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# Isospectral Deformations of Random Jacobi Operators

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**Summary.** We show the integrability of infinite dimensional Hamiltonian systems obtained by making isospectral deformations of random Jacobi operators over an abstract dynamical system. The time 1 map of these so called random Toda flows can be expressed by a $QR$ decomposition.

## 1. Introduction

Toda systems have been studied extensively since their discovery by Toda in 1967. Since then, several approaches for their integration have been found and many generalizations have been invented.

Examples are:

• The *tied* or *aperiodic Toda* lattice describes isospectral deformations of finite dimensional aperiodic tridiagonal Jacobi matrices. The integration is performed by taking a spectral measure as the new coordinate. From this measure, the matrix can be recovered. The measure moves linearly by the Toda flow. For a generic Hamiltonian, the matrices converge to diagonal matrices for $t \to \pm\infty$. The integration of the first flow, which has an interpretation of particles on the line, has first been performed in [Mo 1]. For the other flows see [DNT]. There are Lie algebraic generalizations of this Toda lattice [Bog 3,K, Sy 2] and interpretations as a geodesic flow [P] or a constrained harmonic motion [DLT].

• The *half infinite Toda lattice* is an infinite dimensional generalization of the tied lattice. It describes isospectral deformations of tridiagonal operators on $l^2(\mathbb{N})$. The integration is technically more difficult and has been performed in [DLT 1], [Ber]. It resembles the integration of the tied lattice. Again, the operators converge in general to diagonal operators [DLT 1].

• The *periodic Toda lattice* consists of isospectral deformations of periodic Jacobi matrices. The first flow describes a periodic chain of particles interacting with an exponential potential. The explicit integration uses methods of algebraic geometry [vM 1,vM 2]. The flow is conjugated to a motion of an auxiliary spectrum. Jacobi's map transforms this motion into a linear flow on the Jacobi variety of the hyperelliptic curve attached to the matrix. The system is periodic or quasiperiodic.

• A *rapidly decreasing case* of the Toda lattice is a situation with free particle boundary conditions at infinity. The integration is done by an inverse scattering transform [F,FT]. It is an isospectral deformation of doubly infinite Jacobi matrices decaying at infinity.

In this article, we discuss the Toda lattice with a new boundary condition: a so-called *random boundary condition.* It is a generalization of both the periodic and the aperiodic system and consists of deformations of random Jacobi operators. This generalization is a special case of an abstract generalization found by Bogoyavlensky ([Bog 1,Bog 2, p.172], who suggested to consider such differential equations in an associative algebra.

Another suggestion for investigating such random systems is offered in [CL p. 436], where it is motivated by the problem of finding a complete inverse spectral theory for random Jacobi operators, a project initiated by Carmona and Kotani([CK]).

The random Toda lattice is a discrete version of the Korteweg de Vries equation defined over a flow. A special case of such a KdV equation leads to an isospectral deformation of almost periodic Schrödinger operators. In the work of Johnson and Moser [JM] the Floquet exponent for such Schrödinger operators is introduced and the Floquet exponent is shown to be an invariant of the KdV deformation. It is natural to ask for analogous results in the discrete case.

The paper is organized as follows: In the *second section,* we define isospectral Toda deformations of random Jacobi operators. Random Jacobi operators are selfadjoint elements $L = a\tau + (a\tau)^* + b$ in the crossed product $\mathcal{X}$ of the commutative Banach algebra $L^\infty(X)$ with an abstract dynamical system $(X, T, m)$. The multiplication in $\mathcal{X}$ is just the convolution multiplication of power series $\sum L_n\tau^n$ in the variable $\tau$ with the additional rule that $\tau^k L_n = L_n(T^k)\tau^k$ for all $k, n \in \mathbb{Z}$. The random Jacobi operators form a Banach space $\mathcal{L} \subset \mathcal{X}$. For almost all $x \in X$, one gets *stochastic Jacobi matrices* $[L(x)]_{mn} = L_{n-m}(T^m x)$. We are interested in deformations of Jacobi operators which are given by a differential equation $\dot{L} = [B_H(L), L]$, where $B_H(L)$ is skew symmetric in $\mathcal{X}$ and depending on a Hamiltonian $H$. These Toda flows generalize periodic and aperiodic finite dimensional Toda flows. The periodic case is obtained when the cardinality of the set $X$ is finite. If $a(x) = 0$ on a set of positive measure one gets the aperiodic case. There is a trace in the $C^*$ algebra $\mathcal{X}$ and for each continuous function $f : \mathbb{C} \to \mathbb{C}$ there is an integral $\mathrm{tr}(f(L))$ of the Toda flow. Another integral of the deformations is the mass $\exp(\int_X \log(a)\, dm(x))$. This integral cannot be written in the form $\mathrm{tr}(f(L))$.

In the *third section* we give an integration of the random Toda lattice in the following sense: There is a mapping $\phi$ from $\mathcal{L}$ to an infinite dimensional vector space $\mathcal{G}$, in which the flow is linear. The mapping $\phi$ has a left inverse $\psi$ and the time one maps $\mathrm{Exp}_H, \overline{\mathrm{Exp}}_H$ of the flow in the old and new coordinates are related by $\mathrm{Exp}_H(L) = \psi \circ \overline{\mathrm{Exp}}_H \circ \phi(L)$. The idea is to approximate the random Toda flow by finite dimensional aperiodic Toda lattices which are known to be integrable. This approximation is due to a lemma which says, roughly speaking, that a differential equation $\dot{x} = f(x)$ in a Banach space gives a flow which is also continuous in a weaker topology, if $f$ is continuous with respect to this weaker one.

There can be transient behaviour for the random Toda lattice: The random Toda flow splits into infinitely many aperiodic finite dimensional flows, provided that $a(x)$ is zero on a set of positive measure and the underlying dynamical system is ergodic.

In the *fourth section*, we show that $QR$ decompositions generalize to the infinite dimensional case. Such a generalization is known for half infinite Jacobi operators [DLT 1]. The $QR$ decomposition for invertible matrices can also be used to express the time 1 map for each flow.

In the *fifth section* it is shown that random Jacobi operators $L$ have a determinant $\det(L - E)$ which is an integral of motion for the Toda flows. The *Floquet exponent* $w(E)$ satisfying $\det(L - E) = \exp(-w(E))$ is related by the Thouless formula to the Lyapunov exponent and to the rotation number of the transfer cocycle of $L$. These functions as well as the Taylor coefficients of $w(E)$ calculated at a point $E_0$ outside the spectrum of $L$ are also integrals. Random Jacobi operators appear in a natural way for twist diffeomorphisms. They are the second variation of a Percival functional.

At last, in the *sixth section*, we look at generalizations of random Toda flows, for example at isospectral deformations in the crossed product of any Banach algebra with a dynamical system. In the same way as the random Toda lattice, random singular decomposition flows can be defined.

## 2. Random Jacobi Operators and Random Toda Flows

### *2.1 Random Jacobi operators.*

A *dynamical system (X,T,m)* is an automorphism $T$ of a probability space $(X, m)$. Consider the set of sequences $K_n \in L^\infty(X)$, where $K_n \neq 0$ only for finitely many $n \in \mathbb{Z}$. This forms an algebra with the multiplication

$$(KM)_n(x) = \sum_{k+m=n} K_k(x) M_m(T^k x) .$$

The algebra carries an involution given by

$$(K^*)_n(x) = \overline{K_{-n}}(T^n x) .$$

We denote by $\mathcal{X}$ the completion of this algebra with respect to the norm

$$|||K||| = |\ ||K(x)||\ |_\infty \ ,$$

where $K(x)$ is the bounded operator in $l^2(\mathbb{Z})$ given by the infinite matrix

$$[K(x)]_{mn} = K_{n-m}(T^m x) .$$

The multiplication and involution in $\mathcal{X}$ is defined such that

$$K \in \mathcal{X} \mapsto K(x) \in \mathcal{B}(l^2(\mathbb{Z}))$$

is an algebra homomorphism:

$$KL(x) = K(x)L(x), \ \ K^*(x) = K(x)^* .$$

The algebra $\mathcal{X}$ is a $C^*-$ algebra called the *crossed product* of $L^\infty(X)$ with the dynamical system $(X, T, m)$. Elements in $\mathcal{X}$ are called *random operators*. For $K \in \mathcal{X}$ we define the *trace* by

$$\mathrm{tr}(K) = \int_X K_0 \, dm \; .$$

For all $K, M \in \mathcal{X}$,

$$\begin{aligned} \mathrm{tr}(KM) &= \int \sum_n K_n M_{-n}(T^n) \, dm = \int \sum_n K_{-n} M_n(T^{-n}) \, dm \\ &= \int \sum_n M_n K_{-n}(T^n) \, dm = \mathrm{tr}(MK) \; . \end{aligned}$$

It follows that for any invertible $U \in \mathcal{X}$ and every $K \in \mathcal{X}$,

$$\mathrm{tr}(UKU^{-1}) = \mathrm{tr}(K) \; .$$

In order to simplify the writing and the algebraic manipulations, we will write elements $K \in \mathcal{X}$ in the form

$$K = \sum_n K_n \tau^n$$

and think of $\tau$ just as a symbol. The multiplication in $\mathcal{X}$ is the multiplication of power series with the additional rule $\tau^k K_n = K_n(T^k)\tau^k$ for shifting the $\tau$'s to the right and the requirement that $\tau^* = \tau^{-1}$. If we interpret $\tau$ as a shift operator $f \mapsto f(T)$ in $L^2(X)$ and $K_n$ as a multiplication operator, we have a representation of $\mathcal{X}$ in $\mathcal{B}(L^2(X))$:

$$Kf = \sum_n K_n f(T^n) \; .$$

If the dynamical system $(X, T, m)$ is ergodic, there exists for each $K \in \mathcal{X}$ a set of full measure such that for $x$ in this set, the operators $K(x)$ have the same spectrum $\Sigma(K(x))$ denoted by $\Sigma(K)$. This is a version of *Pastur's theorem*. The proof ([CFKS] p. 168) which is written for a parallel case proves it. In general, when no ergodicity is assumed, define

$$\Sigma(K) = \{E \in \mathbb{C} \mid m(\{x \in X \mid E \in \Sigma(K(x))\}) > 0\} \; .$$

A selfadjoint element $L \in \mathcal{X}$ of the form

$$L = a\tau + (a\tau)^* + b$$

is called a *random Jacobi operator* if $a, b \in L^\infty(X, \mathbb{R})$. Denote by $\mathcal{L}$ the real Banach space of all random Jacobi operators in $\mathcal{X}$. We call

$$M(L) := \exp(\int_X \log(a) \, dm)$$

the *mass* of the operator $L$.

*Remark .* $C^*$ algebra techniques for random Jacobi matrices were promoted in [Bel].

*2.2 Random Toda flows.*

We define the projections

$$K = \sum_{n=-\infty}^{\infty} K_n \tau^n \mapsto K^{\pm} = \sum_{\pm n > 0} K_n \tau^n$$

which yield the decomposition $K = K^- + K_0 + K^+$. For a *Hamiltonian*

$$H \in C^\omega(\mathcal{L}) := \{L \mapsto \mathrm{tr}(h(L)) \mid h \text{ entire}, \; h(\mathbb{R}) = \mathbb{R}\} ,$$

the differential equation

$$\dot{L} = [B_H(L), L] ,$$

with $B_H(L) = h'(L)^+ - h'(L)^-$ defines a flow which we call *a random Toda flow* or *a random Toda lattice*.

**Theorem 2.1.** *For each $H(L) = \mathrm{tr}(h(L)) \in C^\omega(\mathcal{L})$, the flow*

$$\dot{L} = [h'(L)^+ - h'(L)^-, L] = [B_H(L), L]$$

*defined on $\mathcal{L}$ is Hamiltonian, isospectral and every $G \in C^\omega(\mathcal{L})$ as well as the mass $\exp(\int \log(a)\, dm)$ are integrals. The flows are globally defined and commute pairwise.*

*Proof* . *Invariance of the spectrum and local integrals:* In $\mathcal{X}$, the differential equation $\dot{Q} = -B_H Q$ with $Q(0) = 1$ has a unitary solution $Q(t)$, because $B_H$ is skew symmetric. The formula $Q(t)L(t)Q(t)^* = L(0)$ shows that the flows leave invariant the spectrum $\Sigma(L)$. For each $g \in C(\Sigma(L))$ we have

$$G(L(t)) = \mathrm{tr}(Q^*(t) g(L(0)) Q(t)) = \mathrm{tr}(g(L(0))) = G(L(0))$$

giving the *local integrals* $G(L) = \mathrm{tr}(g(L))$ for $g \in C(\Sigma(L))$.

*Global existence:* The local existence of the Toda flows follows from *Cauchy's existence theorem* and the fact that

$$f_H : \mathcal{L} \to \mathcal{L}, L \mapsto f_H(L) = [B_H(L), L]$$

is Fréchet differentiable: denote with $B_R$ the ball with radius $R$ in the Banach space $\mathcal{L}$ and with $Df_H(L)$ the Fréchet derivative of $f_H : \mathcal{L} \to \mathcal{L}$. The operator norm in $\mathcal{B}(\mathcal{L})$ is written as $|||\cdot|||_1$.

*Claim* . $f_H$ is differentiable and $\forall R > 0 \; \exists C_{H,R} > 0$ such that for all $L \in B_R$,

$$|||f_H(L)||| \leq C_{H,R}, \; |||Df_H(L)|||_1 \leq C_{H,R} .$$

*Proof* . For $L \in B_R$ we have necessarily $|a|_\infty, |b|_\infty < R$ and we obtain the rough estimate $|(L^n)_i|_\infty \leq 3^n R^n$, leading to $|||h(L)^{\pm}||| \leq 3Rh'(3R)$ and

$$|||f_H(L)||| \leq 12R^2 h'(3R) .$$

Similarly we obtain with

$$D(h(L)^+ - h(L)^-)U = (h'(L)U)^+ - (h'(L)U)^-$$

the estimate

$$|||Df_H(L)|||_1 \leq 12R^2(h''(3R) + h'(3R)) =: C_{H,R} .$$

□

Since $\Sigma(L)$ is invariant by the flow, the norm $|||L(t)|||$ is constant. This assures global existence of the flow.

*Hamiltonian character of the flow:* We will show that $\mathcal{L}$ is a Poisson manifold and that the Toda lattice with the Hamiltonian $H$ can be written in $C^\omega(\mathcal{L})$ as $\dot{F} = \{F, H\}$. Define the projection $\Delta : \mathcal{X} \to \mathcal{L}$ by

$$K = \sum_{n=-\infty}^{\infty} K_n \tau^n \mapsto K^\Delta = K_{-1}\tau^{-1} + K_0 + K_1 \tau \ .$$

Given $G = \mathrm{tr}(g(L)) \in C^\omega(\mathcal{L})$, we denote with

$$\nabla G = g'(L)^\Delta \in \mathcal{L}$$

the *functional derivative* of $G : \mathcal{L} \to \mathbb{R}$. The Fréchet derivative $DG$ satisfies $DG(L)U = \mathrm{tr}(\nabla G(L)U)$ for all $U \in \mathcal{L}$.

*Claim* . The following formula of Moerbeke ([vM 1]) holds:

$$[h'(L)^+ - h'(L)^-, L] = [L^+ - L^-, h'(L)^\Delta] \ .$$

*Proof* . By linearity in $h$, it is enough to check this formula for $h'(L) = L^n$: For $|k| > 2$ we have

$$[(L^n)^+ - (L^n)^-, L]_k = [L^n, L]_k = 0 \ .$$

We use the notation $L^n = \sum_{k=-n}^{n} l_k \tau^k$ to verify also

$$\begin{aligned}
[(L^n)^+ - (L^n)^-, L]_0 &= [l_1\tau, (a\tau)^*] - [l_{-1}\tau^*, a\tau] \\
&= [a\tau, l_{-1}\tau^*] - [(a\tau)^*, l_1\tau] \\
&= [L^+ - L^-, (L^n)^\Delta]_0
\end{aligned}$$

and

$$\begin{aligned}
[(L^n)^+ - (L^n)^-, L]_1 &= [l_1\tau, b] + [l_2\tau^2, (a\tau)^*] \\
&= -[l_0, a\tau] = [a\tau, l_0] \\
&= [L^+ - L^-, (L^n)^\Delta]_1 \ ,
\end{aligned}$$

where we used the identity

$$0 = [L^n, L]_1 = [l_1\tau, b] + [l_2\tau^2, (a\tau)^*] + [l_0, a\tau] \ .$$

□

With the *Poisson bracket*

$$\{F, G\} := 2 \cdot \mathrm{tr}(\nabla F(L^+ - L^-)\nabla G) = \mathrm{tr}(\nabla F[L^+ - L^-, \nabla G]) \ ,$$

$C^\omega(\mathcal{L})$ is a Lie algebra. An observable $G = \mathrm{tr}(g(L)) \in C^\omega(\mathcal{L})$ is evolving according to

$$\begin{aligned}
\dot{G} &= \frac{d}{dt}\mathrm{tr}(g(L)) = \mathrm{tr}(Dg(L)\dot{L}) = \mathrm{tr}(\nabla G\dot{L}) \\
&= \mathrm{tr}(\nabla G(L)[L^+ - L^-, \nabla H(L)]) = \{G, H\} \ .
\end{aligned}$$

Since every $G \in C^\omega(\mathcal{L})$ is a constant of motion, we have $\{H, G\} = 0$ for all $H, G \in C^\omega(\mathcal{L})$ and so all these Hamiltonian flows commute: with the notation $X_H F = \{F, H\}$ one has using the Jacobi identity

$$\begin{aligned}[X_H, X_G]F &= (X_H X_G - X_G X_H)F = \{\{F, G\}, H\} - \{\{F, H\}, G\} \\ &= \{\{H, G\}, F\} = X_{\{H,G\}}F = 0 .\end{aligned}$$

*Conservation laws for mass and momentum:* The differential equation $\dot{L} = [B_H(L), L]$ is equivalent to

$$\begin{aligned}\frac{d}{dt}\log(a) &= h'(L)_0(T) - h'(L)_0 , \\ \frac{d}{dt}b &= ah'(L)_1 - a(T^{-1})h'(L)_1(T^{-1}) .\end{aligned}$$

These are discrete conservation laws for the mass integral $\log(M) = \int_X \log(a)\, dm$ and the *momentum integral* $\int_X b\, dm$. □

*Remark* . To approach the common notation for Hamiltonian systems, one could use the notation $J_L K := [L^+ - L^-, K]$ for $K, L \in \mathcal{L}$ and $< K, L >= \mathrm{tr}(KL)$. The Toda flows can then be written as

$$\dot{L} = J_L \nabla H(L) ,$$

and the Poisson bracket is

$$\{F, G\}_L =< \nabla F, J_L \nabla G > .$$

One can show that the 2-form $w_L(K, M) =< K, J_L M >$ is degenerate. Like this, $\mathcal{L}$ is not a symplectic manifold but only a Poisson manifold.

*Example* . (The first Toda lattice). For $H(L) = \mathrm{tr}(\frac{L^2}{2})$ one obtains the differential equation $\dot{L} = [L^+ - L^-, L]$. Expressed in the coordinates $a, b$, this gives

$$\begin{aligned}\dot{a} &= a(b(T) - b) , \\ \dot{b} &= 2a^2 - 2a^2(T^{-1}).\end{aligned}$$

For fixed $x \in X$ we write $a_n = a(T^n x), b_n = b(T^n x)$; this leads to

$$\begin{aligned}\dot{a}_n &= a_n(b_{n+1} - b_n) , \\ \dot{b}_n &= 2(a_n^2 - a_{n-1}^2) ,\end{aligned}$$

and reduces to the periodic Toda lattice in the case when $|X|$ is finite.

The Toda flows can sometimes be interpreted as a Hamiltonian flow in the new coordinates $q, p$ (see Flaschka [F]) defined by:

$$4a^2 = e^{q(T)-q}, \ 2b = p .$$

To introduce these coordinates $q, p \in L^\infty(X)$, the function $a^2$ must be a multiplicative coboundary: there must exist $f \in L^\infty(X)$ such that $a^2 = f_H(T)f^{-1}$. Not every $a$ satisfies this. A necessary condition is for example that $\int_X \log(a)\, dm = 0$. The Toda differential equations transform into the Hamilton equations

$$\dot{q} = H_p \ ,$$
$$\dot{p} = -H_q,$$

where $H(q,p) = H(L) = \mathrm{tr}(h(L))$. The first flow, given by $h(E) = \frac{E^2}{2}$, describes an infinite chain of particles with position $q_n = q(T^n x)$ moving according to:

$$\frac{d^2}{dt^2} q_n = e^{q_{n+1}-q_n} - e^{q_n - q_{n-1}} \ .$$

On the Banach space $L^\infty(X) \times L^\infty(X)$ there is a symplectic structure. With $z = (q,p) \in L^\infty(X) \times L^\infty(X)$, the flows can be written as $\dot{z} = J\nabla H(z)$.

## 3. Integration of the Random Toda Lattice

If $(X,T,m)$ is periodic, the integration of the random Toda lattice is known. In the case of positive mass one has a collection of periodic Toda lattices and in the case of zero mass one has a collection aperiodic Toda lattices. In the following, we examine the case of an aperiodic dynamical system.

Denote with $\mathrm{Exp}_H : \mathcal{L} \to \mathcal{L}$ the time 1 map of the flow given by $H$. We want to find new coordinates in an infinite dimensional vector space $\mathcal{G}$ where the time 1 map $\overline{\mathrm{Exp}}$ is easy to calculate.

**Theorem 3.1.** *Assume $(X,T,m)$ is aperiodic.*
*a) There exists an injective map $\phi : \mathcal{L} \to \mathcal{G} = (L^\infty(X) \times L^\infty(X))^{\mathbb{N}}$,*

$$a, b \in L^\infty(X) \mapsto (\lambda, r) = \{(\lambda_i, r_i)\}_{i\in\mathbb{N}}$$

*which has a surjective left inverse $\psi$ defined on the flow invariant subset*

$$\mathcal{H} = \{\overline{\mathrm{Exp}}_H(\phi(\mathcal{L})) \mid H \in C^\omega(\mathcal{L})\} \subset \mathcal{G} \ .$$

*The flow given by the differential equation $\dot{L} = [B_H(L), L]$ has in the new coordinates the form*

$$\overline{\mathrm{Exp}}_{tH}(\lambda, r) = (\lambda, e^{th'(\lambda)} r) = \{(\lambda_i, e^{th'(\lambda_i)} r_i)\}_{i\in\mathbb{N}}$$

*and is conjugated to the flow in $\mathcal{L}$:*

$$\mathrm{Exp}_H(L) = \psi \circ \overline{\mathrm{Exp}}_H \circ \phi(L) \ .$$

*b) Assume $(X,T,m)$ is ergodic. If $L \in \mathcal{L}$ is given such that $Y = \{a(x) = 0\}$ has positive measure, then there exists a generic set in $C^\omega(\mathcal{L})$, such that for $H$ in this set, $L(t)(x)$ converges in the weak operator topology to a diagonal operator for almost all $x \in X$.*

*3.1 Proof of Theorem 3.1 in the case when $\{a(x) = 0\}$ has positive measure.*

*Proof .*
a) Call $T_Y$ the *induced mapping* on $Y = \{a(x) = 0\}$,

$$T_Y(y) = T^{n(y)}(y) ,$$

where $n(y) = min\{n > 0 \mid T^n y \in Y\}$. By Poincaré recurrence, $n(y)$ is finite for almost all $y \in Y$. For fixed $y \in Y$, we write $a_n = a(T^n y), b_n = b(T^n y)$ and call the aperiodic Jacobi matrix

$$L_y = \begin{pmatrix} b_1 & a_1 & 0 & \cdot & \cdot & 0 \\ a_1 & b_2 & a_2 & \cdot & \cdot & \cdot \\ 0 & a_2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & a_{(n(y)-2)} & 0 \\ \cdot & \cdot & \cdot & a_{(n(y)-2)} & b_{(n(y)-1)} & a_{(n(y)-1)} \\ 0 & \cdot & \cdot & 0 & a_{(n(y)-1)} & b_{n(y)} \end{pmatrix}$$

a *block*. It evolves independently from the other part of the matrix $L(y)$. We have like this a measurable matrix-valued function

$$y \in Y \mapsto L_y .$$

The spectrum $\lambda_y^{(1)} < \ldots < \lambda_y^{(n(y))}$ of $L_y$ is simple. We can label each $x \in X$ with a value $\lambda(x)$ by defining

$$\lambda(x) = \lambda_{y(x)}^{(m(x))} ,$$

where $m(x) = min\{n > 0 \mid T^{-n}x \in Y\}$ and $y(x) = T^{-m(x)}x$. The mapping $x \mapsto \lambda(x)$ is measurable and because $\lambda(x) \leq |||L|||$ for almost every $x \in X$, we have $\lambda \in L^\infty(X)$. Call

$$dE_y = \sum_{i=1}^{n(y)} \lambda(T^i y) P(T^i y)$$

the matrix-valued spectral measure of $L_y = \int \lambda dE_y(\lambda)$, where $P(x)$ is the projection on the eigenspace of $\lambda(x)$. Define for $y \in Y$ the probability measure

$$\mu_y = [dE_y]_{11} .$$

If $\delta(z)$ denotes the Dirac measure at $z \in \mathbb{R}$, the measure can be written as

$$d\mu_y = \sum_{i=1}^{n(y)} \frac{r(T^i y)}{R_y(r)} \cdot \delta(\lambda(T^i y)) ,$$

where

$$R_y(r) = \sum_{k=1}^{n(y)} r(T^k y) .$$

The function $r \in L^\infty(X)$ is uniquely defined when we require $R_y(r) = 1$ for $y \in Y$. We have so a map

$$\tilde{\phi} : (a, b) \mapsto (\lambda, r) \in \tilde{\mathcal{G}} = L^\infty(X) \times L^\infty(X) .$$

Define $\phi(L) = (\tilde{\phi}_L, \tilde{\phi}_L, \ldots) \in \mathcal{G}$. (Infinitely many coordinates in $\mathcal{G}$ will be necessary only in the case $a(x) > 0$, a.e. which is treated in Sect. 3.3.) In these new coordinates the flow is linear: $\lambda$ does not change and the weights $r(T^i y)$ of the measure $d\mu_y$ are evolving according to

$$\dot{r}(T^i y) = h'(\lambda(T^i y)) \cdot r(T^i y),\ i = 1, \dots, n(y) .$$

(See [Mo 1] for the first flow and [DLT 1] for all the Toda flows.) Therefore

$$(\lambda(t), r(t)) = (\lambda, e^{h'(\lambda)t} \cdot r) .$$

From $\overline{\mathrm{Exp}}_H(\lambda, r) \in \mathcal{H}$ the operator $\mathrm{Exp}_H(L) \in \mathcal{L}$ can be reconstructed: Take a point $y \in Y$. The values

$$\lambda(T^i y), r(T^i y)\ i = 1, \dots, n(y)$$

determine the measure $d\mu_y$ and from this measure, $a(T^i y), b(T^i y)$ for $i = 1, \dots, n(y)$ can be recovered. We have thus an inverse $\psi$ on $\mathcal{H} \subset \mathcal{G}$ and by construction

$$\mathrm{Exp}_H(L) = \psi \circ \overline{\mathrm{Exp}}_H \circ \phi(L) .$$

b) The spectrum $\Sigma(L(x))$ is countable, and because of ergodicity, $\Sigma(L(x)) = \Sigma(L)$ for almost all $x \in X$. There exists a generic set of entire functions $h$ which are injective on $\Sigma(L)$ because $h$ is injective on $\Sigma(L)$ if and only if it is in the countable intersection

$$\bigcap_{E', E'' \in \Sigma(L)} \{h \mid h(E') \neq h(E'')\}$$

of open dense sets. Denote with $\Sigma_y$ be the spectrum of the block $L_y$. If $h'$ is injective on $\Sigma(L)$, it is injective on $\Sigma_y$ and $L_y(t)$ converges with the Hamiltonian $H(L) = \mathrm{tr}(h(L))$ to a diagonal matrix for $|t| \to \infty$ [DLT 2]. Because this happens for all blocks $L_y$ with $y \in \{a(x) = 0\}$, the operator $L(x)$ converges to a diagonal operator in the weak operator topology. □

*3.2 Weak continuity of a flow in a Banach space.*

The rest of the proof of Theorem 3.1 uses an approximation argument which is based on the following technical result:

Given a Banach space $(\mathcal{M}, ||\cdot||)$ endowed with a second topology which is weaker than the norm topology and metrizable on each ball $B_R = \{||z|| < R\} \subset \mathcal{M}$. Call $Df$ the Fréchet derivative of a differentiable map $f : \mathcal{M} \to \mathcal{M}$.

**Proposition 3.2.** *Assume that a differentiable function $f : \mathcal{M} \to \mathcal{M}$ satisfies*
**(i)** *$f$ restricted to the ball $B_R \subset \mathcal{M}$ is continuous in the weaker topology.*
**(ii)** *There exists a constant $C \in \mathbb{R}$, such that for $z \in B_R$*

$$||Df(z)|| \leq C,\ ||Df(z)f(z)|| \leq C .$$

**(iii)** *The flow given by $\dot{z} = f(z)$ leaves each ball $B_R$ invariant.*
*Then, for each $\tau \in \mathbb{R}$, the time $\tau$ map $\phi_\tau : B_R \to B_R$ of the flow given by the differential equation $\dot{z} = f(z)$ is weakly continuous on $B_R$.*

The next simple lemma will also be useful.

**Lemma 3.3.** *Given a sequence of mappings $g_N : B_R \to B_{R'} \subset \mathcal{M}$ converging uniformly to a mapping $g$ for $N \to \infty$:*

$$\sup_{z \in B_R} ||g_N(z) - g(z)|| \to 0 .$$

*If all the $g_N$ are weakly continuous on $B_R$, then $g$ is weakly continuous on $B_R$.*

*Proof* . Denote with $d$ the metric on $B_R \cup B_{R'} \subset \mathcal{M}$ giving the weak topology. Take a sequence $z_n \in B_R$ which is converging weakly to $z \in B_R$. Given $\epsilon > 0$. Because the norm topology is stronger than the weaker topology

$$d(g(z), g_N(z)) \leq \epsilon/3, \;\; d(g_N(z_n), g(z_n)) \leq \epsilon/3$$

for all $n \in \mathbb{N}$ if $N$ is big enough. Because $g_N$ is weakly continuous, we have $d(g_N(z), g_N(z_n)) \leq \epsilon/3$ for $n$ big enough and thus

$$d(g(z), g(z_n)) \leq d(g(z), g_N(z)) + d(g_N(z), g_N(z_n)) + d(g_N(z_n), g(z_n)) \leq \epsilon .$$

□

Now to the proof of Proposition 3.2:

*Proof* . Assumptions (ii) and (iii) together with Cauchy's existence theorem assure that a unique solution of $\dot{z} = f(z)$ exists for all times. Divide the interval $[0, \tau]$ into $N$ intervals,

$$[t_k, t_{k+1}] = [\frac{k\tau}{N}, \frac{(k+1)\tau}{N}], \;\; k = 0, \ldots, N-1$$

of length $h = \tau/N$ and define recursively the Euler steps

$$z_{k+1} = z_k + hf(z_k), \;\; z_0 = z(0) .$$

Using the Taylor development

$$z(t_{k+1}) = z(t_k) + hf(z(t_k)) + \int_{t_k}^{t_{k+1}} (t - t_k) Df(z(t)) f(z(t)) \, dt ,$$

the deviation $e_k := z_k - z(t_k)$ from the orbit after $k$ steps can be estimated as follows: we have $||e_0|| = 0$ and from the estimates (ii), we obtain

$$||e_{k+1}|| \leq ||e_k|| + hC||e_k|| + \frac{C}{2}h^2 = (1 + hC)||e_k|| + \frac{C}{2}h^2 =: A||e_k|| + B ,$$

and so $||e_0|| = 0, ||e_1|| \leq B$. Inductively, $||e_k|| \leq (A^{k-1} + A^{k-2} + \cdots + A + 1)B$. This yields to

$$||e_k|| \leq \frac{A^k - 1}{A - 1} B = \frac{(1 + hC)^k - 1}{hC} \cdot \frac{C}{2}h^2 \leq \frac{e^{khC} - 1}{C} \cdot \frac{C}{2}h \leq (e^{C\tau} - 1) \cdot \frac{h}{2}$$

as long as $z_k$ stays in $B_R$. The error after $N$ Euler steps is

$$||z(\tau) - z_N|| = ||e_N|| \leq (e^{C\tau} - 1)\frac{\tau}{2N} .$$

This estimate is uniform for $z \in B_R$. (If $N$ is so large that

$$(e^{C\tau} - 1)\frac{\tau}{2N} < R - ||z_0||$$

also the linear interpolation of the points $\{z_k\}_{k=1}^N$ (the Euler polygon) is contained in $B_R$ so that the estimates hold for these points also.)

The orbit $t \in [0, \tau_0] \mapsto z(t)$ is now approximated in norm by a piecewise linear Euler polygon. A simple Euler step $z \mapsto z + hf(z)$ is continuous in the weak topology if $f$ is continuous in the weak topology. So also the mapping $\phi_N : z \mapsto z_N$, which is a composition of finitely many Euler steps is weakly continuous. For $N$ big enough, we have

$$||\phi(y) - \phi_N(y)|| \leq (e^{K\tau} - 1)\frac{\tau}{2N}$$

uniformly for $y \in B_R$. Lemma 3.3 shows that $\phi$ is weakly continuous. □

*3.3 Proof of Theorem 3.1 in the case when $a(x) > 0$ almost everywhere.*

From Proposition 3.2 we get

**Corollary 3.4.** *Given $H(L) \in C^\omega(\mathcal{L})$. The flow on $\mathcal{B}(l^2(\mathbb{Z}))$ defined by the differential equation $\dot{L}(x) = [B_H(L(x)), L(x)]$ is continuous in the weak operator topology restricted to each ball $B_R \subset \mathcal{B}(l^2(\mathbb{Z}))$.*

*Proof*. We can apply Proposition 3.2 to the Banach space $\mathcal{B}(l^2(\mathbb{Z}))$. As a Toda flow is isospectral, it will leave each ball $B_R$ invariant. We check first the weak continuity of the mapping

$$L(x) \mapsto f_H(L(x)) := [B_H(L(x)), L(x)]$$

on $B_R$.

For polynomials $h$, the weak continuity is evident inductively, since the multiplication $\mathcal{B}(l^2) \times \mathcal{L}(x) \to \mathcal{B}(l^2)$ is jointly weakly continuous: each matrix entry of the product $LM$ is the sum of only three elements. If $h$ is analytic, it can be approximated by polynomials $h_n \to h$. Then $f_{H_n}(L(x)) \to f_H(L(x))$ in norm, uniformly on each ball $B_R$. With Lemma 3.3 also $L(x) \mapsto f_H(L(x))$ is weakly continuous.
The right-hand side of the differential equation $\dot{L} = f_H(L)$ satisfies the boundedness conditions of Proposition 3.2: We have seen in Sect. 2, that there exists a constant $C$ dependent only on $|||L|||$ and $h$ such that

$$||f_H(L(x))|| \leq C, \ |||Df_H(L)|||_1 \leq C \ .$$

□

We prove now Theorem 3.1 in the case $a(x) > 0$ almost everywhere:

*Proof*. *Construction of sets with arbitrary big return time:* Rohlin's lemma (see [CFS]) implies that there exists for each $N \in \mathbb{N}$ a measurable set $Z_N \subset X$ of positive measure such that

$$T^{-N}(Z_N), \ldots, Z_N, T(Z_N), \ldots, T^N(Z_N)$$

are pairwise disjoint and such that

$$Y_N = X \setminus \bigcup_{i=-N}^{N} T^i(Z_N)$$

has measure $m(Y_N) \leq 1/N$. The countable set

$$\mathcal{Y} = \{Y_1, Y_2, Y_3 \ldots\}$$

has the property that for almost all $x \in X$ and all $N \in \mathbb{N}$ we can find $Y = Y_{k(N,x)} \in \mathcal{Y}$ such that $T^n(x) \notin Y$ for $n = -N, \ldots, N$.

*A countable set of random Jacobi operators with zero mass.* Given $L \in \mathcal{L}$ with $m\{a(x) = 0\} = 0$. Define for each $Y \in \mathcal{Y}$ the new random Jacobi operator

$$L_Y = (1_{Y^c})a\tau + ((1_{Y^c})a\tau)^* + b\,,$$

where $1_{Y^c}$ is the characteristic function of the set $Y^c = X \setminus Y$. The random Toda flow for $L_Y$ can be integrated according to the already proved case because $Y = \{(L_Y)_1(x) = 0\}$ has positive measure.

*Construction of $\phi$:* There exists a mapping

$$\tilde{\phi}_Y : L \mapsto L_Y \mapsto \tilde{\phi}(L_Y) \in L^\infty(X)^2$$

which linearizes the flow. Define

$$\phi = \{\tilde{\phi}_Y\}_{Y \in \mathcal{Y}} = (\tilde{\phi}_{Y_1}, \tilde{\phi}_{Y_2}, \ldots) : \mathcal{L} \to \mathcal{G}\,.$$

*Construction of the left inverse $\psi$:* Take $\overline{\text{Exp}}_H(\lambda, r) \in \mathcal{H}$. For almost all $x \in X$ there is a sequence $N \mapsto k(N, x)$ such that

$$T^i(x) \notin Y_{k(N,x)}$$

for $i = -N, \ldots, N$. Proposition 3.2 implies that for $N \to \infty$

$$\psi_{k(N,x)}(\overline{\text{Exp}}_H(\lambda, r))(x) \to \text{Exp}_H(L)(x)$$

in the weak operator topology, where $\psi_i(\lambda', r')$ is the operator $L'_{Y_i}$ calculated from the spectral data $(\lambda'_i, r'_i)$ satisfying $\psi_i \circ \tilde{\phi}(L'_{Y_i}) = L'_{Y_i}$. Define

$$\psi(\overline{\text{Exp}}_H(\lambda, r))(x) = \lim_{N \to \infty} \psi_{k(N,x)}(\overline{\text{Exp}}_H(\lambda, r))(x)\,,$$

where the limit is taken in the weak operator topology.

*Conjugation of the flow:* Assume $\phi(L) = (\lambda, r)$. In the weak operator topology we have

$$\begin{aligned}\text{Exp}_H(L)(x) &= \lim_{N \to \infty} \psi_{k(N,x)}(\overline{\text{Exp}}_H(\lambda, r))(x)\\ &= \psi(\overline{\text{Exp}}_H(\lambda, r))(x) = \psi \circ \overline{\text{Exp}}_H \circ \phi(L)(x)\end{aligned}$$

and so

$$\text{Exp}_H(L) = \psi \circ \overline{\text{Exp}}_H \circ \phi(L). \quad \square$$

## 4. QR Decomposition

For a real $n \times n$ matrix $M$, there exists a decomposition $M = QR$, where $Q$ is orthogonal and $R$ is upper triangular. For aperiodic Jacobi matrices, Symes [Sy 1] found that the $QR$ decomposition of $exp(tL)$ integrates the first Toda flow $L(t)$. This was worked out further in [DNT] leading to the observation that the time 1 map of the Hamiltonian $H(L) = \mathrm{tr}(L \cdot \log(L) - L)$ is just one step in the $QR$ algorithm, an algorithm which is used to diagonalize a matrix numerically. More generally, the following fact is known:

**Proposition 4.1.** *Given $h \in C^2(\mathbb{R})$. In the matrix algebra $M(d, \mathbb{R})$, the solution of*

$$\dot{L} = [B_H(L), L], \; L(0) = L_0$$

*is given by* $L(t) = Q^* L_0 Q$, *where* $Q$ *is obtained by the* $QR$ *decomposition* $\exp(th'(L_0)) = QR$.

*Proof* . We can write

$$e^{th'(L_0)} = QR$$

in a unique way, because $e^{th'(L_0)}$ is invertible. The dependence of $Q, R$ on $t$ is differentiable. Differentiation gives

$$h'(L_0)QR = \dot{Q}R + Q\dot{R}$$

and after multiplication from the right with $R^{-1}$ and multiplication from the left with $Q^*$, we get

$$Q^* h'(L_0)Q = Q^*\dot{Q} + \dot{R}R^{-1} ,$$

where $Q^*\dot{Q}$ is skew symmetric and $\dot{R}R^{-1}$ is upper triangular. Call $\tilde{L}(t) = Q^*(t)L_0Q(t)$ with $Q(0) = 1$. It follows

$$h'(\tilde{L}(t)) = Q^* h'(L_0)Q = Q^*\dot{Q} + \dot{R}R^{-1} .$$

We can compare this with the unique decomposition

$$\begin{aligned} h'(\tilde{L}(t)) &= -h'(\tilde{L}(t))^+ + h'(\tilde{L}(t))^- \\ &\quad + 2h'(\tilde{L}(t))^+ + h'(\tilde{L}(t))_0 \end{aligned}$$

into a skew symmetric and an upper triangular part to get

$$Q^*\dot{Q} = -h'(\tilde{L}(t))^+ + h'(\tilde{L}(t))^- = -B_H(\tilde{L}) .$$

Now

$$\begin{aligned} \frac{d}{dt}\tilde{L}(t) &= \dot{Q}^*(t)L_0Q(t) + Q^*(t)L_0\dot{Q}(t) \\ &= (\dot{Q}^*(t)Q(t))Q^*(t)L_0(t)Q(t) + Q^*(t)L_0Q(t)(Q^*(t)\dot{Q}(t)) \\ &= (\dot{Q}^*(t)Q(t))\tilde{L}(t) + \tilde{L}(t)(Q^*(t)\dot{Q}(t)) \\ &= [h'(\tilde{L}(t))^+ - h'(\tilde{L}(t))^-, \tilde{L}(t)] = [B_H(\tilde{L}(t)), \tilde{L}(t)] . \end{aligned}$$

Because $\tilde{L}(t), L(t)$ satisfy the same differential equation as well as the same initial conditions $L(0) = \tilde{L}(0)$, they must coincide. □

This can be generalized to the random case:

**Theorem 4.2.** *For $L \in \mathcal{L}$ and $H(L) = \mathrm{tr}(h(L)) \in C^\omega(\mathcal{L})$, there exists a unitary $Q \in \mathcal{X}$ and $R = \sum_{n \geq 0} R_n\tau^n \in \tau^{-1}\mathcal{X}^+$ such that* $\exp(h'(L)) = QR$.

*If $L(t)$ satisfies $\dot{L} = [B_H(L), L]$, one obtains $L(t) = Q^* L(0) Q$ with a $QR$ decomposition* $\exp(th'(L_0)) = QR$.

Call $\mathcal{T}$ the Banach space of selfadjoint real tridiagonal matrices in $\mathcal{B}(l^2)$ and

$$\tilde{\mathcal{T}} = \{ p(L) \mid p \text{ polynomial}, \ L \in \mathcal{T}\} ,$$
$$\overline{\mathcal{T}} = \{ h(L) \mid h \text{ entire}, \ L \in \mathcal{T}\}.$$

We call the weak operator topology on $\mathcal{B}(l^2)$ in the following also the *weak topology*. Denote with $B_R$ the ball with radius $R$ in the Banach space $\mathcal{B}(l^2)$ and $\mathcal{T}_R = \mathcal{T} \cap B_R, \tilde{\mathcal{T}}_R = \tilde{\mathcal{T}} \cap B_R, \overline{\mathcal{T}}_R = \overline{\mathcal{T}} \cap B_R$. The weak topology is metrizable on each ball $B_R$. We denote this metric by $d$.

**Lemma 4.3.**
*a) Given an entire function $f$ and $R > 0$. The mapping*

$$\mathcal{T}_R \to \mathcal{B}(l^2), \ L \mapsto f(L)$$

*is weakly continuous.*
*b) $B_R \times \overline{\mathcal{T}}_R \to \mathcal{B}(l^2)$, $(L, K) \mapsto L \cdot K$ is weakly continuous.*

*Proof* .
a) Because multiplication $\mathcal{T} \times \mathcal{B}(l^2) \to \mathcal{B}(l^2)$ is weakly continuous, one obtains inductively that $L \mapsto p(L)$ is weakly continuous for every polynomial $p$. Applying Lemma 3.3 gives that $L \mapsto f(L)$ is weakly continuous for an entire function $f$.
b) The multiplication $B_R \times \tilde{\mathcal{T}}_R \to \mathcal{B}(l^2)$ is weakly continuous. We can approximate $L \in \overline{\mathcal{T}}$ in norm by elements in $\tilde{\mathcal{T}}$ and this approximation can be made uniform in the ball $B_R$. Use again Lemma 3.3. □

We prove now Theorem 4.2:

*Proof* . Fix $x \in X$. We can approximate $L(x)$ in the weak operator topology by tridiagonal aperiodic $N \times N$ Jacobi matrices $L^{(N)}$. For such matrices we can form

$$\exp(h'(L^{(N)}(x))) = Q^{(N)}(x) R^{(N)}(x) ,$$

where $Q^{(N)}(x)$ is orthogonal and $R^{(N)}(x)$ is tridiagonal and we know also that $Q^{(N)}(x)^* L^{(N)}(x) Q^{(N)}(x)$ is the time 1 map of the Hamiltonian flow

$$\dot{L}^{(N)}(x) = [B_H(L^{(N)}(x)), L^{(N)}(x)] = f_H(L^{(N)}) .$$

We are deforming $L(x)$ with the same Hamiltonian flow

$$\dot{L}(x) = [B_H(L(x)), L(x)] = f_H(L)$$

to get $\mathrm{Exp}_H L(x) = Q^*(x) L(x) Q(x)$.

*Claim* . $Q^{(N)}(x) \to Q(x)$ in the weak operator topology.

*Proof* . Consider in addition to the above differential equations for $L^{(N)}$ and $L$ also the differential equations

$$\dot{Q}^{(N)}(x) = -B_H(L^{(N)}(x)) Q^{(N)}(x) =: g_H(L^{(N)}, Q^{(N)}) ,$$
$$\dot{Q}(x) = -B_H(L(x)) Q(x) =: g_H(L, Q).$$

We can apply Proposition 3.2 to the system

$$\frac{d}{dt}(L, Q) = (f_H(L), g_H(L, Q))$$

in $\mathcal{B}(l^2(\mathbb{Z}))^2$ in order to show that $Q^{(N)}(x) \to Q(x)$ in the weak operator topology. The assumptions of Proposition 3.2 are readily checked: Because $L \mapsto B_H(L)$ is weakly continuous, we can apply Lemma 4.3 b) to conclude that $g$ is also weakly continuous. There are for $L \in B_R \subset \mathcal{B}(l^2(\mathbb{Z}))$ also the estimates $||g(L(x), Q(x))|| \leq C_{H,R}$ and $|||Dg(L, Q)|||_2 \leq 2C_{H,R}$, where $||| \cdot |||_2$ is the norm in $\mathcal{L} \times \mathcal{X}$. Because $B_H$ is skew symmetric, the norm of $Q(t)$ is a constant. □

From Lemma 4.3 a) we have

$$e^{h'(L^{(N)}(x))} \to e^{h'(L(x))} .$$

With this, the just proved claim and Lemma 4.3 b) one gets

$$(Q^{(N)})^*(x)e^{h'(L^{(N)}(x))} = R^{(N)}(x) \to Q^*(x)e^{h'(L(x))} = R(x)$$

in the weak operator topology. It follows that $R(x)$ is also upper triangular and $e^{h'(L(x))} = Q(x)R(x)$ with $Q^*(x)L_0(x)Q(x) = \mathrm{Exp}_H L(x)$. We have now constructed $Q(x)$ and $R(x)$ pointwise for $x \in X$. There is an upper bound for $[R(x)]_{ij}$ due to the fact that the Toda flows are isospectral. By Lebesgue dominated convergence theorem applied to each function $[Q(x)]_{ij}, [R(x)]_{ij}$ we get also random operators $Q, R \in \mathcal{X}$ which satisfy $QR = \exp(h'(L_0))$ with $Q^* L_0 Q = \mathrm{Exp}_H L$. □

*Remark*. Theorem 4.2 is not yet very helpful in order to understand more about the qualitative behaviour of Toda flows. It has been pointed out to us by a referee that the $QR$ decomposition in $\mathcal{X}$ is not unique: For example, $\tau = \tau \cdot 1 = 1 \cdot \tau$ because $\tau$ is unitary and upper trigonal. We have used the Toda flows to construct a $QR$ decomposition for certain elements in $\mathcal{X}$ and we don't know how to find and perform directly the right $QR$ decomposition which leads to an integration of the Toda flow.

*Remark*. Having the right $QR$ decomposition for infinite matrices, one could calculate the time $\tau$ map of the periodic Toda lattice by a $QR$ decomposition of an infinite but periodic matrix. As the solutions of the periodic Toda lattice can be expressed by Theta functions (see for example [T]), it would be interesting to know whether the $QR$ decomposition is a reasonable way to calculate Theta functions numerically.

## 5. Density of States, Lyapunov Exponent and Rotation Number, Floquet Exponent, Determinant. Entropy and Index of Monotone Twist Maps.

### *5.1 The Floquet exponent as an integral of the Toda flow.*

The functional calculus for a normal element $K$ in the $C^*$ algebra $\mathcal{X}$ defines $f(K)$ for a function $f \in C(\Sigma(K))$. The mapping

$$f \mapsto \mathrm{tr}(f(K))$$

is a bounded linear functional on $C(\Sigma(K))$, and by Riesz representation theorem, there exists a measure $dk$ on $\Sigma(K)$ with

$$\mathrm{tr}(f(K)) = \int_{\Sigma(K)} f(E) dk(E) .$$

This measure $dk$ is called the *density of states* of $K$. Because of

$$1 = \int_X 1 \, dm = \mathrm{tr}(1) = \int_{\Sigma(K)} dk(E) ,$$

the measure $dk$ is a probability measure. For selfadjoint elements $K \in \mathcal{X}$, the density of states $dk$ has its support on $\mathbb{R}$. The integral

$$k(E) = \int_{-\infty}^{E} dk(E')$$

is called the *integrated density of states* of $K$. To an operator $L \in \mathcal{L}$, we attach a complex valued function by means of

$$w(E) := -\mathrm{tr}(\log(L - E)) ,$$

which is a priori defined only for $\mathrm{Im}(E) > 0$. Here the branch of the logarithm is chosen so that $\log(1) = 0$. The function $w$ is called the *Johnson-Moser* function or *Floquet exponent.* This gives also a *determinant*

$$e^{-w(E)} = e^{\mathrm{tr}(\log(L-E))} = \det(L - E) .$$

For the *transfer cocycle*

$$A_E(x) := a^{-1}(T^{-1}x) \begin{pmatrix} E - b(x) & -a^2(T^{-1}x) \\ 1 & 0 \end{pmatrix}$$

of $L = a\tau + (a\tau)^* + b$ the *Lyapunov exponent* is

$$\lambda(A_E) = \lim_{n \to \infty} n^{-1} \int_X \log ||A_E^n(x)|| \, dm(x) ,$$

where $A_E^n(x) = A_E(T^{n-1}x) \dots A(Tx)A_E(x)$ and the *rotation number* is given by $\rho(A_E) = \pi k(E)$. The rotation number can be defined by the cocycle $A_E$ alone [DS].

**Theorem 5.1.** *In the case when $a(x) > 0$ almost everywhere, the Floquet exponent*

$$w(E) = -\mathrm{tr}(\log(L - E))$$

*as well as for $E \in \mathbb{R}$ the Lyapunov exponent $\lambda(A_E)$ and the rotation number $\rho(A_E)$ are integrals of the Toda flows.*

*Proof* . The *Thouless formula* relates the mass $M$ and the Floquet exponent $w(E)$ with the Lyapunov exponent $\lambda(A_E)$ and the rotation number $\rho(A_E)$:
if $L$ has positive mass then

$$-\lambda(A_E) + \mathrm{i}\rho(A_E) = w(E) + \log(M) .$$

For the proof see [CL]. In the case of zero mass, both sides are $-\infty$.
In the case $\mathrm{Im}(E) > 0$, the function $g(z) = -\log(z - E)$ is continuous on the real axis and $w(E) = \mathrm{tr}(g(L))$ is an integral of the Toda flows. The function $E \mapsto w(E)$ is a Herglotz function. Because $w(E)$ is time independent for $E$ in the upper half plane, it is also time independent on the real axis. □

*5.2 Monotone twist maps.*

Random Jacobi operators appear in a natural way when embedding an abstract dynamical system in a monotone twist map. Assume we have given a generating function $l \in C^2(\mathbb{R}^2)$ and $r > 0$, such that

$$\begin{aligned} l_{12}(x, x') &= \frac{\partial}{\partial x}\frac{\partial}{\partial x'} l(x, x') \geq r \;, \\ l(x, x') &= l(x + 1, x' + 1). \end{aligned}$$

If we define

$$\begin{aligned} y(x, x') &= l_1(x, x') = \frac{\partial}{\partial x} l(x, x') \;, \\ y'(x, x') &= -l_2(x, x') = -\frac{\partial}{\partial x'} l(x, x') \;, \end{aligned}$$

$x'$ can be expressed as a function of $x$ and $y$. The mapping

$$S : (x, y) \mapsto (x', y')$$

is called a *monotone twist map*. It leaves invariant the Lebesgue measure $dxdy$ on the cylinder $\mathbb{T} \times \mathbb{R}$ where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Given an abstract dynamical system $(X, T, m)$. In order to have a critical point $q$ of the *Percival functional*

$$\mathcal{L}(q) = \int_X l(q, q(T))\, dm$$

on the Banach manifold $L^\infty(X, \mathbb{T})$, we must have

$$\delta\mathcal{L}(q) = l_1(q, q(T)) + l_2(q(T^{-1}), q) = 0 \;.$$

If there exists a $q$ which satisfies this Euler equation, we have embedded a factor of the given dynamical system inside the twist map. The second variation of $\mathcal{L}$ is the random Jacobi operator

$$L(q) = \delta^2\mathcal{L}(q) = a\tau + (a\tau)^* + b \in \mathcal{L} \;,$$

where

$$\begin{aligned} a(x) &= l_{12}(q(x), q(Tx)) \;, \\ b(x) &= l_{11}(q(x), q(Tx)) + l_{22}(q(T^{-1}x), q(x)). \end{aligned}$$

The twist condition implies that $L$ has positive mass. The random Jacobi operator $L$ obtained as a second variation of the Percival functional and the Floquet exponent $w$ are carrying information about the embedded system. It follows for example with a result of Mather [Ma 1] that the embedded system is a hyperbolic set if and only if $L$ is invertible, because in the resolvent set of $L$, the cocycle $A_E$ is uniformly hyperbolic. In this case there is by the implicit function theorem also a neighborhood of generating functions such that the Percival functional has a critical point near the given critical point.

*Examples* .
• Assume $X$ is a bounded subset on the cylinder $\mathbb{T} \times \mathbb{R}$ which is invariant under the twist map $S$ and has positive but finite Lebesgue measure. Call $T$ the restriction of $S$ on $X$ and $m$ the normalized measure on $X$ induced from the Lebesgue measure. There is a critical point $q(x) = \pi_1 x$, where $\pi_1$ is the projection on the angle coordinate of the cylinder. The real part of $\log(M) - w(0)$ is the entropy of the twist map restricted to $X$. The imaginary part of $w(0)$ is an *index*.
For generating functions of the type

$$l(x, x') = \frac{(x' - x)^2}{2} + V(x) ,$$

the twist map $S$ can be defined on the torus $X = \mathbb{T}^2$. The system $(X, T, m)$ with $T = S$ and $m = dxdy$ can then be taken as the abstract dynamical system and the Floquet exponent $w(E)$ gives

$$w(0) = -\text{entropy} + \mathrm{i} \cdot \text{index} .$$

because $M = 1$ in this case.
• A finite dynamical system is just a cyclic permutation $T$ of a finite set $X$. The Jacobi operator is then periodic of period $|X|$. Finding critical points of the functional $\mathcal{L}$ is equivalent to finding periodic points. Periodic points of period $|X|$ always exist. The rotation number of the Jacobi operator is related to the Morse index of the critical point [Ma 3]. When the Jacobi operator is restricted to the finite dimensional Hilbert space of $|X|-$ periodic sequences in $l^2(\mathbb{Z})$, it is just a periodic Jacobi matrix.

• If $(X, T, m)$ is an ergodic automorphism of the circle, nontrivial critical points correspond to invariant circles or Mather sets. Mather's result [Ma 2] proves the existence of nontrivial critical points. There are examples, where the corresponding random Jacobi operator is invertible because a Mather set can be hyperbolic (see [G]).

• If $X$ is a closed bounded $S$ invariant set of the cylinder and $T$ is the restriction of $S$ onto $X$ there exists a $T$ invariant probability measure on $X$. Again, $a(x) = \pi_1(x)$ is a critical point of the functional. An example: if the twist map has a transverse homoclinic point $x_0$ coming from a hyperbolic fixed point, then $X$ can be chosen as the closure of $\{S^n(x_0) \mid n \in \mathbb{Z}\}$. Again one has a hyperbolic system and the corresponding random Jacobi operator is invertible.

*Remark* . Toda deformations and twist maps are still unrelated. Our motivation to study random Toda systems was the hope to make deformations of cocyles appearing in twist maps like the standard map in order to gain more information about the Lyapunov exponents. It is thinkable that there are deformations which lead to cocycles, where Wojtkowski's cone criterion [W] (a necessary and sufficient condition for positive Lyapunov exponents) is applicable to prove positive Lyapunov exponents.

## 6. Generalizations and Questions

*6.1 Toda lattices over noncommutative dynamical systems.*

Let $\mathcal{A}$ be any $C^*$ algebra and $T : \mathcal{A} \to \mathcal{A}, a \mapsto a(T)$ be an automorphism of this algebra. Assume $\mathcal{A}$ has a trace satisfying trace($ab$) = trace($ba$). The crossed product $\mathcal{X}$ of the algebra $\mathcal{A}$ with the dynamical system is again a $C^*$ algebra. Elements in $\mathcal{X}$ can be written as $K = \sum_n K_n \tau^n$, where $K_n \in \mathcal{A}$. On $\mathcal{X}$ there is also a trace defined by

$$\mathrm{tr}(K) = \mathrm{trace}(K_0) .$$

Define the Banach space $\mathcal{L} = \{L \in \mathcal{X} \mid L_n = 0 \ \ |n| > 1, L = L^*\}$. For $H \in C^\omega(\mathcal{L})$, the differential equation

$$\dot{L} = [h'(L)^+ - h'(L)^-, L] = [B_H(L), L]$$

is a Hamiltonian flow in the subspace $\mathcal{L} \subset \mathcal{X}$. It has the Hamiltonian $H(L) = \mathrm{tr}(h(L))$. Let $Q(t)$ be defined by the differential equation

$$\dot{Q} = -B_H(L)Q$$

with initial conditions $Q(0) = 1$. In $\mathcal{X}$, the equality $L(t) = Q(t)^* L(0) Q(t)$ shows that the flow is isospectral. It has the integral $\mathrm{tr}(f(L))$ for each $f \in C(\mathbb{R})$. Given a representation $x : \mathcal{A} \to \mathcal{B}(H)$, which means that $x(a)$ is a bounded linear operator on the Hilbert space $\mathcal{H}$ for all $a \in \mathcal{A}$. If we use the notation $x(a) = a(x)$ and $x(a(T^n)) = a(T^n x)$, each element $K \in \mathcal{X}$ has a representation $K(x)$ in the Hilbert space $l^2(\mathbb{Z}, \mathcal{H})$ defined by the matrix

$$[K(x)]_{mn} = K_{n-m}(T^m x) .$$

The matrix $K(x)$ is a Jacobi matrix where the entries are linear operators on $\mathcal{H}$.

*Examples .*
• If $\mathcal{A} = L^\infty(X)$ we are in the case discussed already because an automorphism $T$ comes from a dynamical system $(X, T, m)$.
• Assume $X$ is a compact topological space and $\mathcal{A} = C(X)$. We have then a deformation in

$$\mathcal{L}_c := \{L = a\tau + (a\tau)^* + b \mid a, b \in C(X, \mathbb{R})\} \subset \mathcal{L} .$$

Also if $X$ is a compact manifold and $\mathcal{A} = C^r(X, \mathbb{R})$, the space

$$\mathcal{L}_r := \{L = a\tau + (a\tau)^* + b \mid a, b \in C^r(X, \mathbb{R})\} \subset \mathcal{L}_c \subset \mathcal{L}$$

for $r = 1, 2, \ldots$ is kept invariant by the Toda flows.

• $\mathcal{A} = L^\infty(X, M(l, \mathbb{R}))$ gives a quite natural noncommutative generalization of the Toda lattice. A suggestion to study such systems is in [C3]. In the case $|X| < \infty$, this gives new finite dimensional systems which are candidates for being integrable. The flow can be written as an isospectral deformation of operators called *stochastic Jacobi matrices on the strip* [KS]. They arise as second variations of higher dimensional twist maps like the Fröschle map (see [KM]). A nonabelian Toda lattice for half infinite matrices is proposed in [BGS].

*6.2 More general Hamiltonians.*

We have chosen Hamiltonians which are defined by entire functions $h \in C^\omega$. Like this, we could use Cauchy's existence theorem for differential equations in a Banach space. One has to be careful in doing generalizations by choosing functions $h$ which are only analytic in a neighborhood of the spectrum $\Sigma(L)$: it can happen that, for a bounded operator $K$ on $l^2(\mathbb{Z})$, the operator $K^+ - K^-$ is no more bounded [DLT 1]. Nevertheless one can consider functions $h' \in L^\infty(\Sigma(L))$. The functional calculus defines then $h'(L) \in \mathcal{X}$. Even if $B_H(L) = h'(L)^+ - h'(L)^-$ is unbounded, one can obtain like this Toda flows in a *weak sense* ([DLT 1]): For all $u, v \in \{u \in l^2(\mathbb{Z}) \mid \exists n_0 > 0 \; u_n = 0, \; \forall \, |n| > n_0\}$,

$$\frac{d}{dt} < u, Lv >= - < B_H(L)u, Lv > - < Lu, B_H(L)v > \; .$$

*6.3 Deformation of complex Jacobi operators and deformations with complex time.*

We considered only real Jacobi operators and deformations where time is real. If $a, b \in L^\infty(X, \mathbb{C})$ then

$$L = a\tau + \tau^* a + b \in \mathcal{L}_\mathbb{C}$$

is no more selfadjoint in general. Even if we make isospectral deformations, the norm can blow up. There are actually isospectral operators to a given operator which have arbitrary big norm. Given an arbitrary entire function $h$, the differential equation

$$\dot{L} = [h'(L)^+ - h'(L)^-, L]$$

in the *complex* Banach space $\mathcal{L}_\mathbb{C}$ has locally a unique analytic solution $t \mapsto L(t)$ for $t$ in a disc $\mathbb{D}_r(0) \subset \mathbb{C}$.

If we restrict to real time, one can define deformations of complex finite dimensional Jacobi operators (see [C 2]): Also for the random version, the differential equation

$$\dot{L} = [B_H(L), L]$$

with

$$B_H(L) = h'(L)^+ - (h'(L)^+)^* + \mathrm{i} \cdot \mathrm{Im}(h'(L))_0$$

defines an isospectral deformation in $\mathcal{L}_\mathbb{C}$. Exactly in the same way as before, one can decompose $\exp(t \cdot h'(L)) = QR$ into a *unitary* $Q$ and upper triangular $R$ such that $L(t) = Q^*(t)L(0)Q(t)$. The flow does not extend to a flow with complex time.

*6.4 Random Singular-Value-Decomposition flows.*

Toda like deformations have been generalized to arbitrary matrices [DLT 2]. Given any real $n \times n$ matrix $A$ and a real entire functions $h$, one can define the so called *Singular-Value-Decomposition* ($SVD$) flow (see [DDLT])

$$\dot{A} = B_H(AA^*)A - AB_H(A^*A) \; .$$

Under the map $A \mapsto L = A^*A$ the flow goes over in the Toda like flow

$$\dot{L} = [h'(L)^+ - h'(L)^-, L] .$$

Both flows have a unique global solution and preserve the singular values of $A$ respectively the spectrum of $L$. The flow can be integrated explicitly as follows: With the $QR$ decompositions

$$e^{th'(A^*A)} = Q_1(t)R_1(t) ,$$
$$e^{th'(AA^*)} = Q_2(t)R_2(t) ,$$

one gets $A(t) = Q_2^* A Q_1$.

Given now any element $A \in \mathcal{X}$ where $\mathcal{X}$ in the $C^*$ crossed product of the Banach algebra $L^\infty(X)$ with a dynamical system. The deformation

$$\dot{L} = B_H(AA^*)A - AB_H(A^*A)$$

is a random version of the singular decomposition flow. The qualitative behaviour could be investigated in the same way by approximation in the weak operator topology by finite dimensional SVD flows.

*6.5 Deformation of operators with no boundary conditions.*

Why is it useful to study deformations of operators with *random boundary conditions*? The Toda deformations can also be done for any tridiagonal bounded operator $L$ on $l^2(\mathbb{Z})$ and the flow given by the differential equation

$$\dot{L} = [B_H(L), L]$$

can be approximated in the weak operator topology by finite dimensional Toda flows. The advantage of looking at random operators is the *existence of a finite trace* which is the ergodic average of the diagonal of the operator. Most integrals can be expressed by this trace. Such integrals are invariant by the shift

$$L_{ij} \mapsto L_{i+1,j+1} .$$

Considering random operators instead of general operators gives an important *symmetry*, in that shift invariant "macroscopic quantities" exist.

*6.6 Some questions.*

• The spectral and inverse spectral problem for random Jacobi operators is not solved. How does the isospectral set look? For which dynamical systems do random operators with the same mass $M$ and the same Floquet exponent $w(E)$ form a group? Does the determinant of the resolvent $(L - E)^{-1}$ determine the isospectral set? What spectra do occur over a given dynamical system? What kind of spectra do occur generically in $\mathcal{L}$ for fixed dynamical system $(X, T, m)$?
• Can one find the explicit solutions of the random Toda lattice? The integration proposed in this paper here is somehow artificial. The hope is that the solutions can be written in terms of generalized theta functions. What are the properties of the *transcendental hyperelliptic curve*

$$y^2 = \det(L - E) = e^{-w(E)} ,$$

where $w(E)$ is the Floquet exponent? Especially interesting would be some knowledge about infinite dimensional Jacobians. Is there an infinite dimensional generalization of the Jacobi map?

• What is the asymptotic behaviour of the random Toda lattice? What happens for $t \to \pm\infty$ in the case when $\{a(x) = 0\}$ has positive measure? To our knowledge, the general asymptotic behaviour of the tied finite dimensional Toda lattice analogous to the first flow [Mo 2] is not known. What happens for Hamiltonians outside the generic set where one has convergence to diagonal operators? (See [C 1].)
Is there recurrence in the weak operator topology in the case when $a(x) > 0$ almost everywhere?

• Is it possible to deform a random Jacobi operator in a way to make the spectral problem or the problem of calculating the Floquet exponent more easy? Can one deform a twist mapping in a way such that the corresponding random Jacobi operator is deformed in an isospectral way?

• Is any isospectral deformation in the crossed product of any Banach algebra with any dynamical system integrable?

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