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Front Solutions for the Ginzburg–Landau Equation

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Abstract. We prove the existence of front solutions for the Ginzburg–Landau equation

$$\partial_t u(x, t) = \partial_x^2 u(x, t) + (1 - |u(x, t)|^2) u(x, t) ,$$

interpolating between two stationary solutions of the form $u(x) = \sqrt{1 - q^2} e^{iqx}$ with different values of q at $x = \pm \infty$. Such fronts are shown to exist when at least one of the q is in the Eckhaus-unstable domain.

1. Introduction

We consider the Ginzburg–Landau equation (GL)

$$\partial_t u(x, t) = \partial_x^2 u(x, t) + (1 - |u(x, t)|^2) u(x, t) , \tag{1.1}$$

where u is a complex-valued function of $x \in \mathbf{R}$ and $t \in \mathbf{R}_+$. This equation has time-independent periodic solutions of the form

$$u_q(x) = \sqrt{1 - q^2} e^{i\varphi} e^{iqx} , \tag{1.2}$$

where $q \in [-1, 1]$ and $\varphi \in \mathbf{R}$. These stationary solutions are known to be unstable for small amplitudes ($q^2 > 1/3$) and marginally stable for large amplitudes ($q^2 < 1/3$) (*Eckhaus stability*, cf. [CE]).

Our aim is to show the existence of *front solutions* of Eq. (1.1) interpolating between two stationary solutions (1.2). By this, we mean solutions of the form $u(x, t) = U(x, x - ct)$, where $U(x, \xi)$ is a complex function which converges to one of the stationary solutions (1.2), say $u_{q_0}(x)$, as $\xi \to -\infty$ and to another one, say $u_{q_1}(x)$, as $\xi \to +\infty$. Such solutions typically look like a fixed envelope moving to the right with constant velocity $c > 0$, while leaving a periodic pattern (the function u_{q_0}) behind and destroying another one (u_{q_1}) in front, as shown in Fig. 1.

In the case where $u_{q_1} \equiv 0$ ($q_1 = \pm 1$), solutions of this form are easily shown to exist, see e.g., [CE, B]. Indeed, inserting in Eq. (1.1) the ansatz $u(x, t)$

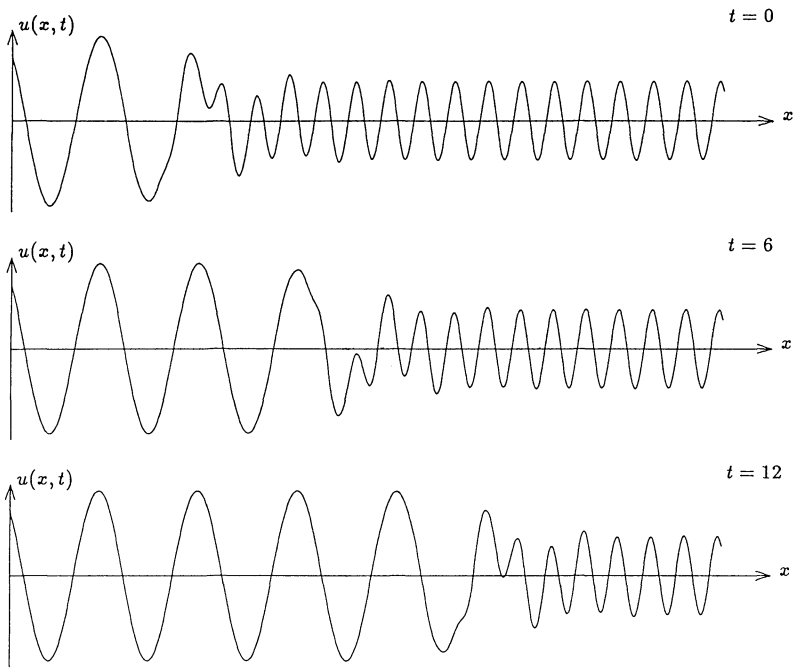

Fig. 1. The real part of a typical front solution of the GL-equation (1.1) is plotted at three different values of t. The parameters are $q_0 = -0.3$, $q_1 = 0.9$ and $c = 5$

$= v(x - ct)e^{iq_0 x}$, where v is a complex-valued function, $q_0 \in (-1, 1)$ and $c > 0$, we obtain an ordinary differential equation for v:

$$v''(\xi) + (c + 2iq_0)v'(\xi) + (1 - q_0^2 - |v(\xi)|^2)v(\xi) = 0 . \tag{1.3}$$

Now, defining

$$H(v, v') = \frac{1}{2}|v'|^2 + \frac{1}{2}(1 - q_0^2)|v|^2 - \frac{1}{4}|v|^4 , \tag{1.4}$$

we can write Eq. (1.3) as a one-dimensional complex Hamiltonian system with Hamiltonian H and (complex) dissipation coefficient $c + 2iq_0$; the fixed points are thus given by the local extrema of the "potential" term in Eq. (1.4). It follows that Eq. (1.3) has a stable fixed point F_1 at $v = 0$, and a circle F_2 of unstable ones ($v = \sqrt{1 - q_0^2}e^{i\varphi}$) which corresponds to the stationary solutions (1.2). In view of the "dissipation law" $dH/d\xi = -c|v'|^2 \leqq 0$, any trajectory entering the region $|v|^2 < 1 - q_0^2$, $H < \frac{1}{4}(1 - q_0^2)^2$, will stay there and converge to the origin. In particular, F_2 is on the boundary of this region, and its unstable manifold intersects this region. Therefore, we can conclude the existence of fronts for Eq. (1.1) connecting any solution (1.2) – no matter whether stable or not – to the origin.

The case where both stationary solutions u_{q_0}, u_{q_1} are non-zero is harder. As a matter of fact, we cannot make the ansatz $u(x, t) = v_0(x - ct)e^{iq_0 x} + v_1(x - ct)e^{iq_1 x}$, for as soon as u contains a superposition of any two different wave-numbers q_0, q_1, the non-linear term $|u|^2 u$ in Eq. (1.1) produces all the "harmonics" $q_n = q_0 + n(q_1 - q_0)$, $n \in \mathbf{Z}$. So, the simplest expression we can hope for is

$$u(x, t) = \sum_{n\in\mathbf{Z}} C_n(x - ct)e^{iq_nx} . \tag{1.5}$$

Inserting in Eq. (1.1), we obtain the following system for the C_n:

$$\begin{aligned} C_n' &= D_n , \\ D_n' &= -(c + 2iq_n)D_n + C_n(q_n^2 - 1) + F_n(C) , \end{aligned} \tag{1.6}$$

where $F = (F_n)_{n\in\mathbf{Z}}$ is the non-linear term

$$F_n(C) = \sum_{p+s+r=n} C_p C_s C_{-r}^* , \tag{1.7}$$

and the symbol $'$ means the derivative with respect to $\xi = x - ct$.

We are thus looking for solutions $C(\xi) = (C_n(\xi))_{n\in\mathbf{Z}}$ of Eq. (1.6) subject to the boundary conditions

$$\lim_{\xi\to-\infty} C(\xi) \in \mathrm{F}_2, \ \lim_{\xi\to+\infty} C(\xi) \in \mathrm{F}_3 , \tag{1.8}$$

where

$$\begin{aligned} \mathrm{F}_2 &= \{(C_n)_{n\in\mathbf{Z}} \,|\, |C_n| = \sqrt{1 - q_0^2}\,\delta_{n,0}\} , \\ \mathrm{F}_3 &= \{(C_n)_{n\in\mathbf{Z}} \,|\, |C_n| = \sqrt{1 - q_1^2}\,\delta_{n,1}\} , \end{aligned}$$

are the circles of fixed points of Eq. (1.6) corresponding to the stationary solutions u_{q_0}, u_{q_1} respectively. More precisely, the question we are interested in is the following: for which $q_0, q_1 \in [-1, 1]$ do solutions of Eqs. (1.6), (1.8) exist? A partial answer is given by the two theorems below, which constitute the main result of this paper.

Theorem 1.1. (*Unstable-Unstable case*) *Let* $0 < \alpha < 1/2$, $c > 0$. *There exists an* $\varepsilon_1 = \varepsilon_1(c) > 0$ *such that, for every* $\varepsilon \leqq \varepsilon_1$, *there is a solution of Eqs.* (1.6), (1.8) *with* $q_0 = -1 + \varepsilon$ *and* $q_1 = 1 - \alpha\varepsilon$. *Moreover,* $\varepsilon_1(c)$ *has a (strictly) positive limit as* $c \to \infty$.

Theorem 1.2. (*Stable-Unstable case*) *Let* $-1/\sqrt{3} < q_0 \leqq 0$. *There exist an* $\varepsilon_1 > 0$ *and a* $c_1 > 0$ *such that, for all* $\varepsilon \leqq \varepsilon_1$ *and all* $c \geqq c_1$, *Eqs.* (1.6), (1.8) *have a solution with* $q_1 = \sqrt{1 - \varepsilon^2}$.

Remark. The problem of constructing front solutions is *phase covariant* in the following sense. The system Eq. (1.6) has two continuous symmetries, which reflect the phase and translation invariance of the GL-equation Eq. (1.1). Indeed, defining the transformation R_φ by $(R_\varphi C)_n = e^{i\varphi}C_n$, we see from Eq. (1.7) that $F \circ R_\varphi = R_\varphi F$ for all $\varphi \in [0, 2\pi]$. Similarly, F commutes with T_δ, where $(T_\delta C)_n = e^{in\delta}C_n$. As a consequence, as soon as any pair of points of F_2 and F_3 are connected by an orbit of Eq. (1.6), the same is true for any other pair, since the two operations R and T allow to rotate the circles F_2, F_3 independently.

We shall briefly comment on the range of validity of the theorems (in q_0, q_1). First of all, a nice application of the Maximum Principle for parabolic equations shows that, if $u(x, t)$ is any solution of Eq. (1.1), the number of zeros of $\mathrm{Re}(u(x, t))$ is (locally in x) non-increasing in time [A]. This means that front solutions can only

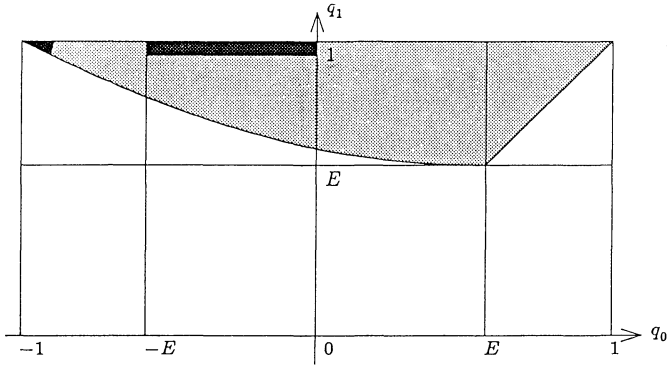

Fig. 2. The shaded region shows the values of q_0, q_1 for which we expect front solutions generically to occur. The black regions are the domains of validity of Theorem 1.1 and Theorem 1.2. The constant $E = 1/\sqrt{3}$ is the threshold of the Eckhaus instability

exist for $|q_1| > |q_0|$, if $c > 0$, as is easily seen from Fig. 1. Moreover, since Eq. (1.1) is invariant under the complex conjugation $u \to u^*$, there is no loss in generality in assuming that $q_1 > |q_0|$. Finally, some genericity considerations which will be explained at the end of Sect. 4.2 lead us to suppose that $(q_0 - q_1)^2 < 6q_1^2 - 2$. Combining these conditions we obtain the shaded region in Fig. 2.

Now, let us choose q_0, q_1 in this shaded region and consider the sequence of wave-numbers $q_n = q_0 + n(q_1 - q_0)$. If the difference $q_1 - q_0$ is sufficiently small, many q_n lie in the interval $[-1, 1]$ and, by Eq. (1.2), there corresponds to each of them a stationary solution u_{q_n}. Thus, as well as between u_{q_0} and u_{q_1}, one can imagine fronts between $u_{q_{-1}}$ and u_{q_1}, $u_{q_{-2}}$ and $u_{q_1}, \ldots$, all of them being solutions of the same system Eq. (1.6) with different boundary conditions (1.8). So, to avoid inessential complications, we restrict ourselves to the case $q_0 \leqq 0$ in which there is only one possibility of constructing a front solution, namely between u_{q_0} and u_{q_1}. In this situation, we expect this solution to exist for all q_0, q_1 in the shaded region, and this is well confirmed by numerical simulations. However, the domain in which we prove it (Theorem 1.1, Theorem 1.2) is much smaller: it is the black region in Fig. 2.

2. Preliminaries

We begin our analysis of the dynamical system Eq. (1.6) by diagonalizing the linear part of the right-hand side. The corresponding operator is already block diagonal with 2×2 blocks labelled by $n \in \mathbf{Z}$; the n^{th} block is just the linear part of the equation for (C_n, D_n), and its eigenvalues are given by

$$\lambda_{n\pm} = \frac{1}{2}\left(-(c + 2iq_n) \pm \sqrt{c^2 - 4 + 4icq_n}\right). \tag{2.1}$$

So, defining the new variables

$$\begin{pmatrix} C_n \\ D_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \lambda_{n+} & \lambda_{n-} \end{pmatrix} \begin{pmatrix} A_n \\ B_n \end{pmatrix}, \tag{2.2}$$

we obtain the following system:

$$\begin{pmatrix} A_n' \\ B_n' \end{pmatrix} = \begin{pmatrix} \lambda_{n+} & 0 \\ 0 & \lambda_{n-} \end{pmatrix} \begin{pmatrix} A_n \\ B_n \end{pmatrix} + \frac{1}{\lambda_{n+} - \lambda_{n-}} \begin{pmatrix} F_n(A+B) \\ -F_n(A+B) \end{pmatrix}, \tag{2.3}$$

where $(A+B)_n \equiv A_n + B_n = C_n$ by Eq. (2.2). In these variables, the circles of fixed points F_2, F_3 become

$$\begin{aligned} &F_2)\ A_0 = \frac{-\lambda_{0-}}{\lambda_{0+} - \lambda_{0-}} \sqrt{1-q_0^2}\, e^{i\varphi}, \quad B_0 = \frac{\lambda_{0+}}{\lambda_{0+} - \lambda_{0-}} \sqrt{1-q_0^2}\, e^{i\varphi}, \quad \varphi \in \mathbf{R}, \\ &F_3)\ A_1 = \frac{-\lambda_{1-}}{\lambda_{1+} - \lambda_{1-}} \sqrt{1-q_1^2}\, e^{i\psi}, \quad B_1 = \frac{\lambda_{1+}}{\lambda_{1+} - \lambda_{1-}} \sqrt{1-q_1^2}\, e^{i\psi}, \quad \psi \in \mathbf{R}, \end{aligned} \tag{2.4}$$

all the other A_n, B_n being equal to zero.

2.1. The Function Space. Let $(\mathscr{H}, (\cdot,\cdot))$ be the Hilbert space

$$\mathscr{H} = \left\{ (A_n)_{n\in\mathbf{Z}} \,\middle|\, \sum_{n\in\mathbf{Z}} (1+|n|)^6 |A_n|^2 < \infty \right\}, \quad (A, \tilde{A}) = \sum_{n\in\mathbf{Z}} (1+|n|)^6 A_n^* \tilde{A}_n,$$

and denote by $\mathscr{H}^2$ the direct sum $\mathscr{H} \oplus \mathscr{H}$. From now on, we mean by a *solution* of the system a $\mathscr{C}^1$ curve $\xi \to (A(\xi), B(\xi))$ in $\mathscr{H}^2$, such that Eq. (2.3) is satisfied. If this is the case, then (by construction) $C_n(\xi) = A_n(\xi) + B_n(\xi)$ is of class $\mathscr{C}^2$ for all n, and Eq. (1.6) is verified. Moreover, since

$$\sum_{n\in\mathbf{Z}} (1+|n|)^2 |C_n(\xi)| \leqq \left(\sum_{n\in\mathbf{Z}} \frac{1}{(1+|n|)^2} \right)^{1/2} \|C(\xi)\|,$$

it is easy to see that $u(x,t)$ defined by the sum (1.5) is $\mathscr{C}^2$ in x and t, and verifies the GL-equation Eq. (1.1).

The space $\mathscr{H}$ is mapped into itself by the non-linear term (1.7). Indeed, a standard result in Sobolev space theory (see e.g., [CE]) says that convolution is a continuous bilinear map from $\mathscr{H} \times \mathscr{H}$ into $\mathscr{H}$. This means that there exists a $K > 0$ such that $\|A * B\| \leqq K \|A\| \, \|B\|$ for all $A, B \in \mathscr{H}$. Now, $F(C)$ is nothing but the double convolution $C * C * \bar{C}$, where $^-: \mathscr{H} \to \mathscr{H}$ is the antilinear isometry defined by $(\bar{C})_n = (C_{-n})^*$. So, $F: \mathscr{H} \to \mathscr{H}$ is $\mathscr{C}^\infty$ and

$$\|F(C) - F(\tilde{C})\| \leqq 3K^2 r^2 \|C - \tilde{C}\|, \tag{2.5}$$

for all $C, \tilde{C} \in \mathscr{H}$ such that $\|C\|, \|\tilde{C}\| \leqq r$. In the sequel, if $\mathscr{E}$ is some normed space and $f: \mathscr{E} \to \mathscr{E}$ some Lipschitz map, we shall denote by $\mathscr{B}_r \subset \mathscr{E}$ the ball of radius r around the origin in $\mathscr{E}$ and by $\mathrm{Lip}(f)$ the Lipschitz constant of f. With these notations, Eq. (2.5) simply means that $\mathrm{Lip}(F) \leq 3K^2r^2$ in $\mathscr{B}_r \subset \mathscr{H}$.

2.2. Spectral Properties and Invariant Manifolds. Figures 3 and 4 show the real part of the spectrum (2.1), plotted as a function of the wave-number q. The points where $q = q_n \equiv q_0 + n(q_1 - q_0)$ for some $n \in \mathbf{Z}$ correspond to the eigenvalues of the system. The two branches (+ and −) cross at $q = 0$ if $c \leq 2$; otherwise, they are separated by a distance growing like c as $c \to \infty$. In all cases, eigenvalues with zero real part only occur if $q_n = \pm 1$ for some $n \in \mathbf{Z}$.

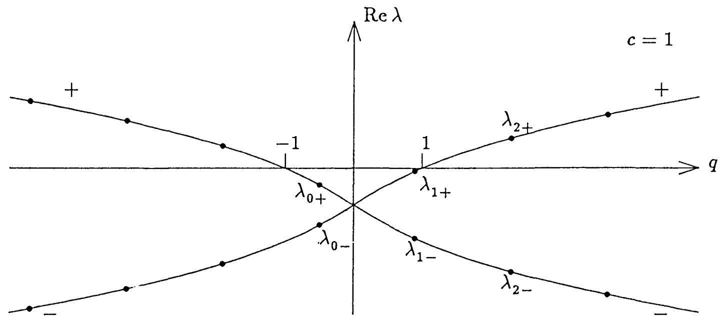

Fig. 3. The real part of the spectrum (2.1), in the case $c < 2$

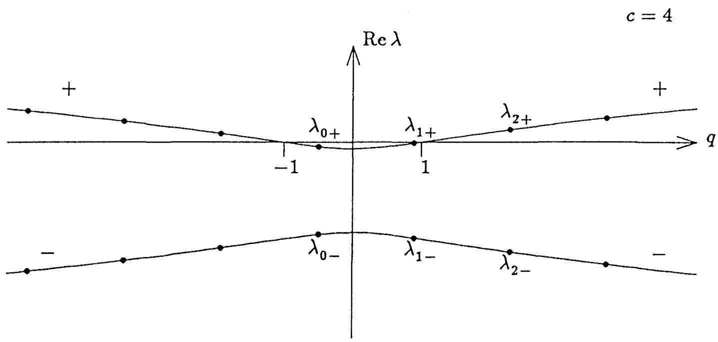

Fig. 4. The real part of the spectrum (2.1), in the case $c > 2$

In view of these spectral properties, our strategies for the proofs of Theorem 1.1 and Theorem 1.2 are very natural. In the "unstable-unstable case" (Sect. 3), we fix $\alpha \in (0, 1/2)$, $c > 0$, and define $q_0 = -1 + \varepsilon$, $q_1 = 1 - \alpha\varepsilon$ for some (small) $\varepsilon > 0$. The spectrum (2.1) thus contains two "central" eigenvalues (that is, $\operatorname{Re}\lambda_{0+}$ and $\operatorname{Re}\lambda_{1+}$ are $\mathcal{O}(\varepsilon)$), while the real parts of all the other ones are bounded away from zero as $\varepsilon \to 0$. Using this information, we consider the evolution of the system (2.3) on the local invariant manifold corresponding to these two central directions. Applying the general theory reported in Appendix A, we shall prove the existence of such a center manifold in a neighborhood of the origin whose size does not depend on ε. Since the circles F_2, F_3 shrink to zero as $\varepsilon \to 0$, all these fixed points will belong to the center manifold if ε is sufficiently small. As a consequence, we shall prove the existence of front solutions connecting F_2 to F_3 by simply studying the resulting flow on the center manifold.

In the "stable-unstable case" (Sect. 4), we choose q_0, q_1 such that $q_0^2 < 1/3$, $1/\sqrt{3} < q_1 < 1$. We do not follow the same procedure as above, because the fixed point F_2 corresponding to q_0 is no longer close to zero, so that we have no guarantee that it would lie on the local center manifold which we would construct. We rather consider the evolution of the system (2.3) on the (infinite-dimensional) invariant manifold corresponding to the upper branch (labelled " + ") of the spectrum. Using c as a parameter, we shall prove the existence of such a center-unstable manifold in a neighborhood of size $\mathcal{O}(c)$ of the origin, thus containing the fixed points F_2, F_3 if c is sufficiently large. We shall then study the resulting semiflow on the manifold, and prove the existence of front solutions.

3. Proof of the Unstable-Unstable Case

As indicated, we fix $\alpha \in (0, 1/2)$, $c > 0$, and define $q_0 = -1 + \varepsilon$, $q_1 = 1 - \alpha\varepsilon$ for some small $\varepsilon > 0$. To avoid complications, we assume from the outset that $\varepsilon \leqq 1/10$.

3.1. Spectral Properties. We first describe in detail the spectrum (2.1) by performing perturbation theory in ε. All calculations are omitted, being straightforward. We find for the two central directions

$$\lambda_{0+} = -\frac{2\varepsilon}{c-2i} + \mathcal{O}\left(\frac{c^2\varepsilon^2}{(c+2)^3}\right), \quad \lambda_{1+} = -\frac{2\alpha\varepsilon}{c+2i} + \mathcal{O}\left(\frac{c^2\varepsilon^2}{(c+2)^3}\right), \tag{3.1}$$

and for the other eigenvalues

$$\operatorname{Re}\lambda_{n+} \geqq \frac{c}{(c+2)^2}, n \neq 0, 1; \quad \operatorname{Re}\lambda_{n-} = -c - \operatorname{Re}\lambda_{n+}, n \in \mathbf{Z}. \tag{3.2}$$

Moreover, using the identity $\lambda_{n+} - \lambda_{n-} = \sqrt{c^2 - 4 + 4icq_n}$, we obtain

$$\frac{1}{|\lambda_{n+} - \lambda_{n-}|} \leqq \frac{2}{c+2}, \quad n \in \mathbf{Z}. \tag{3.3}$$

Finally, for $n = 0, 1$, we also have

$$\frac{1}{\lambda_{0+} - \lambda_{0-}} = \frac{1}{c-2i} + \mathcal{O}\left(\frac{c\varepsilon}{(c+2)^3}\right), \quad \frac{1}{\lambda_{1+} - \lambda_{1-}} = \frac{1}{c+2i} + \mathcal{O}\left(\frac{c\varepsilon}{(c+2)^3}\right). \tag{3.4}$$

3.2. Reduction to the Center Manifold. Using the estimates above, we now reduce the system (2.3) to a center manifold corresponding to the eigenvalues $\lambda_{0+}, \lambda_{1+}$. The first main result of this subsection is:

Proposition 3.1. *There is a $K_0 > 0$ such that, for all $\varepsilon < K_0 c/(c+2)$, the system* (2.3) *defines a flow on a two-dimensional local center manifold of radius $\mathcal{O}(\varepsilon^{1/2})$, which contains the fixed points* F_2, F_3.

Definition. We define $\varepsilon_c = K_0 c/(c+2)$.

Proof. In a first stage, we show the existence of a center-unstable manifold associated with the branch $\{\lambda_{n+}\}_{n\in\mathbf{Z}}$ of the spectrum, by applying Theorem A.1 to the system obained from Eq. (2.3) by reversing the sign of the "time" ξ. Using the notations of Appendix A, we set $\mathscr{E}^{cs} = \{(A_n)_{n\in\mathbf{Z}}\} \cong \mathscr{H}$, $\mathscr{E}^u = \{(B_n)_{n\in\mathbf{Z}}\} \cong \mathscr{H}$, and we define the linear operators A^{cs}, A^u by $(A^{cs}A)_n = -\lambda_{n+}A_n$, $(A^u B)_n = -\lambda_{n-}B_n$. According to Eqs. (3.1), (3.2), we have $\operatorname{Re}(-\lambda_{n+}) \leqq \operatorname{Re}(-\lambda_{0+}) \leqq c/(c^2+4) \leqq c/4$ for all $n \in \mathbf{Z}$, and $\operatorname{Re}(-\lambda_{n-}) \geqq 3c/4$. Thus, the assumption H1 of Theorem A.1 is satisfied with $\lambda^{cs} = c/4$, $\lambda^u = 3c/4$ and $D = 1$ (since A^{cs}, A^u are diagonal). On the other hand, the non-linear terms in Eq. (2.3) are $\mathscr{C}^\infty$, phase covariant, and vanish at the origin together with their derivatives, so that H2 is satisfied. Moreover, if

$\ell^{cs}(r)$, $\ell^{u}(r)$ denote their Lipschitz constants in $\mathscr{B}_r \subset \mathscr{E}^{cs} \oplus \mathscr{E}^{u}$, it follows from Eqs. (2.5), (3.3) that $\ell^{cs}(r), \ell^{u}(r) \leqq Cr^2/(c+2)$ for some $C > 0$. So, choosing $\beta = c/3$ and defining $r_1^2 \equiv K_1 c(c+2)$ for some sufficiently small $K_1 > 0$, we can bound $\sigma(r, \beta)$ in Eq. (A.2) by $\frac{1}{2}(r/r_1)^2$, which is smaller than 1/2 if $r < r_1$. Applying Theorem A.1, we thus obtain:

Lemma 3.2. *There is a* $K_1 > 0$ *such that, for all* $r < r_1 = \sqrt{K_1 c(c+2)}$, *there exists a phase covariant* $\mathscr{C}^{1,1}$ *function* $h: \mathscr{E}^{cs} \to \mathscr{E}^{u}$ *with* $h(0) = 0$, $Dh(0) = 0$, *whose graph (restricted to* $\mathscr{B}_r$*) is a local center-unstable manifold for the system* (2.3). *Moreover,* $\mathrm{Lip}(h) \leqq \frac{1}{2}(r/r_1)^2$.

Thus, the system (2.3) defines on the center-unstable manifold $B = h(A)$ a semi-flow (for $\xi \leqq 0$) whose projection onto $\mathscr{E}^{cs}$ verifies the differential equation

$$A_n' = \lambda_{n+} A_n + \frac{1}{\lambda_{n+} - \lambda_{n-}} F_n(A + h(A)), \quad A \in \mathscr{B}_r \subset \mathscr{H} . \tag{3.5}$$

In a second stage, we reduce the system (3.5) to the two-dimensional center-stable manifold associated with the eigenvalues $\lambda_{0+}, \lambda_{1+}$. Let now $\mathscr{E}^{cs} \equiv \mathscr{H}_0 \cong \mathbf{C}^2$ be the subspace of $\mathscr{H}$ spanned by the two central directions A_0, A_1, and $\mathscr{E}^{u}$ its orthogonal complement in $\mathscr{H}$. Proceeding as above, we define the linear operators $\Lambda^{cs}, \Lambda^{u}$ by $\Lambda^{cs}(A_0, A_1) = (\lambda_{0+} A_0, \lambda_{1+} A_1)$ and $(\Lambda^{u} A)_n = \lambda_{n+} A_n$, $n \neq 0, 1$. In view of Eqs. (3.1), (3.2), the assumption H1 of Theorem A.1 is verified if we take $\lambda^{cs} = 0$, $\lambda^{u} = c/(c+2)^2$, and $D = 1$. On the other hand, denoting by $\ell^{cs}(r)$, $\ell^{u}(r)$ the Lipschitz constants in $\mathscr{B}_r \subset \mathscr{H}$ of the non-linear terms in Eq. (3.5), we see from Eqs. (2.5), (3.3) and Lemma 3.2 that H2 is satisfied, and that $\ell^{cs}(r), \ell^{u}(r) \leqq Cr^2/(c+2)$ for some $C > 0$. So, choosing $\beta = \lambda^{u}/3$ and defining $r_2^2 = K_2 c/(c+2)$ for some sufficiently small $K_2 \leqq K_1$, we have $\sigma(r, \beta) \leqq \frac{1}{2}(r/r_2)^2$ in Eq. (A.2). Applying Theorem A.1, we obtain:

Lemma 3.3. *There is a* $K_2 > 0$ *such that, for all* $r < r_2 = \sqrt{K_2 c/(c+2)}$, *there exists a phase covariant* $\mathscr{C}^{1,1}$ *function* $g: \mathscr{E}^{cs} \to \mathscr{E}^{u}$ *with* $g(0) = 0$, $Dg(0) = 0$, *whose graph (restricted to* $\mathscr{B}_r$*) is a local center-stable manifold for the system* (3.5). *Moreover,* $\mathrm{Lip}(g) \leqq \frac{1}{2}(r/r_2)^2$.

Combining the two lemmas we obtain the existence, if $r < r_2$, of the local center manifold $\Gamma_r = \{(A, B) \in \mathscr{H}^2 \mid A = (a, g(a)), B = h(A), a \in \mathscr{B}_r \subset \mathscr{H}_0\}$. Furthermore, the projection onto $\mathscr{H}_0$ of the flow defined on Γ_r by Eq. (2.3) verifies the differential equation

$$\begin{aligned} A_0' &= \lambda_{0+} A_0 + \frac{1}{\lambda_{0+} - \lambda_{0-}} F_0(a + k(a)) , \\ A_1' &= \lambda_{1+} A_1 + \frac{1}{\lambda_{1+} - \lambda_{1-}} F_1(a + k(a)) , \end{aligned} \tag{3.6}$$

where $a = (A_0, A_1) \in \mathscr{B}_r \subset \mathscr{H}_0$ and $k: \mathscr{H}_0 \to \mathscr{H}$ is defined by the identity $(a, g(a)) + h(a, g(a)) = a + k(a)$. So, k is phase covariant, $k(0) = 0$, $Dk(0) = 0$, and $\mathrm{Lip}(k) \leqq \ell (r/r_2)^2$ for some $\ell > 0$.

We now complete the proof of Proposition 3.1. Consider the fixed points (2.4). Using Eqs. (3.1), (3.3) and recalling that $\sqrt{1 - q_1^2}, \sqrt{1 - q_0^2}$ are $\mathcal{O}(\varepsilon^{1/2})$, we see that there exists an $R > 0$ such that $\mathrm{F}_2, \mathrm{F}_3 \in \mathscr{B}_{R\varepsilon^{1/2}} \subset \mathscr{H}^2$ for all $\varepsilon > 0$. Thus, if

$R\varepsilon^{1/2} < r_2$, these fixed points will lie on the center manifold $\Gamma_{R\varepsilon^{1/2}}$ (see Remark 3 after Theorem A.1). Thus, defining $\varepsilon_c = (r_2/R)^2 = K_0 c/(c+2)$ and noting that $(R\varepsilon^{1/2}/r_2)^2 \equiv \varepsilon/\varepsilon_c$, the proof of Proposition 3.1 is complete. □

The proof gives us the further result:

Corollary 3.4. *The projection onto* $\mathscr{H}_0 \cong \mathbf{C}^2$ *of the flow defined on the center manifold by Eq.* (2.3) *verifies the differential equation Eq.* (3.6) *with* $k: \mathscr{H}_0 \to \mathscr{H}$ *a phase covariant* $\mathscr{C}^{1,1}$ *function verifying* $k(0) = 0$, $Dk(0) = 0$, $\mathrm{Lip}(k) \leqq \ell\varepsilon/\varepsilon_c$.

This result reduces the proof of Theorem 1.1 to the study of the system (3.6) in the ball $\mathscr{B}_{R\varepsilon^{1/2}} \subset \mathscr{H}_0$. In order to extract the relevant terms as $\varepsilon \to 0$, we rescale the amplitudes A_0, A_1 and the parameter ξ, by defining $\eta = -\varepsilon\xi$ and setting $A_0(\xi) = \sqrt{\varepsilon} X_0(\eta)$, $A_1(\xi) = \sqrt{\varepsilon} X_1(\eta)$, $k(\sqrt{\varepsilon} x) = \sqrt{\varepsilon}\, l(x)$ for all $x \in \mathscr{H}_0$. We thus obtain the new system

$$X_0' = -\frac{\lambda_{0+}}{\varepsilon} X_0 - \frac{1}{\lambda_{0+} - \lambda_{0-}} F_0(x + l(x)) ,$$

$$X_1' = -\frac{\lambda_{1+}}{\varepsilon} X_1 - \frac{1}{\lambda_{1+} - \lambda_{1-}} F_1(x + l(x)) , \tag{3.7}$$

where $x = (X_0, X_1) \in \mathscr{B}_R \subset \mathscr{H}_0$ and $'$ denotes the derivative with respect to η. By construction, the function $l: \mathscr{H}_0 \to \mathscr{H}$ has the same properties as k in Corollary 3.4. In view of Eqs. (1.7), (3.1), (3.4) and Corollary 3.4, the formal limit $\varepsilon \to 0$ in Eq. (3.7) yields the simple equations

$$X_0' = \frac{1}{c - 2i} X_0(2 - |X_0|^2 - 2|X_1|^2) ,$$

$$X_1' = \frac{1}{c + 2i} X_1(2\alpha - |X_1|^2 - 2|X_0|^2) . \tag{3.8}$$

We shall study these equations in the next subsection, and come back to the case $\varepsilon > 0$ in Sect. 3.4.

3.3. The Limiting Case $\varepsilon = 0$. We now study the reduced system (3.8) and show that it has front solutions. This system has two circles of fixed points corresponding to Eq. (2.4):

$$\mathrm{F}_2^0 = \{|X_0| = \sqrt{2}, X_1 = 0\}, \quad \mathrm{F}_3^0 = \{X_0 = 0, |X_1| = \sqrt{2\alpha}\} .$$

Lemma 3.5. *Every point of* F_3^0 *is connected by an orbit to* F_2^0.

Proof. We begin by setting $X_0 = \rho_0 e^{i\psi_0}$, $X_1 = \rho_1 e^{i\psi_1}$, with $\rho_0, \rho_1 \in \mathbf{R}_+$ and $\psi_0, \psi_1 \in \mathbf{R}$. Inserting in Eq. (3.8), we obtain the following equations for the amplitudes ρ_0, ρ_1:

$$(c + 4/c)\rho_0' = \rho_0(2 - \rho_0^2 - 2\rho_1^2), \quad (c + 4/c)\rho_1' = \rho_1(2\alpha - \rho_1^2 - 2\rho_0^2) . \tag{3.9}$$

The equations for the phases ψ_0, ψ_1 can be explicitly integrated and yield the relations $\rho_0 = C_0 e^{c\psi_0/2}$, $\rho_1 = C_1 e^{-c\psi_1/2}$, where C_0, C_1 are positive constants determined by the initial conditions. We next eliminate the parameter c from Eq. (3.9) by the transformation $r_0(\eta) = \rho_0((c + 4/c)\eta), r_1(\eta) = \rho_1((c + 4/c)\eta)$, leading to the system

$$r'_0 = r_0(2 - r_0^2 - 2r_1^2), \quad r'_1 = r_1(2\alpha - r_1^2 - 2r_0^2) . \tag{3.10}$$

Since $0 < \alpha < 1/2$ and $r_0 \geqq 0$, $r_1 \geqq 0$, it is straightforward to verify that Eq. (3.10) has exactly three fixed points: $F_1 = (0, 0)$ (a source), $F_2 = (\sqrt{2}, 0)$ (a sink), and $F_3 = (0, \sqrt{2\alpha})$ (a saddle).

We now prove the existence of a trajectory $(r_0, r_1)(\eta)$ of Eq. (3.10) leaving F_3 at $\eta = -\infty$ and reaching F_2 at $\eta = +\infty$. We will do this by showing that the (one-dimensional) unstable manifold $\mathscr{W}$ of F_3 lies in the basin of attraction of F_2. In order to do that, we consider the (closed) domain D in $\mathbf{R}_+ \times \mathbf{R}_+$ bounded by the two curves

$$E_+ = \{(r_0, r_1) | r_0^2 + 2r_1^2 = 2\}, \quad E_- = \{(r_0, r_1) | r_1^2 + 2r_0^2 = 2\alpha\} ,$$

as shown in Fig. 5.

Elementary calculations show that, on both E_+, E_-, the vector field (3.10) points toward the interior of D, whereas it is parallel to the boundary on the two remaining segments of ∂D; this means that the interior $\mathring{D}$ of D is invariant under the flow of Eq. (3.10). Moreover, it is easy to verify that $r'_0 > 0$ and $r'_1 < 0$ everywhere in $\mathring{D}$. As a consequence, since D is compact, any trajectory in $\mathring{D}$ necessarily converges to some (fixed) point in D as $\eta \to \infty$, and by elimination this fixed point must be F_2. So, it remains to show that the unstable manifold $\mathscr{W}$ of F_3 intersects $\mathring{D}$. Writing $\mathscr{W} = \{(r_0, f(r_0)) | r_0 > 0\}$ and $E_- = \{(r_0, \hat{f}(r_0)) | r_0 > 0\}$ near $r_0 = 0$, we easily find

$$f(r_0) = \sqrt{2\alpha}\left(1 - \frac{1}{2}\frac{r_0^2}{1 - \alpha} + \mathcal{O}(r_0^4)\right), \quad \hat{f}(r_0) = \sqrt{2\alpha}\left(1 - \frac{1}{2}\frac{r_0^2}{\alpha} + \mathcal{O}(r_0^4)\right) .$$

Since $1 - \alpha > \alpha$, we have $f(r_0) > \hat{f}(r_0)$ if r_0 is sufficiently small, so that $\mathscr{W}$ lies in $\mathring{D}$ in a neighborhood of F_3. □

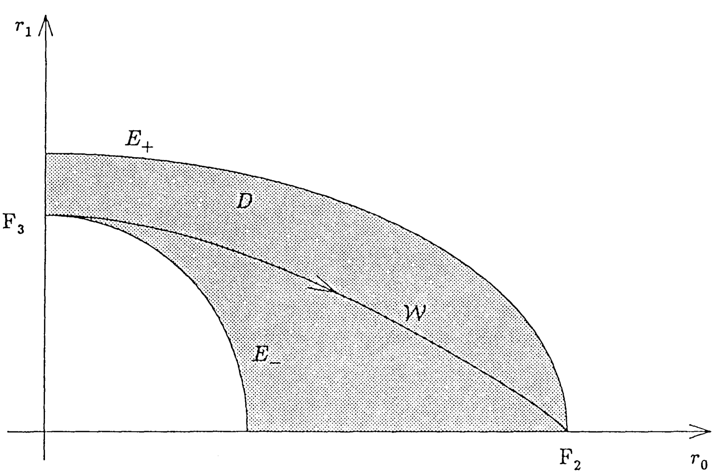

Fig. 5. The domain D of $\mathbf{R}_+ \times \mathbf{R}_+$ bounded by the two ellipses E_+, E_- (shaded region) is attracted to F_2 by the flow of (3.10). In particular, the unstable manifold $\mathscr{W}$ of F_3 intersects F_2

Remark. The condition $\alpha < 1/2$ is essential in this argument: if $1/2 < \alpha < 2$, both fixed points F_2, F_3 are stable, so that no connection can occur between them. Note that the slope of the tangent to the curve $(q_0 - q_1)^2 = 6q_1^2 - 2$ at $q_0 = -1, q_1 = 1$ is exactly $\frac{1}{2}$, see Fig. 2.

3.4. The Case $\varepsilon > 0$. We now come back to the full equations (3.7) and prove, by a perturbation argument, the existence of front solutions for sufficiently small ε, i.e., Theorem 1.1. Although this could be done by direct estimates in this simple finite-dimensional case, we shall use the general methods of Appendix A, as a preparation for the infinite-dimensional situation of Sect. 4.

The perturbation argument is a comparison of the flows Φ_η^ε, Φ_η^0 defined by the vector fields χ^ε, χ^0 of Eqs. (3.7), (3.8) respectively. By construction, the flow Φ_η^ε has two circles of fixed points:

$$\mathrm{F}_2^\varepsilon)\quad X_0 = \frac{-\lambda_{0-}}{\lambda_{0+} - \lambda_{0-}}\sqrt{2 - \varepsilon}\, e^{i\varphi}, \qquad X_1 = 0, \quad \varphi \in \mathbf{R}\,,$$

$$\mathrm{F}_3^\varepsilon)\quad X_1 = \frac{-\lambda_{1-}}{\lambda_{1+} - \lambda_{1-}}\sqrt{2\alpha - \alpha^2\varepsilon}\, e^{i\psi}, \; X_0 = 0, \quad \psi \in \mathbf{R}\,.$$

Using Eqs. (3.1) and (3.3) it is easy to see that $\mathrm{dist}(\mathrm{F}_2^\varepsilon, \mathrm{F}_2^0)$ and $\mathrm{dist}(\mathrm{F}_3^\varepsilon, \mathrm{F}_3^0)$ are $\mathcal{O}(\varepsilon)$ as $\varepsilon \to 0$.

The main steps of the proof are:

i) The flows depend continuously on ε at $\varepsilon = 0$.
ii) The stable and unstable manifolds of F_2^ε, F_3^ε are continuous in ε at $\varepsilon = 0$.

We begin by comparing the flows.

Proposition 3.6. *There exists a* $K_3 > 0$ *such that*

$$\|\Phi_\eta^\varepsilon(x) - \Phi_\eta^0(y)\| \leqq \exp\left(\frac{K_3\eta}{c+2}\right)(\|x - y\| + K_3(\varepsilon/\varepsilon_c))\,, \tag{3.11}$$

for all $x, y \in \mathscr{B}_R \subset \mathscr{H}_0$ *and all* $\eta \in \mathbf{R}_+$.

Proof. We first note that χ^ε, χ^0 are close to each other in the Lipschitz norm:

Lemma 3.7. *Let* $\Delta\chi(x) = \chi^\varepsilon(x) - \chi^0(x)$, $x \in \mathscr{H}_0$. *Then, there exists a* $K_4 > 0$ *such that* $\mathrm{Lip}(\Delta\chi) \leqq (\varepsilon/\varepsilon_c)K_4/(c+2)$ *in* $\mathscr{B}_{2R} \subset \mathscr{H}_0$.

The proof is a calculation which can be found in Appendix B.

We next write Eq. (3.7) in the form $x' = \chi^0(x) + \Delta\chi(x)$, regarding $\Delta\chi$ as an additional non-linear term. From this point of view, the systems (3.7), (3.8) have the same linear part, with spectrum contained in the half-plane $\mathrm{Re}(z) \leqq 2c/(c^2+4)$. The non-linear part of χ^0 has (in $\mathscr{B}_{2R}$) a Lipschitz constant bounded by $C/(c+2)$, for some $C > 0$, and by Lemma 3.7 the same is true for $\chi^\varepsilon = \chi^0 + \Delta\chi$, with C replaced by $C + K_4(\varepsilon/\varepsilon_c)$. So, setting $\mathscr{E}^{cs} = \mathscr{H}_0, \mathscr{E}^u = \{0\}$, $\lambda^{cs} = 2c/(c^2+4)$, $D = 1$, and $\ell^{cs}(2R) = (C + K_4)/(c+2)$, we can apply Theorem A.1 to both systems simultaneously, the condition (A.4) being fulfilled with $\beta = K_3/(c+2)$ for some sufficiently large $K_3 > 0$. It follows that $\Phi^\varepsilon, \Phi^0 \in \mathscr{K}_\beta$, and in particular we have $\|\Phi_\eta^\varepsilon(x) - \Phi_\eta^\varepsilon(y)\| \leqq e^{\beta\eta}\|x - y\|$ for all $x, y \in \mathscr{B}_R$ and all $\eta \in \mathbf{R}_+$. Now, we apply

Theorem A.2 to compare Φ^ε with Φ^0, the condition (A.5) being fulfilled with $A_1^{cs} = A_2^{cs}$, $f_1^{cs} - f_2^{cs} = \Delta\chi$, $\delta = 0$ and $\varepsilon \to \varepsilon/\varepsilon_c$. By Eq. (A.6), we thus have $\| \Phi_\eta^\varepsilon(y) - \Phi_\eta^0(y) \| \leqq (1 - 2\sigma)^{-1} R(\varepsilon/\varepsilon_c) e^{\beta\eta}$ for all $y \in \mathscr{B}_R$ and all $\eta \in \mathbf{R}_+$. Combining these results, we obtain Eq. (3.11). □

We now study the stable and unstable manifolds of F_2^ε, F_3^ε.

Lemma 3.8. *There is a $K_5 > 0$ such that, for $\varepsilon/\varepsilon_c \leqq K_5 c/(c+2)$, the annulus of radius $\rho_2 = K_5 c/(c+2)$ around the circle F_2^ε is attracted to F_2^ε by the flow Φ_η^ε.*

Proof. We first study the geometry in the case $\varepsilon = 0$. Let $x^0 = (\sqrt{2}, 0) \in \mathrm{F}_2^0$ and set $x = x^0 + z$, with $z = (Z_0, Z_1) \in \mathscr{B}_{2\rho} \subset \mathscr{H}_0$ for sufficiently small $\rho > 0$. Inserting in Eq. (3.8), we obtain

$$Z_0' = \frac{-2}{c-2i}(Z_0 + Z_0^*) + f_0(z), \quad Z_1' = \frac{2\alpha - 4}{c+2i} Z_1 + f_1(z)\,, \tag{3.12}$$

where $f: \mathscr{H}_0 \to \mathscr{H}_0$ is $\mathscr{C}^{1,1}$ and $\mathrm{Lip}(f) \leqq C\rho/(c+2)$ in $\mathscr{B}_{2\rho}$ for some $C > 0$. As is easily verified, the linear part of Eq. (3.12) (regarded as an operator in $\mathbf{R}^4$) has one zero and three stable eigenvalues. The gap between the stable and the central part of the spectrum is equal to $2c/(c^2+4)$, and if V_s, V_c denote the corresponding eigenspaces, then $x^0 + V_c$ is just the tangent to the circle F_2^0 at x^0, and V_s is the normal hyperplane. This situation prevails for small ε. To see this, denote by x^ε the unique point of F_2^ε for which $X_0 \in \mathbf{R}_+$. Setting now $x = x^\varepsilon + z$ and inserting in Eq. (3.7), we obtain Eq. (3.12), with $f^\varepsilon(z) = f(z) + \delta\chi(z)$ replacing $f(z)$, where $\delta\chi(z) = \chi^\varepsilon(x^\varepsilon + z) - \chi^0(x^0 + z)$. In Appendix B, we prove:

Lemma 3.9. *There exists a $K_6 > 0$ such that $\mathrm{Lip}(\delta\chi) \leqq (\varepsilon/\varepsilon_c) K_6/(c+2)$ in $\mathscr{B}_R \subset \mathscr{H}_0$.*

Using this bound, we can compare the two systems (i.e., Eq. (3.12) with f or f^ε in the right-hand side) in the ball $\mathscr{B}_\rho \subset \mathscr{H}_0$. Setting $\mathscr{E}^{cs} = V_s$, $\mathscr{E}^u = V_c$, $\lambda^{cs} = -2c/(c^2+4)$, $\lambda^u = 0$, $D = 1$ and $\ell^{cs}(2\rho)$, $\ell^u(2\rho) = (C\rho + K_6\varepsilon/\varepsilon_c)/(c+2)$, the condition (A.3) can be fulfilled if $\varepsilon/\varepsilon_c \leqq \rho$ and $\rho \leqq \rho_2 = K_5 c/(c+2)$ for some (sufficiently small) $K_5 > 0$. Thus, by Theorem A.1, there exist $h^0, h^\varepsilon : \mathscr{B}_{\rho_2} \subset V_s \to V_c$ whose graphs $\mathscr{V}^0$, $\mathscr{V}^\varepsilon$ are (except for a translation) the local stable manifolds of x^0, x^ε for the flows Φ_η^0, Φ_η^ε respectively. Moreover, applying Theorem A.2 with $A_1 = A_2$, $f_1 - f_2 = \delta\chi$, $\delta = 0$ and $\varepsilon \to \varepsilon/\varepsilon_c$, we easily see that $\| h^\varepsilon - h^0 \|_\infty = \mathscr{O}(\varepsilon/\varepsilon_c)$ in $\mathscr{B}_{\rho_2} \subset V_s$.

Note that, in view of the phase covariance of the system, we can obtain from $\mathscr{V}^\varepsilon$ the corresponding stable manifold of any point of F_2^ε by simply applying the transformation $X_0 \to X_0 e^{i\varphi}$, $\varphi \in \mathbf{R}$. The union over $\varphi \in [0, 2\pi]$ of these manifolds is the annulus of radius ρ_2 around the circle F_2^ε. This completes the proof of Lemma 3.8. □

These considerations about the behavior of Φ_η^0, Φ_η^ε near F_2^0, F_2^ε can be repeated in an analogous manner for F_3^0, F_3^ε. Choosing $y^0 = (0, \sqrt{2\alpha}) \in \mathrm{F}_3^0$ and setting $x = y^0 + z$, we obtain instead of Eq. (3.12),

$$Z_0' = \frac{2-4\alpha}{c-2i} Z_0 + g_0(z), \quad Z_1' = \frac{-2\alpha}{c+2i}(Z_1 + Z_1^*) + g_1(z)\,, \tag{3.13}$$

where again $\mathrm{Lip}(g) \leqq C\rho/(c+2)$ in $\mathscr{B}_{2\rho} \subset \mathscr{H}_0$. Recalling that $\alpha < 1/2$, one verifies that the linear part has one stable, one zero and two unstable eigenvalues. The gap between the center-stable and the unstable part of the spectrum is equal to $(2-4\alpha)c/(c^2+4)$, and if W_{cs}, W_u denote the corresponding eigenspaces, then W_u is just the complex line $\{Z_1=0\}$. In the same way, choosing $y^\varepsilon \in \mathrm{F}_3^\varepsilon$ as above and setting $x=y^\varepsilon+z$, we obtain Eq. (3.13) with g replaced by some g^ε such that $\mathrm{Lip}(g^\varepsilon-g) \leqq K_6(\varepsilon/\varepsilon_c)/(c+2)$. So, reversing the sign of the "time" η and setting $\mathscr{E}^{cs}=W_u$, $\mathscr{E}^u=W_{cs}$, $\lambda^{cs}=-(2-4\alpha)c/(c^2+4)$, $\lambda^u=0$, $D=1$, and $\ell^{cs}(2\rho)$, $\ell^u(2\rho)=(C\rho+K_6\varepsilon/\varepsilon_c)/(c+2)$, the condition (A.3) can be fulfilled if $\varepsilon/\varepsilon_c \leqq \rho$ and $\rho \leqq \rho_3 = K_5(1-2\alpha)c/(c+2)$. Thus, there exist $k^0, k^\varepsilon : \mathscr{B}_{\rho_3} \subset W_u \to W_{cs}$ whose graphs $\mathscr{W}^0$, $\mathscr{W}^\varepsilon$ are (except for a translation) the local unstable manifolds of y^0, y^ε; moreover $\| k^\varepsilon - k^0 \|_\infty = \mathcal{O}(\varepsilon/\varepsilon_c)$ in $\mathscr{B}_{\rho_3} \subset W_u$.

Having gained control near the circle of fixed points, we can conclude the proof of Theorem 1.1 by following the flow in the space in between, using Proposition 3.6. Assuming that $\varepsilon/\varepsilon_c \leqq \rho_3$, we choose two points in the unstable manifolds of F_3^ε and F_3^0, defined by

$$P^\varepsilon = y^\varepsilon + (\rho_3, k^\varepsilon(\rho_3)) \in y^\varepsilon + \mathscr{W}^\varepsilon, \quad P^0 = y^0 + (\rho_3, k^0(\rho_3)) \in y^0 + \mathscr{W}^0 .$$

By construction, $\mathrm{dist}(P^\varepsilon, P^0) \leqq \| y^\varepsilon - y^0 \| + |k^\varepsilon(\rho_3) - k^0(\rho_3)| = \mathcal{O}(\varepsilon/\varepsilon_c)$ as $\varepsilon \to 0$. On the other hand, since P^0 lies in the unstable manifold of F_3^0, we have seen in Sect. 3.3 that $\Phi_\eta^0(P^0)$ converges to F_2^0 as $\eta \to \infty$. Thus, there exists an $\eta > 0$ such that $\mathrm{dist}(\Phi_\eta^0(P^0), \mathrm{F}_2^0) \leqq \rho_2/3$, and it follows from Eq. (3.9) that $\eta = (c+4/c)T$, where $T = T(\rho_2, \rho_3)$ does not depend explicitly on c. Finally, we know from Eq. (3.11) that

$$\mathrm{dist}(\Phi_\eta^\varepsilon(P^\varepsilon), \Phi_\eta^0(P^0)) \leqq \exp\left(K_3 \frac{c+4/c}{c+2} T \right) (\mathrm{dist}(P^\varepsilon, P^0) + K_3(\varepsilon/\varepsilon_c)) .$$

Now, let us choose ε so small that $\mathrm{dist}(\Phi_\eta^\varepsilon(P^\varepsilon), \Phi_\eta^0(P^0))$ and $\mathrm{dist}(\mathrm{F}_2^\varepsilon, \mathrm{F}_2^0)$ are smaller than $\rho_2/3$. Then, by the triangle inequality

$$\begin{aligned} \mathrm{dist}(\Phi_\eta^\varepsilon(P^\varepsilon), \mathrm{F}_2^\varepsilon) &\leqq \mathrm{dist}(\Phi_\eta^\varepsilon(P^\varepsilon), \Phi_\eta^0(P^0)) + \mathrm{dist}(\Phi_\eta^0(P^0), \mathrm{F}_2^0) + \mathrm{dist}(\mathrm{F}_2^0, \mathrm{F}_2^\varepsilon) \\ &\leqq \rho_2 . \end{aligned}$$

In view of Lemma 3.8, this means that $\Phi_\eta^\varepsilon(P^\varepsilon) \to \mathrm{F}_2^\varepsilon$ as $\eta \to \infty$, while $\Phi_\eta^\varepsilon(P^\varepsilon) \to y^\varepsilon \in \mathrm{F}_3^\varepsilon$ as $\eta \to -\infty$ since $P^\varepsilon \in y^\varepsilon + \mathscr{W}^\varepsilon$. Thus, we have shown the existence of a solution of Eq. (3.7) connecting $y^\varepsilon \in \mathrm{F}_3^\varepsilon$ to some point of F_2^ε. The various assumptions on ε can be summarized by the single condition $\varepsilon \leqq \varepsilon_1(c)$, where

$$\varepsilon_1(c) = K_7 \left(\frac{c}{c+2} \right)^2 \exp\left(-K_3 \frac{c+2}{c} T(\rho_2, \rho_3) \right),$$

for some $K_7 > 0$. Since ρ_2, ρ_3 have positive limits as $c \to \infty$, so does $\varepsilon_1(c)$. This concludes the proof of Theorem 1.1. □

4. The Stable-Unstable Case

We now study the more interesting case where one of the stationary solutions is (Eckhaus) stable and the other unstable, i.e., we choose two wave-numbers q_0, q_1 such that $q_0^2 < 1/3$, $1/\sqrt{3} < q_1 < 1$. We follow the procedure announced in Sect. 2.2.

4.1. Reduction to the Center-Unstable Manifold. We first investigate the behavior of the spectrum (2.1) as $c \to \infty$. If $n \in \mathbf{Z}$ is fixed and if $c \geqq 8(1 + |q_n|)$, we find by straightforward calculations

$$\lambda_{n+} = \frac{1}{c}(q_n^2 - 1) + \frac{1}{c^2}\mathscr{R}(c, q_n)\,, \tag{4.1}$$

where $|\mathscr{R}(c, q_n)| \leqq 5(1 + |q_n|)^3$. On the other hand, if $c \geqq 2$, it is easy to see that $\operatorname{Re} \lambda_{n+} \geqq -\hat{\lambda}$ for all $n \in \mathbf{Z}$, where

$$\hat{\lambda} = \frac{1}{2}(c - \sqrt{c^2 - 4}) = \frac{1}{c} + \mathcal{O}\left(\frac{1}{c^3}\right). \tag{4.2}$$

Since $\lambda_{n+} + \lambda_{n-} = -(c + 2iq_n)$, it follows that $\operatorname{Re} \lambda_{n-} \leqq -c + \hat{\lambda}$, so that the gap between the two branches of the spectrum is greater than $c - 2\hat{\lambda} = \sqrt{c^2 - 4}$. As a consequence, we have for all $n \in \mathbf{Z}$,

$$\frac{1}{|\lambda_{n+} - \lambda_{n-}|} \leqq \frac{1}{\sqrt{c^2 - 4}} = \mathcal{O}\left(\frac{1}{c}\right). \tag{4.3}$$

Finally, if n is fixed and $c \geqq 8(1 + |q_n|)$, we find

$$\left|\frac{1}{\lambda_{n+} - \lambda_{n-}} - \frac{1}{c}\right| \leqq \frac{4}{c^2}(1 + |q_n|)\,. \tag{4.4}$$

We now follow exactly the same procedure as in Sect. 3.2: reversing the sign of the time ξ in (2.3), we apply Theorem A.1 to show the existence of a center-stable manifold corresponding to the branch $\{-\lambda_{n+}\}_{n \in \mathbf{Z}}$ of the spectrum. Using the same notation, we take $\mathscr{E}^{cs} = \mathscr{E}^u = \mathscr{H}$, $\lambda^{cs} = \hat{\lambda}$, $\lambda^u = c - \hat{\lambda}$, $D = 1$, and $\beta = c/3$. Moreover, if c is sufficiently large, we see from Eqs. (2.5), (4.3) that $\ell^{cs}(r) = \ell^u(r) \leqq Cr^2/c$ for some $C > 0$. So, defining $r_1 = K_1 c$ for some (sufficiently small) $K_1 > 0$, we have $\sigma(r, \beta) \leqq \frac{1}{2}(r/r_1)^2$ in Eq. (A.2), and Lemma 3.2 still holds for $r < r_1$. This shows the existence of the local center-unstable manifold $\Gamma_r = \{(A, B) \in \mathscr{H}^2 \,|\, B = h(A),\, A \in \mathscr{B}_r \subset \mathscr{H}\}$ for the system (2.3). Now, as is easily seen from Eqs. (4.1), (4.3), the fixed points (2.4) have a finite limit as $c \to \infty$; so, we can find an $R > 0$ such that $\mathrm{F}_2, \mathrm{F}_3 \in \mathscr{B}_R \subset \mathscr{H}^2$ for all sufficiently large c. Defining thus $c_0 = R/K_1$ and noting that $\frac{1}{2}(R/r_1)^2 = \frac{1}{2}(c_0/c)^2$, we obtain:

Proposition 4.1. *There exists a $c_0 > 0$ such that, for all $c \geqq c_0$, the system* (2.3) *defines a semiflow (for $\xi \leqq 0$) on the local center-unstable manifold Γ_R, which contains the fixed points* (2.4). *The projection onto $\mathscr{H}$ of this semiflow verifies the differential equation* (3.5), *with $h: \mathscr{H} \to \mathscr{H}$ a phase covariant $C^{1,1}$ function verifying* $h(0) = 0$, $Dh(0) = 0$, $\operatorname{Lip}(h) \leqq \frac{1}{2}(c_0/c)^2$.

This proposition reduces the proof of Theorem 1.2 to the study of the system (3.5) in the ball $\mathscr{B}_R \subset \mathscr{H}$. In order to extract the leading terms as $c \to \infty$, we rescale the time ξ by setting $\xi = -c\eta$. We thus obtain

$$A_n' = \alpha_n(c)A_n - v_n(c)F_n(A + h(A)), \quad A \in \mathscr{B}_R \subset \mathscr{H}\,, \tag{4.5}$$

where $\alpha_n(c) = -c\lambda_{n+}$, $v_n(c) = c/(\lambda_{n+} - \lambda_{n-})$, and $'$ denotes the derivative with respect to η. In view of Eqs. (4.1), (4.4), the formal limit $c \to \infty$ in Eq. (4.5) yields the simpler equations

$$A_n' = \alpha_n A_n - F_n(A), \quad A \in \mathscr{B}_R \subset \mathscr{H}\,, \tag{4.6}$$

where $\alpha_n = 1 - q_n^2$.

4.2. The Limiting Case $c = \infty$. We now study the reduced equations (4.6) and show that they have front solutions. Of course, this system is still infinite-dimensional, so that we cannot hope to show the existence of front solutions just by a simple argument as in Sect. 3.3. For convenience, we suppose from now on that $-1/\sqrt{3} < q_0 \leqq 0$ and that $q_1 = \sqrt{1 - \varepsilon^2}$ for some $\varepsilon \leqq \varepsilon_0 = 1/10$ (Fig. 2, black region). Then, recalling that $q_n = q_0 + n(q_1 - q_0)$ and $\alpha_n = 1 - q_n^2$, we see that α_0, α_1 (and perhaps α_{-1}) are positive, whereas $\alpha_n < 0$ for all $|n| > 1$. This means that most of the variables A_n are exponentially damped by Eq. (4.6), so that only a few modes (A_{-1}, A_0, A_1) will be relevant in our analysis.

We first consider the behavior of the system in a neighborhood of the two circles of fixed points F_2, F_3 corresponding to Eq. (2.4):

F_2) $|A_0| = \sqrt{\alpha_0}$, $A_n = 0$ for all $n \neq 0$,
F_3) $|A_1| = \sqrt{\alpha_1}$, $A_n = 0$ for all $n \neq 1$.

The following results will be proven in Sect. 4.3:

i) The circle F_2 has an annular neighborhood $\mathscr{A}$ which is attracted to F_2 by Eq. (4.6) and whose size does not depend on ε.
ii) Any point $P \in F_3$ has a local unstable manifold $\mathscr{W}_P$ of (complex) dimension 1, which is nearly parallel to the 0-direction (i.e., the direction defined by $A_0 = 1$, $A_n = 0$ for all $n \neq 0$), and whose size does not depend on ε.

To prove that front solutions exist for $c = \infty$ we now show that the continuation of the local unstable manifold $\mathscr{W}_P$ under the semiflow defined by Eq. (4.6) intersects the attractive annular neighborhood $\mathscr{A}$, if ε is sufficiently small. This has to be done by direct estimates; for the sake of clarity, we just explain here the main steps of the calculation, and defer the proofs to Appendix C.

First of all, we write any $A \in \mathscr{H}$ as $A_{||} + A_{\perp}$, where $A_{||} = (A_{-1}, A_0, A_1)$ and $A_{\perp} = (A_n)_{|n|>1}$; the corresponding decomposition of $\mathscr{H}$ will be denoted by $\mathscr{H}_{||} \oplus \mathscr{H}_{\perp}$. We also define the domain $D_\varepsilon = \{A_{||} \,|\, |A_0| \leqq 1, |A_1| \leqq 2\varepsilon, |A_{-1}| \leqq 2\varepsilon\} \subset \mathscr{H}_{||}$, and note that the fixed points F_2, F_3 lie in D_ε, see Fig. 6.

Now, if $A(\eta)$ is a solution of Eq. (4.6) and if ε is sufficiently small, we have the following results:

Lemma 4.2. *There exists a $K_2 > 0$ such that, if $A_{||} \in D_\varepsilon$ and $\| A_\perp \| = K_2 \varepsilon^2$, then* $\frac{d}{d\eta} \| A_\perp \| < 0$.

In other words, as long as $A_{||}$ stays in D_ε, the other components $A_\perp$ of A remain bounded by $K_2\varepsilon^2$ if they were initially. Since we are interested in a trajectory starting from F_3 (where $A_\perp = 0$), we can henceforth assume that $\| A_\perp \| \leqq K_2 \varepsilon^2$, as long as $A_{||} \in D_\varepsilon$.

Lemma 4.3. *If $\| A_\perp \| \leqq K_2 \varepsilon^2$, $A_{||} \in D_\varepsilon$ and $|A_1| = 2\varepsilon$, then* $\frac{d}{d\eta} |A_1| < 0$.

A similar result holds for A_{-1} replacing A_1. Together with Lemma 4.2, this means that a trajectory of the system cannot leave the region $\hat{D}_\varepsilon = \{A_{||} \in D_\varepsilon, \| A_\perp \| \leqq K_2 \varepsilon^2\}$ unless $|A_0| > 1$. In particular, we can assume in our case that $A \in \hat{D}_\varepsilon$, as long as $|A_0| \leqq 1$.

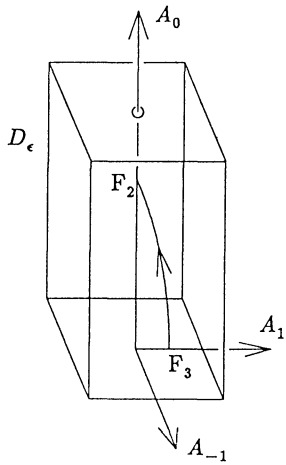

Fig. 6. The parallelepiped D_ε contains the fixed points F_2, F_3. The unstable manifold of F_3 remains in this domain and goes along the A_0-axis until it intersects the basin of attraction of F_2

Lemma 4.4. *If $A \in \hat{D}_\varepsilon$ and $\varepsilon^3 \leqq |A_0| \leqq \sqrt{\alpha_0}(1-\varepsilon)$, then $\frac{d}{d\eta}|A_0|$ is positive and bounded away from zero.*

Now, the main result of this subsection is:

Proposition 4.5. *There is an $\varepsilon_1 \leqq \varepsilon_0$ such that, for all $\varepsilon \leqq \varepsilon_1$, every point of* F_3 *is connected to* F_2 *by an orbit of the system* (4.6).

Proof. If $\mathscr{W}_P$ is the local unstable manifold of $P \in F_3$, we can choose $A \in \mathscr{W}_P \cap \hat{D}_\varepsilon$ such that $|A_0| \geqq \varepsilon^3$. This is always possible if ε is sufficiently small, because $\mathscr{W}_P$ is nearly tangent to the 0-direction and its size does not depend on ε. So, denoting by $A(\eta)$ $(\eta \geqq 0)$ the evolution of A under Eq. (4.6), we know from the preceding lemmas that $A(\eta)$ remains in $\hat{D}_\varepsilon$ and is driven along the 0-direction with non-vanishing velocity until $|A_0(\eta)| = \sqrt{\alpha_0}(1-\varepsilon)$. But, as is easily verified, this last point lies in the annular neighborhood $\mathscr{A}$ of the circle F_2, if ε is sufficiently small. □

Remarks.

1) If $A(\eta)$ is a solution of Eq. (4.6) and if

$$u(x,t) = \sum_{n \in \mathbf{Z}} A_n(t) e^{iq_n x},$$

then it is easy to see that $u(x,t)$ verifies the GL-equation (1.1). So, Proposition 4.5 shows the existence of solutions of Eq. (1.1) satisfying

$$\lim_{t \to -\infty} u(x,t) = u_{q_1}(x), \quad \lim_{t \to +\infty} u(x,t) = u_{q_0}(x),$$

uniformly in x (unlike the front solutions).

2) Let n be the (real) dimension of the unstable manifold of any point of F_3, and let m be the (real) codimension of the stable manifold of the circle F_2. For q_0, q_1 in the range of Theorem 1.2, we have seen that $n = 2$ and $m = 0$, so that the intersection of the two manifolds is generic in the sense that $n > m$. For other values of q_0, q_1, it is not difficult to show that this genericity condition is fulfilled if and only if $(q_0 - q_1)^2 < 6q_1^2 - 2$. This is the shaded region in Fig. 2.

4.3. The Full Case $c < \infty$. We now come back to the full equations (4.5) and show, by the same perturbation argument as in Sect. 3.4, the existence of front solutions for sufficiently large $c \geqq c_0$. To simplify the forthcoming expressions, we rewrite Eqs. (4.5), (4.6) in the form

$$A' = \Lambda^c A - F^c(A), \quad A \in \mathscr{B}_R \subset \mathscr{H}, \tag{4.5'}$$

$$A' = \Lambda A - F(A), \quad A \in \mathscr{B}_R \subset \mathscr{H}, \tag{4.6'}$$

where Λ^c, Λ are the linear operators in $\mathscr{H}$ defined by $(\Lambda^c A)_n = \alpha_n(c) A_n$, $(\Lambda A)_n = \alpha_n A_n$, and $F^c\colon \mathscr{H} \to \mathscr{H}$ is the $\mathscr{C}^{1,1}$ function defined by $(F^c(A))_n = v_n(c) F_n(A + h(A))$.

We begin by comparing the semiflows Φ^c_η, Φ_η of Eq. (4.5′), (4.6′) respectively.

Proposition 4.6. *There exists a $K_3 > 0$ such that,*

$$\| \Phi^c_\eta(A) - \Phi_\eta(B) \| \leqq \exp(K_3 \eta) \left(\| A - B \| + \frac{K_3}{c^{1/4}} \right), \tag{4.7}$$

for all $A, B \in \mathscr{B}_R \subset \mathscr{H}$ and all $\eta \geqq c^{-1/4}$.

Proof. Since (by Eqs. (4.1), (4.2)) $\alpha_n \leqq 1$ and $\operatorname{Re} \alpha_n(c) \leqq c\hat{\lambda}$ for all $n \in \mathbf{Z}$, there exists a $\lambda > 1$ such that $\| e^{\Lambda^c t} \|, \| e^{\Lambda t} \| \leqq e^{\lambda t}$ for all $t \in \mathbf{R}_+$ and all $c \geqq c_0$. On the other hand, using Eqs. (2.5), (4.3) and Proposition 4.1, it is easy to see that $\operatorname{Lip}(F^c)$, $\operatorname{Lip}(F) \leqq K_4$ in $\mathscr{B}_{2R} \subset \mathscr{H}$, for some $K_4 > 0$. So, setting $\mathscr{E}^{cs} = \mathscr{H}$, $\mathscr{E}^u = \{0\}$, $\lambda^{cs} = \lambda$, $D = 1$, and $\ell^{cs}(2R) = K_4$, we can apply Theorem A.1 to both systems (4.5′), (4.6′), the condition (A.3) being fulfilled if $\beta > \lambda + 10\, K_4$. It follows that $\Phi^c, \Phi \in \mathscr{K}_\beta$ for all $c \geqq c_0$.

Now, we want to use Theorem A.2 to compare the semiflows Φ^c, Φ in $\mathscr{B}_R \subset \mathscr{H}$. Before doing this, let us remark that, although (by Eq. (4.1)) $\alpha_n(c) \equiv -c\lambda_{n+}$ converges to $\alpha_n = 1 - q_n^2$ for all n as $c \to \infty$, the convergence is not uniform in n: in fact, $|\alpha_n(c)|$ grows like $\sqrt{n}$ as $n \to \infty$ and $|\alpha_n|$ like n^2, so that $\| \Lambda^c - \Lambda \| = \infty$ for all c. On the other hand, the conditions (A.5) do not involve the operators themselves, but the associated semigroups $e^{\Lambda^c t}$, $e^{\Lambda t}$, in which the large n components are exponentially small if $t > 0$. So, we can hope that $e^{\Lambda^c t}$, $e^{\Lambda t}$ are close to each other if c is sufficiently large and t strictly positive. Indeed, we find:

Lemma 4.7.

$$\sup_{t \geqq c^{-1/4}} \left(e^{-\lambda t} \| e^{\Lambda^c t} - e^{\Lambda t} \| \right) = \mathscr{O}\left(\frac{1}{c^{1/4}} \right), \quad \text{as } c \to \infty .$$

(See Appendix C for the proofs of the lemmas in this section.) The same phenomenon occurs when comparing the non-linear terms F^c, F, for the convergence of the factor $v_n(c) \equiv c/(\lambda_{n+} - \lambda_{n-})$ to 1 as $c \to \infty$ is not uniform in n, cf. Eq. (4.4). As a consequence, the difference $F^c - F$ does not become small, but nevertheless $e^{\Lambda t}(F^c - F)$ does, if $t > 0$:

Lemma 4.8.

$$\sup_{t \geqq c^{-1/2}} \sup_{\substack{A \in B_{2R} \\ A \neq 0}} \left(e^{-\lambda t} \frac{\| e^{\Lambda t}(F^c(A) - F(A)) \|}{\| A \|} \right) = \mathscr{O}\left(\frac{1}{c^{1/2}} \right), \quad \text{as } c \to \infty .$$

Accordingly, setting $A_1^{cs} = \Lambda^c$, $A_2^{cs} = \Lambda$, $f_1^{cs} = -F^c$, $f_2^{cs} = -F$, we see that the assumption (A.5) of Theorem A.2 is fulfilled with $\delta = c^{-1/4}$ and $\varepsilon = Cc^{-1/4}$ for some $C > 0$. In view of Eq. (A.6), it follows that $\| \Phi_\eta^c - \Phi_\eta \|_\infty = e^{\beta\eta} \mathcal{O}(c^{-1/4})$ in $\mathcal{B}_R \subset \mathcal{H}$, for all $\eta \geqq c^{-1/4}$. Combining this with the fact that $\Phi^c, \Phi \in \mathcal{K}_\beta$ and choosing $K_3 \geqq \beta$ sufficiently large, we obtain Eq. (4.7). □

We next study the behavior of the semiflow Φ_η^c around the circles of fixed points corresponding to Eq. (2.4):

$$\mathrm{F}_2^c)\ A_0 = \frac{-\lambda_{0-}}{\lambda_{0+} - \lambda_{0-}} \sqrt{\alpha_0}\, e^{i\varphi}, \quad A_n = 0 \text{ for all } n \neq 0, \quad \varphi \in \mathbf{R}\,,$$

$$\mathrm{F}_3^c)\ A_1 = \frac{-\lambda_{1-}}{\lambda_{1+} - \lambda_{1-}} \sqrt{\alpha_1}\, e^{i\psi}, \quad A_n = 0 \text{ for all } n \neq 1, \quad \psi \in \mathbf{R}\,.$$

Using Eqs. (4.1), (4.3), it is easy to see that F_2^c, F_3^c are close to F_2, F_3 (Sect. 4.2), in the sense that $\mathrm{dist}(\mathrm{F}_2^c, \mathrm{F}_2)$ and $\mathrm{dist}(\mathrm{F}_3^c, \mathrm{F}_3)$ are $\mathcal{O}(1/c^2)$ as $c \to \infty$. Now, let $\hat{A}^c$ be the point of F_2^c corresponding to $\varphi = 0$ in the expression above, and let $A = \hat{A}^c + X$, with $X \in \mathcal{B}_{2\rho} \subset \mathcal{H}$ for some small $\rho > 0$. In order to study the evolution of X, we introduce the (real) subspaces $V_s = \{X \in \mathcal{H} \mid \mathrm{Im}\, X_0 = 0\}$, $V_c = \{X \in \mathcal{H} \mid \mathrm{Re}\, X_0 = 0,\ X_n = 0\ \forall n \neq 0\}$ and we write X as a pair (X_s, X_c) with $X_s \in V_s$, $X_c \in V_c$. With these notations, we have the following result:

Lemma 4.9. *If $A = \hat{A}^c + X$ is a solution of (4.5′) and if c is sufficiently large, then $X = (X_s, X_c)$ verifies the differential equation*

$$X_s' = M_s^c X_s + R_s^c(X), \quad X_c' = M_c^c X_c + R_c^c(X)\,, \tag{4.8}$$

where $M_c^c = 0$, $M_s^c\colon V_s \to V_s$ is a linear operator satisfying $\| e^{M_s^c t} \| \leqq \hat{D} e^{-\frac{1}{2}\alpha_0 t}$ $(t \in \mathbf{R}_+)$ for some $\hat{D} \geqq 1$, and $R^c = (R_s^c, R_c^c)\colon \mathcal{H} \to \mathcal{H}$ is a $\mathcal{C}^{1,1}$ function, vanishing at the origin, such that $\mathrm{Lip}(R^c) \leqq K_5(\rho + c_0/c)$ in $\mathcal{B}_{2\rho} \subset \mathcal{H}$, for some $K_5 > 0$.

It is clear from the proof (see Appendix C) that Lemma 4.9 remains true if $c = \infty$, that is, for the system (4.6′). Indeed, if $\hat{A} \in \mathrm{F}_2$ verifies $\hat{A}_0 = \sqrt{\alpha_0}$ and if $A = \hat{A} + X$, then Eq. (4.6′) for X is simply

$$X_n' = (\alpha_n - 2\alpha_0) X_n - \alpha_0 X_{-n}^* + R_n(X), \tag{4.9}$$

which can be rewritten as

$$X_s' = M_s X_s + R_s(X), \quad X_c' = M_c X_c + R_c(X)\,, \tag{4.10}$$

with M_s, M_c and R_s, R_c as in Lemma 4.9.

So, setting $\mathcal{E}^{cs} = V_s$, $\mathcal{E}^u = V_c$, $\lambda^{cs} = -\frac{1}{2}\alpha_0$, $\lambda^u = 0$, $D = \hat{D}$, $\ell^{cs}(2\rho) = \ell^u(2\rho) \leqq K_5(\rho + c_0/c)$, and $\beta = -\frac{1}{4}\alpha_0$, we can apply Theorem A.1 to both systems (4.8), (4.10), the condition (A.3) being fulfilled if ρ is sufficiently small and if $c_0/c \leqq \rho$. Thus, there exist $\mathcal{C}^{1,1}$ maps $h^c, h\colon \mathcal{B}_\rho \subset V_s \to V_c$, whose graphs $\mathcal{V}^c$, $\mathcal{V}$ are (except for a translation) the local stable manifolds of $\hat{A}^c$, $\hat{A}$ for the semiflows Φ_η^c, Φ_η respectively. In particular, using the phase covariance of the system, we easily obtain the analogue of Lemma 3.8:

Lemma 4.10. *There exists a $\rho_2 > 0$ such that, for all sufficiently large c (including $c = \infty$), the annulus $\mathcal{A}^c$ of radius ρ_2 around the circle F_2^c is attracted to F_2^c by the semiflow Φ_η^c.*

Moreover, in the same way as Lemma 4.7 and Lemma 4.8, one can prove the following continuity result for M_s^c, M_c^c and R_s^c, R_c^c:

Lemma 4.11. *As* $c \to \infty$,

i) $\sup_{t \geqq c^{-1/4}} (e^{\frac{1}{2}\alpha_0 t} \| e^{M_s^c t} - e^{M_s t} \|) = \mathcal{O}\left(\frac{1}{c^{1/4}}\right)$,

ii) $\sup_{t \geqq c^{-1/2}} \sup_{\substack{X \in \mathscr{B}_{2\rho} \\ X \neq 0}} \left(e^{\frac{1}{2}\alpha_0 t} \frac{\| e^{M_s t}(R_s^c(X) - R_s(X)) \|}{\| X \|} \right) = \mathcal{O}\left(\frac{1}{c^{1/2}}\right)$,

iii) $\mathrm{Lip}(R_c^c - R_c) = \mathcal{O}(1/c)$.

So, setting $A_1^{cs} = M_s^c$, $A_2^{cs} = M_s$, $A_1^u = M_c^c = 0$, $A_2^u = M_c = 0$, $f_1^{cs} = R_s^c$, $f_2^{cs} = R_s$, $f_1^u = R_c^c$, $f_2^u = R_c$ and using Lemma 4.11, we see that the assumption (A.5) of Theorem A.2 is fulfilled with $\delta = c^{-1/4}$ and $\varepsilon = Cc^{-1/4}$ for some $C > 0$. Thus, it follows from Eq. (A.6) that the unstable manifold $\mathscr{V}^c$ is continuous in c at $c = \infty$ in the sense that $\| h^c - h \|_\infty = \mathcal{O}(c^{-1/4})$ in $\mathscr{B}_{\rho_2} \subset V_s$.

These considerations about the behavior of Φ_η^c near F_2^c can be repeated in an analogous way for F_3^c; for brevity, we only point out the main differences, and leave the details to the reader. For example, we have instead of Eq. (4.9),

$$X_n' = (\alpha_n - 2\alpha_1)X_n - \alpha_1 X_{2-n}^* + S_n(X), \tag{4.11}$$

where $S: \mathscr{H} \to \mathscr{H}$ has the same properties as R^c in Lemma 4.9. The spectrum of the linear operator in Eq. (4.11) is contained in the half-plane $\mathrm{Re}(z) \leqq 0$, except for a single positive eigenvalue $\hat{\mu} \geqq \alpha_0 - 2\varepsilon^2$, whose eigenvector $\hat{X}$ is "nearly tangent to the 0-direction" (cf. Sect. 4.2) in the sense that $\hat{X}_0 = 1$, $\hat{X}_1 = \mathcal{O}(\varepsilon^2)$ and $\hat{X}_n = 0$ for all $n \neq 0, 2$. So, defining the subspaces $W_u = \mathbf{C}\hat{X}$, $W_{cs} = W_u^\perp$, we can apply Theorem A.1 and obtain the existence of a local one-dimensional unstable manifold $\mathscr{W}$ as the graph of a $\mathscr{C}^{1,1}$ map $g: \mathscr{B}_{\rho_3} \subset W_u \to W_{cs}$, for some (sufficiently small) $\rho_3 > 0$. The same is true for the full system ($c < \infty$), and the manifold $\mathscr{W}^c$ depends continuously on c in the sense that $\mathrm{dist}(\mathscr{W}^c, \mathscr{W}) = \mathcal{O}(c^{-1/4})$ as $c \to \infty$.

Now, let us summarize our results. The semiflow Φ_η^c corresponding to (4.5′) is continuous in c at $c = \infty$ (in the sense of Proposition 4.6), and so are the fixed points F_2^c, F_3^c. Moreover, the circle F_2^c has an attractive annular neighborhood $\mathscr{A}^c$, of radius ρ_2 independent of c (Lemma 4.10), and at each point of F_3^c one can attach a local one-dimensional unstable manifold $\mathscr{W}^c$ which depends continuously on c at $c = \infty$. Combining these facts with Proposition 4.5 in the same way as in Sect. 3.4, we see that there exists a $c_1 \geqq c_0$ such that, for all $c \geqq c_1$ and all $\varepsilon \leqq \varepsilon_1$, the continuation of the local unstable manifold $\mathscr{W}^c$ intersects the attractive neighborhood $\mathscr{A}^c$ of F_2^c. This concludes the proof of Theorem 1.2. □

Appendix A. The Center-Stable Manifold Theorem

In this appendix, we recall (for easy reference) some results of center manifold theory in infinite-dimensional Banach spaces. Proofs of these statements can be found in the companion paper by one of us [G]. They are an extension of results of [EW], Appendix A.

Let $(\mathscr{E}^{cs}, \|\cdot\|_{cs})$, $(\mathscr{E}^u, \|\cdot\|_u)$ be two Banach spaces, and denote by $\mathscr{E}$ the direct sum $\mathscr{E}^{cs} \oplus \mathscr{E}^u$ equipped with the norm $\|z\| = \max(\|z^{cs}\|_{cs}, \|z^u\|_u)$, for $z = (z^{cs}, z^u) \in \mathscr{E}$. We consider the differential equation in $\mathscr{E}$,

$$\frac{dz^{cs}}{dt} = A^{cs}z^{cs} + f^{cs}(z^{cs}, z^u)\,,$$

$$\frac{dz^u}{dt} = A^u z^u + f^u(z^{cs}, z^u)\,, \tag{A.1}$$

and make the following hypotheses:

H1) The linear operators $A^{cs}: \mathscr{E}^{cs} \to \mathscr{E}^{cs}$ and $-A^u: \mathscr{E}^u \to \mathscr{E}^u$ define strongly continuous semigroups $e^{A^{cs}t}$, $e^{-A^u t}$ for $t \geqq 0$. Moreover, there exist real constants λ^{cs}, λ^u, D such that $\lambda^u \geqq 0$, $\lambda^u > \max(\lambda^s, 2\lambda^s)$, and

$$\|e^{A^{cs}t}\|_{cs} \leqq De^{\lambda^{cs}t}, \quad \|e^{-A^u t}\|_u \leqq De^{-\lambda^u t}\,,$$

for all $t \geqq 0$.

H2) $f^{cs}: \mathscr{E} \to \mathscr{E}^{cs}$ and $f^u: \mathscr{E} \to \mathscr{E}^u$ are $\mathscr{C}^{1,1}$ functions vanishing at the origin together with their first derivatives.

H3) The norm $\|\cdot\|_{cs}$ is a $\mathscr{C}^{1,1}$ function on $\mathscr{E}^{cs}$.

Under these hypotheses, the center-stable manifold theorem asserts the existence, in a small neighborhood of the fixed point 0, of a $\mathscr{C}^{1,1}$ manifold Γ which is tangent to the subspace $\mathscr{E}^{cs}$ at the origin, is left invariant by Eq. (A.1) and contains all the trajectories which stay near 0 for all $t \in \mathbf{R}_+$. We shall give here an explicit formulation of this theorem, because in our applications to the system (2.3), we need to know exactly how the manifold depends on the parameters ε and c.

In order to do that, we introduce some more notations. For all $r > 0$, we denote by $\mathscr{B}_r^{cs}$, $\mathscr{B}_r^u$, $\mathscr{B}_r$ the balls of radius r around the origin in $\mathscr{E}^{cs}$, $\mathscr{E}^u$, $\mathscr{E}$ respectively, and we set

$$\ell^{cs}(r) = \sup_{z \in \mathscr{B}_r} \|Df^{cs}(z)\|, \quad \ell^u(r) = \sup_{z \in \mathscr{B}_r} \|Df^u(z)\|\,,$$

where Df^{cs}, Df^u are the derivatives of f^{cs}, f^u. In view of H2, $\ell^{cs}(r), \ell^u(r) \to 0$ as $r \to 0$. Next, for all $\sigma \in [0, 1]$, $\beta \in (\lambda^{cs}, \lambda^u)$, we define the function spaces

$$H_\sigma = \{h: \mathscr{E}^{cs} \to \mathscr{E}^u \mid h(0) = 0;\ \|h(\xi) - h(\tilde{\xi})\|_u \leqq \sigma \|\xi - \tilde{\xi}\|_{cs}\ \forall \xi, \tilde{\xi} \in \mathscr{E}^{cs}\}\,,$$

$$\begin{aligned}\mathscr{K}_\beta = \{&\Phi: \mathbf{R}_+ \times \mathscr{E}^{cs} \to \mathscr{E}^{cs} \mid \Phi_0(\xi) = \xi\ \forall \xi \in \mathscr{E}^{cs};\\ &\Phi_t(0) = 0\ \forall t \in \mathbf{R}_+;\ \Phi \text{ is continuous in } t;\\ &\|\Phi_t(\xi) - \Phi_t(\tilde{\xi})\|_{cs} \leqq De^{\beta t}\|\xi - \tilde{\xi}\|_{cs}\ \forall t \in \mathbf{R}_+, \forall \xi, \tilde{\xi} \in \mathscr{E}^{cs}\}\,.\end{aligned}$$

Finally, defining $\hat{\beta} = \max(\beta, 2\beta)$, we know from H1 that there exists a $\beta \in (\lambda^{cs}, \lambda^u)$ such that $\hat{\beta} \in (\lambda^{cs}, \lambda^u)$. For such a β, we set

$$\sigma = \sigma(r, \beta) = D^2 \max\left(\frac{5\ell^{cs}(2r)}{\beta - \lambda^{cs}}, \frac{5\ell^u(2r)}{\lambda^u - \hat{\beta}}\right). \tag{A.2}$$

With these notations, we have the following result:

Theorem A.1. [G]. *Assume that the hypotheses* H1, H2, H3, *are fulfilled, and choose* r *so small that, in Eq.* (A.2),

$$\sigma < \frac{1}{2D} . \tag{A.3}$$

Then there exist a map $h \in H_\sigma$ *and a semiflow* $\Phi \in \mathscr{K}_\beta$ *with the following properties:*

i) h *is of class* $\mathscr{C}^{1,1}$, $Dh(0) = 0$, *and* h *maps* $\mathscr{D}(A^{cs})$ (*the domain of* A^{cs}) *into* $\mathscr{D}(A^u)$ (*the domain of* A^u).
ii) *For all* $\xi \in \mathscr{B}_r^{cs} \cap \mathscr{D}(A^{cs})$, *the curve* $z(t) = (\Phi_t(\xi), h(\Phi_t(\xi)))$, $t \in \mathbf{R}_+$, *is a solution of Eq.* (A.1) *as long as it remains in* $\mathscr{B}_r$.
iii) *If* $z(t)$ *is any solution of Eq.* (A.1) *such that* $z(t) \in \mathscr{B}_r$ *for all* $t \in \mathbf{R}_+$, *then* $z(t) = (\Phi_t(\xi), h(\Phi_t(\xi)))$ *for some* $\xi \in \mathscr{B}_r^{cs}$.

Thus, denoting by Γ_r the restriction of the graph of h to $\mathscr{B}_r$, we see that Eq. (A.1) defines a *local semiflow* on Γ_r (in the sense of ii). We shall always refer to Γ_r as the (local) *center-stable manifold*, although in the case $\lambda^{cs} < 0$ one rather speaks of a *stable manifold.*

Remarks.

1) In the proof of Theorem A.1, one has to "cut off" the non-linear terms f^{cs}, f^u outside the ball $\mathscr{B}_r^{cs}$. Since (by H3) the norm $\|\cdot\|_{cs}$ is $\mathscr{C}^{1,1}$ on $\mathscr{E}^{cs}$, this is simply done by writing $f(z^{cs}, z^u)\chi(\|z^{cs}\|_{cs}/r)$, where $\chi: \mathbf{R}_+ \to [0,1]$ is some $\mathscr{C}^{1,1}$ function equal to 1 on $[0,1]$, vanishing on $[2, \infty)$, and satisfying $|\chi'(x)| \leqq 2$ for all $x \in \mathbf{R}_+$. This is how the expressions $5\ell^{cs}(2r)$, $5\ell^u(2r)$ arise in Eq. (A.2). For more details, see [G], Sect. 3.2.
2) Except for the assertion $Dh(0) = 0$, Theorem A.1 remains true if Df^{cs}, Df^u are not assumed to vanish at origin, provided that Eq. (A.3) can be satisfied for r sufficiently small.
3) As a consequence of iii), all fixed points of the system (A.1) in $\mathscr{B}_r$ must lie on the center-stable manifold Γ_r.
4) If the non-linear term $f = (f^{cs}, f^u)$ commutes with a linear isometry of $\mathscr{E}$, then h can be chosen to commute with the same isometry.
5) Theorem A.1 also makes sense if $\mathscr{E}^{cs} = \mathscr{E}$ and $\mathscr{E}^u = \{0\}$. In this case, it only asserts that the solutions of the system

$$\frac{dz^{cs}}{dt} = A^{cs}z^{cs} + f^{cs}(z^{cs}) ,$$

define a local semiflow $\Phi \in \mathscr{K}_\beta$ in the ball $\mathscr{B}_r$ around the origin. Setting $\lambda^u = +\infty$, $\ell^u(2r) = 0$, the condition (A.2), (A.3) reduces to

$$\sigma = D^2 \frac{5\ell^{cs}(2r)}{\beta - \lambda^{cs}} < \frac{1}{2D} , \tag{A.4}$$

which is always satisfied if $\beta > \lambda^{cs}$ is sufficiently large.

It is well-known that the center-stable manifold, although generally not unique, can be chosen to depend continuously (for suitable topologies) on the operators A^{cs}, A^u and the functions f^{cs}, f^u in Eq. (A.1). We shall give here a formulation of this result which is sufficient for our applications in Sect. 3 and Sect. 4. Suppose that we are given two pairs of linear operators A_1^{cs}, A_1^u and A_2^{cs}, A_2^u satisfying H1 with the same constants λ^{cs}, λ^u, D. Assume also that we have two couples of functions f_1^{cs}, f_1^u and f_2^{cs}, f_2^u verifying H2, and that the condition (A.3) is fulfilled for both systems 1 and 2. Denoting thus by h_1, h_2 and Φ_1, Φ_2 the maps and semiflows whose existence is asserted by Theorem A.1, we have the following result:

Theorem A.2. *Assume that there exist an $\varepsilon > 0$ and a $\delta \geqq 0$ such that*

$$\sup_{t \geqq \delta} (e^{-\lambda^{cs}t} \| e^{A_1^{cs}t} - e^{A_2^{cs}t} \|_{cs}) \leqq D\varepsilon ,$$

$$\sup_{t \geqq \delta} (e^{\lambda^u t} \| e^{-A_1^u t} - e^{-A_2^u t} \|_u) \leqq D\varepsilon ,$$

$$\sup_{t \geqq \delta} \sup_{\substack{z \in \mathscr{B}_{2r} \\ z \neq 0}} \left(e^{-\lambda^{cs}t} \frac{\| e^{A_2^{cs}t}(f_1^{cs}(z) - f_2^{cs}(z)) \|_{cs}}{\| z \|} \right) \leqq D\ell^{cs}(2r)\varepsilon ,$$

$$\sup_{t \geqq \delta} \sup_{\substack{z \in \mathscr{B}_{2r} \\ z \neq 0}} \left(e^{\lambda^u t} \frac{\| e^{-A_2^u t}(f_1^u(z) - f_2^u(z)) \|_u}{\| z \|} \right) \leqq D\ell^u(2r)\varepsilon . \tag{A.5}$$

Then h_1, h_2 and Φ_1, Φ_2 can be chosen so that

$$\sup_{\xi \neq 0} \frac{\| h_1(\xi) - h_2(\xi) \|_u}{\| \xi \|_{cs}} \leqq \frac{1}{1 - 2\sigma} (D\varepsilon + 3\delta(\lambda^u - \lambda^{cs})) ,$$

$$\sup_{t \geqq \delta} \sup_{\xi \neq 0} \left(e^{-\beta t} \frac{\| \Phi_{1,t}(\xi) - \Phi_{2,t}(\xi) \|_{cs}}{\| \xi \|_{cs}} \right) \leqq \frac{1}{1 - 2\sigma} (D\varepsilon + 3\delta(\lambda^u - \lambda^{cs})) . \tag{A.6}$$

For a proof in the case $\delta = 0$, see [G], Sect. 2.3. The general case is easily proved along the same lines.

Appendix B. Some Proofs (Unstable-Unstable case)

Proof of Lemma 3.7. We first write the vector field $\Delta\chi(x)$ in the form

$$\Delta\chi_0(x) = \left(-\frac{\lambda_{0+}}{\varepsilon} - \frac{2}{c - 2i} \right) X_0 - \frac{\Delta_0(x)}{c - 2i} + \left(\frac{-1}{\lambda_{0+} - \lambda_{0-}} + \frac{1}{c - 2i} \right) F_0(x + l(x)) ,$$

$$\Delta\chi_1(x) = \left(-\frac{\lambda_{1+}}{\varepsilon} - \frac{2\alpha}{c + 2i} \right) X_1 - \frac{\Delta_1(x)}{c + 2i} + \left(\frac{-1}{\lambda_{1+} - \lambda_{1-}} + \frac{1}{c + 2i} \right) F_1(x + l(x)) ,$$

where $\Delta: \mathscr{B}_{2R} \subset \mathscr{H}_0 \to \mathscr{H}$ is defined by $\Delta(x) = F(x + l(x)) - F(x)$. Next, we use the identity

$$F(A + B) = F(A) + G(A, B) + G(B, A) + F(B), \quad A, B \in \mathscr{H} , \tag{B.1}$$

where $G: \mathscr{H} \times \mathscr{H} \to \mathscr{H}$ is the Fréchet derivative of F:

$$G_n(A, B) = \sum_{p+s+r=n} (A_p A_s B_{-r}^* + 2A_p B_s A_{-r}^*) . \tag{B.2}$$

As is easily seen,

$$\| G(A, B) - G(\tilde{A}, \tilde{B}) \| \leqq 3K^2(\| A - \tilde{A} \|(\|A\| + \|\tilde{A}\|)\|B\| + \|\tilde{A}\|^2 \|B - \tilde{B}\|) , \tag{B.3}$$

with K as in (2.5). Now, using (B.1) with $A = x$ and $B = l(x)$, we can write $\Delta(x) = G(x, l(x)) + G(l(x), x) + F(l(x))$. Using (B.3) and recalling that $\mathrm{Lip}(l) \leqq \ell\varepsilon/\varepsilon_c$, we thus obtain

$$\begin{aligned}\mathrm{Lip}(\Delta) &\leqq \mathrm{Lip}(G(\cdot, l(\cdot))) + \mathrm{Lip}(G(l(\cdot), \cdot)) + \mathrm{Lip}(F(l(\cdot)))\\ &\leqq C\varepsilon/\varepsilon_c + C(\varepsilon/\varepsilon_c)^2 + C(\varepsilon/\varepsilon_c)^3 \leqq 3C\varepsilon/\varepsilon_c ,\end{aligned}$$

in $\mathscr{B}_{2R}$, for some $C > 0$. In the same way, it follows from (2.5) that $\mathrm{Lip}(F(\cdot + l(\cdot))) \leqq 3K(2R)^2(1 + \ell)^2$ in $\mathscr{B}_{2R}$. Finally, in view of (3.1), (3.4), we have

$$\left| -\frac{\lambda_{0+}}{\varepsilon} - \frac{2}{c - 2i} \right| = \mathcal{O}\left(\frac{\varepsilon}{c+2}\right), \left| \frac{-1}{\lambda_{0+} - \lambda_{0-}} + \frac{1}{c - 2i} \right| = \mathcal{O}\left(\frac{\varepsilon}{(c+2)^2}\right),$$

and similarly for $\lambda_{1+}, \lambda_{1-}$. Combining these estimates, we easily find

$$\mathrm{Lip}(\Delta\chi) \leqq K_4(\varepsilon/\varepsilon_c)/(c + 2)$$

in $\mathscr{B}_{2R}$, for some $K_4 > 0$. □

Proof of Lemma 3.9. Using the definitions of $\delta\chi$ and $\Delta\chi$, we easily obtain the identity

$$\delta\chi(z) - \delta\chi(\tilde{z}) = \Delta\chi(x^\varepsilon + z) - \Delta\chi(x^\varepsilon + \tilde{z}) + \Delta_\varepsilon(z, \tilde{z}) ,$$

where $\Delta_\varepsilon(z, \tilde{z}) = (\chi^0(x^\varepsilon + z) - \chi^0(x^0 + z)) - (\chi^0(x^\varepsilon + \tilde{z}) - \chi^0(x^0 + \tilde{z}))$. Since $x^\varepsilon \in \mathscr{B}_R \subset \mathscr{H}_0$, it follows from Lemma 3.7 that

$$\| \Delta\chi(x^\varepsilon + z) - \Delta\chi(x^\varepsilon + \tilde{z}) \| \leqq \frac{K_4}{c+2}\frac{\varepsilon}{\varepsilon_c} \| z - \tilde{z} \| ,$$

for all $z, \tilde{z} \in \mathscr{B}_R$; so, it remains to bound the function $\Delta_\varepsilon(z, \tilde{z})$. First, if $D\chi^0$ denotes the derivative of χ^0, it is easy to see that $\mathrm{Lip}(D\chi^0) \leqq C_1/(c + 2)$ in $\mathscr{B}_{2R} \subset \mathscr{H}_0$, for some $C_1 > 0$. Next, since $\lambda_{0\pm}$ and $\sqrt{2 - \varepsilon}$ are smooth functions of ε, the curve $\varepsilon \to x^\varepsilon$ is $\mathscr{C}^1$ and $\| (dx^\varepsilon/d\varepsilon) \| \leqq C_2$ for some $C_2 > 0$. So, $\Delta_\varepsilon(z, \tilde{z})$ is differentiable in ε and

$$\left\| \frac{\partial \Delta_\varepsilon(z, \tilde{z})}{\partial \varepsilon} \right\| = \left\| (D\chi^0(x^\varepsilon + z) - D\chi^0(x^\varepsilon + \tilde{z})) \cdot \frac{dx^\varepsilon}{d\varepsilon} \right\| \leqq \frac{C_1}{c+2} \| z - \tilde{z} \| \cdot C_2 .$$

Since $\Delta_0(z, \tilde{z}) = 0$, we find

$$\| \Delta_\varepsilon(z, \tilde{z}) \| \leqq \varepsilon \sup_\varepsilon \left\| \frac{\partial \Delta_\varepsilon(z, \tilde{z})}{\partial \varepsilon} \right\| \leqq \frac{C_1 C_2 \varepsilon}{c + 2} \| z - \tilde{z} \| \leqq \frac{K_0 C_1 C_2}{c+2} \frac{\varepsilon}{\varepsilon_c} \| z - \tilde{z} \| .$$

This concludes the proof of Lemma 3.9, with $K_6 = K_4 + K_0 C_1 C_2$. □

Appendix C. Some Proofs (Stable-Unstable Case)

Proof of Lemma 4.2. To simplify the notation, we set $A_{\|} = a$ and $A_{\perp} = X$. Inserting $A = a + X$ in (4.6), we obtain for $|n| \geqq 2$:

$$X'_n = -F_n(a) + \alpha_n X_n - G_n(a, X) - G_n(X, a) - F_n(X)\,, \tag{C.1}$$

where G is the derivative of F (cf. (B.1), (B.2)). We want to obtain an upper bound on the quantity $\frac{1}{2}\frac{d}{d\eta}\|X\|^2 = \mathrm{Re}(X, X')$. First of all, using the definition (1.7), we find $F_2(a) = a_1^2 a_0^* + 2a_1 a_0 a_{-1}^*$, $F_3(a) = a_1^2 a_{-1}^*$, $F_n(a) = 0$ for all $n > 3$, and symmetric expressions hold for $n < 0$. If $a \in D_\varepsilon$, we thus have

$$|\mathrm{Re}(X, F(a))| \leqq \|X\| C_1 \varepsilon^2\,, \tag{C.2}$$

for some $C_1 > 0$. Let us now consider the linear terms in (C.1). Since $\alpha_n \leqq \alpha_2 \leqq -3 + 4\varepsilon^2 < 0$ for all $|n| \geqq 2$, we have $\mathrm{Re}(X, \alpha X) \leqq \|X\|^2(-3 + \mathcal{O}(\varepsilon^2))$. On the other hand, a direct calculation yields $G_n(a, X) = 2X_n|a_0|^2 + \hat{G}_n(a, X)$, where $\hat{G}_n(a, X)$ is some complicated expression satisfying $\|\hat{G}(a, X)\| \leqq \|X\|(|a_0|^2 + \mathcal{O}(\varepsilon))$ if $a \in D_\varepsilon$. As a consequence, $\mathrm{Re}(X, -G(a, X)) \leqq \|X\|^2 \mathcal{O}(\varepsilon)$. Combining these estimates, we obtain for the linear terms

$$\mathrm{Re}(X, \alpha X - G(a, X)) \leqq \|X\|^2(-3 + \mathcal{O}(\varepsilon))\,. \tag{C.3}$$

Finally, we just bound the two remaining terms in (C.1) by using (2.5), (B.3); we find

$$|\mathrm{Re}(X, G(X, a))| \leqq \|X\|^3 C_2, \quad |\mathrm{Re}(X, F(X))| \leqq \|X\|^4 K^2\,, \tag{C.4}$$

for some $C_2 > 0$. Now, we summarize (C.2)–(C.4)

$$\frac{1}{2}\frac{d}{d\eta}\|X\|^2 \leqq \|X\| C_1 \varepsilon^2 + \|X\|^2(-3 + \mathcal{O}(\varepsilon)) + \|X\|^3 C_2 + \|X\|^4 K^2\,, \tag{C.5}$$

and choose a $K_2 > 0$ such that $3K_2 > C_1$. Then, assuming that ε is sufficiently small, it is easy to verify that the right-hand side of (C.5) is negative if $\|X\| = K_2 \varepsilon^2$. □

Proof of Lemma 4.3. Let $A = a + X$ with $\|X\| \leqq K_2 \varepsilon^2$, $a \in D_\varepsilon$, $|a_1| = 2\varepsilon$. Inserting in (4.6), we obtain for $n = 1$:

$$a'_1 = \alpha_1 a_1 - F_1(a) - G_1(a, X) - G_1(X, a) - F_1(X)\,. \tag{C.6}$$

Again, we want an upper bound on $\frac{1}{2}\frac{d}{d\eta}|a_1|^2 = \mathrm{Re}(a_1^* a'_1)$. First of all, $F_1(a) = a_1(|a_1|^2 + 2|a_0|^2 + 2|a_{-1}|^2) + a_0^2 a_{-1}^*$, and since $\mathrm{Re}(-|a_1|^2|a_0|^2 + a_1^* a_0^2 a_{-1}^*) \leqq 0$, we have

$$\mathrm{Re}(\alpha_1 |a_1|^2 - a_1^* F_1(a)) \leqq |a_1|^2(\alpha_1 - |a_1|^2 - |a_0|^2)\,. \tag{C.7}$$

On the other hand, a direct calculation yields $G_1(a, X) = a_{-1}^2 X_{-3}^* + 2a_0 a_{-1} X_{-2}^* + 2a_0 a_1^* X_2 + 2a_0^* a_{-1} X_2 + 2a_1^* a_{-1} X_3$, so that

$$|\mathrm{Re}(a_1^* G_1(a, X))| \leqq |a_1| \|X\| (C_3 |a_0| \varepsilon + \mathcal{O}(\varepsilon^2))\,, \tag{C.8}$$

for some $C_3 > 0$. Finally, the two remaining terms in (C.6) are simply bounded by

$$|\mathrm{Re}(a_1^* G_1(X,a))| \leqq |a_1| \, \| X \|^2 C_2, \quad |\mathrm{Re}(a_1^* F_1(X))| \leqq |a_1| \, \| X \|^3 K^2 . \tag{C.9}$$

Now, recalling that $\alpha_1 = \varepsilon^2$, $|a_1| = 2\varepsilon$, $\| X \| \leqq K_2 \varepsilon^2$, we can summarize (C.7)–(C.9) as follows:

$$\frac{1}{2}\frac{d}{d\eta}|a_1|^2 \leqq 4\varepsilon^2(-3\varepsilon^2 - |a_0|^2) + 2C_3 K_2 |a_0| \varepsilon^4 + \mathcal{O}(\varepsilon^5) . \tag{C.10}$$

The maximum over $|a_0|$ of the right-hand side (reached for $|a_0| = \frac{1}{4} C_3 K_2 \varepsilon^2$) is of the form $-12\varepsilon^4 + \mathcal{O}(\varepsilon^5)$, and thus is negative if ε is sufficiently small. □

Proof of Lemma 4.4. As before, let $A = a + X$ with $a \in D_\varepsilon$ and $\| X \| \leqq K_2 \varepsilon^2$. Inserting in (4.6), we obtain for $n = 0$:

$$a_0' = \alpha_0 a_0 - F_0(a) - G_0(a, X) - G_0(X, a) - F_0(X) . \tag{C.11}$$

We now want to find a lower bound on the quantity $\frac{1}{2}\frac{d}{d\eta}|a_0|^2 = \mathrm{Re}(a_0^* a_0)$. First of all, we have $F_0(a) = a_0(|a_0|^2 + 2|a_1|^2 + 2|a_{-1}|^2) + 2a_0^* a_1 a_{-1}$, and since $|a_1| \leqq 2\varepsilon$, $|a_{-1}| \leqq 2\varepsilon$, we can write

$$\mathrm{Re}(\alpha_0 |a_0|^2 - a_0^* F_0(a)) \geqq |a_0|^2(\alpha_0 - |a_0|^2 - 24\varepsilon^2) . \tag{C.12}$$

On the other hand, we have $G_0(a, X) = a_1^2 X_2^* + a_{-1}^2 X_{-2}^* + 2a_1 a_{-1}^* X_{-2} + 2a_1^* a_{-1} X_2$, and thus

$$-\mathrm{Re}(a_0^* G_0(a, X)) \geqq -|a_0| \, \| X \| C_4 \varepsilon^2 , \tag{C.13}$$

for some $C_4 > 0$. Finally, we find as above

$$-\mathrm{Re}(a_0^* G_0(X, a)) \geqq -|a_0| \, \| X \|^2 C_2, \quad -\mathrm{Re}(a_0^*, F_0(X)) \geqq -|a_0| \, \| X \|^3 K^2 . \tag{C.14}$$

Recalling that $\| X \| \leqq K_2 \varepsilon^2$, we thus obtain

$$\frac{1}{2}\frac{d}{d\eta}|a_0|^2 \geqq |a_0|^2(\alpha_0 - |a_0|^2 - 24\varepsilon^2) - |a_0|(K_2 C_4 \varepsilon^4 + \mathcal{O}(\varepsilon^5)) . \tag{C.15}$$

Assuming that ε is sufficiently small, it is easy to verify that the right-hand side is positive and bounded away from zero if $\varepsilon^3 \leqq |a_0| \leqq \sqrt{\alpha_0}(1 - \varepsilon)$. □

Proof of Lemma 4.7. We have to bound the difference $e^{\alpha_n(c)t} - e^{\alpha_n t}$ for sufficiently large $c \geqq c_0$. In view of (4.1), $\alpha_n(c) = -c\lambda_{n+}$ is in fact a function of c and q_n, and we shall deal with the cases $|q_n| \leqq c^{1/4}$, $|q_n| > c^{1/4}$ separately. First, if $|q_n| \leqq c^{1/4}$, we always have $c \geqq 8(1 + |q_n|)$ if c is sufficiently large, so that $|\alpha_n(c) - \alpha_n| \leqq (5/c)(1 + |q_n|)^3 \leqq 40/c^{1/4}$, by (4.1). Since $\alpha_n \leqq 1$ and $\mathrm{Re}\,\alpha_n(c) \leqq c\hat{\lambda} = 1 + \mathcal{O}(1/c^2)$ for all n, it follows that

$$|e^{\alpha_n(c)t} - e^{\alpha_n t}| \leqq t e^{c\hat{\lambda} t} |\alpha_n(c) - \alpha_n| \leqq e^{\lambda t} \sup_{t \geqq 0} (t e^{(c\hat{\lambda} - \lambda)t}) \frac{40}{c^{1/4}} = e^{\lambda t} \mathcal{O}\left(\frac{1}{c^{1/4}}\right) ,$$

for all $t \geqq 0$. Conversely, assume that $|q_n| > c^{1/4}$; then, since $\operatorname{Re} \alpha_n(c)$ is a decreasing function of $|q_n|$ (as can be seen from Fig. 4), we have the bound

$$\operatorname{Re}(\alpha_n(c)) = \frac{c^2}{2} \operatorname{Re}\left(1 - \sqrt{1 - \frac{4}{c^2} + \frac{4iq_n}{c}}\right) \leqq \frac{c^2}{2} \operatorname{Re}\left(1 - \sqrt{1 - \frac{4}{c^2} + \frac{4i}{c^{3/4}}}\right)$$

$$\leqq \frac{c^2}{2}\left(1 - \left(1 + \frac{2}{c^{3/2}} + \mathcal{O}(1/c^2)\right)\right) = -\sqrt{c} + \mathcal{O}(1)\,.$$

As a consequence, $\operatorname{Re} \alpha_n(c) \leqq -\sqrt{c}/2$ if c is sufficiently large, and since $\alpha_n \leqq 1 - \sqrt{c}$, the same is true for $\operatorname{Re} \alpha_n$. Thus, for all $t \geqq c^{-1/4}$, we have $|e^{\alpha_n(c)t} - e^{\alpha_n t}| \leqq 2e^{-\sqrt{c}t/2} \leqq 2e^{-\frac{1}{2}c^{1/4}} \leqq 4/c^{1/4}$. Combining the two cases $|q_n| \leqq c^{1/4}$, $|q_n| > c^{1/4}$, the assertion follows. □

Proof of Lemma 4.8. Using the definitions of Λ, F^c, F, we easily obtain for all $t \in \mathbf{R}_+$,

$$\| e^{\Lambda t}(F^c(A) - F(A)) \| \leqq \| e^{\Lambda t}(F(A + h(A)) - F(A)) \|$$

$$+ \sup_{n \in \mathbf{Z}} (e^{\alpha_n t} | v_n(c) - 1 |)\, \| F(A + h(A)) \|\,. \qquad \text{(C.16)}$$

In view of Proposition 4.1, we have $\| F(A + h(A)) - F(A) \| \leqq K_4 \frac{1}{2}(c_0/c)^2 \| A \|$ and $\| F(A + h(A)) \| \leqq K_4(1 + \frac{1}{2}(c_0/c)^2) \| A \|$ for all $A \in \mathscr{B}_{2R} \subset \mathscr{H}$. So, it remains to bound the difference $e^{\alpha_n t}(v_n(c) - 1)$, and this can be done as above, by dealing with the cases $|q_n| \leqq \sqrt{c}$, $|q_n| > \sqrt{c}$ separately. First, if $|q_n| \leqq \sqrt{c}$, we always have $c \geqq 8(1 + |q_n|)$ if c is sufficiently large, and thus $|v_n(c) - 1| \leqq (4/c)(1 + |q_n|) \leqq (8/c)(1 + \sqrt{c}) \leqq (8/\sqrt{c})$, by (4.4). So, $e^{\alpha_n t}|v_n(c) - 1| \leqq (8/\sqrt{c})e^{\lambda t}$ in this case. Conversely, if $|q_n| > \sqrt{c}$, then $\alpha_n = 1 - q_n^2 \leqq 1 - c$, so that $e^{\alpha_n t} \leqq e^{\lambda t}e^{-ct} \leqq e^{\lambda t}e^{-\sqrt{c}} \leqq e^{\lambda t}/\sqrt{c}$ for all $t \geqq 1/\sqrt{c}$; on the other hand, $|v_n(c) - 1| \leqq 1 + \mathcal{O}(1)$ by (4.3). So, combining the results for $|q_n| \leqq \sqrt{c}$ and $|q_n| > \sqrt{c}$, we obtain

$$\sup_{t \geqq c^{-1/2}} \sup_{n \in \mathbf{Z}} \left(e^{\alpha_n t} | v_n(c) - 1 | e^{-\lambda t} \right) \leqq \frac{C}{\sqrt{c}}\,,$$

for some $C > 0$. Combining this with (C.16), the assertion follows. □

Proof of Lemma 4.9. By definition of $\hat{A}^c$ and by construction of the center-unstable manifold, we have $\hat{A}^c + h(\hat{A}^c) = C \in \mathscr{H}$, where $C_n = \sqrt{\alpha_0}\,\delta_{n,0}$. Thus, if $A = \hat{A}^c + X$, then $A + h(A) = C + X + \Delta h(X)$, where $\Delta h(X) = h(\hat{A}^c + X) - h(\hat{A}^c)$. As a consequence, it follows from (B.1) that $F(A + h(A)) = F(C) + G(C, X) + \hat{R}^c(X)$, where $\hat{R}^c(X) = F(X + \Delta h(X)) + G(X + \Delta h(X), C) + G(C, \Delta h(X))$. On the other hand, using the definitions of F and G, we find $F_n(C) = \alpha_0^{3/2}\delta_{n,0}$, $G_n(C, X) = 2\alpha_0 X_n + \alpha_0 X^*_{-n}$. So, inserting $A = \hat{A}^c + X$ in (4.5) and noting that $\alpha_0(c)\hat{A}^c_0 \equiv \alpha_0^{3/2}$, we obtain the following equation for X:

$$X'_n = \alpha_n(c)X_n - v_n(c)(2\alpha_0 X_n + \alpha_0 X^*_{-n} + \hat{R}^c_n(X))\,, \quad n \in \mathbf{Z}\,. \qquad \text{(C.17)}$$

Obviously, the linear operator in (C.17) is block diagonal. Choosing X_n, X^*_{-n} as (complex) variables, the n^{th} block ($n \neq 0$) is given by the 2×2 matrix:

$$M_n^c = \begin{pmatrix} \alpha_n(c) - 2\nu_n(c)\alpha_0 & -\nu_n(c)\alpha_0 \\ -\nu^*_{-n}(c)\alpha_0 & \alpha^*_{-n}(c) - 2\nu^*_{-n}(c)\alpha_0 \end{pmatrix}.$$

If $n = 0$, we use the identity

$$\begin{aligned} &\alpha_0(c)X_0 - \nu_0(c)(2\alpha_0 X_0 + \alpha_0 X_0^*) \\ &\quad = -\alpha_0(X_0 + X_0^*) + (1 - \nu_0(c))\alpha_0(X_0 + X_0^*) - \nu_0(c)\lambda_{0+}^2 X_0 \\ &\quad = -\alpha_0(X_0 + X_0^*) + \tilde{R}^c(X), \end{aligned}$$

and include the quantity $\tilde{R}^c(X)$ in the non-linear term. Thus, choosing $\operatorname{Re} X_0$, $\operatorname{Im} X_0$ as (real) variables, the 0^{th} block is given by

$$M_0^c = \begin{pmatrix} -2\alpha_0 & 0 \\ 0 & 0 \end{pmatrix}.$$

So, defining $R_n^c(X) = -\nu_n(c)\hat{R}_n^c(X) + \delta_{n,0}\tilde{R}^c(X)$, we can rewrite (C.17) in the form

$$\begin{aligned} \begin{pmatrix} X'_n \\ X^{*\prime}_{-n} \end{pmatrix} &= M_n^c \begin{pmatrix} X_n \\ X^*_{-n} \end{pmatrix} + \begin{pmatrix} R_n^c(X) \\ R^{c*}_{-n}(X) \end{pmatrix}, \\ \begin{pmatrix} \operatorname{Re} X'_0 \\ \operatorname{Im} X'_0 \end{pmatrix} &= M_0^c \begin{pmatrix} \operatorname{Re} X_0 \\ \operatorname{Im} X_0 \end{pmatrix} + \begin{pmatrix} \operatorname{Re} R_0^c(X) \\ \operatorname{Im} R_0^c(X) \end{pmatrix}, \end{aligned} \tag{C.18}$$

which is nothing but (4.8).

Now, we bound the Lipschitz constant of the non-linear term $R^c(X)$ in the ball $\mathscr{B}_{2\rho} \subset \mathscr{H}$. Since $\operatorname{Lip}(\Delta h) \leqq \frac{1}{2}(c_0/c)^2$ by Proposition 4.1, we see from (2.5), (4.3) that

$$\begin{aligned} \operatorname{Lip}(\hat{R}^c) &\leqq \operatorname{Lip}(F(\cdot + \Delta h(\cdot))) + \operatorname{Lip}(G(\cdot + \Delta h(\cdot), C)) + \operatorname{Lip}(G(C, \Delta h(\cdot))) \\ &\leqq C\rho^2 + C\rho + C(c_0/c)^2 \leqq C(2\rho + (c_0/c)^2), \end{aligned}$$

for some $C > 0$, if $\rho \leqq 1$. On the other hand, it follows from (4.1), (4.3), (4.4) that $|\nu_n(c)| = \mathcal{O}(1)$ and $\operatorname{Lip}(\tilde{R}^c) = \mathcal{O}(1/c)$ as $c \to \infty$. Thus, combining these results, we conclude that $\operatorname{Lip}(R^c) \leqq K_5(\rho + (c_0/c))$ in $\mathscr{B}_{2\rho} \subset \mathscr{H}$, for some $K_5 > 0$.

To complete the proof of Lemma 4.9, it remains to show that $\| e^{M_n^c t} \| \leqq \hat{D} e^{-\frac{1}{2}\alpha_0 t}$ for all $t \in \mathbf{R}_+$ and all $n \in \mathbf{Z}^*$, if c is sufficiently large. Let $\mu^c_{n\pm}$ be the eigenvalues of the matrix M_n^c; we claim that $\operatorname{Re} \mu^c_{n\pm} \leqq -\frac{1}{2}\alpha_0$ for all n, if c is sufficiently large. To see this, we first study the limiting matrix

$$M_n = \lim_{c\to\infty} M_n^c = \begin{pmatrix} \alpha_n - 2\alpha_0 & -\alpha_0 \\ -\alpha_0 & \alpha_{-n} - 2\alpha_0 \end{pmatrix},$$

whose eigenvalues are

$$\mu_{n\pm} = \frac{1}{2}\left(\alpha_n + \alpha_{-n} - 4\alpha_0 \pm \sqrt{4\alpha_0^2 + (\alpha_n - \alpha_{-n})^2}\right).$$

Obviously, $\operatorname{Re} \mu_{n\pm} \leqq \max(\alpha_n, \alpha_{-n}) - \alpha_0 \leqq \alpha_1 - \alpha_0 = \varepsilon^2 - \alpha_0$ for all $|n| > 1$. On the other hand, using (4.1), (4.4) and setting $\hat{q}_n = \max(|q_n|, |q_{-n}|)$, it is not difficult to show that $\| M_n^c - M_n \| = \mathcal{O}(1/c)(1 + |\hat{q}_n|)^3$ if $c \geqq 8(1 + |\hat{q}_n|)$. It then follows from perturbation theory [K], that $|\mu_{n\pm}^c - \mu_{n\pm}| \leqq \| M_n^c - M_n \|$, if $\| M_n^c - M_n \| \leqq \alpha_0/2$.

Now, assume that $|\hat{q}_n| \leqq c^{1/4}$. Then $\| M_n^c - M_n \| = \mathcal{O}(c^{-1/4}) \leqq \alpha_0/4$ if c is sufficiently large, and thus $\mathrm{Re}(\mu_{n\pm}^c) \leqq \mathrm{Re}(\mu_{n\pm}) + |\mu_{n\pm}^c - \mu_{n\pm}| \leqq \varepsilon^2 - \alpha_0 + \alpha_0/4 \leqq -\alpha_0/2$. Conversely, if $|\hat{q}_n| > c^{1/4}$, we know from the proof of Lemma 4.7 that $\mathrm{Re}\,\alpha_{\pm n}(c) \leqq -\sqrt{c}/2$. Since the other matrix elements of M_n^c are $\mathcal{O}(1)$ as $c \to \infty$, the same perturbation argument shows that $\mu_{n+}^c = \alpha_n(c) + \mathcal{O}(1)$ and $\mu_{n-}^c = \alpha_{-n}^*(c) + \mathcal{O}(1)$. In particular, Re $\mu_{n\pm}(c) \leqq -\alpha_0/2$ if c is sufficiently large.

To bound the exponential of M_n^c, we let S_n^c be an invertible 2×2 matrix such that $(S_n^c)(M_n^c)(S_n^c)^{-1}$ is diagonal. Then $\| e^{M_n^c t} \| \leqq D_n^c e^{-(\alpha_0/2)t}$ for all $t \in \mathbf{R}_+$, with $D_n^c = \| S_n^c \| \, \| (S_n^c)^{-1} \|$. It is not difficult to see that $\sup_n D_n^c \leqq \hat{D} < \infty$ for some $\hat{D} > 1$, uniformly in c. The reason is that the quantities $\alpha_n(c)$, $\alpha_{-n}^*(c)$ in M_n^c grow at least like $\sqrt{n}$ as $n \to \infty$, whereas the other matrix elements remain bounded; thus, M_n^c is nearly diagonal if n is large, and then $S_n^c \to 1$ as $n \to \infty$. The details of this argument are completed by separating the cases $|\hat{q}_n| \leqq c^{1/4}$, $|\hat{q}_n| > c^{1/4}$. □

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