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Fusion and Singular Vectors in $A_1^{(1)}$ Highest Weight Cyclic Modules

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Abstract. We show how the interplay between the fusion formalism of conformal field theory and the Knizhnik-Zamolodchikov equation leads to explicit formulae for the singular vectors in the highest weight representations of $A_1^{(1)}$.

I. Introduction

Infinite dimensional Lie algebras occur everywhere in the study of 2-d conformal field therories: the Virasoro algebra and the affine algebras are the most common examples. However, the construction of the irreducible representations of these algebras is quite involved. Singular vectors are important because they indicate the existence of subrepresentations in a given representation. In the affine case, Kac and Kazhdan [12] gave the criterion for the reducibility or irreducibility of the Verma modules and Malikov, Feigin, and Fuks [16] found a formula for the singular vectors. This formula looks very simple, but involves an analytic continuation to make sense, which makes it very difficult to use.

Apart from the purely mathematical description, several approaches motivated by physics have been proposed, based on vertex operators (see [18] for a general reference dealing with $A_1^{(1)}$), bosonization and variants of the Feigin and Fuks construction and BRST cohomology [4]. In the physical context, the importance of singular vectors comes from Ward identities: to calculate a correlation function involving a descendent of a primary field, one simply applies a linear operator to the correlation function of the primary [2]. A singular vector is a descendent that is set to zero in an irreducible representation, with the consequence that the correlation functions of the corresponding primary satisfy closed linear relations, leading to a contour integral representation.

One of the aims of this paper is to show that elementary methods of conformal field theory allow us to understand some important features of the structure of

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representations of these algebras. Our inspiration comes from remarks at the end of the seminal paper of Belavin, Polyakov, and Zamolodchikov (see Appendix B in [2]). We restricted our attention to the $A_1^{(1)}$ algebra not only for simplicity (although generalization is not straightforward we believe that the same methods applied to other affine algebras will lead to interesting results), but also because we hoped to get a better understanding of the construction made in [1] by Bauer, Di Francesco, Itzykson, and Zuber for the singular vectors in Virasoro Verma modules.

The basic idea is the following: the symmetries of conformal field theories are so large that they determine "almost" completely the structure of the operator product expansion of primary fields. A remarkable homogeneous linear system, the system of descent equations (see Sect. IV.3), encodes this structure. The singular vectors are in the kernel of the descent equations, and by duality, they also appear as an obstruction to solve the linear system. This can be used to compute them.

The $A_1^{(1)}$ case has its own peculiarities, but is in a sense easier to deal with than the case of the Virasoro algebra, and a more complete treatment is possible. We still expect a precise connection between the two cases via Hamiltonian reduction [5], although as yet we have only been able to work out some simple examples.

The organization of this paper is as follows. We begin with a short reminder of the basic notions in the representation theory of affine algebras in our particular case. We introduce Verma modules, singular vectors, and the contravariant form. This is standard material, included only for the sake of completeness. For a more detailed and pedagogical presentation, see [13]. The next section quotes (without proofs and again restricting to the $A_1^{(1)}$ case) the results of Kac and Kazhdan [12], and the formula for singular vectors given by Malikov, Feigin, and Fuks [16]. In Sect. IV we introduce the notion of Verma primary fields and explain fusion from a naive point of view. This leads to the "descent equations," which summarize the structure of the operator product expansion. We end this section with some comments showing the relation with a more mathematical definition of fusion. In Sect. V we derive important consequences of the descent equations, using the contravariant form as a fundamental tool. This leads to the existence of fusion rules. In Sect. VI we recast the descent equations in triangular form, and point out the role played by the so-called Knizhnik-Zamolodchikov equation. This allows us to calculate recursively all the descendants of a primary field in a fusion process. We use this recursive form in Sect. VII to obtain explicit recursion relations or matrix forms to calculate the singular vectors. The next section is devoted to some simple comments related to our initial motivations, i.e. the relation with the case of the Virasoro algebra via Hamiltonian reduction. Some technical details are treated in appendix.

II. Basic Definitions

II.1. The $A_1^{(1)}$ Algebra

The $A_1^{(1)}$ algebra (which we shall also denote simply by $\mathscr{A}$) can be presented as a current algebra with generators k and J_n^a, $n \in \mathbf{Z}$, $a \in \{-, 0, +\}$. The non-vanishing commutators are:

$$[J_m^0, J_n^\pm] = \pm J_{m+n}^\pm \qquad [J_m^0, J_n^0] = \frac{k}{2} m\delta_{n+m}\,, \qquad\qquad (1)$$
$$[J_m^+, J_n^-] = km\delta_{n+m} + 2J_{m+n}^0\,.$$

This algebra is doubly graded if we define

$$\bar{d}(J_n^a) = a\,, \qquad \bar{d}(k) = 0\,, \qquad \underline{d}(J_n^a) = n\,, \qquad \underline{d}(k) = 0\,.$$

The so-called principal gradation $d = 2\underline{d} + \bar{d}$ is used to define several subalgebras needed to construct the $A_1^{(1)}$ Verma modules. We remark that the commutation relations with J_0^0 simply calculate the $\bar{d}$ gradation, i.e. $ad(J_0^0)$ is multiplication by $\bar{d}$. It is also useful to add to $\mathscr{A}$ a generator called $\underline{D}$ with analogous properties with respect to $\underline{d}$, that is

$$[\underline{D}, J_n^a] = nJ_n^a\,, \qquad [\underline{D}, k] = 0\,.$$

The Jacobi identities are still true because $\mathscr{A}$ is graded by $\underline{d}$. Shifting $\underline{D}$ by a constant does not change the commutation relations. We set $\hat{\mathscr{A}} = \mathscr{A} \oplus \mathbf{C}\underline{D}$. In physical applications, the Sugawara construction will provide an explicit form for $\underline{D}$, so adding it to $\mathscr{A}$ is not completely artificial. Up to an additive constant, $-\underline{D}$ will be the energy operator, which we require to be bounded below in representations.

We write

$$\hat{\mathscr{A}} = \bigoplus_{i \in \mathbf{Z}} \mathscr{E}_i = \mathscr{E}_- \oplus \mathscr{E}_0 \oplus \mathscr{E}_+\,,$$

where $\mathscr{E}_i$ is the subspace on which $d = 2\underline{d} + \bar{d}$ takes the value i and $\mathscr{E}_-$ (resp. $\mathscr{E}_+$) is the direct sum of the $\mathscr{E}_i$'s for negative (resp. positive) i's. Finally we let $\mathscr{B} = \mathscr{E}_0 \oplus \mathscr{E}_+$. The dimension of $\mathscr{E}_0$ is 3 and the dimension of $\mathscr{E}_i$, $i \neq 0$ is 1 or 2 depending on whether i is even or odd. It is easy to check that the smallest Lie subalgebra of $\mathscr{A}$ containing $\mathscr{E}_{-1}$ (resp. $\mathscr{E}_1$) is $\mathscr{E}_-$ (resp. $\mathscr{E}_+$). Furthermore $\mathscr{E}_{-1} \oplus \mathscr{E}_1$ generates $\mathscr{A}$. This last observation can be generalized (see [11]) to give an axiomatic definition of affine algebras by generators and relations, leading to a theory very akin to the theory of finite dimensional complex semi-simple Lie algebras.

We introduce now the basic tools to study a certain class of representations of $\hat{\mathscr{A}}$. We begin by recalling some useful concepts. For the rest of this section, we more or less follow [11].

II.2. Verma Modules

Let $\mathscr{G}$ be a Lie algebra. We shall denote by $U(\mathscr{G})$ its universal enveloping algebra. Representations of $\mathscr{G}$ and left $U(\mathscr{G})$-modules have the same meaning.

We recall two results which we shall need later on.

- The first one is the Poincaré-Birkhoff-Witt (PBW) theorem: fix a basis γ_i of $\mathscr{G}$ as a vector space, where i belongs to some ordered set I, then monomials of the form $\gamma_{i_1} \ldots \gamma_{i_n}$, where $i_1 \leq \ldots \leq i_n$, form a basis of $U(\mathscr{G})$ as a vector space.
- The second one is the fact that $U(\mathscr{E}_-)$ does not contain zero divisors.

For an elementary and lucid account on universal enveloping algebras, see [14].

Verma modules are usually defined by giving properties that characterize them. The starting point is a one dimensional representation of $\mathscr{E}_0$, a maximal Abelian subalgebra of $\hat{\mathscr{A}}$. In this representation, J_0^0 and k act by scalars which we denote generically by j and $t-2$. By analogy with the finite dimensional Lie algebra A_1, we shall sometimes call j the spin of the representation. The value of $\underline{D}$ is immaterial, we take it to be 0. This space is a one dimensional representation of B if we let $\mathscr{E}_+$ act as 0. We denote this representation by $\mathbf{C}^{(j,t)}$. The Verma module is the induced representation $U(\hat{\mathscr{A}}) \bigotimes_{U(\mathscr{B})} \mathbf{C}^{(j,t)}$. As an $U(\hat{\mathscr{A}})$-module this is isomorphic to the quotient of $U(\hat{\mathscr{A}})$ by the left ideal generated by $J_0^0 - j$, $k - (t-2)$, $\underline{D}$, and the J_n^a's in $\mathscr{E}_+$. Any element in $U(\hat{\mathscr{A}})$ can be written as a linear combination of terms of the form $x_- x_0 x_+$ with $x_a \in U(\mathscr{E}_a)$ for $a \in \{-, 0, +\}$. Hence $V^{j,t}$ is isomorphic to $U(\mathscr{E}_-)$ as an $U(\mathscr{E}_-)$-module. If $x \in U(\hat{\mathscr{A}})$ we denote its image in the quotient by $|x\rangle$. The module property is simply that $x|y\rangle = |xy\rangle$, and we call $|1\rangle$ the highest weight vector, a terminology borrowed from the theory of semi-simple Lie algebras. Later, when we need to manipulate several Verma modules at the same time, we shall use the notation $|j,t\rangle$ for the highest weight vector in $V^{(j,t)}$. The Verma module $V^{(j,t)}$ has the following properties:

1. The module $V^{(j,t)}$ contains a one dimensional subspace $V_{0,0}$ carrying a representation of $\mathscr{B}$ isomorphic to $\mathbf{C}^{j,t}$.
2. The smallest subspace of $V^{(j,t)}$ stable under the action of $\hat{\mathscr{A}}$ and containing $V_{0,0}$ is $V^{(j,t)}$ itself.
3. Any representation of $\hat{\mathscr{A}}$ satisfying the first two properties is isomorphic to a quotient of $V^{(j,t)}$.

Representations satisfying properties one and two are called cyclic representations.

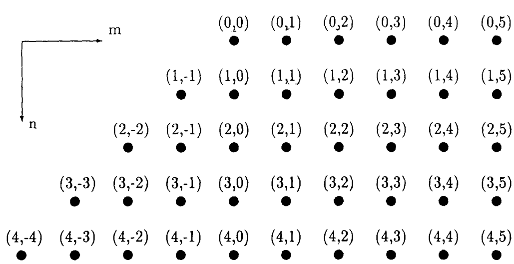

Fig. 1. The set I

Let us finally remark that $V^{(j,t)}$ is a doubly graded representation. In fact the PBW theorem implies that the monomials

$$\prod_{i=1}^{+\infty}(J_{-i}^{+})^{p_{i,+}}\prod_{i=1}^{+\infty}(J_{-i}^{0})^{p_{i,0}}\prod_{i=1}^{+\infty}(J_{-i}^{-})^{p_{i,-}}|1\rangle \tag{2}$$

(where all but a finite number of the integers p's are zero) form a basis of the Verma module. The values of $-\underline{d}$ and $-\bar{d}$ on such a monomial are respectively $n = \sum_{i,a} ip_{i,a}$ and $m = -\sum_{i,a} ap_{i,a}$, and we see that n is always non-negative and m is never less than $-n$. We denote by I (see Fig. 1) the set of couples (n,m) and end up with a decomposition

$$V^{(j,t)} = \bigoplus_{(n,m)\in I} V_{n,m}\,.$$

Highest weight cyclic modules are quotients of Verma modules. Thus they are doubly graded.

II.3. Singular Vectors, the Contravariant Form and Representation Theory

It is important to know whether $V^{(j,t)}$ is irreducible as a $U(\hat{\mathscr{A}})$-module or not. Two important tools allow us to reformulate this question and will also prove useful later on when we discuss fusion:
• Vectors lying in $V^{(j,t)}$ with vanishing projection on $V_{0,0}$ and annihilated by $\mathscr{E}_+$, called singular vectors,
• A bilinear symmetric form on $V^{(j,t)}$ called the contravariant form.

As a consequence of an elementary algebraic lemma, any linear subspace F of $V^{(j,t)}$ stable under the action of $\underline{D}$ and J_0^0 can be decomposed as

$$F = \bigoplus_{(n,m)\in I} F_{n,m}$$

with $F_{n,m} = V_{n,m} \cap F$. It follows that Verma module is irreducible if and only if it contains no singular vector. If it is reducible it contains a unique maximal submodule M_S, the smallest submodule containing the linear subspace of singular vectors.

We are now going to recover M_S from another object, the covariant bilinear form on $V^{(j,t)}$. We extend the linear anti-automorphism σ of $\hat{\mathscr{A}}$ of order two defined by

$$\sigma(J_n^a) = J_{-n}^{-a} \qquad \sigma(k) = k\,, \qquad \sigma(\underline{D}) = \underline{D}$$

to $U(\hat{\mathscr{A}})$. We let x in $U(\hat{\mathscr{A}})$ act on $V_{0,0}$ and take the projection of the result on $V_{0,0}$. This defines an endomorphism of the one dimensional space $V_{0,0}$, i.e. a complex number $l(x)$, linear in x. We can now define $b(x,y) = l(\sigma(x)y)$ for $x, y \in U(\hat{\mathscr{A}})$. The form b is bilinear symmetric and factors through a bilinear symmetric form on $V^{(j,t)}$. We use the notation $\langle x|y\rangle$ for this bilinear form called the contravariant form. The kernel of this bilinear form is M_S. We denote by $|x\rangle^*$ the linear form associating to $|y\rangle$ the complex number $\langle x|y\rangle$.

To summarize

Theorem II.1. *The following properties are equivalent:*
1. *The module $V^{(j,t)}$ is irreducible.*
2. *The module $V^{(j,t)}$ contains no singular vector.*
3. *The contravariant form on $V^{(j,t)}$ is non degenerate.*

II.4. The Sugawara Construction

The idea that in some quantum field theories, the energy-momentum tensor is a suitably renormalization bilinear combination of the currents proved to have many applications in the representation theory of affine algebras (see for instance [11]). We shall see several examples in the rest of this paper.

Let us define elements C_n for integral n by the following formulae:

$$C_n = \tfrac{1}{2} \sum_{m=-\infty}^{+\infty} (J_{n-m}^+ J_m^- + J_{n-m}^- J_m^+ + 2J_{n-m}^0 J_m^0) \quad \text{for } n \neq 0\,,$$

$$C_0 = \tfrac{1}{2}(J_0^+ J_0^- + J_0^- J_0^+ + 2J_0^0 J_0^0) + \sum_{m=1}^{+\infty} (J_{-m}^+ J_m^- + J_{-m}^- J_m^+ + 2J_{-m}^0 J_m^0)\,.$$

The expression for C_0 is some normal ordered version of the generic expression. These operators live in some completion of $U(\mathscr{A})$ but have a well-defined action on $V^{(j,t)}$. As such it is well known that they satisfy the following commutation relations:

$$[C_m, J_n^a] = -tnJ_{m+n}^a\,, \qquad [\underline{D}, C_n] = -nC_n\,,$$
$$[C_m, C_n] = t(m-n)C_{m+n} + \tfrac{1}{4}t(t-2)\,(m^3-m)\delta_{m+n}\,.$$

So, for $t \neq 0$, $V^{(j,t)}$ carries automatically a representation of the Virasoro algebra with central charge $c = 3(t-2)/t$ and conformal weight $h_j = j(j+1)/t$ if one sets $L_n = C_n/t$. The currents are primary fields of weight one. As a byproduct, we remark that the enlargement of $\mathscr{A}$ in $\hat{\mathscr{A}}$ is also automatic in the class of representations we studying. We simply use L_0 instead of $\underline{D}$.

III. Fundamental Results

We introduce some notations. The set of couples $(n, m) \in I$ such that $m \neq 0$ and n is a multiple of m is denoted by $I^{(\mathrm{sing})}$ (see Fig. 2). The elements in $I^{(\mathrm{sing})}$ are in one to one correspondence with the elements of the set $J^{(\mathrm{sing})}$ of couples of integers (α, β) such that $\alpha \neq 0$, $\beta \geq 0$, and $\alpha + |\alpha|\beta \geq 0$, by the map $(\alpha, \beta) \to (|\alpha|\beta, \alpha)$. We shall often use this parametrization of $I^{(\mathrm{sing})}$. For $(\alpha, \beta) \in J^{(\mathrm{sing})}$, we define $j_{\alpha,\beta}(t)$ to be the solution of

$$t|\alpha|\beta + \alpha(2j_{\alpha,\beta}(t) + 1 - \alpha) = 0\,.$$

The first theorem, due to Kac and Kazhdan, localizes the singular vectors in certain subspaces $V_{n,m}$.

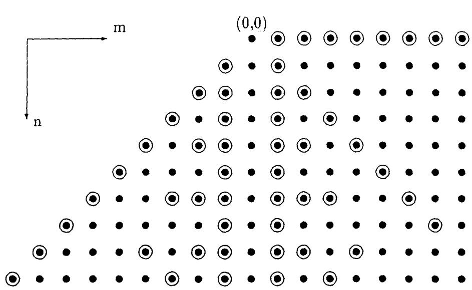

Fig. 2. The subset I^{sing} of I

Theorem III.1 (Kac-Kazhdan, [12]). *For nonzero t the Verma module $V^{(j,t)}$ contains a singular vector at level (n, m) if and only if there is a couple of integers $(\alpha, \beta) \in J^{(\mathrm{sing})}$ such that $(n, m) = (|\alpha|\beta, \alpha)$ and $j = j_{\alpha,\beta}(t)$. Then the dimension of $S_{n,m}$ is exactly one, i.e. the singular vector is unique up to an overall factor.*

This is an immediate consequence of the following lemma.

Lemma III.2 (Kac-Kazhdan, [12]). *The determinant $D_{n,m}$ of the contravariant form in $V_{n,m}$ (defined up to a non-vanishing basis dependent overall factor) is* *proportional to*

$$\Sigma_{\alpha\geq 1,\beta\geq 1} \dim V_{n-\alpha\beta,m} \prod_{(\alpha,\beta)\in J^{\mathrm{sing}}} (t|\alpha|\beta + \alpha(2j+1-\alpha))^{\dim V_{n-|\alpha|\beta,m-\alpha}} .$$

Armed with this result, it is possible to look for "explicit" expressions. This was done by Malikov, Feigin, and Fuks. We quote their result for $A_1^{(1)}$.

Theorem III.3 (Malikov-Feigin-Fuks, [16]). *Fix a nonzero t. The vector*

$$(J_0^-)^{|\alpha|+t\beta}(J_{-1}^+)^{|\alpha|+t(\beta-1)}(J_0^-)^{|\alpha|+t(\beta-2)}(J_{-1}^+)^{|\alpha|+t(\beta-3)}\dots(J_0^-)^{|\alpha|-t\beta}|j_{\alpha,\beta}(t),t\rangle \quad (3)$$

for positive α (resp. the vector

$$\begin{aligned}(J_{-1}^+)^{|\alpha|+t(\beta-1)}&(J_0^-)^{|\alpha|+t(\beta-2)}(J_{-1}^+)^{|\alpha|+t(\beta-3)}\\ &\times (J_0^-)^{|\alpha|+t(\beta-4)}\dots(J_{-1}^+)^{|\alpha|+t(\beta-1)}|j_{\alpha,\beta}(t),t\rangle \end{aligned} \quad (4)$$

for negative α) is a non-trivial element of $S_{|\alpha|\beta,\alpha}$ in $V^{(j_{\alpha,\beta}(t),t)}$, i.e. is a singular vector.

These are expressions involving complex exponents of the operators J_0^- and J_{-1}^+, and they do not make sense a priori. Malikov, Feigin, and Fuks are able to prove that they make sense by using the following trick: they prove identities relating products of integral powers of generators of $\mathscr{E}_-$, and observe that these identities admit an analytic continuation for complex powers. Starting from the above expression, by repeated application of these identities, they end up with a well-defined expression belonging to $U(\mathscr{E}_-)$ and depending polynomially on t. Moreover, naive manipulations using the commutation relations as if the exponents where non-negative integers "show" that the above expressions are singular vectors. Uniqueness of the analytic continuation integers "show" that the above expressions are singular vectors. Uniqueness of the analytic continuation ensures that this is indeed the case.

In the case when α is a positive integer and $\beta = 0$, there is no analytic continuation to implement, because (3) reduces to $(J_0^-)^\alpha|j_{\alpha,\beta}(t),t\rangle$. One recovers the well-known singular vector for the A_1-subalgebra $\{J_0^-, J_0^0, J_0^+\}$. The simplest non-trivial case where analytic continuation is needed is $(\alpha,\beta) = (1,1)$. We treat this example in Appendix A.1 to illustrate the method.

It is fair to say that explicit calculations of singular vectors remain quite complicated, but these compact formulae exhibit naturally many non-trivial properties. Among these, we quote

- The singular vectors are naturally normalized. We denoted by $\mathscr{E}_-$ the Lie algebra of generators of degree (with respect to the principal gradation d) less than 0. The generators of degrees less than -1 form an ideal in $\mathscr{E}_-$, and we can consider the quotient Lie algebra. In this quotient J_0^- and J_{-1}^+ commute, and the operators acting on $|j_{\alpha,\beta}(t),t\rangle$ to give the singular vectors reduce to $(J_0^-)^{\alpha+|\alpha|\beta}(J_{-1}^+)^{|\alpha|\beta}$.
- Another useful property of the singular vectors is that with the above normalization they are polynomial in t.

In the rest of this paper we shall give alternative formulae for the singular vectors. They are quite efficient and have an intuitive physical interpretation. They are connected with fusion rules. However, we have neither been able to show the relation between the two approaches, nor to check directly the above properties.

IV. Primary Fields and Fusion

We first give some motivation for our abstract definitions, considering for a while general properties of quantum and conformal field theories. In a Euclidean quantum field theory, the short distance singularities in the correlation functions can be understood in terms of operator product expansions: when the spatial arguments of two local operators almost coincide, we can replace their product by some asymptotic expansion in local operators with functions as coefficients. In 2-d conformal field theory, the operator product expansion, also called fusion, has a much stronger status. Its convergence is only limited by the position of the nearest operator in the correlation function under study. The symmetries of the theory are rich enough to determine almost completely the structure of the operator product expansion. This in turn leads to a purely algebraic or geometric study of the fusion.

IV.1. Motivations

Our starting point will be a naive definition of fusion based on elementary properties of the operator product expansion in 2-d conformal field theory. We shall concentrate on the holomorphic part of an unspecified conformal field theory but similar statements hold for the antiholomorphic part. A chiral field $\Phi(w)$ is called a primary field of weight h if its operator product expansion with the holomorphic component of the stress tensor T reads

$$T(z)\Phi(w) = \left(\frac{h}{(z-w)^2} + \frac{1}{(z-w)}\,\partial_w\right)\Phi(w) + \text{regular terms} \tag{5}$$

expressing in the formalism of quantum field theory that $\Phi(w)$ is an h form in the language of complex geometry. The fields appearing in this expansion are also scaling fields. They have in general more singular terms in their short distance expansion with T and $\bar{T}$. All the fields one gets by repeated operator product expansions of T with a given primary are called its descendants and they form what is called a conformal family.

When one brings two scaling fields $F_1(z)$ and $F_2(w)$ close together, one expects that in some weaks sense (for instance after insertion in a correlation function) there is an expansion

$$F_1(z)F_2(w) = \sum c^F_{F_1,F_2}(z-w)F(w)\,, \tag{6}$$

where the sum is over all scaling fields and the coefficients $c^F_{F_1,F_2}$ are functions. We can split this sum by putting together scaling fields belonging to the same conformal family. If (6) is to be true, both sides of the equality should have the same geometric properties, i.e. change in the same way under a change of coordinates (see [1]). In the field theoretic language, they should have the same operator product expansion with the components of the stress-energy tensor (which generates changes of coordinates). This is only a necessary condition, but it is very powerful as we shall see. To go from a formalism of correlation functions to an operator formalism, we use radial quantization and write $T(z) = \sum\limits_{-\infty}^{+\infty} L_n z^{-n-2}$. A simple application of the Cauchy residue theorem gives an operator version of (5),

$$[L_m, \Phi(z)] = (h(m+1)z^m + z^{m+1}\partial)\Phi(z)\,. \tag{7}$$

The operator product expansion of the stress tensor with itself gives of course the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{n+m}\,. \qquad (8)$$

In particular L_{-1} generates translations and L_0 generates dilatations.

Similar considerations apply in the case when holomorphic currents associated to some semisimple finite dimensional Lie algebra $\mathscr{G}$ are present. In this case primary fields have several components. The translation into operator language of the operator product expansion gives the commutation relations of the untwisted affine algebra associated to $\mathscr{G}$ for the commutators of the currents [that is (1) in the particular case $\mathscr{G} = A_1$]. For the commutator of a current with a primary field, we get

$$[J_n^a, \Phi_i(z)] = -z^n (R^a)_i^j \Phi_j(z)\,, \qquad (9)$$

where the matrices R^a carry a representation of $\mathscr{G}$. So we see that, apart from a minus sign, the commutator acts as the loop algebra in some representation. This time a descendant is obtained by repeated operator product expansion of the currents with a primary. It should be stressed that although the Sugawara construction leads (in a Verma module) from the $\mathscr{G}$ commutation relations to those of the Viraroso algebra for suitable central charge, the commutation relations (9) do not imply that the components of Φ are primary fields for the Sugawara stress tensor. An explicit calculation shows that one has to postulate the correct commutation relations with one of the L_n's and then the others follow. The usual choice is L_{-1}, leading to the Knizhnik-Zamolodchikov equation, which really is a dynamical equation, and not a mere tautology. We see that descendants of a primary field can split into several conformal families. By repeated use of these commutation relations (7) and (9) we can evaluate the commutator of any product of primary fields with the components of the stress-energy tensor or of the currents, i.e. in a more geometric language the behavior of such a product under a conformal or a gauge transformation.

If $|\Omega\rangle$ denotes the vacuum state (annihilated by all the L_n's with $n \geq -1$ and all the J_n^a's with $n \geq 0$), we can create new states by applying a primary field. The states $\Phi_j(0)\,|\Omega\rangle$ carry a representation of $\mathscr{G}$ and we can build on this a representation of the associated affine algebra. We expect $e^{zL_{-1}}\Phi_j(0)\,|\Omega\rangle$ to coincide with $\Phi_j(z,\bar{z})\,|\Omega\rangle$.

All these statements made sense in some a priori known conformal field theory, where operator products were assumed to be well-defined. This is a naive approach, but as we shall see, what we do is close to a more axiomatic approach.

IV.2. Verma Primary Fields

It is time now to return to the $A_1^{(1)}$ case. We let t be a fixed nonzero complex number (sectors of distinctg central charges are decoupled).

First of all we ought to define a vacuum sector. We look for a state annihilated by all the J_n^a's for $n \geq 0$. It is to be found in a cyclic module and has properties of a highest weight state. As it should be annihilated by J_0^0 the obvious candidate is the highest weight vector $|0,t\rangle$ in $V^{(0,t)}$. It not annihilated by J_0^- but clearly $J_0^-\,|0,t\rangle$ is a singular vector, so we choose for the vacuum sector the resulting quotient and denote the image of $|0,t\rangle$ (i.e. the vacuum state) by $|\Omega\rangle$.

We want now to associate a primary field to an arbitrary Verma module $V^{(j,t)}$. As we saw above, the components of this field should carry a representation of the finite dimensional A_1 algebra generated by J_0^a, $a = -, 0, +$. The subspace $\bigoplus_m V_{(0,m)}$ of $V^{(j,t)}$ carries such a representation. It is finite dimensional, spanned by the Taylor coefficients of the family of states $e^{xJ_0^-}|j,t\rangle$ (parametrized by a complex number x). On this family of states J_0^- acts as $D_j^- \equiv \partial/\partial_x$, J_0^0 as $D_j^0 \equiv j - x\partial_x$, and J_0^+ as $D_j^+ \equiv 2jx - x^2\partial_x$. Hence the natural primary field to introduce ought to depend on one extra variable x, with commutation relations

$$[J_n^a, \Phi_j(z,x)] = z^n D_j^a \Phi_j(z,x). \tag{10}$$

We call such a primary field a Verma primary field. A closely related construction was proposed in [19]. This leads to define the action of Φ_j on the vacuum by the formula

$$\Phi_j(z,x)\,|\Omega\rangle \equiv e^{zL_{-1}+xJ_0^-}\,|j,t\rangle.$$

Then we can use repeatedly the commutation relations (10) to define the action of $\Phi_j(z,x)$ on the whole vacuum sector. For fixed z and x, $e^{zL_{-1}+xJ_0^-}|j,t\rangle$ is not a state in $V^{(j,t)}$ but rather in some completion. Of course, if $V^{(j,t)}$ is not irreducible, we can replace it by a quotient module.

Let us mention a more algebraic point of view. The differential operators $\mathscr{J}_n^a \equiv -z^n D_j^a$ [resp. $\mathscr{L}_n \equiv -h_j(m+1)z^m - z^{m+1}\partial_z$] satisfy formally the commutation relations of the (non-anomalous) current (resp. Virasoro) algebra. Hence the tensor product of $V^{(j,t)}$ with a suitable space of functions of the variables x and z will carry a graded representation of $A_1^{(1)}$ and of the Virasoro algebra with the correct anomaly. Thus we can interpret $\Phi_j(z,x)\,|\Omega\rangle$ as an element of this tensor having the properties of the vacuum [i.e. it is annihilated by the same left ideal of $U(\mathscr{A})$]. We shall see a similar phenomenon when we analyze fusion.

IV.3. Fusion and Descent Equations

We shall now try to understand the structure of the operator product expansion of our Verma primary fields. Suppose that we bring Φ_{j_1} and Φ_{j_0} close together and look for their operator product expansion. For our purpose it is sufficient to consider the following state

$$\Phi_{j_1}(z,x)\Phi_{j_0}(0,0)\,|\Omega\rangle \equiv \Phi_{j_1}(z,x)\,|j_0,t\rangle. \tag{11}$$

We postulate the following expansion, which is the analogue in the operator formalism of the short distance expansion (6)

$$\Phi_{j_1}(z,x)\,|j_0,t\rangle = \sum_j |j,t,z,x\rangle, \tag{12}$$

where $|j,t,z,x\rangle$ is a (z,x) dependent state in $V^{(j,t)}$.

Covariance (with respect to the symmetries generated by the current algebra) implies non-trivial constraints for the right-hand side of this expansion. This leads to the following theorem, which is crucial for the rest of our discussion.

Theorem IV.1. *The covariance of the operator product expansion has the following consequences:*

1. *It fixes the* (z,x) *dependence of* $|j,t,z,x\rangle$ *to be*

$$|j,t,z,x\rangle = \sum_{(n,m)\in I} z^{h-h_0-h_1+n} x^{j_0+j_1-j+m} |n,m\rangle_j ,$$

with $|n,m\rangle_j \in V_{n,m}$.
2. *It leads to relations among the coefficients* $|n,m\rangle_j$,

$$J_1^- |n,m\rangle_j = (-j+j_0+j_1+m+1)\,|n-1,m+1\rangle_j \,, \tag{13}$$

$$J_0^+ |n,m\rangle_j = -(-j+j_0-j_1+m-1)\,|n,m-1\rangle_j \,. \tag{14}$$

To find these constraints, we use the following trick: the left ideal in $U(\hat{\mathscr{A}})$ generated by $J_0^0 - j_0$, $k-(t-2)$, $L_0 - h_0$, and the J_n^a's in $\mathscr{E}_+$ annihilates $|j_0,t\rangle$. Then by using the commutators (10) we get relations that the right-hand side of (12) has to satisfy. For instance $(J_0^0 - j_0)\,|j_0,t\rangle = 0$ implies $\Phi_{j_1}(z,x)\,(J_0^0 - j_0)|j_0,t\rangle = 0$ and after commutation we get

$$(J_0^0 - D_{j_1}^0 - j_0)\Phi_{j_1}(z,x)\,|j_0,t\rangle = 0 \,. \tag{15}$$

In the same way we obtain also

$$(L_0 - h_0 - z\partial_z - h_1)\Phi_{j_1}(z,x)\,|j_0,t\rangle = 0 \tag{16}$$

and

$$(J_n^a - z^n D_{j_1}^a)\Phi_{j_1}(z,x)\,|j_0,t\rangle = 0 \;\forall J_n^a \in \mathscr{E}_+ \,.$$

As we noticed before, the corresponding constraints on the right-hand side of (12) do not mix different values of j, and they apply to each term in the sum separately. So we fix j and decompose $|j,t,z,x\rangle = \sum_{n,m} |j,t,z,x,n,m\rangle$ according to the eigenvalues of L_0 and J_0^0. Then Eq. (15) and (16) imply that

$$(j - m - j_0 + x\partial_x - j_1)\,|j,t,z,x,n,m\rangle = 0$$

and

$$(h + n - h_0 - z\partial_z - h_1)\,|j,t,z,x,n,m\rangle = 0 \,,$$

so they determine completely the x and z dependence. We write

$$|j,t,z,x,n,m\rangle = z^{h-h_0-h_1+n} x^{j_0+j_1-j+m} |n,m\rangle ,$$

with $|n,m\rangle_j \in V_{n,m}$. Then we obtain for the other constraints

$$J_p^- |n,m\rangle_j = (-j+j_0+j_1+m+1)\,|n-p,m+1\rangle_j \quad \text{for } p \geq 1 \,, \tag{17}$$

$$J_p^0 |n,m\rangle_j = -(-j+j_0+m)\,|n-p,m\rangle_j \quad \text{for } p \geq 1 \,, \tag{18}$$

$$J_p^+ |n,m\rangle_j = -(-j+j_0-j_1+m-1)\,|n-p,m-1\rangle_j \quad \text{for } p \geq 0 \,. \tag{19}$$

This will be the starting point of the definition of fusion.

We expect that these equations, called the "descent equations", are compatible. A formal proof of this leads to the definition of a family (parametrized by j_0, j_1, and j) of graded representations of $\mathscr{B}$. The vector space V on which they act is a direct sum of copies of $\mathbf{C}$ indexed by couples $(n,m) \in I$, that is $V = \oplus \mathbf{C}_{(n,m)}$. We denote by $\Psi_{n,m}$ the vector with component 1 in $\mathbf{C}_{(n,m)}$ and 0 elswhere. The action of $\mathscr{B}$

on V is as follows. The vectors $\Psi_{n,m}$ are eigenvectors of L_0 and J_0^0 with eigenvalue $h+n$ and $j-m$ respectively. Moreover

$$J_p^-\Psi_{n,m} = (-j+j_0+j_1+m+1)\Psi_{n-p,m+1} \quad \text{for } p \geq 1\,,$$
$$J_p^0\Psi_{n,m} = -(-j+j_0+m)\Psi_{n-p,m} \quad \text{for} \quad p \geq 1\,,$$
$$J_p^+\Psi_{n,m} = -(-j+j_0-j_1+m-1)\Psi_{n-p,m-1} \quad \text{for } p \geq 0\,.$$

Note that we did just mimic the descent equations. It is easy to check that we indeed get a representation whatever the parameters j_0, j_1, and j are. We denote these representations by $R^j_{j_1,j_0}$. The representation property implies that Eq. (17, 18, 19) are compatible. Then they are consequences of (13, 14), because J_1^- and J_0^+ generate $\mathscr{E}_+$ by repeated commutations. □

We introduce the notation $\mu^a(j-j_0,j_1,m)$ for the scalar factors on the right-hand side of the descent equations, that is

$$J_p^a\Psi_{n,m} \equiv \mu^a(j-j_0,j_1,m)\Psi_{n-p,m-a} \quad \forall (n,m)\in I\,, \qquad \forall J_p^a \in \mathscr{E}_+\,.$$

The striking fact is that $\mu^a(j-j_0,j_1,m)$ does not depend on the L_0 degree.

In the formalism of correlation functions, mutually local fields commute. If they are not mutually local, they do not commute, but after fusion in a given sector, they commute up to a phase. Thus, in the spirit of radial quantization we expect that $\Phi_{j_1}(z,x)\,|j_0,t\rangle$ has exactly the same covariance properties as (notice the change in the operator ordering)

$$e^{zL_{-1}+xJ_0^-}\,\Phi_{j_0}(-z,-x)\,|j_1,t\rangle. \tag{20}$$

We give the proof in Appendix B. This property allows these two states to be identified, as far as covariance is concerned.

According to this discussion, we propose the following definition of fusion.

Fusion of the Verma modules $V^{(j_1,t)}$ and $V^{(j_0,t)}$ in $V^{(j,t)}$ is possible if and only if the descent equations (17–19) have a non-trivial solution. The dimension of the vector space $E^j_{j_1,j_0}$ of solutions of the set of linear equations (17–19) for the family of vectors $|n,m\rangle_j \in V_{n,m}$ is called the multiplicity of the fusion. A solution of the descent equations is said to be proper if $|0,0\rangle_j \neq 0$.

This deserves some comments.

• The first point is that we could look for analogous definitions involving quotient modules of non-irreducible Verma modules.

1. Equations (17–19) still make sense in any quotient module of $V^{(j,t)}$ and we can look for solutions in this smaller space, modifying the definition of $E^j_{j_1,j_0}$ accordingly. We shall use this generalized definition freely in the following.

2. The case when we consider a quotient module of $V^{(j_1,t)}$ or $V^{(j_0,t)}$ is more complicated. We have to introduce new constraints because the ideal annihilating the highest weight state is bigger. We shall see examples of this in Appendix C.

• The second point is concerned with the relation between our construction and the existence of intertwiners between representations. As we saw above in the definition of Verma primary fields, the differential operators $\mathscr{J}_n^a \equiv -z^n D^a_{j_1}$ [resp. $\mathscr{L}_n \equiv -h_{j_1}(m+1)z^m - z^{m+1}\partial_z$) satisfy formally the commutation relations of the (non-anomalous) current (resp. Virasoro) algebra. Hence the tensor product (denoted by $V^{(j,t)}[z,x]$) of $V^{(j,t)}$ with a suitable space of functions of the variables x and z will carry a graded representation of $A_1^{(1)}$ and of the Virasoro algebra with the correct anomaly. The covariance constraints ensure that the state $\sum\limits_{n,m} z^{h-h_0-h_1+n}x^{j_0+j_1-j+m}|n,m\rangle_j$

associated to a non-trivial element of R is a highest weight state with highest weight j_0 in this representation. As such it generates a highest weight module. Hence there is an intertwiner between $V^{(j_0,t)}$ and $V^{(j,t)}[z,x]$. In the same way one can construct an intertwiner between $V^{(j_1,t)}$ and $V^{(j,t)}[z,x]$. Admittedly this is very formal. We do not attempt to define what we mean by "suitable space of functions" and this prevents us from elucidating the structure the tensor product representation. But this suggests that our definition of fusion is reasonably close in spirit to what is usually done. Let us also observe that solving the descent equations, i.e. finding $E^j_{j_1,j_0}$, is also an intertwiner problem, because it amounts to find graded linear maps from $R^j_{j_1,j_0}$ to $V^{(j,t)}$ commuting with the action of $\mathscr{B}$.

• The third point is that we do not impose the absence of short distance singularities in x-space, that is we do not restrict to the case when $j_1 + j_0 - j$ is a nonnegative integer. This is quite unconventional but well suited to our purposes. As we shall see in Appendix C, when j_1 or j_0 are positive integers of half-integers, the singularities in x-space disappear. This is related to the existence of singular vectors (see the first remark above).

Bearing all this in mind, we can now proceed with the consequences of our definitions. To summarize, we shall give two reformulations of the descent equations. The first one will be used to show that if $V^{(j,t)}$ contains a singular vector, the descent equations cannot be solved unless j_1 and j_0 satisfy some non-trivial polynomial relation. The second reformulation will recast the descent equations in triangular form. If $V^{(j,t)}$ contains a singular vector, it appears up to a factor as an obstruction to the recursive solution of this triangular system. The first reformulation identifies the factor as (non-trivial) fusion rules, and if they do not allow the fusion of j_1 and j_0 to give j, the second reformulation of the descent equation is equivalent to a recursive formula for the singular vector.

V. First Reformulation of the Descent Equations

As they stand, the descent equations are not very tractable. For given j_1, j_0, and j, it is not at all clear whether or not they do have non-trivial solutions. However, we have the following simple bound.

Lemma V.1. *The vector space of solutions of the descent equations in an irreducible highest weight cyclic module has dimension at most one.*

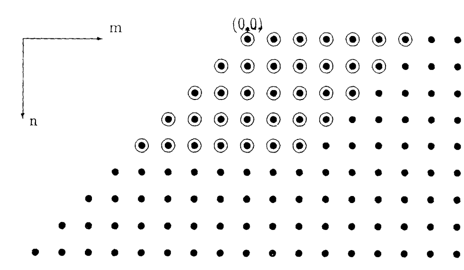

Fig. 3. The couples (n,m) satisfying $(n,m) \leq (4,2)$

We introduce a partial ordering on the couples (n,m) by the rule $(n,m) \leq (n',m')$ if and only if $n \leq n'$ and $n+m \leq n'+m'$ (see example on Fig. 3). With respect to this ordering $\mathscr{E}_+$ decreases the degree. This implies that if the descent equations do have a non-trivial solution in a highest weight cyclic module, then the nonzero $|n,m\rangle_j$ with minimal (n,m) have to be annihilated by $\mathscr{E}_+$. If the module is irreducible, the vectors annihilated by $\mathscr{E}_+$ form a one dimensional subspace generated by the highest weight state. Hence any two solutions of the descent equations are proportional. □

We are going to see that if $V^{(j,t)}$ is irreducible, the vector space of solutions of the descent equations is exactly one dimensional.

Using the representations $R^j_{j_1,j_0}$, we shall derive consequences of the descent equations which are much easier to deal with. We define a family of linear forms on $U(\mathscr{E}_+)$. Let x_+ be in $U(\mathscr{E}_+)$. If we denote by π the projection on $\mathbf{C}_{(0,0)}$ in $R^j_{j_1,j_0}$, the composition πx_+ defines a linear map from $\mathbf{C}_{(n,m)}$ into $\mathbf{C}_{(0,0)}$, i.e. (we identify the endomorphisms of $\mathbf{C}$ with $\mathbf{C}$ itself) a complex number $u_{n,m}(x_+)$, clearly linear in x_+. Then $u_{n,m}\circ\sigma$ defines a linear form on $U(\mathscr{E}_-)$, thus on $V^{j,t)}$. We denote this form by $\tilde{u}_{n,m}$ and observe that it acts non-trivialy only on $V_{n,m}$. As $|0,0\rangle_j$ is proportional to the highest weight of $V^{(j,t)}$, by applying repeatedly Eqs. (17–19) until we end at level $(0,0)$ we do in fact calculate up to a factor the "scalar products" between $|n,m\rangle_j$ and arbitrary elements of $V^{(j,t)}$. More precisely we have shown

Lemma V.2. *The descent equations imply that*

$$|n,m\rangle_j^* = \langle j,t\,|\,0,0\rangle_j \tilde{u}_{n,m} \qquad \forall (n,m)\in I\,. \tag{21}$$

If we replace the Verma module $V^{(j,t)}$ by a quotient module, we have to be careful since $\tilde{u}_{n,m}$ does not always descent to this quotient. The obstruction is clearly that $\tilde{u}_{n,m}$ should vanish on the submodule with respect to which we take the quotient. However, the former reasoning shows that if it does not, the descent equations cannot have a solution in the quotient module.

V.1. Preliminaries

To use the full strength of (21), we need to know some properties of the linear forms $\tilde{u}_{n,m}$. The action of $\tilde{u}_{n,m}$ on $V^{(j,t)}$ is simple. We begin with

Lemma V.3. *If x_- is a homogeneous element of $U(\mathscr{E}_-)$ of degree (n,m), $\tilde{u}_{n,m}(x_-|j,t\rangle)$ contains a factor $\prod_{i=1}^{m}(j-j_0+j_1-m+i)$ if $m>0$ and $\prod_{i=1}^{-m}(-j+j_0+j_1+m+i)$ if $m<0$.*

Without loss of generality, we can assume that x_- is a monomial in the generators of $\mathscr{E}_-$. It is homogeneous in the double gradation, and we call (n,m) its degree. We associate to x_- an oriented walk on the set I. The starting point is the pair (n,m). The operator $\sigma(x_-)$ is a product of generators of $\mathscr{E}_+$. Each of these generators defines a step on I according to the double gradation, and the walk ends at $(0,0)$. Knowing the walk allows x_- to be reconstructed. Relative to the ordering on I, the walk consists of a decreasing sequence. Now $\sigma(x_-)$ acts on $\mathbf{C}_{n,m}$ in $R^j_{j_1,j_0}$, and if our sole purpose is to calculate $\tilde{u}_{n,m}(x_-\,|\,j,t\rangle)$, we only need to know the projection of the oriented walk on the second factor (i.e. the space of eigenvalues of J_0^0) because the descent equation does not depend on the projection on the first factor (i.e. the space of eigenvalues of L_0). This new oriented walk goes from m to 0 and we observe that each step changes the eigenvalue of J_0^0 of at most one unit. Hence if m is strictly positive, this walk contains at least once the steps $i \to i-1$ for $i = 1, \ldots, m$. For the same reasons, if m is strictly negative this walk contains at least once the steps $i \to i+1$ for $i = m, \ldots, -1$. This leads to the announced factors. $\square$

In general, no other factor is expected, because there is always at least one monomial x_- whose associated walk consists (after projection on the second factor) only of decreasing steps if $m > 0$ and increasing steps if $m < 0$.

To go one step further in the calculation, we use the particular basis (2). Consider the monomial

$$x_- = \prod_{i=1}^{+\infty} (J^+_{-i})^{p_{i,+}} \prod_{i=1}^{+\infty} (J^0_{-i})^{p_{i,0}} \prod_{i=0}^{+\infty} (J^-_{-i})^{p_{i,-}}$$

and set $m_- = \sum_i p_{i,-}$, $m_0 = \sum_i p_{i,0}$, $m_+ = \sum_i p_{i,+}$, $\sum_{i,a} i p_{i,a} = n$, and $m_- - m_+ = m$ (then $x_- |j,t\rangle$ belongs to $V_{n,m}$). Define polynomials

$$P_{m_-,m_0,m_+}(u,v) = (u - m_-)^{m_0} \prod_{i=1}^{m_+} (v - u + m_- - m_+ + i) \prod_{i=1}^{m_-} (v + u - m_- + i) .$$

Then we have

Lemma V.4. *The linear form $\tilde{u}_{n',m'}$ takes the value $\delta_{n,n'} \delta_{m,m'} P_{m_-,m_0,m_+}(j - j_0, j_1)$ on $x_- |j,t\rangle$.*

This is a simple application of the descent equations. $\square$

As we remarked above, this "scalar product" has no dependence on the L_0 gradation, with the consequence that, in general, several monomials x_- lead to the same result. However, $(J^+_{-1})^n (J^-_0)^{n+m}$ is the only monomial having $m_- = n + m$ and $m_+ = n$. The next lemma will allow us to prove the existence of fusion rules

Lemma V.5. *For fixed m, the family of polynomials $P_{m_-,m_0,m_- - m}$ indexed by m_- and m_0 is linearly independent.*

Suppose $\sum_{m_-,m_0} \lambda_{m_-,m_0} P_{m_-,m_0,m_- - m}$ is some vanishing linear combination of these polynomials. We can group terms to get

$$\sum_{m_-} \left(\sum_{m_0} \lambda_{m_-,m_0} (u - m_-)^{m_0} \right) \prod_{i=1}^{m_- - m} (v - u + m + i) \prod_{i=1}^{m_-} (v + u - m_- + i) = 0 .$$

The degree of the polynomials

$$\prod_{i=1}^{m_- - m} (v - u + m + i) \prod_{i=1}^{m_-} (v + u - m_- + i)$$

in v is $2m_- - m$, thus they are linearly independent as polynomials in v. This implies that $\sum_{m_0} \lambda_{m_-,m_0} (u - m_-)^{m_0} = 0 \ \forall m_-$. This in turn implies that the initial linear combination was trivial, i.e. that the λ's were all zero. $\square$

V.2. Fusion in Irreducible Verma Modules

Lemma V.6. *If $V^{(j,t)}$ is irreducible, the vector space of solutions of the descent equations is exactly one dimensional. Equivalently, fusion of $V^{(j_0,t)}$ and $V^{(j_1,t)}$ in an irreducible $V^{(j,t)}$ is always possible and unique.*

In the case when $V^{(j,t)}$ is irreducible, the contravariant form is non-degenerate. Hence Eq. (21) have a unique solution if we fix the value of $\langle j,t\,|\,0,0\rangle_j$. The solution is also a solution of the descent equations. The check is easy. It is enough to check scalar products. Let x_- belong to $U(\mathscr{E}_-)$ and J^a_p belong to $\mathscr{E}_+$. We have

$$\begin{aligned}\langle x_-\,|\,(J^a_p\,|\,n,m\rangle_j) &= \langle \sigma(J^a_p)x_-\,|\,n,m\rangle_j\\ &= \tilde{u}_{n,m}(\sigma(J^a_p)x_-)\\ &= u_{n,m}(\sigma(x_-)J^a_p)\\ &= \mu^a(j-j_0,j_1,m)u_{n-p,m-a}(\sigma(x_-))\\ &= \mu^a(j-j_0,j_1,m)\,\langle x_-\,|\,n-p,m-a\rangle_j\,. \quad \square \end{aligned} \tag{22}$$

In the sequel, we shall normalize the solution by taking $|0,0\rangle_j = |j,t\rangle$.

V.3. Fusion in Reducible Verma Modules

In the case when $V^{(j,t)}$ is reducible, the contravariant form is degenerate on M_S which is a submodule, i.e. is stable under the action of $U(\mathscr{E}_-)$. This implies that the direct sum of the subspaces $V_{n,m}$ on which the contravariant form is non-degenerate (we call I' the set of couples (n,m) such that this is true, and although I' depends on j and t, we shall not mention this dependence explicitly) is a $U(\mathscr{E}_+)$-module. Hence the descent equations make sense when restricted to this subspace, and by the former reasoning, the vectors $|n,m\rangle_j$ for $(n,m)\in I'$ are completely determined once the value of $\langle j,t\,|\,0,0\rangle_j$ has been fixed, and satisfy the descent equations restricted to this subspace.

However, this solution cannot always be extended to define the states $|n,m\rangle_j$ for $(n,m)\in I\backslash I'$. This means that fusion rules have made their appearance. We shall examine them shortly. They have interest in themselves, but they will also be of use later on when we shall give formulae for the singular vectors. A word of caution is needed here. For generic values of t, there is no hope of building a respectable conformal field theory, and the word fusion we use here is an extension of what is usually meant.

Lemma V.7. *If $V^{(j,t)}$ is reducible, fusion is not always possible. The descent equation has no proper (i.e. such that $|0,0\rangle_j \neq 0$) solution in general. A necessary condition for fusion to be possible is that j_0 and j_1 satisfy non-trivial polynomial relations.*

We mentioned in Sect. III a crucial property of singular vectors, called normalization. We can rephrase it by saying that if $V^{(j,t)}$ contains a singular vector at level (n,m) [there is no need at this point to be more precise, but we recall that (n,m) cannot be arbitrary in I] and if we expand it in the basis (2) the coefficient of $(J^+_{-1})^n(J^-_0)^{n+m}$ is nonzero and can be rescaled to one (this is the normalization we find if we use the Malikov, Feigin, and Fuks expressions). The result on linear independence (Lemma V.5) proved in the preliminaries shows that the value of $\tilde{u}_{n,m}$ on this singular vector is a non-zero polynomial in j_0 and j_1. Hence (21) implies that fusion is not possible unless either j_0 and j_1 satisfy a non-trivial relation containing t as a parameter, or $\langle j, t\,|0,0\rangle_j$ is taken to be zero. □

These are a priori only necessary conditions. The second one means that the operator product expansion, if possible, is less singular than expected. Of course, if $V^{(j,t)}$ contains several singular vectors, each one contributes a (possibly redundant) constraint on fusion.

If $V^{(j,t)}$ is reducible, it contains at least one non-trivial submodule, and we can look for solutions of the descent equations in the quotient module. As any submodule contains a singular vector, the proof of the above lemma shows that there is in general an obstruction to extending the linear forms $\tilde{u}_{n,m}$ to the quotient (see remark after Lemma V.2), with the consequence that the fusion rules are also non-trivial in this case.

V.4. Truncation of the Descent Equations

We shall now see that the descent equation can be truncated in several ways.

Lemma V.8. *If $-j+j_0+j_1$ is a nonnegative integer i_+, it is possible to restrict the descent equations to the subspaces $V_{n,m}$ such that $m \geq -i_+$. If $-j+j_0-j_1$ is a nonpositive integer i_-, it is possible to restrict the descent equations to the subspaces $V_{n,m}$ such that $-i_- \geq m$.*

In the first case, the descent equations connecting the domain $m \geq -i_+$ with the rest of I state that $J_p^+|n,-i_+\rangle_j = 0$. In the second case, the descent equations connecting the domain $-i_- \geq m$ with the rest of I state that $J_p^-|n,-i_-\rangle_j = 0$. Hence the announced truncation is possible. □

In fact, we have a more precise result, stating that in the rest of I', the solution of the descent equations is identically 0.

Lemma V.9. *If $-j+j_0+j_1$ is a nonnegative integer i_+ and if $(n,m) \in I'$ is such that $m < -i_+$ then $|n,m\rangle_j = 0$. If $-j+j_0-j_1$ is a nonpositive integer i_- and if $(n,m) \in I'$ is such that $-i_- < m$ then $|n,m\rangle_j = 0$.*

This is a simple application of Lemma V.3 and the fact that the contravariant form is non-degenerate on $V_{n,m}$, $(n,m) \in I'$. □

If both the above conditions are satisfied, (in which case $j_1 = (i_+ - i_-)/2$ is a nonnegative integer or half-integer) this truncation is related to the fusion of quotients of Verma modules. This is shown in Appendix C, where a derivation, using our technique, of the (well known) fusion rules for the unitary models is also given.

V.5. Algebraic Structure of the Solutions of the Descent Equations

To close this section, we make some comments on the behavior of the solutions of the descent equation as functions of the parameters j, j_0, j_1, and t.

We already remarked that all Verma modules are isomorphic to $U(\mathscr{E}_-)$ as $U(\mathscr{E}_-)$-modules. This allows us to consider them in a uniform way.

Lemma V.10. *The action of $\hat{\mathscr{A}}$ (hence of $U(\hat{\mathscr{A}})$) on $V^{(j,t)}$ is polynomial in j and t.*

To give a precise content to this lemma, we use our preferred basis (2) in $V^{j,t}$ to write down the matrices of the linear maps J_p^a mapping $V_{n,m}$ into $V_{n-p,m-a}$. That the matrix elements are polynomial in j and t (in fact of degree ≤ 1) is an immediate consequence of the constitutive commutation relations (1). The same property of course holds if we choose another j and t independent basis of $U(\mathscr{E}_-)$. $\square$

If $V^{(j,t)}$ is irreducible, we have seen that the descent equations have exactly one normalized solution and we can interpret the normalized sequence $|n,m\rangle_j$ as a sequence $x_-^{n,m}$ in $U(\mathscr{E}_-)$.

Lemma V.11. *Each $x_-^{n,m}$ is rational in j and t and polynomial in j_0 and j_1. The poles in j can occur only at zeroes $j_{\alpha,\beta}(t)$ of the determinant of the contravariant form.*

According to Theorem III.1, for fixed (n,m) and $t \neq 0$, there is only a finite number of values of j such that the contravariant form is degenerate on $V_{n,m}$ in $V^{(j,t)}$. The determinant of the contravariant form is polynomial in j and t and the linear forms $\tilde{u}_{n,m}$ evaluated at members of the basis (2) depend on j, j_0, and j_1 polynomially. Hence the solution of the system (21), whose determinant is the determinant of the contravariant form at level (n,m), has the announced properties. $\square$

The singularities of $x_-^{n,m}$ as a function of j and t may depend on the value of j_0 and j_1. The two above lemmas lead to the following

Corollary V.12. *If for a certain choice of (α,β), j_0 and j_1, each and every $x_-^{n,m}$ has a limit when j goes to $j_{\alpha,\beta(t)}$, then the image of the limit of $x_-^{n,m}$ in $V^{(j_{\alpha,\beta}(t),t)}$ gives a solution of the descent equations.*

VI. Second Reformulation of the Descent Equations

We are now going to derive the most useful consequences of the descent equations. Then, we shall give a geometric interpretation to our computations.

VI.1. Triangular Form of the Descent Equations

The fundamental result is

Lemma VI.1. *Any solution of the descent equations satisfies*

$$\begin{aligned}
&(tn + m(2j+1-m))\,|n,m\rangle_j \\
&\quad = (-j + j_0 + j_1 + m + 1)\sum_{p=1}^{n} J^+_{-p}|n-p,m+1\rangle_j \\
&\qquad - 2(-j + j_0 + m)\sum_{p=1}^{n} J^0_{-p}|n-p,m\rangle_j \\
&\qquad - (-j + j_0 - j_1 + m - 1)\sum_{p=0}^{n} J^-_{-p}|n-p,m-1\rangle_j
\end{aligned} \tag{23}$$

for $(n,m) \neq (0,0)$.

Multiply the descent equations (17), (18), and (19) by J^+_{-p}, J^0_{-p}, and J^-_{-p} respectively. Then the sum $\sum_{p=1}^{n} J^+_{-p}(17) + 2\sum_{p=1}^{n} J^0_{-p}(18) + \sum_{p=0}^{n} J^-_{-p}(19)$ gives on the right-hand side of the equality the right-hand side of (23). On the left-hand side, one recognizes the definition of $(C_0 - J_0^0(J_0^0+1))\,|n,m\rangle_j$, which is nothing but the left-hand side of (23). □

For $(n,m)=(0,0)$ it is natural to interpret (23) as the empty relation $0=0$.
It will be useful later on to separate Eq. (23) to get a system

$$\begin{cases} \overline{|n,m\rangle_j} = (-j+j_0+j_1+m+1)\sum\limits_{p=1}^{n} J^+_{-p}|n-p,m+1\rangle_j \\ \qquad\qquad - 2(-j+j_0+m)\sum\limits_{p=1}^{n} J^0_{-p}|n-p,m\rangle_j \\ \qquad\qquad - (-j+j_0-j_1+m-1)\sum\limits_{p=0}^{n} J^-_{-p}|n-p,m-1\rangle_j \\ (tn+m(2j+1-m))\,|n,m\rangle_j = \overline{|n,m\rangle_j} \end{cases}$$

The important property of Eq. (23) is its triangular structure. The appearance of the prefactor $tn+m(2j+1-m)$ should not come as a surprise. If this prefactor does not vanish, the state $|n,m\rangle_j$ is expressed in terms of lower degree (we still use the same ordering in I). Hence, if j and t are such that $tn+m(2j+1-m)$ vanishes for no non-trivial value of (n,m) [this is more restrictive than demanding that j is not a $j_{\alpha,\beta}(t)$], (23) has a unique proper normalized solution, whatever the values of j_0 and j_1 are. By unicity, this solution has to be a solution of the descent equations. However, we can show a little more.

For fixed value of j and t, we call I'' the subset of I containing the set of pairs (n',m') such that $tn+m(2j+1-m)\neq 0$ for any $(n,m)\in I\backslash(0,0)$ such that $(n,m)\leq(n',m')$. The set I'' contains $(0,0)$.

Lemma VI.2. *Equation* (23) *restricted to* I'' *has a unique normalized solution, and this solution satisfies the descent equations.*

By the definition of I'', the direct sum $\bigoplus\limits_{(n,m)\in I''} V_{n,m}$ is a $U(\mathscr{E}_+)$-module. Hence Eq. (23) and the descent equations make sense when restricted to this subspace of $V^{(j,t)}$. It is clear from the triangular structure of (23) that the restricted equation has a unique solution. As I'' is included in I', we know that the descent equations also have a unique normalized solution for $(n,m)\in I''$. These solutions have to coincide. □

We also have a weaker result when (n,m) is "as close as possible" to I''.

Lemma VI.3. *Let* $(n,m)\in I$ *be such that* $(n',m')<(n,m)$ *implies* $(n',m')\in I''$. *Then*

$$J^a_p\overline{|n,m\rangle_j} = (tn+m(2j+1-m))\mu^a(j-j_0,j_1,m)\,|n-p,m-a\rangle_j \quad \forall J^a_p\in\mathscr{E}_+.$$

Let us first note that, with the hypotheses of the lemma, either (n,m) belongs to I'' or $tn+m(2j+1-m)=0$. According to Lemma VI.2, $\overline{|n,m\rangle_j}$, which is expressed only in terms of vectors $|n',m'\rangle_j$ with $(n',m')\in I''$, is well-defined. To prove the lemma, it is enough to check the cases $J^a_p=J^+_0$ and $J^a_p=J^-_1$. We do the calculation in detail for J^+_0, and leave the other verification to the motivated reader. Using the commutation relations (1) we obtain

$$J_0^+|\overline{n,m}\rangle_j = (-j+j_0+j_1+m+1)\sum_{p=1}^{n} J_{-p}^+ J_0^+ |n-p,m+1\rangle_j$$
$$-2(-j+j_0+m)\sum_{p=1}^{n}(J_{-p}^0 J_0^+ - J_{-p}^+)\,|n-p,m\rangle_j$$
$$-(-j+j_0-j_1+m-1)\sum_{p=0}^{n}(J_{-p}^- J_0^+ + J_{-p}^0)\,|n-p,m-1\rangle_j \,. \tag{24}$$

On the right-hand side, the descent equations are valid, because we can simply invoke Lemma VI.2. (Notice that we might also argue by induction as follows. The vector $|0,0\rangle_j$ always satisfies the descent equations. We assume that the descent equations are valid for the predecessors of (n,m) and we follow the rest of the proof of Lemma VI.3. Then if (n,m) belongs to I'', $tn + m(2j+1-m)$ does not vanish and we infer that $|n,m\rangle_j$ is well-defined and satisfies the descent equations, completing the induction step and giving an alternative proof of VI.2.). Using the descent equations we get

$$J_0^+|\overline{n,m}\rangle_j = -(-j+j_0+j_1+m+1)(-j+j_0-j_1+m)\sum_{p=1}^{n} J_{-p}^+|n-p,m\rangle_j$$
$$+2(-j+j_0+m)(-j+j_0-j_1+m-1)\sum_{p=1}^{n} J_{-p}^0|n-p,m-1\rangle_j$$
$$+2(-j+j_0+m)\sum_{p=1}^{n} J_{-p}^+|n-p,m\rangle_j$$
$$+(-j+j_0-j_1+m-1)(-j+j_0-j_1+m-2)$$
$$\times\sum_{p=0}^{n} J_{-p}^-|n-p,m-2\rangle_j$$
$$-(-j+j_0-j_1+m-1)\sum_{p=0}^{n} J_{-p}^0|n-p,m-1\rangle_j \,. \tag{25}$$

We recognize many terms of the right-hand side of (23) for the couple $(n,m-1)$. We obtain

$$J_0^+|\overline{n,m}\rangle_j = -(-j+j_0-j_1+m-1)(tn+(m-1)(2j+2-m))\,|n,m-1\rangle_j$$
$$-2(-j+j_0+m)\sum_{p=1}^{n} J_{-p}^+|n-p,m\rangle_j$$
$$+2(-j+j_0-j_1+m-1)\sum_{p=1}^{n} J_{-p}^0|n-p,m-1\rangle_j$$
$$+2(-j+j_0+m)\sum_{p=1}^{n} J_{-p}^+|n-p,m\rangle_j$$
$$-2(-j+j_0-j_1+m-1)\sum_{p=0}^{n} J_{-p}^0|n-p,m-1\rangle_j \,. \tag{26}$$

There are many cancellations on the right-hand side, and except for the first line and the term $p = 0$ in the last line, everything disappears. But J_0^0 acts on $|n-p, m-1\rangle_j$ as multiplication by $j - m + 1$, and we finally obtain

$$J_0^+ \overline{|n,m\rangle}_j = -(-j + j_0 - j_1 + m - 1)(tn + m(2j + 1 - m)) \, |n, m - 1\rangle_j \quad \square$$

We deduce the following result, which is reminiscent of Corollary V.12. For fixed nonzero t, we can consider the solution of Eq. (23) as a function of j, j_0, and j_1. A given couple (n, m) belongs to I'' for all but a finite number of values of j, and the form of Eq. (23) gives another proof that the vectors $x_-^{n,m} \in U(\mathscr{C}_-)$, introduced in Sect. V.5, are rational in j and t and polynomial in j_0 and j_1. However, the prefactor $tn + m(2j + 1 - m)$ in (23) leads to consider "spurious" poles for $x^{n,m}$. We know that the true poles are the zeroes of the determinant of the contravariant form. Hence, the only couples (n, m) that contribute to the poles are of the form $(|\alpha|\beta, \alpha)$ for $(\alpha, \beta) \in J^{\text{sing}}$.

Corollary VI.4. *Let* $(n, m) \in I$ *be such that for* $j = -t\dfrac{n}{2m} + \dfrac{m-1}{2}$, $(n', m') < (n, m)$ *implies* $(n', m') \in I''$. *If* $\overline{|n,m\rangle}_{-t\frac{n}{2m}+\frac{m-1}{2}} = 0$, *then* $x_-^{n,m}$ *has a limit when* $j \to -t\dfrac{n}{2m} + \dfrac{m-1}{2}$. *The image of this limit in* $V^{\left(-t\frac{n}{2m}+\frac{m-1}{2}, t\right)}$ *satisfies the descent equations at degree* (n, m).

We are interested in the behaviour of $|n, m\rangle_j$ near $j = -t\frac{t}{2m} + \frac{m-1}{2}$. The vector $\bar{x}_-^{n,m} \in U(\mathscr{C}_-)$ (corresponding to $\overline{|n,m\rangle}_j \in V^{(j,t)}$) is well-defined and analytic in j in a neighbourhood of $-t\dfrac{n}{2m} + \dfrac{m-1}{2}$. Hence the vanishing of $\overline{|n,m\rangle}_{-t\frac{n}{2m}+\frac{m-1}{2}}$ implies that $(tn + m(2j + 1 - m))^{-1}\bar{x}_-^{n,m}$ has a limit when $j \to -t\dfrac{n}{2m} + \dfrac{m-1}{2}$. We take this limit to be $x_-^{n,m}$ at the point $j = -t\dfrac{n}{2m} + \dfrac{m-1}{2}$. The proof that this limit satisfies the descent equations at degree (n, m) is the same as the proof of Corollary V.12. $\square$

VI.2. The Knizhnik-Zamolodchikov Equation

Although the derivation of (23) is simple, its physical meaning is not clear. We shall now show that (23) is a consequence of the Knizhnik-Zamolodchikov equation, illuminating the geometrical origin of the descent equations and their associated triangular form.

Lemma VI.5. *Equation* (23) *is the constraint on* $\Phi_{j_1}(z, x)\,|j_0, t\rangle$ *coming from the fact that* $tL_{-1} - J_{-1}^+ J_0^- - 2J_{-1}^0 J_0^0$ *annihilates the state* $|j_1, t\rangle$.

The proof is a straightforward but tedious computation. Remark that $(tL_{-1} - J_{-1}^+ J_0^- - 2J_{-1}^0 J_0^0)\,|j_1, t\rangle = 0$ comes from the definition of L_{-1} by the Sugawara construction. On $|j_1, t\rangle$, J_0^0 acts as multiplication by j_1, so we start with

$$(tL_{-1} - J_{-1}^+ J_0^- - 2j_1 J_{-1}^0)\,|j_1, t\rangle = 0$$

and multiply on the left by $e^{zL_{-1}+xJ_0^-}\Phi_{j_0}(-z,-x)$. We use the commutation relations (7) and (10) to get

$$\begin{aligned}
&e^{zL_{-1}+xJ_0^-}(t(L_{-1}+\partial_z)-(J_{-1}^+-z^{-1}D_{j_0}^+)(J_0^-+D_{j_0}^-)\\
&\quad -2j_1(J_{-1}^0+z^{-1}D_{j_0}^0))e^{zL_{-1}-xJ_0^-}e^{zL_{-1}+xJ_0^-}\Phi_{j_0}(-z,-x)\,|j_1,t\rangle=0\,.
\end{aligned}$$

We have checked in Lemma B.1 that, as far as covariance is concerned, it is not possible to distinguish $e^{zL_{-1}+xJ_0^-}\Phi_{j_0}(-z,-x)\,|j_1,t\rangle$ and $\Phi_{j_1}(z,x)\,|j_0,t\rangle$. Hence we have to compute

$$\begin{aligned}
&e^{zL_{-1}+xJ_0^-}(t(L_{-1}+\partial_z)-(J_{-1}^+-z^{-1}D_{j_0}^+)(J_0^-+D_{j_0}^-)\\
&\quad -2j_1(J_{-1}^0+z^{-1}D_{j_0}^0))e^{-zL_{-1}-xJ_0^-}\,.
\end{aligned}\tag{27}$$

This is done by repeated use of the basic commutation relations. We compute

$$\begin{aligned}
e^{zL_{-1}+xJ_0^-}J_{-1}^+e^{-zL_{-1}-xJ_0^-}&=e^{zL_{-1}}(J_{-1}^+-2xJ_{-1}^0-x^2J_{-1}^-)e^{-zL_{-1}}\\
&=\sum_{p=1}^{\infty}z^{p-1}(J_{-p}^+-2xJ_{-p}^0-x^2J_{-p}^-)\,,\\
e^{zL_{-1}+xJ_0^-}J_{-1}^0e^{-zL_{-1}-xJ_0^-}&=e^{zL_{-1}}(J_{-1}^0+xJ_{-1}^-)e^{-zL_{-1}}\\
&=\sum_{p=1}^{\infty}z^{p-1}(J_{-p}^0+xJ_{-p}^-)\,,\\
e^{zL_{-1}+xJ_0^-}J_0^-e^{-zL_{-1}-xJ_0^-}&=J_0^-\,.
\end{aligned}$$

We define $\tilde{J}^+(z)=\sum_{p=1}^{\infty}z^{p-1}J_{-p}^+$, $\tilde{J}^0(z)=\sum_{p=1}^{\infty}z^{p-1}J_{-p}^0$, and $\tilde{J}^-(z)=\sum_{p=0}^{\infty}z^{p-1}J_{-p}^+$. We can interpret these expressions as the "negative part" of the currents, the part which acts non-trivially on the highest weight state. The $p=0$ part of $\tilde{J}^-(z)$ appears in the computation of

$$\begin{aligned}
e^{zL_{-1}+xJ_0^-}D_{j_0}^+e^{-zL_{-1}-xJ_0^-}&=D_{j_0}^++x^2J_0^-\,,\\
e^{zL_{-1}+xJ_0^-}D_{j_0}^0e^{-zL_{-1}-xJ_0^-}&=D_{j_0}^++xJ_0^-\,.
\end{aligned}$$

It is now a simple matter of regrouping terms to check that (27) is equal to

$$(t\partial_z+z^{-1}(D_{j_1}^+D_{j_1}^--2j_0D_{j_1}^0))-(\tilde{J}^+(z)D_{j_1}^-+2\tilde{J}^0(z)D_{j_1}^0+\tilde{J}^-(z)D_{j_1}^+)\,.\tag{28}$$

The exchange of j_0 and j_1 is somewhat unexpected, but in fact $D_{j_1}^+D_{j_1}^--2j_0D_{j_1}^0=D_{j_0}^+D_{j_0}^--2j_1D_{j_0}^0$. If we apply (28) to the short distance expansion projected on the j-sector

$$\sum_{n,m}z^{h-h_0-h_1+n}x^{j_0+j_1-j+m}|n,m\rangle_j$$

we know that we obtain zero. Term by term identification of the powers of z and x leads to (23). □

By abuse of language, we call (23) the fused Knizhnik-Zamolodchikov equation.

The fundamental role played by the Knizhnik-Zamolodchikov equation, or its fused version (23), is not really a surprise. It is well known that this equation is related to the existence of integral representations (i.e. quite explicit forms) for the correlation functions of minimal $A_1^{(1)}$ Wess-Zumino-Witten models (see for instance [17]). This shows that it is related to the fusion, but also to the structure of singular vectors. We shall see shortly that this is indeed true.

VII. Singular Vectors

We are finally in position to propose an effective way to compute singular vectors.

VII.1. General Construction

We fix a nonzero t.

Lemma VII.1. *Let $(n,m) \in I$ be such that for $j = -t\,\dfrac{n}{2m}+\dfrac{m-1}{2}$, $(n',m') < (n,m)$ implies $(n',m') \in I''$. Then $\overline{|n,m\rangle}_{-t\frac{n}{2m}+\frac{m-1}{2}}$ is annihilated by $U(\mathscr{E}_+)$.*

If (n,m) satisfies the hypotheses, $\overline{|n,m\rangle}_{-t\frac{n}{2m}+\frac{m-1}{2}}$ is well-defined. The lemma is then a direct consequence of Lemma VI.3. $\square$

Corollary VII.2. *Under the same hypotheses, if $\overline{|n,m\rangle}_{-t\frac{n}{2m}+\frac{m-1}{2}}$ does not vanish, it is a singular vector.*

Clear from the definition of the singular vector. $\square$

Corollary VII.3. *Under the same hypotheses, if (n,m) is not of the form $(|\alpha|\beta,\alpha)$ for some $(\alpha,\beta) \in J^{\text{sing}}$, $\overline{|n,m\rangle}_{-t\frac{n}{2m}+\frac{m-1}{2}}$ does vanish.*

Clear because in this case $V^{\left(-t\frac{n}{2m}+\frac{m-1}{2}\right)}$ contains no singular vector. $\square$

Lemma VII.4. *Let $(|\alpha|\beta,\alpha) \in I$ be such that for $j = j_{\alpha,\beta}(t)$, $(n',m') < (|\alpha|\beta,\alpha)$ implies $(n',m') \in I''$. As a polynomial in j_0 and j_1, $\overline{|\beta|\alpha|,\alpha\rangle}_{j_{\alpha,\beta}(t)}$ cannot vanish identically.*

As we have seen in the proof of Lemma V.7, if $V^{(j,t)}$ contains a singular vector at level (n,m), the equation

$$|n,m\rangle_j^* = \langle j,t|0,0\rangle_j \tilde{u}_{n,m} \tag{29}$$

cannot have a solution, unless j_0 and j_1 satisfy non-trivial relations. But Corollary VI.4 shows that whenever $\overline{|\beta|\alpha|,\alpha\rangle}_{j_{\alpha,\beta}(t)}$ vanishes (for a particular value of j_0 and j_1), it is possible to define a solution of the descent equations at level $|\alpha|\beta,\alpha)$ by analytic continuation. This solution is automatically a solution of (29). $\square$

This leads to the important

Theorem VII.5. *Let t be irrational. Unless j_0 and j_1 satisfy non-trivial fusion rules, the vector $\overline{|\beta|\alpha|,\alpha\rangle}_{j_{\alpha,\beta(t)}}$ is a non-vanishing singular vector in $V^{(j_{\alpha,\beta(t)},t)}$ at level $(|\alpha|\beta,\alpha)$.*

We demand that t be irrational to be sure that the condition $(n',m') < (|\alpha|\beta,\alpha)$ implies $(n',m') \in I''$ is satisfied.

The values of j_0 and j_1 leading to a vanishing vector are restricted by polynomial equations. Hence, we can choose j_0 and j_1 almost arbitrarily to get the singular vector. We shall illustrate this point below.

VII.2. Some Matrix Forms for Singular Vectors

In Eq. (23), it is possible to put the vectors $|n,m\rangle_{j_{\alpha,\beta(t)}}$ for $(n,m) < (|\alpha|\beta,\alpha)$ together to build a column vector with $((|\alpha|\beta+\alpha+1)(|\alpha|\beta+1)-1)$ components. We have to choose a total ordering for the couples $(n,m) < (|\alpha|\beta,\alpha)$. We can even arrange things to make this total ordering compatible with the partial ordering we had before (but there is no canonical way to do this). We write for instance $\vec{f} = (|\beta|\alpha|,\alpha-1\rangle_{j_{\alpha,\beta(t)}},\ldots,|0,0\rangle_{j_{\alpha,\beta(t)}})^{\mathrm{tr}}$ and $\vec{F} = (\overline{|\beta|\alpha|,\alpha\rangle}_{j_{\alpha,\beta(t)}},0,\ldots,0)^{\mathrm{tr}}$. Equation (23) is then recast in a matrix form $\vec{F} = \mathbf{M}\vec{f}$. The matrix elements of $\mathbf{M}$ are of course operators.

We shall also use the notation $|\ \rangle^{(\mathrm{sing})}_{j_{\alpha,\beta(t)}}$ for the state $\overline{|\beta|\alpha|,\alpha\rangle}_{j_{\alpha,\beta(t)}}$. The matrix $\mathbf{M}$ is triangular.

In certain circumstances, a simpler matrix form is available. This is based on the truncation of the descent equations (see Sect. V.4 and Appendix C). If α is positive, we choose j_0 and j_1 such that $j_0-j_1 = j_{\alpha,\beta}(t)-\alpha$ and $j_0+j_1 = j$, i.e. $2j_0 = -t\beta-1$, $2j_1 = \alpha$. In this case, we know that the couples (n,m) with $m<0$ or $m>\alpha$ do not contribute. This leads to a matrix form for the singular vector, involving only the states $|n,m\rangle_{j_{\alpha,\beta(t)}}$ with $0\le m\le\alpha$ and $0\le n\le\alpha\beta$. The number of components of the vectors is reduced to $((\alpha\beta+1)(\alpha+1)-1)$. A similar construction is also possible if α is negative. To be sure that we obtain the singular vector, we ought to prove that the values of j_0 and j_1 do not satisfy the fusion rules. We conjecture that this is true.

The case, when $\alpha = 1$ is interesting. We remark that $j_{1,\beta}(t) = -\dfrac{t\beta}{2}$. The family of Eqs. (23) can be restricted to

$$\begin{aligned} tn|n,0\rangle_{-\frac{t\beta}{2}} &= \sum_{p-1}^{n} J^{+}_{-p}|n-p,1\rangle_{-\frac{t\beta}{2}} + \sum_{p=1}^{n} J^{0}_{-p}|n-p,0\rangle_{-\frac{t\beta}{2}}\,, \\ t(n-\beta)|n,1\rangle_{-\frac{t\beta}{2}} &= -\sum_{p-1}^{n} J^{0}_{-p}|n-p,1\rangle_{-\frac{t\beta}{2}} + \sum_{p=0}^{n} J^{-}_{-p}|n-p,0\rangle_{-\frac{t\beta}{2}} \end{aligned} \tag{30}$$

We recall that the singular vector is given by the right-hand side of the degenerate equation corresponding to the singular level $(n,m) = (\beta,1)$. The associated matrix form can be written explicitly. We give an example in Sect. A.2. These expressions play the same role for $A_1^{(1)}$ as do the matrix expressions (see [1]) of the Benoit-Saint Aubin formulae (see [3]) for the Virasoro algebra. We shall comment on this in the next section.

In this case, we have computed the overlap function (see Appendix D) $\Gamma_{\beta,1}$ for $(j_0, j_1) = \left(\frac{-t\beta - 1}{2}, \frac{1}{2}\right)$ and (j_0', j_1') for small values of β. This leads to

Conjecture VII.6. *When $j = j_{1,\beta}(t)$, a necessary condition for fusion from $V^{(j_0,t)}$ and $V^{(j_1,t)}$ in $V^{(j,t)}$ to be possible is the vanishing of the polynomial*

$$\prod_{m=1}^{\beta}(j_0 + j_1 - j + 1 - mt)\prod_{m=0}^{\beta}(j_0 - j_1 - j - mt).$$

If t is irrational, the vanishing of this polynomial is also a sufficient condition.

VII.3. Projection of the Recursion Relations

The family of Eq. (23) involves only $U(\mathscr{E}_-)$. We have already emphasized several times that $\mathscr{E}_-$, which consists of generators of degree less than 0 with respect to the principal gradation, contains the generators of degree less than -1 as an ideal. The quotient is a commutative Lie algebra with J_0^- and J_{-1}^+ as generators. Its universal enveloping algebra is still graded by n and m, and there is a single generator at level (n,m), $(J_{-1}^+)^n(J_0^-)^{n+m}$. We can write Eq. (23) in the quotient, replacing $|n,m\rangle_j$ by $C_{n,m}(J_{-1}^+)^n(J_0^-)^{n+m}$. The coefficients $C_{n,m}$ are complex numbers satisfying

$$\begin{aligned}(tn + m(2j+1-m))C_{n,m} &= (-j + j_0 + j_1 + m + 1)C_{n-1,m+1}\\ &\quad - (-j + j_0 - j_1 + m - 1)C_{n,m-1}\,.\end{aligned}$$

The initial condition for a proper solution is $C_{0,0} = 1$. It follows from the previous considerations that, as a function of j for fixed t, $C_{n,m}$ is rational, with poles only at the zeroes of the contravariant form. The residues at the poles give the fusion rules (this is a consequence of the normalization property of the singular vectors.) The non-appearance of the spurious poles is highly non-obvious. Hence, this innocent-looking recursion relation contains a lot of information, and it would be of great value to be able to study it independently. We have not been able to do so, and leave it as an open problem. This is an appropriate point to close this section.

VIII. Some Comments on Hamiltonian Reduction

We make some comments related to our initial motivations.

There is a close connection between the structure of the representations of the $A_1^{(1)}$ algebra and the Virasoro algebra. It uses quantum Hamiltonian reduction (see for instance [5] for references in the quantum case and [6] for the classical one, also [9] and [10] seem to be dealing with this problem using a different approach). We recall the basic steps of the construction. The idea is to introduce on Verma modules for $A_1^{(1)}$ a modified Virasoro algebra. From now on, we denote by $L_m^{(S)}$ the Virasoro generators obtained by the Sugawara construction. We set $L_m^{(N)} = L_m^{(S)} - (m+1)J_m^0$. We observe that there is no modification for $m = -1$. It is easy to check that

$$[L_m^{(N)}, L_n^{(N)}] = (m-n)L_{m+n}^{(N)} + \frac{m^3 - m}{12}(15 - 6t - 6t^{-1})\delta_{m+n}\,.$$

With respect to this new Virasoro algebra, we obtain

$$[L_m^{(N)}, J_{n-1}^+] = -(m+n)J_{n+m-1}^+ ,$$
$$[L_m^{(N)}, J_{n+1}^-] = (m-n)J_{n+m+1}^- .$$

Hence, $J^+(z) = \sum_{-\infty}^{+\infty} J_n^+ z^{-n-1}$ and $J^-(z) = \sum_{-\infty}^{+\infty} J_n^- z^{-n-1}$ are primary fields of respective weights 0 and 2. However,

$$[L_m^{(N)}, J_n^0] = -nJ_{n+m}^0 - \frac{t-2}{2} m(m+1)\delta_{n+m}$$

leading to

$$[L_m^{(N)}, J^0(z)] = ((m+1)z^m + z^{m+1}\partial_z)J^0(z) - \frac{t-2}{2} m(m+1)z^{m-1} .$$

Hence $J^0(z)$ is a scaling field of weight 1 but not a primary field.

If we replace the above commutators by Poisson brackets, the system becomes classical. If we take $J^+(z)$ as a dynamical variable, the fact it has conformal weight 0 makes it possible to reduce the phase space by the constraint $J^+(z) = 1$ without losing conformal invariance. The correct way to treat this problem in quantum field theory is to introduce ghosts.

To the bc system with commutation relations

$$\{c_m, b_n\} = \delta_{m+n} , \qquad \{c_m, c_n\} = \{b_m, b_n\} = 0$$

we associate a graded Fock space. There are two states at level 0, $|\uparrow\rangle$ and $|\downarrow\rangle$ such that $c_0|\downarrow\rangle = |\uparrow\rangle$ and $b_0|\uparrow\rangle = |\downarrow\rangle$. The states $|\uparrow\rangle$ and $|\downarrow\rangle$ are annihilated by b_n and c_n for positive n. By definition, the Fock space is the representation obtained by acting on the states at level 0 with any combination of the generators. The Fock space can be turned into a representation of the Virasoro algebra by choosing an arbitrary parameter s and taking

$$L_m^{(G)} = \sum_{n=-\infty}^{+\infty} (m(s-1)-n)b_{m+n}c_{-n} \quad \text{for } m \neq 0 ,$$
$$L_0^{(G)} = \sum_{n=1}^{+\infty} n(b_{-n}c_n + c_{-n}b_n) + \frac{s(1-s)}{2}.$$

Then one can check that

$$[L_m^{(G)}, L_n^{(G)}] = (m-n)L_{m+n}^{(G)} + \frac{m^3-m}{12}(12s(1-s)-2)\delta_{m+n} ,$$
$$[L_m^{(G)}, b_n] = (m(s-1)-n)b_{n+m} \qquad [L_m^{(G)}, c_n] = (m((1-s)-1)-n)c_{n+m} .$$

Then the fields

$$b(z) = \sum_{-\infty}^{+\infty} b_n z^{-n-s} \quad \text{and} \quad c(z) = \sum_{-\infty}^{+\infty} c_n z^{-n-1+s}$$

are primary fields of weight s and $1-s$, respectively. Remark that the regularity of $b(z)|\uparrow\rangle$ and $c(z)|\uparrow\rangle$ at $z=0$ is equivalent to the defining properties of $|\uparrow\rangle$ if and only if $s = 0$. We shall see below another reason to fix s to be zero. We define the ghost number to be 1 for $c(z)$ and -1 for $b(z)$. This leads to define the ghost number operator U by $U = \sum_{n=1}^{+\infty}(c_{-n}b_n - b_{-n}c_n) - b_0c_0$.

We can now study the tensor product of this Fock space with a highest weight cyclic $A_1^{(1)}$-module. The generator J_0^0 commutes with the Virasoro algebra (with generators $L_m^{(\mathrm{tot})} = L_m^{(N)} + L_m^{(G)}$)) and can still be diagonalized in the tensor product. To impose a quantum analog of the constraint $J^+(z) = 1$, we define $Q = \sum_{n=-\infty}^{+\infty} c_n(J_{-n-1}^+ - \delta_n, 0)$. Then $[U, Q] = Q$, i.e. Q has ghost number 1. It is easy to check that $Q^2 = 0$. The operator Q commutes with the Virasoro algebra (with generators $L_m^{(\mathrm{tot})} = L_m^{(N)} + L_m^{(G)}$) if and only if $s = 0$. We assume that $s = 0$ in the following. Then Q is proportional to $\oint c(z)\,(J^+(z) - 1)$ which is geometrically well-defined, showing clearly the relation with the appropriate constraint. Moreover, the representation of the Virasoro algebra in the tensor product has central charge $c = 13 - 6t - 6t^{-1}$, and the eigenvalue of L_0 acting on $|j,t\rangle \otimes |\uparrow\rangle$ is $h = \dfrac{j(j+1)}{t} - j = \dfrac{(2j+1-t)^2 - (1-t)^2}{4t}$. This state is clearly annihilated by the L_n's for positive n. The fundamental remark is that if we take $j = j_{\alpha,\beta}(t)$ with α positive, we get

$$h_{\alpha,\beta}(t) = \frac{(\alpha - t(\beta+1))^2 - (1-t)^2}{4t}$$

and these are just the weights for which the Virasoro Verma module is not irreducible and contains a singular vector at level $\alpha(\beta+1)$. The cohomology of Q is graded by U, and at a given degree, the cohomology space carries a representation of the Virasoro algebra. Clearly, the state $|j,t\rangle \otimes |\uparrow\rangle$ has ghost number 0 and is in the kernel of Q. It is never Q-exact. This is because the only states at level 0 for the Virasoro algebra are obtained by repeated action of J_{-1}^+ and b_0 on $|j,t\rangle \otimes |\uparrow\rangle$. But Q commutes with J_{-1}^+ and $Qb_0|j,t\rangle \otimes |\uparrow\rangle = (J_{-1}^+ - 1)\,|j,t\rangle \otimes |\uparrow\rangle$. Hence, no finite linear combination can lead to $|j,t\rangle \otimes |\uparrow\rangle$ by application of Q (we note, however, that the ill-defined $-\sum_0^{\infty}(J_{-1}^+)^n b_0|j,t\rangle \otimes |\uparrow\rangle$ would formally do the job). Hence the cohomology at ghost number 0 is non-trivial.

We believe that there is no cohomology at non-zero ghost number and that if the $A_1^{(1)}$-module is a Verma module, the cohomology at ghost number zero is a Verma module for the Virasoro algebra. This result probably exists already in the literature, but we have neither been able to find it written in an accessible language for us, nor to build a proof, although we think there should be some elementary argument.

It is easy to check that a singular vector in an $A_1^{(1)}$-module tensored with $|\uparrow\rangle$ is annihilated by Q. Our hope was then to prove that the singular vectors for $A_1^{(1)}$ with α positive could be easily rewritten as polynomials in the generators of the Virasoro algebra modulo a Q-exact term. Remark that the operator $-c_0$ has a trivial cohomology and that Q is the sum of $-c_0$ and a term decreasing the eigenvalue of J_0^0 by one. This ensures that a state annihilated by Q which is a finite linear combination of eigenstates of J_0^0 with eigenvalues greater than j is always equivalent to an eigenstate of J_0^0 with eigenvalue j modulo a Q-exact term. Hence the situation is not hopeless. But we have not been able to proceed further except in very special examples. For instance, if $j = 0$,

$$tL_{-1}^{(N)}|0,t\rangle = J_{-1}^{+}J_{0}^{-}|0,t\rangle, \qquad L_{-1}^{(G)}|\uparrow\rangle = 0 .$$

Hence

$$\begin{aligned} tL_{-1}^{(\mathrm{tot})}|o,t\rangle \otimes |\uparrow\rangle &= J_{-1}^{+}J_{0}^{-}|0,t\rangle \otimes |\uparrow\rangle \\ &= J_{0}^{-}|0,t\rangle \otimes |\uparrow\rangle + (J_{-1}^{+} - 1)J_{0}^{-}|0,t\rangle \otimes |\uparrow\rangle \\ &= J_{0}^{-}|0,t\rangle \otimes |\uparrow\rangle + Qb_0 J_{0}^{-}|0,t\rangle \otimes |\uparrow\rangle \end{aligned}$$

showing that this particular singular vector for $A_1^{(1)}$ flows to the singular vector for the Virasoro algebra under Hamiltonian reduction. If we could do this more systematically, we would probably understand much better the construction (see [1]) of singular vectors in Virasoro Verma modules. The special case $\alpha = 1$ is promising and interesting because the relation with the Benoit-Saint Aubin formulae (see [3]), but has nevertheless eluded us.

Moreover, a precise solution to these questions would give an interesting shortcut for the usual proof (see [5] and for the mathematically inclined reader [8]) that Hamiltonian reduction relates the minimal models for the $A_1^{(1)}$ and the Virasoro algebra. The usual method is quite indirect and involves bosonization in two places, with the necessity of introducing other Q operators. A direct proof would be much more illuminating. We leave this as an open problem.

IX. Conclusions and Remarks

The interplay between fusion, fusion rules and singular vectors has been used to construct these singular vectors explicitly. It is not clear for us whether these expressions can be used in other theoretical applications, but we think that the relationship between these aspects, although not unexpected, was not recognized to be so intimate. The proper interpretation of the Knizhnik-Zamolodchikov equation in our context has been of great importance. On the other hand unitarity played no role in our discussion. Some fusion rules have been computed, and a general calculation should be possible. However, many questions remain open. Among these we would like to emphasize two.

The generalization to other affine algebras would be interesting. There are serious technical difficulties, but they should not be insuperable. Much more intricate seems to be the extension to other chiral algebras. The Virasoro algebra is an example which stil needs to be better understood, and we are back to Hamiltonian reduction.

We have concentrated on purely algebraic aspects, but geometry certainly plays a fundamental role. We have some hints that a geometrical interpretation of the formulae (30) exists, and is related to the analogous geometrical interpretation of the Benoit-SaintAubin formulae in terms of covariant differential equations given in [1], inspired by [6]. We observe that the two cases are related by Hamiltonian reduction.

We hope that these questions will motivate further work.

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A. The Singular Vector at Level $(1, 1)$

The singular vector for $(\alpha, \beta) = (1, 1)$ and $j = -\dfrac{t}{2}$ is the simplest non-trivial singular vector. We compute it in two different ways.

A.1. The Method of Malikov, Feigin, and Fuks

To illustrate the technique of analytic continuation, we do the calculation in detail for $(\alpha, \beta) = (1, 1)$. So, we are trying to make sense of

$$(J_0^-)^{1+t} J_{-1}^+ (J_0^-)^{1-t}.$$

The fact that J_{-1}^+ already appears raised to an integral power (in fact 1) makes the situation comparatively easy. However, the general computation follows analogous patterns. The starting point is the identity

$$e^{xJ_0^-} J_{-1}^+ e^{-xJ_0^-} = J_{-1}^+ - 2xJ_{-1}^0 - x^2 J_{-1}^-$$

which is proved for instance by differentiation. Then we expand

$$e^{xJ_0^-} J_{-1}^+ = (J_{-1}^+ - 2xJ_{-1}^0 - x^2 J_{-1}^-) e^{-xJ_0^-}$$

in powers of x to get

$$(J_0^-)^p J_{-1}^+ = J_{-1}^+ (J_0^-)^p - 2pJ_{-1}^0 (J_0^-)^{p-1} - p(p-1) J_{-1}^- (J_0^-)^{p-2}, \qquad p = 0, 1, \ldots.$$

We observe that the coefficients are polynomial in p, and we extend these identities for complex p. Both sides are ill-defined. We take $p = 1 + t$ and multiply the identity by $(J_0^-)^{1-t}$ on the right. This leads to

$$\begin{aligned}(J_0^-)^{1+t} J_{-1}^+ (J_0^-)^{1-t} = {} & (J_{-1}^+ (J_0^-)^{1+t} - 2(1+t) J_{-1}^0 (J_0^-)^t \\ & - t(1+t) J_{-1}^- (J_0^-)^{t-1}) (J_0^-)^{1-t}.\end{aligned}$$

If we assume that the usual rules for multiplication of powers of J_0^- can be extended to complex powers, we end up with

$$(J_0^-)^{1+t} J_{-1}^+ (J_0^-)^{1-t} = J_{-1}^+ (J_0^-)^2 - 2(1+t) J_{-1}^0 J_0^- - t(1+t) J_{-1}^-.$$

The right-hand side gives a definition of the left-hand side. We remark that the left-hand side was already well-defined for $t \in \{-1, 0, 1\}$. It is easy to check that at these special values, the two definitions coincide.

Of course, we could have started with an identity for $J_{-1}^+ (J_0^-)^p$. We do not prove that the result is the same. This is a consequence of the general theory of Malikov, Feigin, and Fuks [16].

It is clear that even when $\alpha = 1$, if $\beta > 1$ formula (3) contains more factors, making the computation more and more complicated. This is to be contrasted with the form given in Eq. (30).

A.2. The Matrix Form

In the case when $(\alpha,\beta)=(1,1)$, our method leads to the following computation. The family of equations (30) reduces to

$$\begin{aligned} -t|0,1\rangle_{-t/2} &= J_0^-|0,0\rangle_{-t/2}\,,\\ t|1,0\rangle_{-t/2} &= J_{-1}^+|0,1\rangle_{-t/2}+J_{-1}^0|0,0\rangle_{-t/2}\,,\\ |\;\rangle_{-t/2}^{(\mathrm{sing})} &= -J_{-1}^0|0,1\rangle_{-t/2}+J_0^-|1,0\rangle_{-t/2}+J_{-1}^-|0,0\rangle_{-t/2}\,. \end{aligned}$$

It is easy to recast this in a matrix form. We write

$$\begin{pmatrix} |\;\rangle_{-t/2}^{(\mathrm{sing})}\\ 0\\ 0\end{pmatrix}=\begin{pmatrix} J_0^- & -J_{-1}^0 & J_{-1}^-\\ t & -J_{-1}^+ & -J_{-1}^0\\ 0 & t & J_0^-\end{pmatrix}\begin{pmatrix}|1,0\rangle_{-t/2}\\ |0,1\rangle_{-t/2}\\ |0,0\rangle_{-t/2}\end{pmatrix}.$$

We solve this triangular system and obtain

$$|\;\rangle_{-t/2}^{(\mathrm{sing})}=-\frac{1}{t^2}\left(J_0^-J_{-1}^+J_0^- - t(J_0^-J_{-1}^0+J_0^-J_{-1}^0)-t^2J_{-1}^-\right)|0,0\rangle_{-t/2}\,.$$

Using the commutation relations to rewrite the right-hand side of this equation in the basis (2), it is easy to check that the different expressions for the singular vector are proportional to each other. The analogous computations for $\beta>1$ become more and more tedious, but they are much simpler than the ones involved in the computation by analytic continuation. There is some intuitive explanation for this: our recursion formulae define the singular vector, without specifying a basis of $U(\mathscr{E}_-)$, with the consequence that in a sense "the singular vector itself chooses the way it wants to be expressed."

B. Further Covariance Constraints

We are going to study the covariance properties of the state (20) of Sect. IV.3 with respect to the current algebra. So we apply our method to the state

$$e^{zL_{-1}+xJ_0^-}\Phi_{j_0}(-z,-x)\,|j_1,t\rangle. \tag{31}$$

The left ideal annihilating $|j_1,t\rangle$ is generated by $J_0^0-j_1$, $k-(t-2)$, L_0-h_1, and the J_n^a's in $\mathscr{E}_+$. We use once more the commutators (10) and then conjugate with $e^{zL_{-1}+xJ_0^-}$ to get

$$\begin{aligned} &e^{zL_{-1}+xJ_0^-}(J_0^0-D_{j_0}^0-j_1)e^{-zL_{-1}-xJ_0^-}e^{zL_{-1}+xJ_0^-}\\ &\quad\times\Phi_{j_0}(-z,-x)\,|j_1,t\rangle=0\,,\\ &e^{zL_{-1}+xJ_0^-}(L_0-h_1-z\partial_z-h_0)e^{-zL_{-1}-xJ_0^-}e^{zL_{-1}+xJ_0^-}\\ &\quad\times\Phi_{j_0}(-z,-x)\,|j_1,t\rangle=0\,,\\ &e^{zL_{-1}+xJ_0^-}(J_n^a-(-)^{n+a}z^nD_{j_0}^a)e^{-zL_{-1}-xJ_0^-}e^{zL_{-1}+xJ_0^-}\\ &\quad\times\Phi_{j_0}(-z,-x)\,|j_1,t\rangle=0\quad\forall J_p^a\in\mathscr{E}_+\,. \end{aligned}$$

We can now prove

Lemma B.1. *The covariance constraints on* (11) *and* (20) *coincide.*

We use the commutation relations between the stress-energy tensor and the currents to check that

$$e^{zL_{-1}+xJ_0^-}(J_0^0 - D_{j_0}^0 - j_1)e^{-zL_{-1}-xJ_0^-} = J_0^0 - D_{j_1}^0 - j_0\,,$$
$$e^{zL_{-1}+xJ_0^-}(L_0 - h_0 - z\partial_z - h_1)e^{-zL_{-1}-xJ_0^-} = L_0 - h_0 - z\partial_z - h_1\,.$$

It is quite tedious to show directly that the operators

$$e^{zL_{-1}+xJ_0^-}(J_n^a - (-)^{n+a}z^n D_{j_0}^a)e^{-zL_{-1}-xJ_0^-}$$

for J_n^a in $\mathscr{E}_+$ are [(z,x) dependent] linear combinations of the operators appearing in the constraints for $\Phi_{j_1}(z,x)\,|j_0,t\rangle$. Happily, as we emphasized above, two particular constraints generate them all. So we are left with two simple computations

$$e^{zL_{-1}+xJ_0^-}(J_{-1}^- - zD_{j_0}^-)e^{-zL_{-1}-xJ_0^-} = J_{-1}^- - zD_{j_1}^-$$

and

$$\begin{aligned} e^{zL_{-1}+xJ_0^-}(J_0^+ + D_{j_0}^+)e^{-zL_{-1}-xJ_0^-} &= J_0^+ - 2xJ_0^0 + 2j_0x - x^2\partial_x \\ &= (J_0^+ - D_{j_1}^+) - 2x(J_0^0 - D_{j_1}^0 - j_0)\,. \end{aligned}$$

This concludes the proof that the covariance constraints on (11) and (20) are the same. □

It is in this sense that we can identify these two states.

C. Fusion of Quotients of Verma Modules

We give an interpretation of Lemma V.8. This will also lead to some illustrations of the comments we made after the definition of fusion. This section is very close in spirit to the computation of the fusion rules in [19]. We note that if j_1 is a nonnegative integer or half-integer, $(J_0^-)^{2j_1+1}|j_1,t\rangle$ is a singular vector in $V^{(j_1,t)}$. This singular vector generates a submodule, and we can take the quotient. In this quotient the left ideal of $U(\hat{\mathscr{A}})$ annihilating $|j_1,t\rangle$ contains $(J_0^-)^{2j_1+1}$. So if we try to implement fusion of this quotient module with $V^{(j_0,t)}$ to get $V^{(j,t)}$ there is a new constraint.

Lemma C.1. *This constraint is simply that*

$$\partial_x^{2j_1+1}\Phi_{j_1}(z,x)\,|j_0,t\rangle = 0\,,$$

where this time Φ_{j_1} stands for the primary field associated to the quotient module.

We multiply the relation $(J_0^-)^{2j_1+1}|j_1,t\rangle = 0$ on the left by

$$e^{zL_{-1}+xJ_0^-}\Phi_{j_0}(-z,-x)\,.$$

Then a simple application of the commutation relations (10) leads to

$$\partial_x^{2j_1+1}e^{zL_{-1}+xJ_0^-}\Phi_{j_0}(-z,-x)\,|j_1,t\rangle = 0\,.$$

But, as far as covariance is concerned, we have shown that it is possible to identify $e^{zL_{-1}+xJ_0^-}\Phi_{j_0}(-z,-x)\,|j_1,t\rangle$ with $\Phi_{j_1}(z,x)\,|j_0,t\rangle$. □

From this we deduce that in the j-sector

$$\partial_x^{2j_1+1}\sum_{n,m} z^{h-h_0-h_1+n}x^{j_0+j_1-j+m}|n,m\rangle_j = 0\,.$$

Hence for any $(n,m)\in I$,

$$(j_0+j_1-j+m)(j_0+j_1-j+m-1)\dots(j_0+j_1-j+m-2j_1)\,|n,m\rangle_j = 0\,. \quad (32)$$

In particular, either $|0,0\rangle_j = 0$ or $j\in\{j_0+j_1, j_0+j_1-1,\dots,j_0-j_1\}$. This is a fusion rule. It looks quite familiar. If we define i_+ and i_- by $j_1=(i_+-i_-)/2$, $j_0-j=(i_++i_-)/2$ then the content of (32) is equivalent to the truncations of the descent equations obtained in V.8. We observe that the use of the quotient module of $V^{(j_1,t)}$ to define fusion imposes that the operator product expansion has no singularity in x-space.

Using the same method one shows

Lemma C.2. *Assume j_0 is a nonnegative integer or half-integer. In the fusion of $V^{(j_1,t)}$ with the quotient module of $V^{(j_0,t)}$, the new constraint is*

$$(J_0^- - \partial_x)^{2j_0+1}\Phi_{j_1}(z,x)|j_0,t\rangle = 0$$

leading to the fusion rule $j\in\{j_1+j_0, j_1+j_0-1,\dots,j_1-j_0\}\}$.

Lemma C.3. *Assume $t/2-j_0-1$ is a nonnegative integer or half-integer. Then $(J_{-1}^+)^{t-2j_0-1}|j_0,t\rangle$ is a singular vector in $V^{(j_0,t)}$. In the fusion of $V^{(j_1,t)}$ with the quotient module of $V^{(j_0,t)}$, the new constraint is*

$$(J_{-1}^+ - z^{-1}D_{j_1}^+)^{t-2j_0-1}\Phi_{j_1}(z,x)\,|j_0,t\rangle = 0$$

leading to the fusion rule $j\in\{t-2-j_0-j_1, t-2-j_0-j_1-1\dots j_0-j_1\}$. If $t/2-j_1-1$ is a nonnegative integer or half-integer the fusion rule is $j\in\{t-2-j_1-j_0, t-2-j_1-j_0-1,\dots,j_1-j_0\}$.

Putting all these results together, we obtain the usual conditions for fusion.

- If both j_0 and j_1 are nonnegative integers or half-integers, we simply recover the law of composition of spins. The spin j has to belong to $\{j_1+j_0, j_1+j_0-1,\dots,|j_1-j_0|\}$. Thus it, too, is an integer or half-integer, and $V^{(j,t)}$ contains a singular vector. We have only obtained a necessary condition for fusion to be possible. But it is not difficult to show that the fusion involving the three quotient is possible and unique if t is irrational. We do not give the proof here.
- If moreover $t-2$ is a positive integer, we recover the full set of fusion rules for the unitary models. Note that unitarity played no direct role in the discussion. This is common in the representation theory of finite dimensional semi-simple Lie algebras, where the requirement of finite dimensionality of a representation implies its unitarity.

Let us stress once more that, although our definitions did not prevent short distance singularities in x-space, these singularities disappear when we consider fusion of quotients of appropriate Verma modules.

D. The Overlap Function

In the definition of fusion, j_0 and j_1 play the role of parameters, and it is interesting to have some kind of measure of how much the solutions of the descent equations differ at level (n, m) when j_0 and j_1 vary. The contravariant form gives such a "measure." We define the "overlap" between two solutions of the descent equations, corresponding to distinct couples (j_0, j_1) and (j'_0, j'_1) but of course with the same value of j and t to be

$$\Gamma_{n,m}(j_0, j_1, j'_0, j'_1, j, t) \equiv {}_j\langle n, m|n, m\rangle'_j .$$

Using the method of Sect. V and VI, it is easy to show that the overlap satisfies recursion relations.

Lemma D.1. *The overlap $\Gamma_{n,m}$ satisfies*

$$\begin{aligned}(tn + m(2j + 1 - m))\Gamma_{n,m} &\\ = (-j + j_0 + j_1 + m + 1)(-j + j'_0 + j'_1 + m + 1)\sum_{p=1}^{n} \Gamma_{n-p,m+1} &\\ + 2(-j + j_0 + m)(-j + j'_0 + m)\sum_{p=1}^{n} \Gamma_{n-p,m} &\\ + (-j + j_0 - j_1 + m - 1)(-j + j'_0 - j'_1 + m - 1)\sum_{p=0}^{n} \Gamma_{n-p,m-1} . &\qquad (33)\end{aligned}$$

To prove this, we first use (23) for $|n, m\rangle'_j$, and then we use the descent equations for ${}_j\langle n, m|$. This procedure in not symmetric, but the final formula treats (j_0, j_1) and (j'_0, j'_1) in a symmetric way. □

These relations are quite complicated as they stand, but by using truncation, it is possible to use them to compute for instance fusion rules.

Theorem VII.5 allows us to say something about the structure of the overlap $\Gamma_{|\alpha|\beta,\alpha}$ when $j = j_{\alpha,\beta}(t)$. In fact, for these very special values of the indices, the right-hand side of (D.1) has to split as a product of the fusion rules for (j_0, j_1) and (j'_0, j'_1). This is because $\overline{|\beta|\alpha|, \alpha\rangle}_{j_{\alpha,\beta}(t)}$ and $\overline{|\beta|\alpha|, \alpha\rangle}'_{j_{\alpha,\beta}(t)}$ are both proportional to the singular vector, and the right-hand side of (D.1) computes the obstruction to the solving of the descent equation at level $(|\alpha|\beta, \alpha)$.

Hence, the overlap equation provides a method to compute the fusion rules. When $\alpha = 1$ and $j = j_{1,\beta}(t) = -\beta\frac{t}{2}$ for instance, it is possible to compute $\Gamma_{\beta,1}$ for small values of β, taking $(j_0, j_1) = \left(\frac{-t\beta-1}{2}, \frac{1}{2}\right)$ (these are the values leading to the truncation of the descent equations) and (j'_0, j'_1). This leads to Conjecture VII.6.

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