *that leaves the differential Eqs.* (3.44) *invariant is a linear combination of the above.*

Proof. 1) One makes use of the inverse Laplace transform. Write

$$f^A(z) = \int_{a-i\infty}^{a+i\infty} d\zeta e^{\zeta z} \tilde{f}^A(\zeta)\,, \tag{3.45}$$

where a is to the right of the singularities of $\tilde{f}^A$. It follows from Definition 12 that ($H(\zeta,[z])$ is defined in Eq. (3.43)

$$f^A([z]) = \int d\zeta e^{H(\zeta,[z])} \tilde{f}^A(\zeta)\,. \tag{3.46}$$

Making us of Eqs. (3.43), (3.45), one may rewrite the variation of $f^A([z])$ under the form

$$\delta_\varepsilon^{(\ell)} f(z) = \int_{a-i\infty}^{a+i\infty} d\zeta e^{H(\zeta,[z])} \varepsilon(-\partial_\zeta)(\zeta^\ell \tilde{f}(\zeta))\,, \tag{3.47}$$

which is indeed a solution on the differential equations (3.34).
2) Conversely, consider a variation $\delta z^{(r)} = \varrho^{(r)}([z])$.
If the variation of f^A is a solution, one should be able to write

$$\delta e^{H(\zeta,[z])} \equiv \sum_{r=0}^{\infty} \varrho^{(r)}([z]) \zeta^r e^{H(\zeta,[z])} = \sum_{q=0}^{\infty} \zeta^q P_q(\partial_\zeta) e^{H(\zeta,[z])},$$

where $P_q(\partial_\zeta)$ is a differential operator with constant coefficients. To each P_q, and making use of the generation function Eq. (3.43), we associate a family of functions $P_q^{(s)}([z])$ by the equations

$$e^{-H(\zeta,[z])} P_q(\partial_\zeta) e^{H(\zeta,[z])} = \sum_{s=0}^{\infty} \zeta^s P_q^{(s)}([z])\,, \tag{3.48}$$

and we obtain

$$\begin{aligned} \delta e^{H(\zeta,[z])} &= \sum_q \zeta^q \sum_s \zeta^s P_q^{(s)}([z])\,, \\ \delta z^{(r)} &= \sum_q \delta_{P_q}^{(q)}(z) = \sum_{q\le r} P_q^{(r-q)}([z])\,. \end{aligned} \tag{3.49}$$

This completes the proof. Q.E.D.

Next, if follows from Eq. (3.43) that Eq. (3.44) gives

$$\delta_\varepsilon^{(\ell)} f^A([z]) = \sum_{s=0}^{\infty} \varepsilon^{(s)}([z]) \frac{\partial}{\partial z^{(l+s)}} f^A([z]) = \sum_{s=0}^{\infty} \varepsilon^{(s)}([z]) \partial^{(\ell+s)} f^A([z])\,. \tag{3.50}$$

It is easy to see that, on the W surface (that is for $z^{(1)} = z$, $z^{(0)} = z^{(2)} = \cdots = z^{(n)} = 0$), Eq. (3.43) gives $\varepsilon^{(s)} = 0$, for $s \neq 0$ and $\varepsilon^{(0)}(z) = \varepsilon(z)$. Thus the function $\varepsilon(z)$ specifies the variation of the embedding functions themselves:

$$\delta_\varepsilon^{(\ell)} f^A(z) = \varepsilon(z) \partial^{(\ell)} f^A(z)\,. \tag{3.51}$$

Clearly the W-transformations introduced by Definition 11 are linear transformations of such variations. Lemma 2 thus leads to the

Theorem 11. W-***Diffeomorphisms.*** *Each* W*-transformation has a unique local extension to* $\mathscr{E}^{n+1}$ *that leaves the differential equations Eqs.* (3.34) *invariant.*

The above discussion moreover shows that we have

$$\delta_W f^A([z]) = \sum_r W^r \partial_r f^A([z])\,, \quad W^r = \sum_{s=0}^{r} w^{r-s\,(s)}\,. \tag{3.52}$$

The functions W^r should be regarded as the components of the tangent vector associated with the W-transformation considered.

The Lie algebra of W-transformations coincides with the bracket algebra of the associated tangent vectors. The corresponding change of coordinates is $\delta_W z^{(r)} = W^r$, and one sees that *the* W*-surface is moved by the* W*-transformations.* For a W-surface, there is no covariant separation between intrinsic and extrinsic geometries.

Our next topic is the local rescaling introduced in Eq. (2.37), that is, the transformation $f^A(z) \to \varrho(z) f^A(z)$. Infinitesimal transformations of this type may be regarded as special cases of Definition 11, where $w^j = 0$, for $j \neq 0$. Moreover, one may check that the covariance properties Theorem 3 of the moving frame are extended away from the W-surface by the definition of the W-parametrization. The above discussion thus gives the

Corollary 4. ***Local rescaling.*** *The infinitesimal rescaling* $\delta f^A(z) = \sigma(z) f^A(z)$ *is equivalent to the following change of* W*-parameters:*

$$\delta z^{(s)} = \sigma^{(s)}([z])\,, \quad e^{-H(\zeta,[z])}\sigma(\partial_\zeta) e^{H(\zeta,[z])} = \sum_{s=0}^{\infty} \zeta^s \sigma^{(s)}([z])\,. \tag{3.53}$$

The generalized moving frame introduced by Definition 14 is covariant under these transformations.

The fact that the rescaling is equivalent to a change of W-coordinates is understood by noting that $e^{H(\zeta,[z])}$ is of the form $\exp(z^{(0)})$ times a factor that does not depend upon this variable. Thus $z^{(0)}$ is really the scaling factor of the W-coordinates.

Next we deal with the case where the dimension of the target-space is finite and equal to n. We prove that the modification is given by

Theorem 12. W-***Diffeomorphism for finite*** n***.*** *For finite* n*, the* W*-reparametrizations Eq.* (3.52) *become*

$$\delta_W f^A([z]) = \sum_{r=0}^{n} \left((W^r + \sum_{s>0} W^{n+s} \lambda_r^{(s)}([z]) \right) \partial_r f^A([z])\,. \tag{3.54}$$

Similarly, the reparametrization Eq. (3.44) *is to be rewritten as*

$$\delta_\varepsilon^{(\ell)} z^{(s)} = \varepsilon^{(s-\ell)}([z]) + \sum_{t=1}^{\infty} \varepsilon^{(n-\ell+t)}([z]) \lambda_s^{(t)}([z])\,, \quad (s = 0, \ldots, n)\,, \tag{3.55}$$

where, according to Eq. (3.44), $\varepsilon^{(r)}$ *is defined to be zero for* $r < 0$. *The notations of this theorem are explained in the proof.*

Proof. In this case there are relations between the embedding functions. They are derived from the equations ($A = 0, \ldots, n$)

$$\begin{vmatrix} f^0(z) & \cdots & f^n(z) & f^A(z) \\ \vdots & & \vdots & \vdots \\ f^{(n)0}(z) & \cdots & f^{(n)n}(z) & f^{(n)A}(z) \\ f^{(n+s)0}(z) & \cdots & f^{(n+s)n}(z) & f^{(n+s)A}(z) \end{vmatrix}$$
$$\equiv \left\{ U_0(z)\partial^{(n+s)} - \sum_{t=0}^{n} U_t^{(s)}(z)\partial^{(t)} \right\} f^A(z) = 0\,. \tag{3.56}$$

U_0 which is the Wronskian of the functions $f^0, \ldots, f^n$, does not vanish at regular points of Σ. Thus we may eliminate the higher derivatives by the relation

$$\partial^{(n+s)} f^A(z) = \sum_{t=0}^{n} \lambda_t^{(s)}(z)\partial^{(t)} f^A(z)\,, \quad \lambda_t^{(s)}(z) = U_t^{(s)}(z)/U_0(z)\,. \tag{3.57}$$

When the W-parametrizations are defined according to Definition 12, it is easy to see that this last condition is extended by construction. Indeed, one has

$$\begin{vmatrix} \partial_0 f^0([z]) & \cdots & \partial_0 f^n([z]) & f^A([z]) \\ \vdots & & \vdots & \vdots \\ \partial_n f^0([z]) & \cdots & \partial_n f^n([z]) & \partial_n f^A([z]) \\ \partial_{n+s} f^0([z]) & \cdots & \partial_{n+s} f^n([z]) & \partial_{n+s} f^A([z]) \end{vmatrix}$$
$$\equiv \left\{ U_0([z])\partial_{n+s} - \sum_{t=0}^{n} U_t^{(s)}([z])\partial_t \right\} f^A([z]) = 0 \tag{3.58}$$

and the W-parameters automatically satisfy the conditions

$$\partial_{n+s} f^A([z]) = \sum_{t=0}^{n} \lambda_t^{(s)}([z])\partial_t f^A([z])\,, \quad \lambda_t^{(s)}([z]) = U_t^{(s)}([z])/U_0([z])\,. \tag{3.59}$$

By this, we can eliminate all dependence in the higher coordinates $z^{(k)}$, with $k > n$. Q.E.D.

The reparametrization is an explicit function of f. This reflects the fact that the W parametrization depends upon the W-surface considered. This is why the embedding functions f^A transform nonlinearly.

Finally, it is clear that, since we have worked in a way which is covariant under the local rescaling defined by Corollary 4, we have the

Corollary 5. W-***parametrization of*** $\mathscr{C}P^n$. *The* W*-parameters of Definition 12 give a parametrization of* $\mathscr{C}P^n$ *if one identifies any two points which are connected by a transformation of the form Eq.* (3.53) *introduced in Corollary 4.*

3.3. Extended Frenet-Serret Formula and Toda Hierarchy

In this section, we study the generalization of the Frenet-Serret equations Eqs. (2.11), that include the KP coordinates introduced above to define W-parametrizations. As expected, such study leads us to the Lax pair for the Toda *hierarchy*. We shall deal with the $\mathscr{C}P^n$ W-surface, by working in $\mathscr{C}^{n+1}$ with the generalized homogeneous coordinates introduced in the previous section (Corollary 5). Our result is expressed by the [15]

Theorem 13. ***Frenet-Serret Formulae for* KP *Coordinates.*** *Consider the chiral vectors*

$$\tilde{u}_\ell = \frac{1}{\tau_\ell}\tilde{v}_\ell\,,\ \ \tilde{\bar{u}}_\ell = \frac{1}{\tau_{\ell+1}}\tilde{\bar{v}}_\ell\,,\quad (\tilde{\bar{u}}_\ell, \tilde{u}_{\ell'}) = \delta_{\ell\ell'}\,, \tag{3.60}$$

where $\tilde{v}_\ell$ and $\tilde{\bar{v}}_\ell$ are given by Eqs. (3.42). *It follows from the differential equations Eqs.* (3.34) *that they obey the differential equations*

$$\partial_p \tilde{u}_\ell = \sum_{\ell'=\ell}^{\ell+p} \tau_\ell^{-1}(F^p)_{\ell\ell'}\tau_{\ell'}\tilde{u}_{\ell'}\,, \tag{3.61}$$

$$\bar{\partial}_p \tilde{u}_\ell = -\sum_{\ell'=\ell-p}^{\ell-1} \tau_{\ell+1}(G^p)_{\ell\ell'}\tau_{\ell'+1}^{-1}\tilde{u}_{\ell'}\,, \tag{3.62}$$

$$\bar{\partial}_p \tilde{\bar{u}}_\ell = \sum_{\ell'=\ell+1}^{\ell+p} \tilde{\bar{u}}_{\ell'}\tau_{\ell'+1}(G^p)_{\ell'\ell}\tau_{\ell+1}^{-1}\,, \tag{3.63}$$

$$\partial_p \tilde{\bar{u}}_\ell = -\sum_{\ell'=\ell-p}^{\ell} \tilde{\bar{u}}_{\ell'}\tau_{\ell'}^{-1}(F^p)_{\ell'\ell}\tau_\ell\,, \tag{3.64}$$

where F^p and G^p, which are the p^{th} power of matrices $F^1 \equiv F$ and $G^1 \equiv G$, respectively, are given by

$$\begin{aligned}(F^p)_{\ell\ell'} &= \frac{1}{\tau_{\ell'}\tau_{\ell'+1}}\sum_{s=0}^{\infty}\langle \ell'+1|\mathscr{G}\psi_{s+p}^{+}|\ell'\rangle\langle\ell|\mathscr{G}\psi_s|\ell+1\rangle\,,\\ (G^p)_{\ell'\ell} &= \frac{1}{\tau_{\ell'}\tau_{\ell'+1}}\sum_{s=0}^{\infty}\langle \ell+1|\psi_s^{+}\mathscr{G}|\ell\rangle\langle\ell'|\psi_{s+p}\mathscr{G}|\ell'+1\rangle\,.\end{aligned} \tag{3.65}$$

Using a matrix-notation which is self-explanatory, one may view the general structure of the above equations as follows,

$$\begin{aligned} \partial_p\tilde{u} &= H_1^{-1}(F^p)^{+}H_1\tilde{u}\,, & \bar{\partial}_p\tilde{u} &= -H_2(G^p)^{-}H_2^{-1}\tilde{u}\,,\\ \bar{\partial}_p\tilde{\bar{u}} &= \tilde{\bar{u}}H_2(G^p)^{-}H_2^{-1}\,, & \partial_p\tilde{\bar{u}} &= -\tilde{\bar{u}}H_1^{-1}(F^p)^{+}H_1\,,\\ (H_1)_{\ell\ell'} &= \tau_\ell\delta_{\ell\ell'}\,, & (H_2)_{\ell\ell'} &= \tau_{\ell+1}\delta_{\ell\ell'}\,. \end{aligned} \tag{3.66}$$

The Lax operators which appear in Eqs. (3.61)–(3.64) are exactly those of the A_n-type Toda hierarchy. The integrability conditions for them, therefore, quite naturally give the famous Zakharov-Shabat equations. Thus we have the

[15] From now on, we omit the arguments $[z]$, $[\bar{z}]$ unless they are explicitly needed

Corollary 6. ***Solutions of Zakharov-Shabat Equations.*** *The integrability conditions of the generalized Frenet-Serret formulae of Theorem 13 are*

$$\begin{aligned}
&[\partial_p - H_1^{-1}(F^p)^+ H_1\,, \quad \bar{\partial}_q + H_2(G^q)^- H_2^{-1}] = 0\,,\\
&[\partial_p - H_1^{-1}(F^p)^+ H_1\,, \quad \partial_q - H_1^{-1}(F^q)^+ H_1] = 0\,,\\
&[\bar{\partial}_p + H_2(G^p)^- H_2^{-1}\,, \quad \bar{\partial}_q + H_2(G^q)^- H_2^{-1}] = 0\,,
\end{aligned} \tag{3.67}$$

which coincide with the Zakharov-Shabat equations.

This is the most explicit proof of the relation between the extrinsic geometry of the W-surfaces and the Toda hierarchy.

According to the general scheme, of Sect. 3.1., F and G are given by derivatives of tau-functions. Equations (3.38)–(3.40) give

Proposition 8.

$$\begin{aligned}
(F^p)_{\ell\ell'} &= \frac{1}{\tau_{\ell'}\tau_{\ell'+1}} \sum_{s=0}^{\ell-\ell'+p} \left(\chi_s^{\rm Sch}([\partial])\tau_{\ell'+1}\right)\left(\chi_{\ell-\ell'+p-s}^{\rm Sch}(-[\partial])\tau_\ell\right),\\
(G^p)_{\ell'\ell} &= \frac{1}{\tau_{\ell'}\tau_{\ell'+1}} \sum_{s=0}^{\ell-\ell'+p} \left(\chi_s^{\rm Sch}([\bar{\partial}])\tau_{\ell'+1}\right)\left(\chi_{\ell-\ell'+p-s}^{\rm Sch}(-[\bar{\partial}])\tau_\ell\right).
\end{aligned} \tag{3.68}$$

The first few terms are very simple,

$$F_{\ell\ell+1} = \frac{\tau_\ell}{\tau_{\ell+1}}\,, \quad F_{\ell\ell} = \partial_1 \ln(\tau_{\ell+1}/\tau_\ell)\,, \quad F_{\ell\ell-1} = \frac{\tau_\ell}{\tau_{\ell+1}}\partial_1^2\tau_\ell \quad \cdots\,. \tag{3.69}$$

The rest of this section is devoted to the detailed proof of Theorem 13. In Subsect. 3.3.1, and since we need to treat the higher KP coordinates systematically, we first translate the moving-frame equations into the free-fermion language. In Subsect. 3.3.2, explicit formulae for the Lax operators F, G are given. In Subsect. 3.3.3, we finally spell out the actual derivation of the generalized Frenet-Serret equations.

3.3.1. Free-Fermion Representation of the Moving Frame. In order to automatically solve the differential equations Eqs. (3.34), we re-write all expressions of the moving-frame equations, in the free-fermion operator-formalism. The basic point is that the Hirota equation (3.13) remains valid when the higher coordinates are included. Indeed, using the same argument as in Sect. 3.1, we can again reduce the derivation to the point $z^{(k)} = \bar{z}^{(k)} = 0$, $\forall k$. Thus the full power of the fermionic method is still on. From Eqs. (3.42), it is straightforward to derive the following neat expressions for $\tilde{v}_\ell$ and $\tilde{\bar{v}}_\ell$,

$$\begin{aligned}
\tilde{v}_\ell &= \sum_{s=0}^{\ell} f^{(s)}\langle \ell|\mathscr{G}\psi_s|\ell+1\rangle\,,\\
\tilde{\bar{v}}_\ell &= \sum_{s=0}^{\ell} \bar{f}^{(s)}\langle \ell+1|\psi_s^+\mathscr{G}([z],[\bar{z}])|\ell\rangle\,.
\end{aligned} \tag{3.70}$$

Instead of working with the vectors $\tilde{v}_\ell$ and $\tilde{\bar{v}}_\ell$, it is more useful to introduce the ket- and bra-fermionic-states which correspond to them:

$$\tilde{v}_\ell \leftrightarrow \sum_{s=0}^{\ell} e^{\sum_0^n J_s z^{(s)}} \psi_s^+|\emptyset\rangle\langle\ell|\mathscr{G}\psi_s|\ell+1\rangle \equiv |v_\ell\rangle\,,$$

$$\tilde{\bar{v}}_\ell \leftrightarrow \sum_{s=0}^{\ell}\langle\ell+1|\psi_s^+\mathscr{G}|\ell\rangle\langle\emptyset|\psi_s e^{\sum_0^n \bar{J}_t\bar{z}^{(t)}} \equiv \langle\bar{v}_\ell|\,. \tag{3.71}$$

These states satisfy

$$\langle\bar{v}_\ell|\mathscr{G}([0],[0])|v_{\ell'}\rangle = \tilde{\bar{v}}_\ell^{\bar{A}}\tilde{v}_{\ell'}^{A}\delta_{A\bar{A}} = \tau_\ell\tau_{\ell+1}\delta_{\ell\ell'}\,. \tag{3.72}$$

It is easy to derive the equations of $\tilde{v}$, $\tilde{\bar{v}}$ from those of $\langle\bar{v}_\ell|$ and $|v_\ell\rangle$.

3.3.2. Definition of the Lax operator. Using the free-fermion representation of the moving frame, we first justify the introduction of the Lax operators F, G. Since one has $\partial_p e^{\sum_0^n J_s z^{(s)}} = J_p e^{\sum_0^n J_s z^{(s)}}$, we first study the action of the J_p's on the states $\langle\bar{v}_\ell|$ and $|v_\ell\rangle$. Define F and G by

$$J_1|v_\ell\rangle \equiv \sum_{\ell'=0}^{\ell+1} F_{\ell\ell'}|v_{\ell'}\rangle\,, \quad \langle\bar{v}_\ell|\bar{J}_1 \equiv \sum_{\ell'=0}^{\ell+1}\langle\bar{v}_{\ell'}|G_{\ell'\ell}\,. \tag{3.73}$$

Since $|v_\ell\rangle$ and $\langle\bar{v}_\ell|$ are one-particle states, we can easily derive the following lemma,

$$J_p|v_\ell\rangle \equiv \sum_{\ell'=0}^{\ell+p}(F^p)_{\ell\ell'}|v_{\ell'}\rangle\,, \quad \langle v_\ell|\bar{J}_p \equiv \sum_{\ell'=0}^{\ell+p}\langle\bar{v}_{\ell'}|(G^p)_{\ell'\ell}\,. \tag{3.74}$$

We can calculate the explicit formulae for F and G by using the Hirota equation (Theorem 9). For example,

$$\begin{aligned} F_{\ell\ell'} &= \frac{1}{\tau_{\ell'}\tau_{\ell'+1}}\langle\bar{v}_{\ell'}|\mathscr{G}J_1|v_\ell\rangle \\ &= \frac{1}{\tau_{\ell'}\tau_{\ell'+1}}\sum_{s=0}^{\infty}\sum_{s'=0}^{\infty}\langle\ell'+1|\psi_{s'}^+\mathscr{G}|\ell'\rangle\langle\ell|\mathscr{G}\psi_s|\ell+1\rangle\langle\emptyset|\psi_{s'}\mathscr{G}\psi_s^+|\emptyset\rangle \\ &= \frac{1}{\tau_{\ell'}\tau_{\ell'+1}}\sum_{s=0}^{\infty}\langle\ell'+1|\mathscr{G}\psi_{s+1}^+|\ell'\rangle\langle\ell|\mathscr{G}\psi_s|\ell+1\rangle\,. \end{aligned} \tag{3.75}$$

Similarly,

$$G_{\ell'\ell} = \frac{1}{\tau_{\ell'}\tau_{\ell'+1}}\sum_{s=0}^{\infty}\langle\ell+1|\psi_s^+\mathscr{G}|\ell\rangle\langle\ell'|\psi_{s+1}\mathscr{G}|\ell'+1\rangle\,. \tag{3.76}$$

The summations in Eqs. (3.75)–(3.76) can be taken from $\ell'-1$ to ℓ since the other terms vanish. Due to the lemma (3.74), the powers of F, G have similar forms,

$$(F^p)_{\ell\ell'} = \frac{1}{\tau_{\ell'}\tau_{\ell'+1}}\sum_{s=0}^{\infty}\langle\ell'+1|\mathscr{G}\psi_{s+p}^+|\ell'\rangle\langle\ell|\mathscr{G}\psi_s|\ell+1\rangle\,,$$

$$(G^p)_{\ell'\ell} = \frac{1}{\tau_{\ell'}\tau_{\ell'+1}}\sum_{s=0}^{\infty}\langle\ell+1|\psi_s^+\mathscr{G}|\ell\rangle\langle\ell'|\psi_{s+p}\mathscr{G}|\ell'+1\rangle\,. \tag{3.77}$$

3.3.3. Proof of the Theorem. We give a proof of (3.61)–(3.65) by direct computations. Before that, it is useful to observe that, by differentiation of Eq. (3.72),

$$\partial_p(\tau_\ell \tau_{\ell+1})\delta_{\ell\ell'} = (\partial_p \langle \bar{v}_\ell|)\mathscr{G}([0],[0])|v_{\ell'}\rangle + \langle \bar{v}_\ell|\mathscr{G}([0],[0])(\partial_p|v_{\ell'}\rangle)\,. \tag{3.78}$$

The first term on the RHS vanishes if $\ell' > \ell$. Hence the second term should also vanish in this case. By the orthogonality of $\langle \bar{v}|$, and $|v\rangle$, we can conclude that $\partial_p|v_{\ell'}\rangle$ is the linear combination of $|v_{\ell'+p}\rangle$, $|v_{\ell'+p-1}\rangle$, $\cdots$, $|v_{\ell'}\rangle$. On the other hand, a direct computation gives

$$\begin{aligned}\partial_p|v_\ell\rangle &= J_p|v_\ell\rangle + \sum_{s=0}^{\infty} e^{\sum_0^n J_t z^{(t)}} \psi_s^+|\emptyset\rangle\langle \ell|\mathscr{G} J_n \psi_s|\ell+1\rangle \\ &= \sum_{\ell'=0}^{\ell+p}(F^p)_{\ell\ell'}|v_{\ell'}\rangle + \sum_{s=0}^{\infty} e^{\sum_0^n J_t z^{(t)}} \psi_s^+|\emptyset\rangle\langle \ell|\mathscr{G} J_n \psi_s|\ell+1\rangle\,.\end{aligned} \tag{3.79}$$

We remark that the second term is a linear combination of $|v_0\rangle, \cdots, |v_\ell\rangle$. From these two arguments, we can conclude that the second term cancels with $(F^p)_{\ell\ell'}|v_{\ell'}\rangle$ for $\ell' = 0, 1, \cdots, \ell-1$. This fact can be confirmed from the following explicit calculations. By using the Hirota equation (Theorem 9), one obtains

$$\begin{aligned}&\sum_{s=0}^{\infty}\langle \bar{v}_m|\mathscr{G}\psi_s^+|\emptyset\rangle\langle \ell|\mathscr{G} J_p \psi_s|\ell+1\rangle \\ &= \begin{cases} 0 & (\ell < m) \\ \tau_{\ell+1}\partial_p \tau_\ell & (\ell = m) \\ -\displaystyle\sum_{q=0}^{\infty}\langle m+1|\mathscr{G}\psi_{q+p}^+|m\rangle\langle \ell|\mathscr{G}\psi_q|\ell+1\rangle & (\ell > m)\,. \end{cases}\end{aligned} \tag{3.80}$$

The $\ell = m$ term can be alternatively written as

$$-\sum_{q=0}^{\infty}\langle \ell+1|\mathscr{G}\psi_{q+p}^+|\ell\rangle\langle \ell|\mathscr{G}\psi_q|\ell+1\rangle + \tau_\ell \partial_p \tau_{\ell+1}\,. \tag{3.81}$$

By combining, Eqs. (3.77) with Eqs. (3.79)–(3.81), we get the desired result,

$$\begin{aligned}\partial_p|v_\ell\rangle &= \sum_{\ell'=\ell+1}^{\ell+p}(F^p)_{\ell\ell'}|v_{\ell'}\rangle + (\partial_p \tau_{\ell+1})|v_\ell\rangle \\ &= \sum_{\ell'=\ell}^{\ell+p}(F^p)_{\ell\ell'}|v_{\ell'}\rangle + (\partial_p \tau_\ell)|v_\ell\rangle\,.\end{aligned} \tag{3.82}$$

If we combine this with Eq. (3.78), we get another formula,

$$\begin{aligned}\partial_p\langle \bar{v}_\ell| &= -\sum_{\ell'=\ell-p}^{\ell-1}\langle \bar{v}_{\ell'-1}|(\tau_{\ell'}\tau_{\ell'-1})^{-1}(F^p)_{\ell'\ell}\tau_\ell\tau_{\ell-1} + (\partial_p \ln \tau_\ell)\langle \bar{v}_\ell| \\ &= -\sum_{\ell'=\ell-p}^{\ell}\langle \bar{v}_{\ell'-1}|(\tau_{\ell'}\tau_{\ell'-1})^{-1}(F^p)_{\ell'\ell'}\tau_\ell\tau_{\ell-1} + (\partial_p \ln \tau_{\ell+1})\langle \bar{v}_\ell|\,.\end{aligned} \tag{3.83}$$

Equations (3.61) and (3.64) can be obtained from these formulae by scaling the moving frame appropriately s.t. we remove the diagonal terms. The proof of Eqs. (3.62), (3.63) is exactly parallel. This completes our proof of the Frenet-Serret formula including the higher coordinates. Q.E.D.

3.4. Generalized WZNW *Equations and Riemannian Geometry*

In the present section we show that the W-parametrization of the target-spaces $\mathscr{C}^{n+1}$ and $\mathscr{C}P^n$, which were defined in Sect. 3.2 (Definition 12), are such that the correspondence between W-surfaces and conformally reduced WZNW has a natural multi-dimensional extension away from the W-surface Σ. This will also lead to an extension of the Drinfeld-Sokolov equations. These extensions show the intimate connection between these equations and the Riemannian geometry of the target-space. First we have the

Theorem 14. ***Christoffel Connection for* W*-parametrization.*** *With the* W*-parametrization, the Christoffel symbols are chiral and given by*

$$\begin{cases} (\gamma_p([z]))^k_j = \delta_{p+j,k}\,, & \text{if } p+j \le n\,, \\ \qquad\qquad = \lambda_j^{(p+j-n)}([z])\,, & \text{if } p+j > n \end{cases}\;; \qquad \begin{cases} (\bar\gamma_{\bar p}([\bar z]))^{\bar k}_{\bar j} = \delta_{\bar p+\bar j,\bar k}\,, & \text{if } \bar p+\bar j \le n\,, \\ \qquad\qquad = \bar\lambda_{\bar j}^{(\bar p+\bar j-n)}([\bar z])\,, & \text{if } \bar p+\bar j > n \end{cases}\,, \tag{3.84}$$

where λ, $\bar\lambda$ *are defined in Eqs.* (3.56) *and* (3.57).

Proof. With the W-parameters, the metric tensors is given by Eq. (3.41), that is, $\eta_{i\bar j} = \sum_{A,\bar B} \delta_{A\bar B}\partial_i f^A([z])\bar\partial_{\bar j}\bar f^{\bar B}([z])$. The Christoffel symbols are such that its covariant derivatives vanish. Since this metric tensor is factorized into a product of two chiral matrices, we immediately get

$$\partial_p \eta_{i\bar j} = \sum_j (\gamma_p([z]))^j_i \eta_{j\bar j}\,, \quad \bar\partial_{\bar p}\eta_{i\bar j} = \sum_{\bar k} (\bar\gamma_{\bar p}(\bar z))^{\bar k}_{\bar j}\ {}_{i\bar k}\,, \tag{3.85}$$

where the Christoffel symbols γ and $\bar\gamma$ are such that

$$\partial_p(\partial_\ell f) = (\gamma_p)^{\ell'}_\ell(\partial_{\ell'} f)\,, \quad \bar\partial_p(\bar\partial_\ell \bar f) = (\bar\partial_{\ell'}\bar f)(\bar\gamma_{\bar p})^{\ell'}_\ell\,. \tag{3.86}$$

Making use of Eqs. (3.50) together with its anti-chiral counterpart, one easily deduces the explicit expressions Eqs. (3.84). Clearly the Christoffel connection satisfies

$$\bar\partial_{\bar q}(\gamma_p)^j_i = \partial_q(\bar\gamma_{\bar p})^{\bar k}_{\bar j} = 0\,. \tag{3.87}$$

It is thus chiral, and this completes the proof. Q.E.D.

Next it is clear that Eqs. (3.86) may be considered as multi-variable extensions of the Drinfeld-Sokolov equation (2.62). Thus we introduce the

Definition 15. ***Generalized Drinfeld-Sokolov Equations.*** They are defined as a set of n partial-differential equations of the form

$$\mathscr{L}_p \Upsilon([z]) = 0\,, \quad \mathscr{L}_p \equiv \partial_p - \mathscr{Y}^{(p)}([z])\,, \tag{3.88}$$

where Υ is a column-vector $\{\Upsilon_\ell, 0 \le \ell \le n\}$, and where the $(n+1)\times(n+1)$ matrices $\mathscr{Y}^{(p)}$ are such that

$$(\mathscr{Y}^{(p)})_k^j = 0\,, \text{ if } p+j > k\,, \quad (\mathscr{Y}^{(p)})_k^j = 1\,, \text{ if } p+j = k\,. \tag{3.89}$$

Next it seems appropriate to generalize the link between WZNW and Drinfeld-Sokolov equations as well. Thus we introduce the

Definition 16. ***Generalized*** **WZNW *Equations.*** They are partial-differential equations of the form

$$\bar\partial_{\bar p}(\theta^{-1}([z],[\bar z])\partial_q\theta([z],[\bar z])) = \partial_q((\bar\partial_{\bar p}\theta([z],[\bar z]))\theta^{-1}([z],[\bar z])) = 0\,, \tag{3.90}$$

where θ is a $(n+1)\times(n+1)$ matrix which is real for Minkowski z and $\bar z$ coordinates.

Of course, in the same way as in Sect. 2.3, the present definition includes the $gl(1)$ factor, so that we do not assume that the determinant of θ is equal to one. The W-parametrizations of $\mathscr{C}^{n+1}$ automatically give solutions of these equations, and one easily verifies the

Theorem 15. WZNW ***from Christoffel Connection.*** *The matrix $\theta = \eta^{-1}$, where η is the metric tensor of the* W*-coordinates, is a solution of the generalized* WZNW *equations, satisfying the following constraints:*

$$\begin{aligned} \mathrm{Tr}(\theta^{-1}\partial_p\theta E_{j+p,j}) &= -1\,, \quad 0 \le j \le n-p\,, \\ \mathrm{Tr}((\bar\partial_{\bar p}\theta)\theta^{-1} E_{j,j+\bar p}) &= -1\,, \quad 0 \le j \le n-\bar p\,, \end{aligned} \tag{3.91}$$

and $\mathrm{Tr}(\theta^{-1}\partial_p\theta E_{-\alpha}) = 0$ (resp. $\mathrm{Tr}((\bar\partial_{\bar p}\theta)\theta^{-1}E_\alpha) = 0$) *for all other positive roots (the step operator E_{jk} is defined by $(E_{jk})_a^b = \delta_{a,j}\delta_{k,b}$).*

It seems appropriate to call this last system the conformally reduced WZNW equation, since they are the direct generalizations of the standard notion. Finally it is tempted to give the following

Conjecture 1. ***Equivalences.*** *There is a one-to-one correspondence between the* W*-parametrizations of $\mathscr{C}^{n+1}$* (resp. $\mathscr{C}P^n$) *and the generalized* WZNW *and Drinfeld-Sokolov equations (Definitions 16 and 15) for $gl(n+1)$* (resp. *for A_n).*

As we have observed in Sect. 2.3, the Frenet-Serret formulae give the geometrical interpretation of the Gauss decomposition of the metric $\eta_{j\bar k}(z,\bar z) = C^{-1}_{js}B^{-1}_{ss'}A^{-1}_{s'k}$ on the W-surface. The triangular matrices C and A give the relation between the vectors $\tilde e_\alpha$, $\tilde{\bar e}_\alpha$, and the W-frame $f^{(a)}$ and $\bar f^{(a)}$ [see Eqs. (2.63) and (2.64)]. Making use of the method developed in Sect. 3.2, one immediately sees that the argument may be extended to the target-space, where the Gauss decomposition of the matrix $\theta([z],[\bar z])$ gives the general relationship between the moving frame (=vielbein) span by the vectors $\tilde e^a([z],[\bar z])$, $\tilde{\bar e}^a([z],[\bar z])$, and the W-frame span by $f^{(a)}([z])$, and $\bar f^{(a)}([\bar z])$. In terms of the vectors $\tilde u$ and $\tilde{\bar u}$ defined by Eqs. (3.60), and Eqs. (3.42) one has

$$\begin{aligned} f^{(a)}([z]) &= \sum_b C^{-1}_{ab}([z],[\bar z])\tilde u_b([z],[\bar z])\,, \\ \bar f^{(a)}([\bar z]) &= \sum_b (AB)^{-1}_{ab}([z],[\bar z])\tilde{\bar u}_b([z],[\bar z])\,. \end{aligned} \tag{3.92}$$

In the previous section, we have derived the generalized Frenet-Serret equations,

$$\begin{aligned} \partial_p \tilde{u}_\ell &= (\omega_p)_{\ell\ell'} \tilde{u}_{\ell'}\,, \qquad & \bar{\partial}_{\bar{p}} \tilde{u}_\ell &= (\omega_{\bar{p}})_{\ell\ell'} \tilde{u}_{\ell'}\,, \\ \bar{\partial}_{\bar{p}} \tilde{\bar{u}}_\ell &= \tilde{\bar{u}}_{\ell'} (\bar{\omega}_{\bar{p}})_{\ell\ell'}\,, \qquad & \partial_p \tilde{\bar{u}}_\ell &= \tilde{\bar{u}}_{\ell'} (\bar{\omega}_p)_{\ell'\ell}\,. \end{aligned} \tag{3.93}$$

The matrices ω, $\bar{\omega}$ take the form

$$\begin{aligned} \omega_p &= (H_1^{-1} F^p H_1)^+\,, \qquad & \omega_{\bar{p}} &= -(H_2 G^{\bar{p}} H_2^{-1})^-\,, \\ \bar{\omega}_{\bar{p}} &= (H_2 G^{\bar{p}} H_2^{-1})^-\,, \qquad & \bar{\omega}_p &= -(H_1^{-1} F^p H_1)^+\,. \end{aligned} \tag{3.94}$$

where H_i, $(i = 1, 2)$ is the diagonal matrices $F_{\ell\ell'}$, and $G_{\ell'\ell}$ vanish if $\ell > \ell' + 1$.

Conversely, if a local lorentz frame of any kind satisfies this type of equations, their integrability condition is equivalent to the Zakharov-Shabat equation. The general argument of the Toda theory [19] tells us that there exists a tau-function such that the coefficients of the Zakharov-Shabat equation are given in the form Eq. (3.66). Since the tau-function is defined by the embedding operator $\mathcal{G}$, this argument shows that the local lorentz frame can be identified with the moving frame of a W-surface and their coordinates can be identified with the higher coordinates. Actually, what we are doing in this paper is a reversal procedure of the whole scheme of the Toda equation, i.e. we start from the geometry of the explicit solution (W-surface) to get the equation of motion (Zakharov-Shabat equation). Thus we reach an important conclusion that

Theorem 16. ***Characterization of* W*-Parametrizations.*** *The reparametrization of the* KP *coordinates can be identified with those coming from the* W*-transformations if and only if they do not change the form of Frenet-Serret equation.*

Combining the above formulae, one sees that C^{-1} and $(AB)^{-1}$ which appear in the Gauss decomposition give the transformation between the Lax operator of the Toda hierarchy Eqs. (3.93), (3.90), and the generalized Drinfeld-Sokolov equations Eqs. (3.87), (3.88). Indeed one finds that

$$\begin{aligned} (\partial_p - \gamma_p) C^{-1} &= C^{-1} (\partial_p - \omega_p)\,, \\ \bar{\partial}_{\bar{p}} C^{-1} &= C^{-1} (\bar{\partial}_{\bar{p}} - \omega_{\bar{p}})\,, \\ (AB)^{-1} \partial_p &= (\partial_p - \bar{\omega}_p)(AB)^{-1}\,, \\ (AB)^{-1} (\bar{\partial}_p - \bar{\gamma}_{\bar{p}}) &= (\bar{\partial}_{\bar{p}} - \bar{\omega}_{\bar{p}})(AB)^{-1}\,. \end{aligned} \tag{3.95}$$

This equation without the higher coordinates was discussed previously in [17]. Clearly the integrability conditions of the Frenet-Serret and Drinfeld-Sokolov equations are equivalent.

4. Global Structure of the Embedding

4.1. Associated Mappings

In this last part we deal with singular points of W-surfaces. We shall mostly be interested in the global aspects. They will be described by n topological numbers which will be related to the ramification indices of the singularities by a global Plücker equation that generalizes the Gauss-Bonnet formula. Our guideline is the beautiful discussion of [22]. For these purposes, we need to change the viewpoint which we took until now. So far, we have discussed the extrinsic geometry of the

W-surfaces using the moving frame. We derived the Toda field equations from the Frenet-Serret formulae and Gauss-Codazzi equations. In defining the global indices, this method is not so convenient at present, since we do not yet know how to make use of higher topological invariants of the target space.

The way to replace the extrinsic geometry by the intrinsic one is to introduce the

Definition 17. ***Associated Mappings.*** Consider the family of osculating hyperplanes with contact of order k denoted $\mathscr{O}_k$ $(k = 1, \ldots, n)$ to the original W-surface. With $\mathscr{C}P^n$ as the target space, this family defines an embedding into the Grassmannian $G_{n+1,k+1}$, which we call the k^{th} associated mappings to the original W-surface.

This formulation looks different, but is equivalent to the construction of the moving frame and only uses the intrinsic geometries of the induced metrics for $k = 1, \ldots, n$. In practice, what this means is that, instead of forming moving-frame vectors e_k out of $f, \ldots, f^{(k)}$ $(k = 1, \ldots, n)$, we consider the nested osculating planes $\mathscr{O}_1 \subset \mathscr{O}_2 \subset \cdots \subset \mathscr{O}_n$. It is obvious that those two have the same information. The rest of this section is devoted to the explicit form of these mappings.

For pedagogical purpose, we recall the well-known Grassmannian aspect of the hyperplanes in $\mathscr{C}P^n$. The Grassmannian manifold $G_{n+1,k}$ is the set of $(n+1) \times k$ matrices $\mathscr{F}_k$ with the equivalence relation $\mathscr{F}_k \sim a\mathscr{F}_k$, where a is a $k \times k$ matrix. In our case, it is natural to consider another set of $(n+1) \times k$ matrices $\bar{\mathscr{F}}_k$ simultaneously, in order to deal with each chiral component independently. For arbitrarily given $\mathscr{F}_k$ and $\bar{\mathscr{F}}_k$, we can uniquely define hyperplanes in $\mathscr{C}P^n$ by equations of the form $X^A(t) = \sum_j f^{j,A} t_j$, $\bar{X}^{\bar{A}}(\bar{t}) = \sum_j \bar{f}^{j,\bar{A}} \bar{t}_j$, where t_j and $\bar{t}_j$ are arbitrary parameters. $f^{j,A}$ and $\bar{f}^{j,\bar{A}}$ are the matrix elements of $\mathscr{F}_k$ and $\bar{\mathscr{F}}_k$. The equivalence relation $\mathscr{F} \sim a\mathscr{F}$ and $\bar{\mathscr{F}} \sim \bar{a}\bar{\mathscr{F}}$ simply expresses the fact that the geometrical hyperplane does not change if we replace t_j and $\bar{t}_j$ by linear combinations. Thus the Grassmannian $G_{n+1,k}$ is the space of k-dimensional hyperplanes in $\mathscr{C}P^n$. Following [28], it is natural to base its Kähler structure on the potential $\mathscr{K}_k \equiv \ln(\det \mathscr{F}_k \bar{\mathscr{F}}_k^T)$ which coincides with the Kähler potential of $\mathscr{C}P^n$ for $k = 1$.

Consider an embedding of 2D surface into $G_{n+1,k}$ with chiral parametrization – this time, however, we do not introduce its extrinsic geometry. It is defined by its chiral components $\mathscr{F}_k(z)$ and $\bar{\mathscr{F}}_k(\bar{z})$ which are respectively given by

$$\begin{pmatrix} f^{0,0}(z) & \cdots & f^{0,n}(z) \\ \vdots & & \vdots \\ f^{k,0}(z) & \cdots & f^{k,n}(z) \end{pmatrix}, \quad \begin{pmatrix} \bar{f}^{0,0}(\bar{z}) & \cdots & \bar{f}^{0,n}(\bar{z}) \\ \vdots & & \vdots \\ \bar{f}^{k,0}(\bar{z}) & \cdots & \bar{f}^{k,n}(\bar{z}) \end{pmatrix}. \tag{4.1}$$

The difference with the usual situation is that, for us, $\bar{f}^{i,j}$ is not assumed to be the complex conjugate of $f^{i,j}$. It is immediate that the metric induced on this surface is derivable from the potential $\mathscr{K}^{(k)}$ which is such that

$$e^{\mathscr{K}^{(k)}} = \sum_{0 \le i_1 < \cdots < i_k \le n} \begin{vmatrix} f^{0,i_1}(z) & \cdots & f^{0,i_k}(z) \\ \vdots & & \vdots \\ f^{k-1,i_1}(z) & \cdots & f^{k-1,i_k}(z) \end{vmatrix} \begin{vmatrix} \bar{f}^{0,i_1}(\bar{z}) & \cdots & \bar{f}^{0,i_k}(\bar{z}) \\ \vdots & & \vdots \\ \bar{f}^{k-1,i_1}(\bar{z}) & \cdots & \bar{f}^{k-1,i_k}(\bar{z}) \end{vmatrix}. \tag{4.2}$$

The construction of $\mathscr{O}_{k-1}$ at each point z, $\bar{z}$ goes as follows. One lets

$$\begin{cases} f^{s,l} = \partial^{(s)} f^l(z) \\ \bar{f}^{s,l} = \bar{\partial}^{(s)} \bar{f}^l(\bar{z}) \end{cases}, \quad \text{for } s = 0, \dots, k-1\,, \;\; l = 0, \dots, n\,. \tag{4.3}$$

For a fixed k, the k-planes generate a surface in $G_{n+1,k+1}$ called the k^{th} associated surface. In technical terms: from the embedding $\Sigma \to \mathscr{C}P^n$, we have canonically constructed the k^{th} associated embedding $\Sigma \to G_{n+1,k+1} \hookrightarrow P(\Lambda^{k+1}\mathscr{C}^n)$.

Next, according to Eq. (4.2), the induced metric on the k^{th} associated surface in $G_{n+1,k+1}$ is simply,

$$g^{(k)}_{z\bar{z}} = \partial\bar{\partial} \ln \tau_{k+1}(z, \bar{z})\,, \quad g^{(k)}_{zz} = g^{(k)}_{\bar{z}\bar{z}} = 0\,, \tag{4.4}$$

so that the Toda field $\Phi_{k+1} \equiv -\ln(\tau_{k+1})$ appears naturally. Thus $-\Phi_{k+1}$ is equal to the Kähler potential of the k^{th} associated surface. At this point, it is very clear that by considering the associated surfaces, we can restrict ourselves to intrinsic geometries.

In the discussion of the main section 2, the Toda equation came out from the Gauss-Codazzi equation. Here it is equivalent to the local Plücker formula as we shall see.

4.2. The Instanton-Numbers of a W*-Surface*

A key point in the coming discussion is to use topological quantities that are instanton-numbers. As a preparation, we recall the fact, pointed out in [1], that W-surfaces are instantons of the associated non-linear σ-model. The general situation is as follows. W-surfaces are characterized by their chiral parametrizations which thus satisfy the Cauchy-Riemann relations. These are self-duality equations so that the coordinates of a W-surface define fields that are solutions of the associated non-linear σ-model, with an action equal to the topological instanton number. For a general Kähler manifold M with coordinates ξ^μ and $\bar{\xi}^{\bar{\mu}}$, and metric $h_{\mu\bar{\mu}}$, the action associated with 2D manifolds of M with equations $\xi^\mu = \varphi^\mu(z, \bar{z})$, and $\bar{\xi}^{\bar{\mu}} = \bar{\varphi}^{\bar{\mu}}(z, \bar{z})$ is

$$S = \frac{1}{2} \int d^2x\, h_{\mu\bar{\mu}}\, \partial_j \varphi^\mu \partial_j \bar{\varphi}^{\bar{\mu}}\,. \tag{4.5}$$

In this short digression we let $z = x_1 + ix_2$, and $\partial_j = \partial/\partial x_j$. The instanton-number is defined by

$$Q = \frac{i}{2\pi} \int d^2x\, \epsilon_{jk}\, h_{\mu\bar{\mu}} \partial_j \varphi^\mu \partial k \bar{\varphi}^{\bar{\mu}}\,. \tag{4.6}$$

For W-surfaces and their associated surfaces, $\bar{\partial}\varphi^\mu = \partial\bar{\varphi}^{\bar{\mu}} = 0$, and one has $S = \pi Q$. Q is proportional to the integral of the determinant of the induced metric. Applying the last formula to the k^{th} associated surface, we get

Definition 18. ***Higher Instanton-Numbers of the* W*-Surface.*** The k^{th} instanton number of the W-surface Q_{k+1} is defined by,

$$Q_{k+1} \equiv \frac{i}{2\pi} \int_{\Sigma} dz\, d\bar{z}\, g^{(k)}_{z\bar{z}}\,, \quad k = 1, \ldots, n-1\,. \tag{4.7}$$

Its topological nature is obvious from Eq. (4.4) which shows that the integrand is indeed a total derivative. The collection of the (k^{th}) instanton-numbers together with the original one $Q \equiv Q_1$ gives a set of topological quantities which characterize the global properties of the original W-surface.

4.3. Singular Points of Embeddings

In the main section 2, we have constructed the moving frames at the point where the tau-functions are regular. When those functions become irregular, we meet an obstruction to derive the moving frames. In the WZNW language, this signals that there appears a global obstruction to the Gauss decomposition. Toda equations should be modified at these points. In the following, we study the structure of such singularities and the behavior of the tau-functions.

Let us discuss the $\mathscr{C}P^n$ case, where the structure of such singularity was already studied in detail in mathematics [22]. As always, we use the notation $f^A(z)$ (resp. $\bar{f}^{\bar{A}}(\bar{z})$) to describe the chiral (resp. anti-chiral) part of the embedding. In the following, we only discuss the chiral components explicitly. Consider a point z_0 which is a singular point of the embedding. We assume, as one does in mathematics [22], that we may reduce the problem to the case where there is no branch point around z_0. (If there was, for instance, a non-trivial monodromy-matrix acting on the f^A's around z_0 to begin with, one would assume that this matrix is diagonalizable and that its eigenvalues are rational. In this way, by taking a finite-covering, one would be reduced for the case we are discussing.) Now we remark first that if some f^A's blow up at z_0, we can remove that singularity by applying a local rescaling $f^A(z) \to \varrho(z) f^A(z)$, with $\varrho(z_0) = 0$. The idea is that one divides by the most singular behavior so that every $f^A(z)$ has a finite limit at $z = z_0$. Now the study of the singularity structure is replaced by the study of the zeros of f, and of its derivatives at z_0. By appropriate reshuffling,

$$f^A(z) \to \tilde{f}^A(z) = \sum_B S^A_B f^B(z)\,, \tag{4.8}$$

with a suitable constant matrix S, we can get the following *normal form* for f at $z = z_0$,

$$\begin{aligned}
&\tilde{f}^0(z) = O(1)\,,\\
&\tilde{f}^1(z) = O((z - z_0)^{1+\beta_1(z_0)})\,,\\
&\tilde{f}^2(z) = O((z - z_0)^{2+\beta_1(z_0)+\beta_2(z_0)})\,, \ldots\\
&\tilde{f}^n(z) = O((z - z_0)^{n+\beta_1(z_0)+\cdots+\beta_n(z_0)})\,.
\end{aligned} \tag{4.9}$$

We define

Definition 19. ***Ramification Indices.*** The non-negative integers $\beta_\ell(z_0)$ ($\ell = 1, \dots, n$) which appear in Eq. (4.9) describe the local behavior of the embedding function at $z = z_0$. We call these numbers *ramification indices* following the terminology of the mathematical literature [22]. We introduce similar indices $\bar{\beta}$ to describe the behavior of the anti-chiral embedding functions. We define here also the *total ramification index* β_ℓ, $\bar{\beta}_\ell$ as follows,

$$\beta_\ell = \sum_z \beta_\ell(z)\,, \quad \bar{\beta}_\ell = \sum_{\bar{z}} \bar{\beta}_\ell(\bar{z})\,, \tag{4.10}$$

where z and $\bar{z}$ run over all the singular points of Σ.

The β's are integer since we assumed that there were no branch points. Regular points of the embedding are characterized by the vanishing of all ramification indices.

From this explicit form of the local behavior of the embedding functions, it is easy to calculate the behavior of the tau-functions at $z = z_0$, which is explicitly obtained by,

Theorem 17. ***Behavior of the Tau-Functions.***

$$\tau_\ell = O((z - z_0)^{\beta_{\ell-1} + 2\beta_{\ell-2} + \cdots + (\ell-1)\beta_1} (\bar{z} - \bar{z}_0)^{\bar{\beta}_{\ell-1} + 2\bar{\beta}_{\ell-2} + \cdots + (\ell-1)\bar{\beta}_1})\,. \tag{4.11}$$

Proof. The explicit computations of the first few ones look as follows,

$$\begin{aligned}
\tau_1 &\sim \tilde{f}_0 \tilde{\bar{f}}_0 + \textit{higher order terms} \\
&\sim O(1) \\
\tau_2 &\sim \begin{vmatrix} 1 & (z-z_0)^{1+\beta_1} \\ 0 & (*)(z-z_0)^{\beta_1} \end{vmatrix} \begin{vmatrix} 1 & (\bar{z}-\bar{z}_0)^{1+\bar{\beta}_1} \\ 0 & (*)(\bar{z}-\bar{z}_0)^{\bar{\beta}_1} \end{vmatrix} + \cdots \\
&\sim O((z-z_0)^{\beta_1} (\bar{z}-\bar{z}_0)^{\bar{\beta}_1}) \\
\tau_3 &\sim \begin{vmatrix} 1 & (z-z_0)^{1+\beta_1} & (z-z_0)^{2+\beta_1+\beta_2} \\ 0 & (*)(z-z_0)^{\beta_1} & (*)(z-z_0)^{1+\beta_1+\beta_2} \\ 0 & (*)(z-z_0)^{\beta_1 - 1} & (*)(z-z_0)^{1+\beta_1+\beta_2} \end{vmatrix} \\
&\quad \times \begin{vmatrix} 1 & (\bar{z}-\bar{z}_0)^{1+\bar{\beta}_1} & (\bar{z}-\bar{z}_0)^{2+\bar{\beta}_1+\bar{\beta}_2} \\ 0 & (*)(\bar{z}-\bar{z}_0)^{\bar{\beta}_1} & (*)(\bar{z}-\bar{z}_0)^{1+\bar{\beta}_1+\bar{\beta}_2} \\ 0 & (*)(\bar{z}-\bar{z}_0)^{\bar{\beta}_1 - 1} & (*)(\bar{z}-\bar{z}_0)^{\bar{\beta}_1+\bar{\beta}_2} \end{vmatrix} + \cdots \\
&\sim O((z-z_0)^{2\beta_1+\beta_2} (\bar{z}-\bar{z}_0)^{2\bar{\beta}_1+\bar{\beta}_2})\,.
\end{aligned} \tag{4.12}$$

Here we omit (z_0) in the β's and $(*)$'s are some non-vanishing numerical constants. Obviously, this type of computation can be performed for every tau-function. Q.E.D.

Special combinations of τ-functions appear in the Toda equations. They behave as

$$\frac{\tau_{\ell+1}\tau_{\ell-1}}{\tau_\ell^2} = O((z-z_0)^{\beta_\ell(z_0)} (\bar{z}-\bar{z}_0)^{\bar{\beta}_\ell(\bar{z}_0)})\,. \tag{4.13}$$

In Eqs. (2.40), we need to divide vectors by tau-functions. Unless all ramification index vanish, we get divergence in those formulae. In the WZW language, it shows that the Gauss decomposed matrices A and C become singular at these points. In terms of the Toda equation, it leads to

Theorem 18. ***Modified Toda Equation.*** *At the singular point, the Toda equations have an extra δ-function source-term given by*

$$\partial\bar{\partial}\Phi_\ell + e^{\sum_s K_{\ell s}\Phi_s} + \pi \sum_{z_0,\bar{z}_0 \in \Sigma} \sum_{t=1}^{\ell-1} (\ell - t)(\beta_t(z_0) + \bar{\beta}_t(\bar{z}_0))\delta^{(2)}(z - z_0) = 0\,. \quad (4.14)$$

Proof. The behavior at the singularity can be evaluated directly through the behavior Eq. (4.11) of the tau-functions, and the well-known formula

$$\partial\bar{\partial}\ln(z - z_0) = \partial\bar{\partial}\ln(\bar{z} - \bar{z}_0) = \pi\delta^{(2)}(z - z_0)\,. \quad (4.15)$$

Q.E.D.

4.4. Ramification Indices and DS *Operator*

In the following, we study the relation between the ramification-indices and the singularity of the Drinfeld-Sokolov operator (DS operator). This will be important since it is known that the phase-space of the classical W-algebra is described by the gauge equivalence class of the DS operator. For regular points, this indeed results from the discussion of Sect. 2.3. Ultimately, we should clarify the relation between this phase-space and the space of W-surfaces. A related motivation is to give a direct bridge with the generalization to the quantum situation. In the previous section (4.3), we saw that the local rescaling symmetry is essential to eliminate the singularity so that is should not be fixed globally. In the same way, we have to start from the DS operator of the Sect. 2.3 with additional h_n generator. Let us recall it from the Sect. 2.3,

$$\mathscr{L} = \partial - I - \lambda\,, \quad (4.16)$$

where I is given in Eq. (2.60) and λ is the lower triangle matrix, (Borel subalgebra). The gauge symmetry of $\mathscr{L}$ is generated by the strictly lower-triangular matrices. It follows from Sect. 2.3 that the chiral component of the metric η_R is a solution of the Drinfeld Sokolov equation (DS equation),

$$\mathscr{L}\eta_R = 0\,, \quad (4.17)$$

with λ given in Eq. (2.61). For our purpose, it will be better to switch to the diagonal gauge where only nonvanishing elements of λ are the diagonal elements. We call Υ the gauge transformed of η_R.

Inspired by the instanton-solutions of $\mathscr{C}P^1$ non-linear σ-model, we shall study the solution of the DS equation where the singular part of the DS potential λ has the following form,

$$\lambda = \begin{pmatrix} \partial\varphi_0 & 0 & 0 & \cdots & 0 \\ 0 & \partial\varphi_1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & \cdots & \partial\varphi_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & \partial\varphi_n \end{pmatrix}, \quad (4.18)$$

$$\partial\varphi_i = \frac{\alpha_i - \alpha_{i+1} + \alpha_S}{z} + O(1)\,, \quad \alpha_0 = \alpha_{n+1} \equiv 0\,. \quad (4.19)$$

This α_S is the reflection of the fact that $gl(1)$ local-scale is not yet fixed. We will calculate the singular behavior of the solution of DS equation $\mathscr{L}\Upsilon = 0$ at $z = 0$ and find the relation between α and the ramification index.

Let us start from the simplest situation, i.e. the $n = 2$ case. Then, the singular part of the DS equation is simply,

$$\partial \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} = \begin{pmatrix} -\alpha_1/z & 1 \\ 0 & \alpha_1/z \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \end{pmatrix}, \tag{4.20}$$

where $(f_k)_i = z^{-\alpha_S}\Upsilon_{ki}$. In the following, we suppress the index k. This equation reduces to the second order scalar differential equation,

$$(\partial^2 + u(z))f_0 = 0\,, \quad u(z) = \frac{1}{z^2}(\alpha_1^2 + \alpha_1) + O\Big(\frac{1}{z}\Big)\,. \tag{4.21}$$

This is a well-known example of the differential equation with regular holonomy. The behavior of its solution $f = \binom{f_0}{f_1}$ at $z = 0$ is determined by

$$f_0 \sim z^{-\alpha_1} \qquad f_1 \sim z^{\alpha_1+1} \qquad \Big(\alpha_1 \neq -\frac{1}{2}\Big)\,, \tag{4.22}$$

$$f_0 \sim z^{1/2} \qquad f_1 \sim z^{1/2}\ln z \qquad \Big(\alpha_1 = -\frac{1}{2}\Big)\,. \tag{4.23}$$

Converting to Υ's, we can choose the overall scaling factor α_S in order to remove the singularities. If we use this solution as the embedding into $\mathscr{C}P^1$, we can get the relation between α_1 and ramification index β_1. We find that

$$\beta_1 = 2\alpha_1 \ (\alpha_1 \geq 0)\,, \quad \beta_1 = 2|\alpha_1 + 1| \ (\alpha_1 \leq -1)\,. \tag{4.24}$$

Some of its features should be noticed,

1. For each ramification index, there are two possible values for α_1.
2. When $\alpha_1 = -1$, although the DS operator has apparent singularity at $z = 0$, corresponding solution is perfectly regular.
3. When $\alpha_1 = -1/2$, the monodromy matrix is not diagonalizable, and this case is not covered by the present analysis. This is the so-called parabolic case.

As for the first point, we have met this situation elsewhere. In the free-field approach to CFT, we represent primary fields by means of vertex operators $\exp(\alpha_1\varphi)$. Its conformal dimension is then given by $\alpha_1(\alpha_1 + 1)$. Hence for each conformal dimension, we have two vertex operators which have the dimension with α_1 and $-1-\alpha_1$. Since our φ fields will be replaced by free boson operator, the two situations have actually same origin. The second point of our remark can be understood similarly. In CFT, we have non-trivial operators with vanishing weight apart from the trivial operator 1. Our DS operator with false singularity is apparently similar. In any case, we can show that these two situations are actually gauge equivalent, which is clear from the fact that they have same potential $u(z)$ in (4.21). In the quantum case, rational theories will indeed lead to rational values for α_1. In the classical case these cuts may be removed by switching to the covering space whenever we meet cuts. This is why we only consider the situations where the α's are integers.

Let us generalize our discussion to the $gl(n+1)$ DS operator. In this case, the singular part of the differential equation which generalizes Eq. (4.20) is given by,

$$\left(\partial - \frac{\alpha_n}{z}\right)\left(\partial + \frac{\alpha_n - \alpha_{n-1}}{z}\right)\cdots\left(\partial + \frac{\alpha_2 - \alpha_1}{z}\right)\left(\partial + \frac{\alpha_1}{z}\right)f_0 = 0 . \tag{4.25}$$

The behavior of the solutions of this equation is given by $f_0 \sim z^{\alpha}$ where α satisfies following characteristic equation,

$$(\alpha + \alpha_1)(\alpha - 1 + \alpha_2 - \alpha_1)\cdots(\alpha - n + 1 + \alpha_n - \alpha_{n-1})(\alpha - n - \alpha_n) = 0 \tag{4.26}$$

If any two solution of this equation are equal, we meet the logarithmic singularities, which do not fit in the present analysis. Otherwise, the behavior of the embedding function is found to be

$$\begin{aligned} f_0 &= O(z^{-\alpha_1}) , & f_1 &= O(z^{1+\alpha_1-\alpha_2}) \\ f_2 &= O(z^{2+\alpha_2-\alpha_3}) , \quad \cdots , & f_n &= O(z^{n+\alpha_n}) . \end{aligned} \tag{4.27}$$

There are $n!$ different sets of α which produce the same ramification index. Since they are all gauge equivalent, we can restrict our analysis to one of them. To specify the choice, we can postulate that f_i/f_{i+1} is regular at $z = 0$ for every i. The ramification index in this situation is simply given by

$$\beta_i = \sum_{j=1}^{n} K_{ij}^{A_n} \alpha_i , \tag{4.28}$$

where $K_{ij}^{A_n}$ is the Cartan matrix of A_n. In this way, we can get a clear group-theoretical correspondence between the ramification-indices of the DS operator. This result should be considered as the local version of the Plücker formula we will encounter in the coming section.

4.5. *Plücker Formulae*

The method will be to exhibit relations between the curvatures $R_{z\bar{z}}^{(k)}$ and metric tensors of the associated embeddings. There are two versions of these formulae. First, as a direct consequence of the Toda equation, we obtain the

Theorem 19. ***Infinitesimal Plücker Formulae.*** *At the regular points of the embedding one has*

$$R_{z\bar{z}}^{(k)}\sqrt{g_{z\bar{z}}^{(k)}} = -g_{z\bar{z}}^{(k+1)} + 2g_{z\bar{z}}^{(k)} - g_{z\bar{z}}^{(k-1)} . \tag{4.29}$$

Proof. This is derived by computing the curvature

$$R_{z\bar{z}}^{(k)}\sqrt{g_{z\bar{z}}^{(k)}} \equiv -\partial\bar{\partial}\ln g_{z\bar{z}}^{(k)} = -\partial\bar{\partial}\ln\left(\frac{\tau_{k+2}\tau_k}{\tau_{k+1}^2}\right) . \tag{4.30}$$

The first equality is a simple consequence of the general form Eq. (4.2) of the Kähler potential. The second is a consequence of the specific mapping of Σ in $G_{n+1,k+1}$ [Eq. (4.3)] which is such that the Toda equation is verified automatically. Making use of Eq. (4.4) one easily completes the proof. Q.E.D.

These infinitesimal Plücker formulae give us the following global relations:

Theorem 20. ***Global Plücker Formulae.*** *The genus of a* W*-surface is related to its instanton-numbers and ramification-indices by the relations*

$$2-2g+\beta_k = 2Q_k - Q_{k+1} - Q_{k-1}\,, \quad \left|\begin{array}{c} k=1,\dots,n \\ Q_{n+1}\equiv 0\,, \quad Q_0 \equiv 0 \end{array}\right. . \tag{4.31}$$

Proof. First we apply the Gauss-Bonnet theorem for each of the k^{th} associated surfaces by computing $\int_{\Sigma_\varepsilon} R^{(k)}_{z\bar z}\sqrt{g^{(k)}_{z\bar z}}$. The integral is first computed over Σ_ε where small neighborhoods of singularities are removed. The ramification indices at singularity was previously defined so that at a singular point the induced metric of the k^{th} associated surface behaves as

$$g^{(k)}_{z\bar z} \sim (z-z_0)^{\beta_k(z_0)}(\bar z-\bar z_0)^{\bar\beta_k(\bar z_0)}\tilde g^{(k)}_{z\bar z}\,, \tag{4.32}$$

where $\tilde g^{(k)}_{z\bar z}$ is regular at z_0, $\bar z_0$. Since we do not assume that $\overline{f(z)} = \bar f(\bar z)$, $\beta_k(z_0)$ and $\bar\beta_k(\bar z_0)$ may be different. By letting $\varepsilon \to 0$, one sees that the contribution of the singularities to the Gauss-Bonnet formula is proportional to the k^{th} ramification index

$$\beta_k \equiv \frac{1}{2}\sum_{(z_0,\bar z_0)\in\Sigma}(\beta_k(z_0)+\bar\beta_k(\bar z_0))\,. \tag{4.33}$$

The contribution of the regular part does not depend upon k, since changing k there, is equivalent to using a different complex structure, while the result is equal to the Euler characteristic that does not depend upon it. The Gauss-Bonnet theorem for the k^{th} associated surface finally gives

$$\frac{i}{2\pi}\int_\Sigma dz\,d\bar z R^{(k)}_{z\bar z}\sqrt{g^{(k)}_{z\bar z}} = 2-2g+\beta_k\,. \tag{4.34}$$

Combining these last relations with Eqs. (4.7), completes the derivation. Q.E.D.

Using these formulae, we find that there are n independent topological numbers $(Q_1,\cdots,Q_n)$, which characterize the global topology of W-surfaces. A direct consequence of this observation is that W_{n+1}-string have n coupling constants which play the same rôle as the genus for the usual string theories. Equation (2.9) may be understood as the index theorem for W-surfaces.

4.6. Relation with Self-Intersection Numbers

A few years ago, Polyakov [21] introduced a modified Goto-Nambu action, with a topological term involving the extrinsic geometry of the string-manifold. In this section, we connect his discussion with the one carried out in the present article.

The Goto-Nambu action, is proportional to $\int dz\,d\bar z\sqrt{-\det(\hat g)}$. $\hat g$ is the induced metric which takes the Kähler form. This gives

$$S_{GN} \propto i\int dz\,d\bar z\,\hat g_{z\bar z} = i\int dz\,d\bar z\sum_A \partial f^A\bar\partial\bar f^A\,, \tag{4.35}$$

which is indeed a topological integral, since $\hat{g}_{z\bar{z}} = \partial\bar{\partial}\sum_A f^A \bar{f}^{\bar{A}}$. As expected it coincides with the instanton-number Q of Eq. (4.6) associated with $\hat{g}$. Since the target space is topologically trivial, Q actually vanishes for any $\mathscr{C}^n$-W-surfaces, contrary to the situation of $\mathscr{C}P^n$-W-surface.

In [21], an additional topological term was defined from the second fundamental form. Let denote by $D \equiv \partial - \partial(\ln \Delta_1)$ and $\bar{D} \equiv \bar{\partial} - \bar{\partial}(\ln \Delta_1)$, the covariant derivatives on the W-surface (with the notations of the previous sections $\hat{g}_{z\bar{z}} = \Delta_1$). It follows from Eq. (2.26) that

$$D(\partial f) = \sum_{a=2}^{n} \Omega^a_{zz} e_a\,, \quad \bar{D}(\bar{\partial}\bar{f}) = \sum_{a=2}^{n} \bar{\Omega}^a_{\bar{z}\bar{z}} \bar{e}_a\,, \tag{4.36}$$

where, according to Eqs. (2.4), and (2.5),

$$\partial f = \sqrt{\Delta_1} e_1 \quad \bar{\partial}\bar{f} = \sqrt{\Delta_1} \bar{e}_1\,. \tag{4.37}$$

The self-intersection number of [21] is

$$\nu_1 = \frac{i}{2\pi} \int dz\, d\bar{z} \Omega^a_{zz} \bar{\Omega}^a_{\bar{z}\bar{z}} (\hat{g}^{z\bar{z}})^2 \sqrt{-\det(\hat{g})} = \frac{i}{2\pi} \int dz\, d\bar{z} \frac{\Delta_2}{\Delta_1^2}\,. \tag{4.38}$$

It was originally defined in the case when the target space is $\mathscr{C}^2$, where a W-surface intersects with itself at several isolated points. The index ν_1 counts the number of these isolated points with suitable signs determined by their mutual orientations at intersection points. This explains the original terminology, "self-intersection number."

From our viewpoint, it is clear that there is a close analogy between ν_1 and Q_1 in Eq. (4.7). The only difference between them is that we need to replace Δ_a by the τ_a. This analogy suggests a re-interpretation of ν_1 as the generalized instanton-number of a certain associated mapping. It enables us to obtain a generalization of that index for the W-surface in higher $\mathscr{C}^n$ target spaces where there seems to be no interpretation of "self-intersection."

This time, for each W-surface in $\mathscr{C}^n$, we define the k^{th} associated mapping as being from Σ into $G_{n,k}$ $(k = 1, \cdots, n-1)$. Each point $z \in \Sigma$, is mapped into an osculating frame spanned by $f^{(1)}, \cdots, f^{(k)}$ at z which defines a point in $G_{n,k}$. Although the original $\mathscr{C}^n$-W-surface has only vanishing instanton-number, these associated mappings give nontrivial indices, which obviously are analogous to those of the $\mathscr{C}P^n$ W-surfaces and of their 1 associated surfaces. The only difference is that in $\mathscr{C}P^n$ case we constructed an osculating frame out of $f^{(0)} = f, \cdots, f^{(k-1)}$. Since Δ_a can be obtained from τ_a by replacing $f^{(a-1)}$ by $f^{(a)}$, we obtain,

Definition 20. ***Generalized 'Intersection Numbers.'*** They are defined by the instanton number ν_k of the k^{th} associated mapping of $\mathscr{C}^n$-W-surface,

$$\nu_k = \frac{i}{2\pi} \int dz\, d\bar{z}\, g^{(k)}_{z\bar{z}} = \frac{i}{2\pi} \int dz\, d\bar{z}\, \partial\bar{\partial}\Delta_k = \frac{i}{2\pi} \int dz\, d\bar{z}\, \frac{\Delta_{k+1}\Delta_{k-1}}{\Delta_k^2}\,. \tag{4.39}$$

These $n-1$ integrals have exactly the same topological meanings as those of higher instanton numbers in CP^{n-1}-W-surface. Polyakov's index corresponds to the *first* index, i.e. the instanton number of the associated mapping into CP^{n-1}.

It is easy to express our new indices out of the third fundamental forms. Rewrite Eq. (2.27), with $a \neq 1$ as

$$\left[D - \frac{1}{2}\sum_{l=1}^{a-1} \partial g_{z\bar{z}}^{(\ell)}\right]\left(\sqrt{\Delta_1} e_a\right) = L_a \sqrt{\Delta_1} e_{a+1},$$

$$\left[\bar{D} - \frac{1}{2}\sum_{l=1}^{a-1} \bar{\partial} g_{z\bar{z}}^{(\ell)}\right]\left(\sqrt{\Delta_1} \bar{e}_a\right) = L_a \sqrt{\Delta_1} \bar{e}_{a+1}, \tag{4.40}$$

Since one has $L_a = \sqrt{g_{z\bar{z}}^{(a)}}$, it follows that the numbers ν_a are such that

$$\nu_a = \frac{i}{2\pi} \int dz\, d\bar{z}\, L_a^2 . \tag{4.41}$$

5. Outlook

This article has gone quite a way towards describing W-geometries. Yet, many problems remain untouched, many more aspects deserve our attention. Let us mention a few of them.

Concerning the A_n case itself, it will be interesting to consider the Poisson-bracket structure and its relation with the Lie-brackets of tangent vectors to $\mathscr{C}P^n$. This should be straightforward. Another point is to derive the light-cone formulation of W-gravity in the present frame-work. It should correspond to a particular parametrization of $\mathscr{C}P^n$.

Clearly the next problem is to derive the other Toda dynamics and WZNW theories from W-geometries.

A much more difficult task is of course to consider quantum W-geometries. It is our expectation that the quantum group structure already exhibited [29, 30] for W gravity will emerge. Indeed, in the present classical discussion, the algebra $gl(n+1)$ plays a crucial rôle. In the same way as for Toda theories [30], it is likely that quantum effects will lead to its associated mathematical "quantum" deformation.

The similarity between $\mathscr{C}P^n$ W-geometry and matrix-models, shows that the method just discussed is very general and may be a unifying framework for all the problems related to conformal theories and strings. In particular it may be convenient in the search for the true string-ground-state.

We may forsee interesting progress in the future.

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A. Appendix

A.1. *Frenet-Serret Equations for $\mathscr{C}P^n$*

In Sect. 2.2, we wrote down the Frenet-Serret formulae for $\mathscr{C}P^n$ through the approach which is covariant by local rescaling. We have seen that the additional Toda field in $\mathscr{C}^{n+1}$ W-surface can be cancelled using this covariance. However, in order to accomplish it, we had to include the zero$^{\text{th}}$ derivative term of the embedding functions to define the moving frame. The correspondence with the second and third fundamental forms became not so direct due to this strategy. Moreover, this method seems to be very specific to the A_n Toda theories, and it is useful to apply the general strategy we put forward in our letter [1].

The standard description of $\mathscr{C}P^n$ makes use of the Fubini-Study metric Eq. (2.35), denoted G_{AB}, together with the so-called inhomogeneous coordinate system, which satisfy

$$X^0 = \bar{X}^0 = 1, . \tag{A.1}$$

Thus the embedding functions are supposed to obey the conditions $f^0(z) = 1$, and $\bar{f}^0(\bar{z}) = 1$. This is clearly not always possible since, starting from any parametrization, one goes to the present one by dividing the coordinates by X^0 or $\bar{X}^0$. This scale choice may be made only if f^0 and $\bar{f}^0$ have no zero. Let us assume that this is true, in the present section, in order to proceed. We define an analogue of the metric tensor Eq. (2.3)

$$\tilde{g}_{i\bar{j}} = \sum_{A,\bar{B}} G_{A\bar{B}} \partial^{(i)} f^A(z) \bar{\partial}^{(\bar{j})} \bar{f}^{\bar{B}}(\bar{z}) . \tag{A.2}$$

The apparent drawback to use the strategy of Sect. 2.1 for the curved space is that the higher order derivatives of the embedding functions do not transform covariantly under the target space reparametrizations. Hence formulae like Eq. (A.2) make sense only when we work with a particular coordinate system similar to the W-parametrization of Sect. 3.2. The interesting point is that every argument in Sect. 2.1 is valid with only minor modifications to this situation, which is to be treated following our earlier general scheme [1]. In the present case, the moving frame case is given by Eqs. (2.4), (2.5) after replacing $g_{i\bar{j}}$ by $\tilde{g}_{i\bar{j}}$. We define $\widetilde{\Delta}$ and $\tilde{\phi}$ as in Eqs. (2.6) and (2.18) by using the same replacement. A minor modification is needed in definition of the derivatives ∂e and $\bar{\partial}\bar{e}$. They should be consistently replaced by the covariant derivative,

$$\begin{aligned} (\nabla e_a)^A &\equiv \sum_B f^{(1)B} \left(\frac{\partial e_a^A}{\partial X^B} + \Gamma^A_{BC} e_a^C \right) , \\ (\bar{\nabla} \bar{e}_a)^{\bar{A}} &\equiv \sum_{\bar{B}} \bar{f}^{(1)\bar{B}} \left(\frac{\partial \bar{e}_a^{\bar{A}}}{\partial \bar{X}^{\bar{B}}} + \Gamma^{\bar{A}}_{\bar{B}\bar{C}} \bar{e}_a^{\bar{C}} \right) , \end{aligned} \tag{A.3}$$

throughout the discussion. This modification is needed in order to keep the condition Eq. (2.13) where the metricity is used. The Christoffel symbols take simple forms for the Fubini-Study metric,

$$\Gamma^A_{BC} = -\frac{\delta_{AC}\bar{X}^B + \delta_{AB}\bar{X}^C}{\sum\limits_{D=0}^{n} X^D \bar{X}^D} , \quad \Gamma^{\bar{A}}_{\bar{B}\bar{C}} = -\frac{\delta_{\bar{A}\bar{C}} X^{\bar{B}} + \delta_{\bar{A}\bar{B}} X^{\bar{C}}}{\sum\limits_{D=0}^{n} X^D \bar{X}^D} . \tag{A.4}$$

The use of covariant derivatives in Eqs. (A.3) ensures the validity of the discussion that leads to Eqs. (2.14). The Gauss-Codazzi equations become, in agreement with [1]

$$[\nabla, \bar{\partial}] e_a = \sum_b \widetilde{F}^b_{z\bar{z}a} e_b , \tag{A.5}$$

$$([\nabla, \bar{\partial}] e_a)^A = \sum f^{(1)B} \bar{f}^{(1)\bar{B}} ([\nabla_{\bar{B}}, \bar{\partial}_{\bar{B}}] e_a)^A = \sum f^{(1)B} \bar{f}^{(1)\bar{B}} \mathscr{R}^b_{B\bar{B}a} e^A_b \tag{A.6}$$

with

$$\widetilde{F}_{z\bar{z}} = \sum_{i=1}^{n} h_i \partial\bar{\partial}\tilde{\phi}_i + \sum_{i=1}^{n-1} h_\imath \exp\left(\sum_{j=1}^{n} K_{ij}^{gl(n+1)} \tilde{\phi}_j\right). \tag{A.7}$$

As we see here, due to the contribution of the target-space curvature, we do not directly get the exact form of the Toda equations. However, $\mathscr{C}P^n$ is known to possess a constant sectional-curvature, that is to say, the curvature tensor satisfies[16]

$$(\bar{a}, \mathscr{R}(b, \bar{c})d) = (\bar{a}, d)(\bar{c}, b) - (\bar{a}, b)(\bar{c}, d). \tag{A.8}$$

Writing

$$\begin{aligned} \widetilde{F}_i &= \partial\bar{\partial}\tilde{\phi}_i + \exp\left(\sum_{j=1}^{n} K_{ij}^{gl(n+1)} \tilde{\phi}_j\right), \quad (i = 1, \ldots, n-1), \\ \widetilde{F}_n &= \partial\bar{\partial}\tilde{\phi}_n . \end{aligned} \tag{A.9}$$

Equation (A.6) is more explicitly rewritten as

$$\begin{aligned} \widetilde{F}_\ell - \widetilde{F}_{\ell-1} &= (\bar{e}_\ell, \mathscr{R}(\partial f, \bar{\partial}\bar{f})e_\ell) = \widetilde{\Delta}_1, \quad \ell = 2, \ldots, n, \\ \widetilde{F}_1 &= \widetilde{\Delta}_1 . \end{aligned} \tag{A.10}$$

Solving these equations, we get

$$\widetilde{F}_\ell = \ell \widetilde{\Delta}_1 . \tag{A.11}$$

The relation between the present discussion and the one of Sect. 2.2 is clarified by establishing the following

Proposition 9. *The relation between the tau-functions and $\widetilde{\Delta}$-functions is given by*

$$\widetilde{\Delta}_\ell = \tau_{\ell+1}/\tau_1^{\ell+1} . \tag{A.12}$$

Proof. First, we observe that the inner product in Eq. (A.2) is given by the free-fermion inner product (here $\mathscr{G}$ stands for $\mathscr{G}(z, \bar{z})$)

$$\begin{aligned} f^{(i)A} \bar{f}^{(\bar{\jmath})\bar{A}} G_{A\bar{A}} &= (\tau_1 \partial^{(i)} \bar{\partial}^{(\bar{\jmath})} \tau_1 - \partial^{(i)}\tau_1 \bar{\partial}^{(\bar{\jmath})}\tau_1)/\tau_1^2 \\ &= \frac{(\langle\emptyset|\psi_0 \mathscr{G} \psi_0^+|\emptyset\rangle\langle\emptyset|\psi_{\bar{\jmath}} \mathscr{G} \psi_i^+|\emptyset\rangle - \langle\emptyset|\psi_{\bar{\jmath}} \mathscr{G} \psi_0^+|\emptyset\rangle\langle\emptyset|\psi_0 \mathscr{G} \psi_i^+|\emptyset\rangle)}{\tau_1^2} \\ &= \frac{\langle 1|\psi_{\bar{\jmath}} \mathscr{G} \psi_i^+|1\rangle}{\tau_1^2} . \end{aligned} \tag{A.13}$$

In the last line, we applied Wick's theorem. Further use of this theorem gives the proof of the proposition,

$$\Delta_a \equiv \tau_1^{-2a} \det_{i,\jmath=1,\ldots,a} \langle 1|\psi_{\bar{\jmath}} \mathscr{G} \psi_i^+|1\rangle = \tau_1^{-a-1}\langle a+1|\mathscr{G}|a+1\rangle = \tau_{a+1}\tau_1^{-a-1}. \tag{A.14}$$

Q.E.D.

This leads us to replace the Toda-like field $\tilde{\phi}_\ell$ by the true Toda field $\Phi_\ell = -\ln \tau_\ell$. It is such that

$$\tilde{\phi}_\ell = \Phi_{\ell+1} - (\ell+1)\Phi_1 . \tag{A.15}$$

[16] The notation b, d ($\bar{a}, \bar{c}$) represents tangent vectors which only have chiral (anti-chiral) nonvanishing components

In terms of this Toda field, Eqs. (A.10) and (A.11) become,

$$\begin{aligned} \partial\bar{\partial}\Phi_{n+1} &= (n+1)(\partial\bar{\partial}\Phi_1 + \exp(2\Phi_2 - \Phi_1))\,, \\ \partial\bar{\partial}\Phi_\ell + \exp(2\Phi_\ell - \Phi_{\ell+1} - \Phi_{\ell-1}) &= \ell(\partial\bar{\partial}\Phi_1 + \exp(2\Phi_2 - \Phi_1))\,, \\ &(\ell = 2, \dots, n)\,. \end{aligned} \tag{A.16}$$

This result becomes closer to the well-known A_n Toda equation. However, the coincidence is still not exact. Clearly the above equations are consequences of Toda equations, that is $\partial\bar{\partial}\Phi_1 + \exp(2\Phi_2 - \Phi_1)) = 0$, $\partial\bar{\partial}\Phi_\ell + \exp(2\Phi_\ell - \Phi_{\ell+1} - \Phi_{\ell-1}) = 0$, $2 \leq \ell \leq n$, $\partial\bar{\partial}\Phi_{n+1} = 0$. However we missed the first and the last. On the other hand, Proposition 9 can be used to complete the derivation. Indeed, it was shown in [2] that, apart from the last one, the above Toda equations are automatically true if the ℓ^{th} Toda field is equal to τ_ℓ. Combining this last observation with Eqs. (A.16), one gets the desired result. Finally, we make use of Proposition 9 to derive the

Proposition 10. ***Liouville Solutions.*** *The induced metric of* W*-surfaces embedded in* $\mathscr{C}P^1$ *satisfies Liouville's equation.*

Proof. The induced metric is $\tilde{g}_{11}$ [see Eq. (A.2)]. According to Proposition 9, and Eq. (2.48), one may write

$$\tilde{g}_{11} = \frac{\tau_2}{\tau_1^2} = \frac{U_0(z)\bar{U}_0(\bar{z})}{\tau_1^2}\,. \tag{A.17}$$

With the present choice of coordinates Eq. (A.1), $f^0(z) = \bar{f}^0(\bar{z}) = 1$, and the Wronskians $U_0(z)$, and $\bar{U}_0(\bar{z})$ are respectively equal to $f^{(1)1}(\bar{z})$. One gets

$$\tilde{g}_{11} = f^{(1)1}(z)\bar{f}^{(1)1}(\bar{z})\Big/(1 + f^1(z)\bar{f}^1(\bar{z}))^2, \tag{A.18}$$

which takes the form of Liouville's general solution. Q.E.D.

This solves the mystery pointed out in Sect. 2.1.

Although the approach of this appendix has drawbacks, it certainly has the merit of being applicable to arbitrary Kähler target-manifolds. Moreover, in this approach the relationship with extrinsic geometry is manifest.

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