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A Combinatorial Approach to Topological Quantum Field Theories and Invariants of Graphs*

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Dedicated to H. Araki and E. Lieb on the occasion of their 60th birthdays

Abstract. The combinatorial state sum of Turaev and Viro for a compact 3-manifold in terms of quantum $6j$-symbols is generalized by introducing observables in the form of coloured graphs. They satisfy braiding relations and allow for surgeries and a discussion of cobordism theory. Application of these techniques give the dimension and an explicit basis for the vector space of the topological quantum field theory associated to any Riemann surface with arbitrary coloured punctures.

1. Introduction

Since the early days of topological quantum field theories there was the question whether such field theories have a lattice formulation analogous to lattice gauge theory. The reason is that one would like to work in a context with mathematically well defined quantities instead of more or less formal functional integrals. This question has been answered affirmatively in part by the work of Turaev and Viro [TV]. Invoking the $6j$-symbols for the quantum group $U_q(sl(2, \mathbf{C})$ with q being a $2r^{\text{th}}$ primitive root of unity they constructed invariants $Z(M)$ of closed, compact 3-manifolds M. In [KMS] this construction was extended to compact 3-manifolds with boundary. For orientable 3-manifolds, the case we shall exclusively be dealing with in this article, these invariants, called state sums or partition functions, in the case $\partial M = \emptyset$ satisfy the relation

$$Z(M) = \tau(M)\tau(M^*) = |\tau(M)|^2 \,, \tag{1.1}$$

where $\tau(M)$ is the partition function for the $SU(2)$-Chern Simons topological quantum field theory at level $k = r - 2$ [T1]. Originally $\tau(M)$ was introduced and discussed on a formal level based on functional integration methods [Wi2]. However, introducing the theory of coloured ribbon graphs, Reshetikhin and Turaev [RT] have provided a mathematical construction of $\tau(M)$ having all the desired properties. Now the

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Turaev-Viro construction starts with a finite triangulation X of M and associates to X a partition function $Z(X)$. The main result of [TV] is that $Z(X)$ does not change under (a sufficiently large class of local) subdivisions and hence defines an invariant $Z(M)$ of M. This is a particular attractive feature from the point of view of topological quantum field theories initiated by Witten [Wi1] and axiomatized by Atiyah in [At1] (for a review see also [At2]). Indeed, since such theories have trivial dynamics, a triangulation, which corresponds to the introduction of a high energy cut-off, should have no influence. In other words, the renormalization group transformation (i.e. a block spin transformation) defined by a subdivision should be trivial. Mathematically there is another motivation for a combinatorial approach. In analogy to algebraic topology but in contrast to quantum field theories with nontrivial dynamics, topological quantum field theories are supposed to give rise to finite dimensional vector spaces V^{Σ} with certain structures which provide information on those 3-dim. compact manifolds which are bounded by a fixed closed 2-manifold Σ. Hence a finite set of data should suffice. Yet another nice feature of this approach is that configurations are labelled by representations (customarily called colours) living on the 1-simplexes instead of by group elements as is the case in lattice gauge theories.

Basically the Turaev-Viro construction goes as follows. Using q-dimensions of colours on 1-simplexes and $q-6j$-symbols on 3-simplexes to define local weights, their product essentially defines the Gibbs weight $W(\underline{j})(X)$, where $\underline{j}$ labels the configuration. The partition function, also called state sum, is then given in the usual way as the sum over all configurations

$$Z(X)=\sum_{\underline{j}} W(\underline{j})(X). \tag{1.2}$$

The reason Eq. (1.2) is well defined for $U_q(sl(2,\mathbf{C}))$ (q a $2r^{\text{th}}$ primitive root of unity) is that there the set $\mathscr{J}=\left\{0,\frac{1}{2},\ldots,\frac{r}{2}-1\right\}$ of colours is finite making the sum in (1.2) finite. Cobordism theory was introduced in [TV] by freezing the configuration $\underline{j}$ on the boundary ∂X to define

$$Z(X,\underline{\alpha})=\frac{1}{W_{\underline{\alpha}}}\sum_{\underline{j}|\partial X=\underline{\alpha}} W(\underline{j})(X) \tag{1.3}$$

with a certain weight factor $W_{\underline{\alpha}}$. In view of relation (1.2) this gives in particular

$$Z(X)=\sum_{\underline{\alpha}} W_{\underline{\alpha}} Z(X_1,\underline{\alpha})Z(X_2,\underline{\alpha}) \tag{1.4}$$

whenever X may be cut along a 2-dimensional simplicial complex $\partial X_1=\partial X_2$ into two disjoint parts X_1 and X_2 [see Eqs. (2.14)–(2.16) below for explicit definitions]. Arguments like these we will call surgery techniques. Relation (1.4) is the manifold analogue of transfer matrix (i.e. semigroup) multiplication techniques in statistical physics. Consider in particular the case where $M=\Sigma\times I$ (Σ a compact closed two manifold I, the unit interval). Let X be a triangulation of M inducing a triangulation ∂X of the boundary $\partial M=\Sigma\times\{0,1\}$. If $\underline{\alpha}_l$ and $\underline{\alpha}_r$ denote the colourings on the two connected components ∂X_l and ∂X_r of ∂X, then one has the result that the matrix $Z(X,\underline{\alpha}_l,\underline{\alpha}_r)$ indexed by $\underline{\alpha}_l$ and $\underline{\alpha}_r$ acts as a projection operator, provided the triangulations ∂X_l of $\Sigma\times\{0\}$ and ∂X_r of $\Sigma\times\{1\}$ agree. Its trace, which is equal to $Z(\Sigma \times S^1)$ therefore gives the dimension of the vector space V^{Σ} for the underlying topological quantum field theory associated to Σ. This property, namely that the transfer matrix is just a projection operator, is again a reflection of the fact that topological quantum field theories have trivial dynamics. The drawback of this discussion based on definition (1.3) is that it is triangulation dependent.

The aim of this article is to establish surgery relations which are manifestly independent of the triangulation and which in particular allow for calculations of explicit examples such as $Z(\Sigma \times S^1)$. In fact, we expect that the surgery methods presented in this article should also allow the calculation of the state sum $Z(M)$ for other 3-manifolds like compact hyperbolic spaces (for examples see e.g. [Vin]). By relation (1.1) the calculation of $Z(M)$ for all large r could give some information on the Ray-Singer torsions for the flat connections involved (see e.g. [FG, Wi3]). The idea we use and which will be worked out in this article is to construct observables in the form of coloured graphs, i.e. certain embedded 1-dimensional simplicial complexes, whose 1-simplexes carry colour. These coloured graphs live on ∂M or on the boundary of tubular neighborhoods of embedded 1-dim. simplicial complexes in $\operatorname{int} M$. The resulting state sums will be homotopy invariants of the embedding and will in addition satisfy braiding relations giving rise to knot invariants (see also [T2]). Responsible for these braiding relations is an addtional property of abstract $6j$-symbols called the Racah relation, which we postulate and which holds for $U_q(sl(2,\mathbf{C}))$. As is well known this Racah relation combined with the Biedenharn-Elliot relation ensures the existence of an R-matrix with the ensuing Yang-Baxter and fusion equation (see e.g. [KR]).

The technique we use is the following. In addition to the configurations $\underline{j}$ of Turaev and Viro in [KMS] we introduced configurations $\underline{J}$ on ∂X called vertex colourings. They allowed to associate a certain additional weight factor to ∂X leading to a partition functor $Z(X)$ which was independent of the particular triangulation of X thus leading to a state sum $Z(M)$ for manifolds with boundary. We will generalize the definition of $\underline{J}$. This will allow us to associate weight factors to coloured graphs $|G|_{\underline{x}}$ on ∂M leading to state sums $Z(M, |G|_{\underline{x}})$. These state sums will be homotopy invariants of the embedding. This notation will be generalized to coloured graphs $\mathscr{G}_{\underline{x}}$ in M, which in addition will satisfy braiding relations. Techniques which in spirit are similar to ours have been used by Turaev [T2]. As a result we obtain the surgery formula when M is cut along Σ into M_1 and M_2,

$$Z(M) = \sum_{x} W_{\underline{x}}^{\Sigma} Z(M_1 |G|_{\underline{x}}^{\Sigma^*}) Z(M_2, |G|_{\underline{x}}^{\Sigma}), \tag{1.5}$$

which is a triangulation independent form of relation (1.4). Here Σ is supposed to be oriented and Σ^* is the same space with opposite orientation such that $\partial M_1 = \Sigma^*$ and $\partial M_2 = \Sigma$. $|G|_{\underline{x}}^{\Sigma}$ is a canonical graph (to be defined in Sect. 6) with colours $\underline{x}$ on Σ. $W_{\underline{x}}^{\Sigma}$ is a certain weight depending on $\underline{x}$ only. In addition we construct an explicit basis of the vector space V^{Σ} of the underlying topological quantum field theory associated to Σ. Here Σ may be a surface with coloured punctures. In addition these coloured punctures may each be given an orientation. This additional structure is again a reflection of relation (1.1).

In Sect. 2 we recall and extend the basic data needed in the construction of Turaev and Viro and its generalizations to be presented in this article. In Sect. 3 we introduce coloured graphs $|G|_{\underline{x}}$ on the boundary of 3-manifolds and construct state sums involving such coloured graphs. The notion of a reduced graph by which planar parts of a coloured graph $|G|_{\underline{x}}$ are collapsed is introduced in Sect. 4. In Sect. 5 we establish elementary cutting rules for state sums with coloured graphs and introduce what will turn out to be the central concept of a meridian. The notion of coloured graphs in the interior of a 3-manifold is introduced in Sect. 6 leading to braiding relations and knot invariants on the level of the associated state sums. Section 7 contains the main results of this article where we establish surgery formulas along Riemann surfaces (with oriented coloured punctures). We elaborate on these state sums in relation to topological quantum field theory. We have attempted to make the reader familiar with the concepts introduced here by calculating explicit examples.

2. Basic Definitions

In this section we collect and extend the basic notions and definitions of the Turaev-Viro state sum for 3-manifolds with boundary [TV, KMS].

Let K be a commutative ring with unit, denoted by 1. By K^* we denote the set of invertible elements in K. Let $\mathscr{T}$ be a finite set with a distinguished element 0. The elements in $\mathscr{T}$ will be called colours. Let $i \mapsto w_i$ be a map from $\mathscr{T}$ into K^* such that $\sum_{i \in \mathscr{T}} w_i^4$ is equal to the square of an element $w \in K^*$ and such that $w_0 = 1$. We assume there is given a nonempty set of unordered triples $(i, j, k) \in \mathscr{T}$ called admissible. For any unordered triple $(i, j, k) \in \mathscr{T}$ we set $\delta_{ijk} = 1$ if (i, j, k) is admissible and zero otherwise.

We assume

$$\delta_{0jk} = \delta_{jk} \,. \tag{2.1}$$

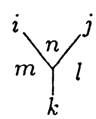
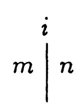
Now the relation

$$\sum_i w_i^2 \delta_{ijk} = w_j^2 w_k^2 \tag{2.2}$$

is supposed to hold for all $j, k \in \mathscr{T}$. This implies that

$$w^2 = w_k^{-2} \sum_{i,j} w_i^2 w_j^2 \delta_{ijk} \tag{2.3}$$

holds for all $k \in \mathscr{T}$. The ordered 6-tuple (i, j, k, l, m, n) is called admissible, if all the 4 triples (i, j, k), (k, l, m), (i, m, n), and (j, l, n) are admissible. To each such admissible 6-tuple we assume there is associated an element of K, the abstract $6j$-symbol, describing a 3-vertex, denoted by

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} =$$  , where  $= \delta_{imn}$ (2.4)

(here the elements $i, j, k \in \mathscr{T}$ are associated to the lines and $l, m, n \in \mathscr{T}$ to the plaquettes or sectors between the lines). The $6j$-symbols are supposed to satisfy the following symmetry relations:

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} = \begin{vmatrix} j & i & k \\ m & l & n \end{vmatrix} = \begin{vmatrix} i & k & j \\ l & n & m \end{vmatrix} = \begin{vmatrix} i & m & n \\ l & j & k \end{vmatrix} , \tag{2.5}$$

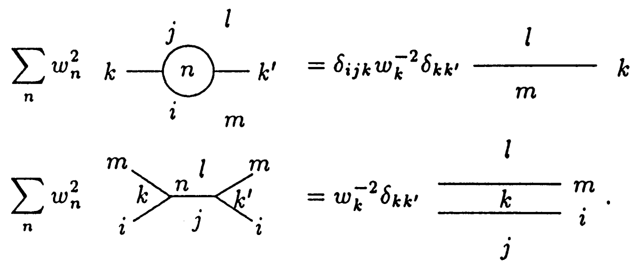
as well as

$$\begin{vmatrix} i & j & k \\ l & m & 0 \end{vmatrix} = \delta_{im}\delta_{ijk}\delta_{jl}\,\frac{1}{w_i w_j}\,, \tag{2.6}$$

and the following orthogonality relations

$$\sum_{n\in\mathscr{I}} w_n^2 \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}\begin{vmatrix} i & j & k' \\ l & m & n \end{vmatrix} = \delta_{ijk}\delta_{lmk}w_k^{-2}\delta_{kk'}\,. \tag{2.7}$$

The summation is such that all $6j$-symbols involved are defined (a convention which we shall employ throughout this article whenever we write out relations involving $6j$-symbols), i.e. both 6-symbols (i,j,k,l,m,n) and (i,j,k',l,m,n) are supposed to be admissible. We give these relations the following two alternative graphical interpretations:

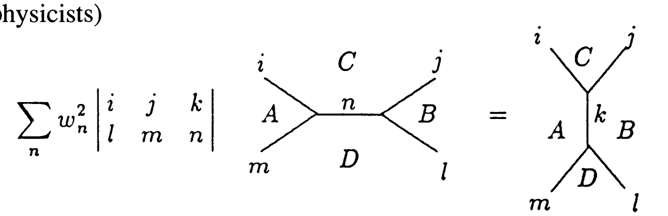


(2.7′)

These $6j$-symbols are also supposed to satisfy the abstract Biedenharn-Elliot relations in the form

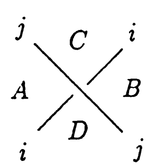
$$\sum_n w_n^2 \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}\begin{vmatrix} i & m & n \\ D & A & C \end{vmatrix}\begin{vmatrix} j & l & n \\ D & C & B \end{vmatrix} = \begin{vmatrix} i & j & k \\ B & A & C \end{vmatrix}\begin{vmatrix} k & l & m \\ D & A & B \end{vmatrix}. \tag{2.8}$$

We give this relation the following graphical interpretation (called a Fierz transformation by physicists)



(2.8′)

Finally we assume there are elements $q_i \in K^*$ $(i \in \mathscr{I})$ with $q_0 = 1$ such that the R-matrix

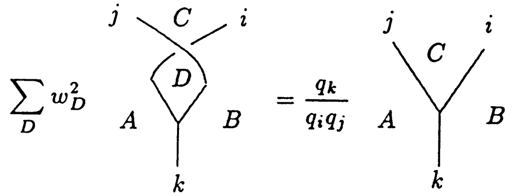


$$= \frac{q_A q_B}{q_C q_D}\begin{vmatrix} A & i & D \\ B & j & C \end{vmatrix} \tag{2.9}$$

satisfies the abstract Racah relation

$$\sum_D w_D^2 \frac{q_A q_B}{q_C q_D}\begin{vmatrix} A & i & D \\ B & j & C \end{vmatrix}\begin{vmatrix} i & j & k \\ B & A & D \end{vmatrix} = \frac{q_k}{q_i q_j}\begin{vmatrix} i & j & k \\ A & B & C \end{vmatrix}. \tag{2.10}$$

We depict this graphically as



$$\sum_D w_D^2 \quad = \frac{q_k}{q_i q_j} \qquad (2.10')$$

All these conditions are met in the case of the quantum group $U_q(sl(2, \mathbf{C}))$ with $q = \exp(i\pi s/r)$ (r and $s \in \mathbf{Z}$ relatively prime) where we have $K = \mathbf{C}$, $\mathscr{I} = \left\{0, \frac{1}{2}, 1, \ldots, \frac{r}{2} - 1\right\}$,

$$\begin{aligned} w_i^2 &= (-1)^{2i} \frac{q^{2i+1} - q^{-2i-1}}{q - q^{-1}} = (2i+1)_{-q}, \\ w^2 &= \frac{-2r}{(q - q^{-1})^2}, \\ q_i &= (-1)^{2i} q^{i(i+1)}, \\ \delta_{ijk} &= \begin{cases} 1 & \text{if } k \le i + j,\ j \le i + k,\ i \le k + j,\ r - 2 \ge i + j + k \in \mathbf{Z}, \\ 0 & \text{otherwise} \end{cases} \end{aligned} \qquad (2.11)$$

and where the $q - 6j$-symbols are given in [KR, K, TV].

The state sum of Turaev and Viro for a compact 3-manifold M with boundary ∂M is defined via a triangulation X of M (inducing a triangulation ∂X of ∂M) as follows [TV, KMS].

Let $\underline{j}: \sigma^1 \in X \mapsto j(\sigma^1) \in \mathscr{I}$ be any map from the (nonoriented) 1-simplexes of X into $\mathscr{I}$ and $\underline{J}: \sigma^0 \in \partial X \mapsto J(\sigma^0) \in \mathscr{I}$ any map from the vertices of ∂X into $\mathscr{I}$. Any such pair $(\underline{j}, \underline{J})$ of maps will be called a configuration. For any (nonoriented) 3-simplex σ^3 in X set

$$(6\underline{j})(\sigma^3) = \begin{vmatrix} j(\sigma_1^1) & j(\sigma_2^1) & j(\sigma_3^1) \\ j(\sigma_4^1) & j(\sigma_5^1) & j(\sigma_6^1) \end{vmatrix}, \qquad (2.12)$$

where σ_i^1 and σ_{i+3}^1 ($i = 1, 2, 3$) are the pairwise opposite 1-simplexes in $\partial\sigma^3$. For any (nonoriented) 2-simplex σ^2 in ∂X set

$$(6\underline{j}, \underline{J})(\sigma^2) = \begin{vmatrix} j(\sigma_1^1) & j(\sigma_2^1) & j(\sigma_3^1) \\ J(\sigma_1^0) & J(\sigma_2^0) & J(\sigma_3^0) \end{vmatrix}, \qquad (2.13)$$

where the vertices σ_i^0 in $\partial\sigma^2$ are opposite to the 1-simplexes σ_i^1 in $\partial\sigma^2$ ($i = 1, 2, 3$). By (2.5) both definitions (2.12) and (2.13) make sense. The Gibbs weight factor of a given configuration is defined to be

$$\begin{aligned} W(\underline{j}, \underline{J})(X) &= \prod_{\sigma^0 \in X} w^{-2} \prod_{\sigma^1 \in X} w_{j(\sigma^1)}^2 \prod_{\sigma^3 \in X} (6\underline{j})(\sigma^3) \\ &\quad \times \prod_{\sigma^0 \in \partial X} w_{J(\sigma^0)}^2 \prod_{\sigma^2 \in \partial X} (6\underline{j}, \underline{J})(\sigma^2) \\ &= W(\underline{j})(X) W(\underline{j}, \underline{J})(\partial X). \end{aligned} \qquad (2.14)$$

Also $W_{\underline{\alpha}}$ in Eq. (1.3) is given as

$$W_{\underline{\alpha}} = \prod_{\sigma^1 \in \partial X} w^2_{j(\sigma^1)} \,. \tag{2.15}$$

For given triangulation the state sum is now defined to be given by

$$Z(X) = \sum_{\underline{j},\underline{J}} W(\underline{j},\underline{J})(X) \,. \tag{2.16}$$

The basic result in [TV] for the case $\partial M = \phi$ and in [KMS] for the general case is that $Z(X)$ is independent of the particular choice of the triangulation and hence defines an invariant of M.

Examples 2.1. In [KMS] we calculated the following state sums

$$Z(D^3) = 1 \,, \qquad Z(\text{handlebody}) = 1 \tag{2.17}$$

(the second equation follows from the first one since the cutting of handles does not change the state sum),

$$Z(M \backslash D^3) = w^2 Z(M) \,, \tag{2.18}$$

$$Z(S^3) = w^{-2} \,, \qquad Z(S^2 \times S^1) = 1 \,. \tag{2.19}$$

Also for a connected sum the relation

$$Z(M_1 \# M_2) = w^2 Z(M_1) Z(M_2) \tag{2.20}$$

is valid.

3. State Sums of Coloured Graphs on ∂M

In what follows G will denote a finite 1-dimensional simplicial complex without boundary and $|G|$ the associated normal Hausdorff space (see e.g. [Sp]). In this section we will in addition make the restriction that every vertex $\sigma^0 \in G$ is contained in the boundary of $n = n(\sigma^0)$ 1-simplexes with $2 \leq n \leq 4$. According to the value of $n(\sigma^0)$ we will speak of an n-vertex. Note that the notion of being an n-vertex with $n \geq 3$ is independent of the particular triangulation G of $|G|$. We will call the set of stars $st(\sigma^0)$ in G the elementary stars in G. Also we will introduce the following additional structure at a 4-vertex $\sigma^0 \in G$ by paring the 4 1-simplexes meeting at σ^0 into two unordered paris. The 1-simplexes in one pair are called opposite to each other. In addition one of the pairs is given the name "above" and the other pair is given the name "below." We depict this structure geometrically in Fig. 1. By the above remark this additional structure is again independent of the particular triangulation G of $|G|$. By abuse of notation we view $|G|$ as the associated space equipped with this additional structure.

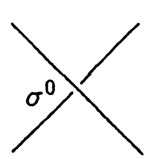


Fig. 1. A 4-vertex

Let $\underline{x}:\sigma^1 \mapsto x(\sigma^1)$ be a map from the set of nonoriented 1-simplexes in G into $\mathscr{T}$ with the following properties:
1) If two 1-simplexes σ_1^1 and σ_2^1 join at a 2-vertex, then $x(\sigma_1^1) = x(\sigma_2^1)$.
2) If two 1-simplexes σ_2^1 and σ_2^1 are opposite to each other at a 4-vertex then $x(\sigma_1^1) = x(\sigma_2^1)$ (compare Fig. 1).

If $\operatorname{sd} G$ is a subdivision of G, then such a map $\underline{x}$ induces a map $\operatorname{sd}\underline{x}$ on $\operatorname{sd} G$ with similar properties by setting $\operatorname{sd} x(\sigma^{1'}) = x(\sigma^1)$ whenever $|\sigma^{1'}| \subset |\sigma^1|$ ($\sigma^{1'} \in \operatorname{sd} G$, $\sigma^1 \in G$). We say that $\underline{x}$ on G and $\underline{x}'$ on G' with $|G| = |G'|$ are equivalent, if they induce the same maps on a common subdivision. An equivalence class is called a coloured graph and is denoted by $|G|_{\underline{x}}$. Any set ℓ in $|G|$ homeomorphic to an interval will be called a line. Given $|G|_{\underline{x}}$ obviously to each line ℓ we may associate a colour $x = x(\ell)$. By definition a coloured graph on a 2-manifold Σ is a pair $(|G|_{\underline{x}}, \varphi)$ where φ is a homeomorphism of $|G|$ into Σ with the following additional property. Near the image $\varphi(\sigma^0)$ of a 4-vertex σ^0, the images of the two open opposite 1-simplexes in one pair are separated by the image of the closed 1-simplexes in the other pair (as pictured in Fig. 1). Two coloured graphs $(|G|_{\underline{x}}, \varphi)$ and $(|G|_{\underline{x}}, \varphi')$ on Σ are called homotopic if there is a homotopy φ_t $(0 \le t \le 1)$ of the maps φ and φ' such that $(|G|_{\underline{x}}, \varphi_t)$ are coloured graphs on Σ for all $0 \le t \le 1$.

The aim of these section is to define a state sum $Z(M, |G|_{\underline{x}}, \varphi)$, which is a homotopy invariant of the coloured graph $(|G|_{\underline{x}}, \varphi)$ on ∂M. In analogy to the strategy reviewed in Sect. 2 this will be achieved by a construction starting with a triangulation X of M. We say that the triangulation G of the coloured graph $(|G|_{\underline{x}}, \varphi)$ is adapted to ∂X if the image under φ of the k^{th} skeleton of G is contained in the k^{th} skeleton of ∂X $(k = 0, 1)$. ∂X obviously induces a triangulation of $|G|$ which is a subdivision of G and which will be denoted by $G(\partial X, \varphi)$. Also to every coloured graph $(|G|_{\underline{x}}, \varphi)$ on ∂M with a triangulation G there is a triangulation X of M such that G is adapted to ∂X. We will generalize the definition of a vertex colouring $\underline{J}$ on ∂X as explained in Sect. 2 to what we will call a sector colouring $\underline{\tilde{J}}$ on ∂X. Away from $\varphi(|G|)$, $\underline{\tilde{J}}$ and $\underline{J}$ will agree. This will enable us to define weight factors in the form

$$W(\underline{j}, \underline{\tilde{J}})(X, |G|_{\underline{x}}, \varphi) = W(\underline{j}, \underline{\tilde{J}})(X) \prod_{\sigma^0 \in G(\partial X, \varphi)} W(\underline{x}, \underline{\tilde{J}})(\sigma^0) \times \prod_{\sigma^1 \in G(\partial X, \varphi)} W(\underline{x}, \underline{j}, \underline{\tilde{J}})(\sigma^1) \tag{3.1}$$

whenever G is adapted to ∂X. Ths will give us a state sum

$$Z(X, |G|_{\underline{x}}, \varphi) = \sum_{\underline{j}, \underline{\tilde{J}}} W(\underline{j}, \underline{\tilde{J}})(X, |G|_{\underline{x}}, \varphi)\,. \tag{3.2}$$

The main result to be proven in this section is that for given $(|G|_{\underline{x}}, \varphi)$ this state sum is independent of the particular choice of the triangulation X of M. Thus we obtain a quantity $Z(M, |G|_{\underline{x}}, \varphi)$ which we simultaneously will prove to be a homotopy invariant of φ. The technique we use is to define configurations in addition to the 1-simplex colouring $\underline{j}$ on X of Turaev and Viro and the vertex colouring $\underline{J}$ on ∂X introduced in [KMS]. These additional configurations live near the support of the coloured graphs and intuitively serve as "ski wax" to allow "gliding," i.e. homotopy invariance and braiding of the state sums for these coloured graphs. It remains to define all quantities on the right-hand side of (3.1) and we start with introducing $\underline{\tilde{J}}$.

For the purpose of easier notation we will identify $G(\partial X, \varphi)$ with its image in ∂X under φ. For given orientation of ∂M and $\sigma^0 \in G(X, \varphi)$ consider the set of $n(\sigma^0)$ 1-simplexes $\{\mu_i^1(\sigma^0), i = 0, \ldots, n(\sigma^0) - 1\}$ in $G(\partial X, \varphi)$ having σ^0 in their boundary and enumerated with respect to their counterclockwise ordering in the open star $st(\sigma^0) \subset G(\partial X, \varphi)$ of σ^0 (see also Fig. 3 below for the three possible cases of $n(\sigma^0) = 2, 3, 4$). A sector colouring $\underline{\tilde{J}}$ on ∂X is by definition a map $(\sigma^0, i) \mapsto \tilde{J}(\sigma^0, i) \in \mathcal{I}$ $(0 \le i \le n(\sigma^0) - 1)$, where σ^0 is a vertex in ∂X and where $n(\sigma^0)$ is set equal to 1 if σ^0 is not a vertex in $G(\partial X, \varphi)$. With $\underline{j}$ being an edge colouring of X as discussed in Sect. 2 we set

$$W(\underline{j}, \underline{\tilde{J}})(X) = \prod_{\sigma^0 \in X} w^{-2} \prod_{\sigma^1 \in X} w^2_{j(\sigma^1)} \prod_{\sigma^3 \in X} (6\underline{j})(\sigma^3) \times \prod_{\substack{\sigma^0 \in \partial X \\ 0 \le i \le n(\sigma^0)-1}} w^2_{\tilde{J}(\sigma^0, i)} \prod_{\sigma^2 \in \partial X} (6\underline{j}, \underline{\tilde{J}})(\sigma^2) \tag{3.3}$$

to be the modification of $W(\underline{j}, \underline{J})(X)$ as defined in Sect. 2. Here $(6\underline{j}, \underline{\tilde{J}})(\sigma^2)$ is the following modification of $(6\underline{j}, \underline{J})(\sigma^2)$. If σ^0 is a vertex in σ^2 not in $G(\partial X, \varphi)$, then $J(\sigma^0)$ is replaced $\tilde{J}(\sigma^0, 0)$. If on the other hand σ^0 is a vertex in $G(\partial X, \varphi)$ then $J(\sigma^0)$ is replaced by $\tilde{J}(\sigma^0, i)$ where i is chosen such that the sector at σ^0 defined by the 1-simplexes $\mu_i^1(\sigma^0)$, $\mu^1_{i+1(\mathrm{mod}\, n(\sigma^0))}(\sigma^0)$ intersects σ^2,

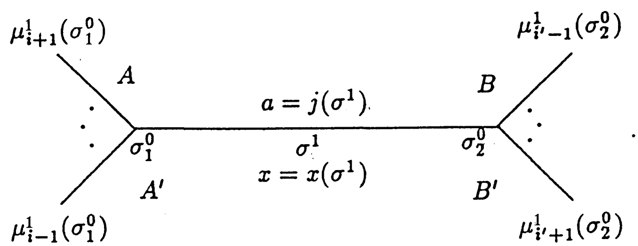


Fig. 2. An elementary line (colour x) of a graph associated to a 1-simplex (colour a)

Furthermore we define a weight for $\sigma^1 \in G(\partial X, \varphi)$,

$$W(\underline{x}, \underline{j}, \underline{\tilde{J}})(\sigma^1) = \begin{vmatrix} A & a & B \\ B' & x & A' \end{vmatrix} \tag{3.4}$$

with $a = j(\sigma^1)$, $x = x(\sigma^1)$. Also $A, A', B, B' \in \mathcal{I}$ are given in terms of $\underline{\tilde{J}}$ as follows. For given $\sigma^1 = [\sigma_1^0, \sigma_2^0]$ let i and i' be such that $\sigma^1 = \mu_i^1(\sigma_1^0) = \mu_{i'}^1(\sigma_2^0)$. Then (compare Fig. 2)

$$\begin{aligned} A &= \tilde{J}(\sigma_1^0, i), \qquad A' = \tilde{J}(\sigma_1^0, i - 1(\mathrm{mod}\, n(\sigma_1^0))), \\ B &= \tilde{J}(\sigma_2^0, i' - 1(\mathrm{mod}\, n(\sigma_2^0))), \qquad B' = \tilde{J}(\sigma_2^0, i). \end{aligned} \tag{3.5}$$

It remains to define the weights for the vertices σ^0 in $G(\partial X, \varphi)$. For the three possible forms of the open star at σ^0, we have the situation depicted in Fig. 3. Here A, B, C, D denote the relevant sector and x, y, z the relevant line colourings respectively. The weights will depend on the orientation of M which induces an orientation of ∂M. If this orientation is counterclockwise on ∂M, we asociate weights

$$\begin{aligned} w(v_2) &= \delta_{xAB}\,, \\ w(v_3) &= \begin{vmatrix} x & y & z \\ A & B & C \end{vmatrix}, \\ w(v_4) &= \frac{q_A q_B}{q_C q_D} \begin{vmatrix} A & x & D \\ B & y & C \end{vmatrix}. \end{aligned} \tag{3.6}$$

If the orientation is clockwise, we associate weights denoted by $w(v_i^*)$. They are obtained from the weights $w(v_i)$ replacing all q_j by $\frac{1}{q_j}$. This means that the weights $w(v_i^*)$ are equal to $w(v_i)$ if $i = 1$ or 3. If $i = 4$ we replace $\frac{q_A q_B}{q_C q_D}$ by its inverse. The vertices v_i^* have the same geometry as the v_i in Fig. 3. In the following we write all relations for counterclockwise oriented boundaries of 3-manifolds. The transition to the opposite orientation will become relevant in Sect. 7. This concludes the definition of the right-hand side of Eq. (3.1),

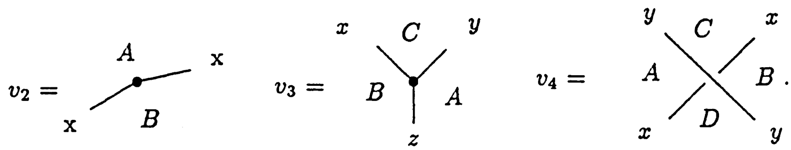


Fig. 3. The elementary vertices

Theorem 3.1. *For sufficiently fine triangulations X of M the state sum* (3.2) *for a coloured graph* $(|G|_{\underline{x}}, \varphi)$ *on ∂M is invariant under isotopies on the* 1*-skeleton of ∂X of the coloured graph* $(|G|_{\underline{x}}, \varphi)$ *and invariant under local subdivisions (such as Alexander moves* [Al]*) of X. Thus the state sums $Z(X, |G|_{\underline{x}}, \varphi)$ for all such X are equal and hence define a state sum $Z(M, |G|_{\underline{x}}, \varphi)$ which is a homotopy invariant* w.r.t. φ.

Remark 3.2. Having established this fact, in order to abbreviate notation we shall also write $Z(M, |G|_{\underline{x}}, \varphi)$ simply as $Z(M, |G|_{\underline{x}})$ with the understanding that now $|G|_{\underline{x}}$ itself is viewed as a coloured graph on ∂M. Also we will write $G(\partial X)$ instead of $G(\partial X, \varphi)$, etc.

To prove this theorem, we note that the first part is a direct consequence of

Lemma 3.3. *For a fixed sufficiently fine triangulation X of M the state sum* (3.2) *is invariant under the following* 3 *elementary isotopies of* $(|G|_{\underline{x}}, \varphi)$ *on the* 1*-skeleton of ∂X.*

a) *A line with colour x passing a* 2*-simplex $\sigma^2 \in \partial X$ along two* 1*-simplexes in its boundary may be shifted to the other* 1*-simplex in its boundary as depicted in Fig. 4a.*

b) *An elementary v_3 graph at a vertex $\sigma^0 \in \partial\sigma^1$ may be shifted to the other vertex $\sigma^{0'} \in \partial\sigma^1$ as in Fig. 4b.*

c) *An elementary v_4 graph at a vertex $\sigma^0 \in \partial\sigma^1$ may be shifted to the other vertex $\sigma^{0'} \in \partial\sigma^1$ as in Fig. 4c (depicted for the case that the colour x is below the colour y).*

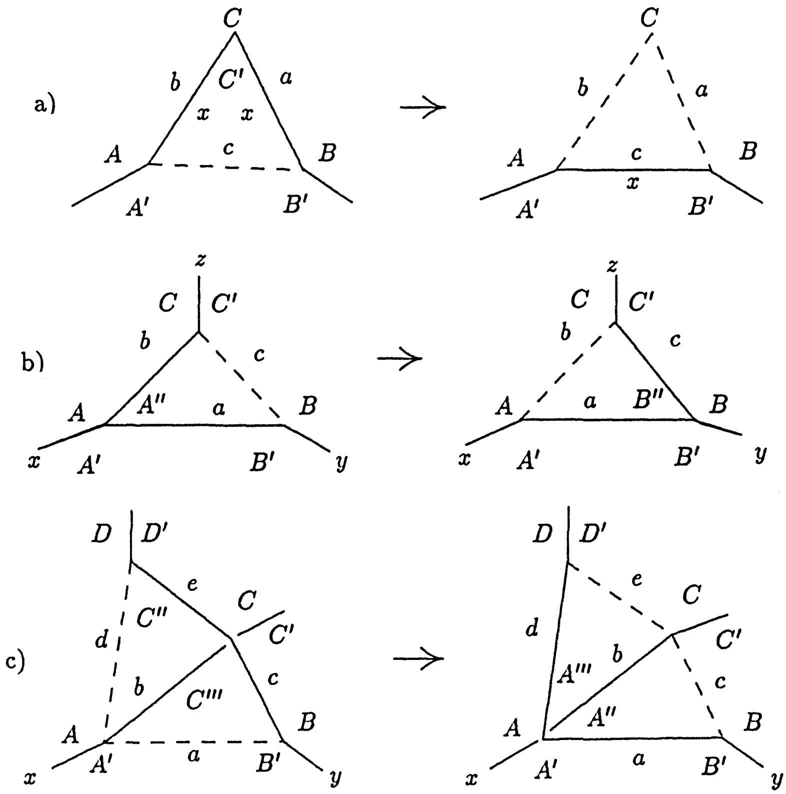


Fig. 4. Elementary deformations of graph

Proof. The proof of a) follows directly from the Biederharn-Elliot relations in the form

$$\sum_{C'} w_{C'}^2 \begin{vmatrix} a & b & c \\ A' & B' & C' \end{vmatrix} \begin{vmatrix} a & C & B \\ x & B' & C' \end{vmatrix} \begin{vmatrix} b & A & C \\ x & C' & A' \end{vmatrix}$$
$$= \begin{vmatrix} a & b & c \\ A & B & C \end{vmatrix} \begin{vmatrix} c & A & B \\ x & B' & A' \end{vmatrix}. \tag{3.7a}$$

The claim b) means that

$$\sum_{A''} w_{A''}^2 \begin{vmatrix} a & b & c \\ C' & B & A'' \end{vmatrix} \begin{vmatrix} a & A'' & B \\ y & B' & A' \end{vmatrix} \begin{vmatrix} b & A & C \\ z & C' & A'' \end{vmatrix} \begin{vmatrix} x & y & z \\ A'' & A & A' \end{vmatrix}$$
$$= \sum_{B''} w_{B''}^2 \begin{vmatrix} a & b & c \\ C & B'' & A \end{vmatrix} \begin{vmatrix} a & A & B \\ x & B' & A' \end{vmatrix} \begin{vmatrix} c & C' & B \\ z & B'' & C \end{vmatrix} \begin{vmatrix} x & y & z \\ B & B'' & B' \end{vmatrix}. \tag{3.7b}$$

This follows by applying the Biedenharn-Elliot relations twice in the form

$$\begin{vmatrix} a & b & c \\ C' & B & A'' \end{vmatrix} \begin{vmatrix} b & A & C \\ z & C' & A'' \end{vmatrix}$$
$$= \sum_{B''} w_{B''}^2 \begin{vmatrix} a & b & c \\ C & B'' & A \end{vmatrix} \begin{vmatrix} c & C' & B \\ z & B'' & C \end{vmatrix} \begin{vmatrix} a & A & B'' \\ z & B & A'' \end{vmatrix}$$

and

$$\sum_{A''} w_{A''}^2 \begin{vmatrix} x & y & z \\ A'' & A & A' \end{vmatrix} \begin{vmatrix} a & A & B'' \\ z & B & A'' \end{vmatrix} \begin{vmatrix} a & A'' & B \\ y & B' & A' \end{vmatrix}$$
$$= \begin{vmatrix} x & y & z \\ B & B'' & B' \end{vmatrix} \begin{vmatrix} a & A & B'' \\ x & B' & A' \end{vmatrix}.$$

The claim c) for the case of Fig. 4c) is that

$$\sum_{C'',C'''} w_{C''}^2 w_{C'''}^2 \begin{vmatrix} b & A & C'' \\ x & C''' & A' \end{vmatrix} \begin{vmatrix} e & C'' & D \\ y & D' & C \end{vmatrix} \begin{vmatrix} c & B' & C''' \\ y & C' & B \end{vmatrix}$$
$$\times \frac{q_{C'}q_{C''}}{q_C q_{C'''}} \begin{vmatrix} C'' & x & C''' \\ C' & y & C \end{vmatrix} \begin{vmatrix} b & d & e \\ D & C'' & A \end{vmatrix} \begin{vmatrix} a & b & c \\ C''' & B' & A' \end{vmatrix}$$
$$= \sum_{A'',A'''} w_{A''}^2 w_{A'''}^2 \begin{vmatrix} b & A''' & C \\ x & C' & A'' \end{vmatrix} \begin{vmatrix} d & A & D \\ y & D' & A''' \end{vmatrix} \begin{vmatrix} a & A'' & B \\ y & B' & A' \end{vmatrix}$$
$$\times \frac{q_A q_{A''}}{q_{A'} q_{A'''}} \begin{vmatrix} A & x & A' \\ A'' & y & A''' \end{vmatrix} \begin{vmatrix} b & d & e \\ D' & C & A''' \end{vmatrix} \begin{vmatrix} a & b & c \\ C' & B & A'' \end{vmatrix}. \qquad (3.7c)$$

To prove (3.7c) we first use the Biedenharn-Elliot relations in the form

$$\begin{vmatrix} a & b & c \\ C''' & B' & A' \end{vmatrix} \begin{vmatrix} c & B' & C''' \\ y & C' & B \end{vmatrix}$$
$$= \sum_{A''} w_{A''}^2 \begin{vmatrix} c & b & a \\ A'' & B & C' \end{vmatrix} \begin{vmatrix} b & A' & C''' \\ y & C' & A'' \end{vmatrix} \begin{vmatrix} a & A'' & B \\ y & B' & A' \end{vmatrix}.$$

Then the Yang-Baxter equations in the form

$$\sum_{C'''} \frac{q_{C'}q_{C''}}{q_C q_{C'''}} w_{C'''}^2 \begin{vmatrix} C'' & x & C''' \\ C' & y & C \end{vmatrix} \begin{vmatrix} b & A & C'' \\ x & C''' & A' \end{vmatrix} \begin{vmatrix} b & A' & C''' \\ y & C' & A'' \end{vmatrix}$$
$$= \sum_{A'''} \frac{q_A q_{A''}}{q_{A'''} q_{A'}} w_{A'''}^2 \begin{vmatrix} b & C'' & A \\ y & A''' & C \end{vmatrix} \begin{vmatrix} b & A''' & C \\ x & C' & A'' \end{vmatrix} \begin{vmatrix} A & x & A' \\ A'' & y & A''' \end{vmatrix}$$

and finally again the Biedenharn-Elliot relations in the form

$$\sum_{C''} w_{C''}^2 \begin{vmatrix} b & d & e \\ D & C'' & A \end{vmatrix} \begin{vmatrix} e & C'' & D \\ y & D' & C \end{vmatrix} \begin{vmatrix} b & C'' & A \\ y & A''' & C \end{vmatrix}$$
$$= \begin{vmatrix} d & A & D \\ y & D' & A''' \end{vmatrix} \begin{vmatrix} b & d & e \\ D' & C & A''' \end{vmatrix}$$

prove the claim (3.7c). The case where the line with colour x lies above the line with colour y is treated analogously. This concludes the proof of Lemma 3.3.

To conclude the proof of Theorem 3.1 we adapt the strategy used in [KMS]. Consider a local subdivision in $Y \subset X$. By the first part of the theorem before

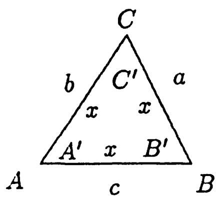


Fig. 5. A triangulation of the coloured graph $|G|_x \cong S^1_x$

performing the subdivision if necessary we may move the graph $(|G|_{\underline{x}}, \varphi)$ away from Y without changing (3.2). Then we may perform the subdivision which by the arguments in [KMS] again does not change the state sum (3.2) proving the theorem.

The present restriction to the case of the elementary vertices of Fig. 3 with $n(\sigma^0) \leq 4$ will be removed in the next section when we discuss what we will call reductions of coloured graphs $(|G|_{\underline{x}}, \varphi)$.

Example 3.4. Let the graph $|G|_x \cong S^1_x$ be contractible in ∂M and triangulated as in Fig. 5. Using Eqs. (2.8), (2.7), and (2.2) one finds for the contribution from the 2-simplex (ABC) of Fig. 5 to the state sum the expression

$$\begin{aligned}
&\sum_{A'B'C'} w_{A'}^2 w_{B'}^2 w_{C'}^2 \begin{vmatrix} a & b & c \\ A' & B' & C' \end{vmatrix} \begin{vmatrix} a & C & B \\ x & B' & C' \end{vmatrix} \begin{vmatrix} b & A & C \\ x & C' & A' \end{vmatrix} \begin{vmatrix} c & A & B \\ x & B' & A' \end{vmatrix} \\
&= \begin{vmatrix} a & b & c \\ A & B & C \end{vmatrix} \sum_{A'B'} w_{A'}^2 w_{B'}^2 \begin{vmatrix} c & A & B \\ x & B' & A' \end{vmatrix} \begin{vmatrix} c & A & B \\ x & B' & A' \end{vmatrix} \\
&= w_A^{-2} \begin{vmatrix} a & b & c \\ A & B & C \end{vmatrix} \sum_{A'} w_{A'}^2 \delta_{xAA'} = \begin{vmatrix} a & b & c \\ A & B & C \end{vmatrix} w_x^2 .
\end{aligned} \tag{3.8}$$

The $6j$-symbol is just the contribution of the 2-simplex (ABC) to the state sum $Z(M)$ without the graph $|G|_x \cong S^1_x$ so this means

$$Z(M, S^1_x) = Z(M) w_x^2 . \tag{3.9}$$

In Sect. 4 we derive a generalization of this formula to an arbitrary, what we will call planar graph.

4. Reduced Graphs

With the techniques to be developed in this chapter, we shall establish the following results.

Theorem 4.1. *The state sum $Z(M, |G|_{\underline{x}})$ is invariant under the following local changes of $|G|_{\underline{x}}$ on ∂M as depicted in Fig. 6,*

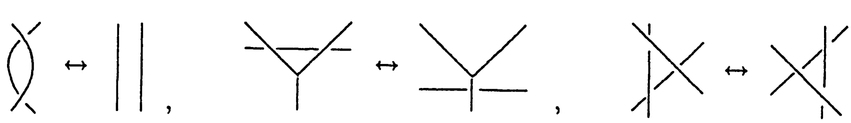


Fig. 6. Local changes of graphs

Any closed subset V of $|G|$ is called planar, if it is equal to the intersection of $|G|$ with a closed set in ∂M homeomorphic to a disc.

Theorem 4.2. *For a planar coloured graph $|G|_{\underline{x}}$ on ∂M the state sum factorizes*

$$Z(M, |G|_{\underline{x}}) = Z(M) Z(|G|_{\underline{x}}), \tag{4.1}$$

where $(Z(|G|_{\underline{x}})$, called the invariant of the planar graph, is independent of M. For a coloured circle S^1_x,

$$Z(S^1_x) = Z(\bigcirc^x) = w_x^2 . \tag{4.2}$$

This theorem states that as observables, planar coloured graphs on ∂M are uninteresting for probing ∂M. Only nonplanar graphs give rise to interesting information. Note that for the case that ∂M is the disjoint union of spheres S^2 every connected component of a coloured graph is planar. Relation (4.1) then reflects the fact that the vector space V^{S^2} of the associated topological quantum field theory is one dimensional as has been proved for the present context in [KMS]. The following remark deals with the general case.

Remark 4.3. Theorem 4.2 will be generalized as follows. To each oriented 2-manifold Σ there is a canonical coloured graph $|G|^{\Sigma}_{\underline{x}}$ on Σ and for any M with $\partial M = \Sigma$ one can write the state sum $Z(M, |G|_{\underline{y}})$ as a linear combination of the form

$$Z(M, |G|_{\underline{y}}) = \sum_{\underline{x}} W^{\Sigma}_{\underline{x}} (|G|_{\underline{y}}, |G|^{\Sigma^*}_{\underline{x}}) \, Z(M, |G|^{\Sigma}_{\underline{x}}) \tag{4.3}$$

for a certain weight factor $W^{\Sigma}_{\underline{x}}$ and with coefficients $(|G|_{\underline{y}}, |G|^{\Sigma^*}_{\underline{x}})$ which are independent of M. In the language of topological quantum field theory we will see that the graphs $|G|^{\Sigma^*}_{\underline{x}}$ define a complete set (although in general not linearly independent) of vectors $v(\underline{x})$ in the finite dimensional vector space V^{Σ} associated to Σ. In Sect. 7 we will construct these canonical graphs explicitly and write the above coefficients as state sums of coloured graphs on $\Sigma \times I$.

The notion of reduced graphs to be developed now also will serve to eliminate the topologically nonrelevant parts of any coloured graph. For a given triangulation G of $|G|_{\underline{x}} \subset \partial M$ let $\{V_\kappa\}_{\kappa \in \mathscr{K}}$ be a finite family of pairwise disjoint planar subsets of $|G|_{\underline{x}}$ with the following properties:

a) every vertex in G is contained in some V_κ and conversely each V_κ contains at least a vertex of G.

b) The intersection of any closed 1-simplex $\bar{\sigma}^1$ in G with any V_κ is either empty or a closed interval with nonempty interior in $\bar{\sigma}^1$ containing at least one vertex in the boundary of σ^1.

The sets V_κ ($\kappa \in \mathscr{K}$) will be called generalized vertices. More precisely, if we identify points in $|G|_{\underline{x}}$ which lie in the same V_κ we obtain a 1-dimensional simplicial complex G^r with vertices $\{\sigma^0_\kappa\}_{\kappa \in \mathscr{K}}$ (the images of V_κ under this identification) and 1-simplexes which may be identified with those 1-simplexes in G not completely contained in any V_κ. Stated in a different way $|G^r| \subset \partial M$ is the deformation retract of $|G|$ in ∂M where each disc defining a V_κ is retracted to a point. With the above identification of 1-simplexes in G^r with certain 1-simplexes in G we obtain a coloured graph $|G^r|_{\underline{x}^r}$ from $|G|_{\underline{x}}$ and in addition an induced colouring $\underline{x}_\kappa$ of each V_κ.

There are of course incidence relations between $\underline{x}^r$ and the $\underline{x}_\kappa$. Taking into account of this it is easy to see that $|G|_{\underline{x}}$ may be reconstructed up to homotopy from $|G^r|_{\underline{x}^r}$ and the $(V_\kappa)_{\underline{x}_\kappa}$. To abbreviate notation we let $|G^r|_{\underline{x}}$ denote the collection of the data $|G^r|_{\underline{x}^r}$ and the $(V_\kappa)_{\underline{x}_\kappa}$ and call it a reduction of $|G|_{\underline{x}}$. In analogy to the definition of $Z(X, |G|_{\underline{x}})$ we will now define $(Z(X, |G^r|_{\underline{x}})$ for any reduction. In fact the definition is analogous to (3.1) and (3.2), now with the proviso that ∂X is adapted to G^r and $\underline{\tilde{J}}$ is defined w.r.t. the induced subdivision $G^r(\partial X)$ of G^r. The only modification necessary is a definition of $w(\underline{x}, \underline{\tilde{J}})(\sigma^0)$ for $\sigma^0 \in G^r(\partial X)$. If σ^0 is not equal to any σ^0_κ then necessarily $n(\sigma^0) = 2$ and its open star is of the form v_2 (see Fig. 3) and we give it the corresponding weight. If σ^0 is a σ^0_κ then we define its weight in terms of the associated coloured graph $(V_\kappa)_{\underline{x}_\kappa}$. For this purpose we define a map $\underline{A}_\kappa : c \mapsto A_\kappa(c)$ into $\mathscr{T}$ from the set of those connected components of $\partial M \backslash |V_\kappa|$ which are disconnected from the complement of the closed disc in terms of which V_κ is defined. $\underline{A}_\kappa$ may be called a plaquette (or cell) colouring of V_κ. These data supplement the sector colours $\tilde{J}(\sigma^0_\kappa, i)$ $(0 \le i \le n(\sigma^0_\kappa) - 1)$. Together with the colours $\underline{x}_\kappa$ of V_κ they allow us to define the weight of the vertex σ^0_κ as the product of the weights $w(v)$ of the open stars v of the form v_2, v_3, and v_4, out of which V_κ is composed, multiplied by $\prod_c w^2_{A_\kappa(c)}$ and summed over all $A_\kappa(c)$:

$$w(V_\kappa) = w(\underline{x}, \underline{\tilde{J}})(\sigma^0_\kappa) = \sum_{A_\kappa(c)} \prod_c w^2_{A_\kappa(c)} \prod_{v \in V_\kappa} w(v) . \tag{4.4}$$

If $n = n(\sigma^0_\kappa)$, we call V_κ a generalized n-vertex which we picture as a disc in the figures below. The definition (4.4) is motivated by and implies the following lemma.

Lemma 4.4. *The local reductions depicted in Fig. 7 do not change the corresponding state sums. In Fig. 7d the weights $w(V_n)$ and $w(V_{n-2})$ are related by*

$$\begin{aligned} &w(V_{n-2})(A_1, \ldots, A_{n-2}; \underline{y}) \\ &\quad = \sum_{A_n} w^2_{A_n} \delta_{xA_1A_n} w(V_n)(A_1, \ldots, A_{n-2}, A_1, A_n; x, \underline{y}) \end{aligned} \tag{4.5}$$

for $n > 2$ and

$$w(V_0) = w^2_x \tag{4.6}$$

for $n = 2$ and $V_2 = v_2$ the elementary 2-vertex defined by (3.6). *This last case reflects the appearance of a "Markov trace" for closed loops. In relations* (4.5) *and* (4.6) *we have only listed the dependence on the "exterior colours" of the generalized vertices involved.*

Proof. Parts a)–c) of the lemma are analogous to those of Lemma 3.3 and have the same proof. It only remains to prove part d). The claim is that

$$\begin{aligned} &\sum_{A_n, A_{n-1}, B', C'} w^2_{A_n} w^2_{A_{n-1}} w^2_{B'} w^2_{C'} \begin{vmatrix} a & b & c \\ A_n & B' & C' \end{vmatrix} \begin{vmatrix} b & A_1 & C \\ x & C' & A_n \end{vmatrix} \\ &\quad \times \begin{vmatrix} a & C & B \\ x & B' & C' \end{vmatrix} \begin{vmatrix} c & B & A_{n-1} \\ x & A_n & B' \end{vmatrix} w(V_n)(A_1, \ldots, A_n; x, \underline{y}) \\ &= \begin{vmatrix} a & b & c \\ A_1 & B & C \end{vmatrix} w(V_{n-2})(A_1, \ldots, A_{n-2}; \underline{y}) . \end{aligned} \tag{4.7}$$

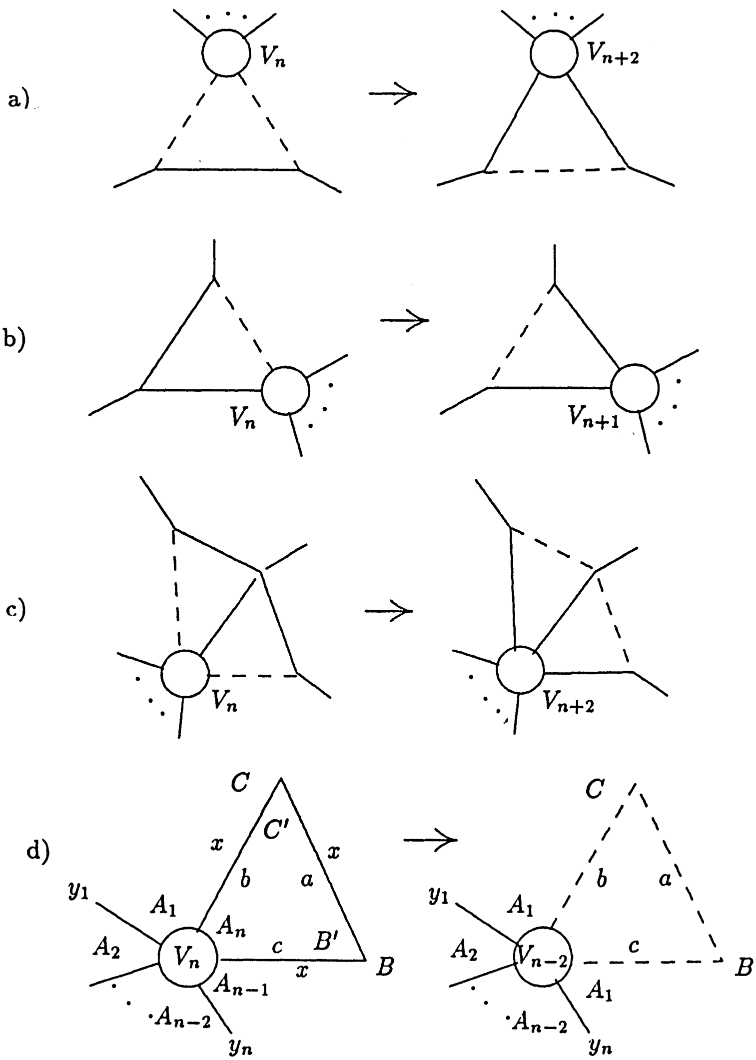


Fig. 7. The four elementary reductions of graphs

Now by the Biedenharn-Elliot relations and the orthogonality relations the sums over B' and C' can be performed on the left-hand side of (4.7) (cf. example 3.4) and one finds

$$\text{l.h.s.}(4.7) = \begin{vmatrix} a & b & c \\ A_1 & B & C \end{vmatrix} \sum_{A_{n-1}} w^2_{A_{n-1}} w^{-2}_{A_{n-1}} \delta_{A_1 A_{n-1}}$$

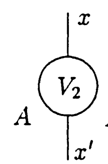
$$\times \sum_{A_n} w^2_{A_n} \delta_{xA_1A_n} w(V_n)(A_1, \ldots, A_{n-2}, A_1, A_n; \underline{y}), \qquad (4.8)$$

proving the claim for $n > 2$. For the case $n = 2$ the sum over A_{n-1} in Eq. (4.8) does not exist and instead we have $A_1 \equiv A_{n-1}$. Therefore inserting (3.6) into (4.8) we obtain the claim (4.6) due to (2.2). Notice that the other vertices in Fig. 7 are considered elementary vertices, i.e. of the type v_2, v_3, and v_4. However, this lemma may easily be generalized to include generalized vertices. Therefore we immediately obtain

Lemma 4.5. *For all reductions $|G^r|_{\underline{x}}$ of $|G|_{\underline{x}}$ such that G^r is adapted to ∂X, the partition functions $Z(X, |G^r|_{\underline{x}})$ agree.*

Proof of Theorem 4.1. The relations of Fig. 6 now follow by considering them inside a generalized vertex. But there they follow easily from relations (2.7), (2.8), and (2.10).

Proof of Theorem 4.2. By Lemma 4.4 we may find a suitable triangulation X and a reduction of $|G|_{\underline{x}}$ which has the form of Fig. 7d for the case $n = 2$. Equation (4.6) can be generalized for the case of an arbitrary 2-vertex. From a Wigner-Eckart type relation (see Appendix A) we conclude



$$\leftrightarrow w(V_2; x, x') = \delta_{xx'}\delta_{xAA'}f(x). \tag{4.9}$$

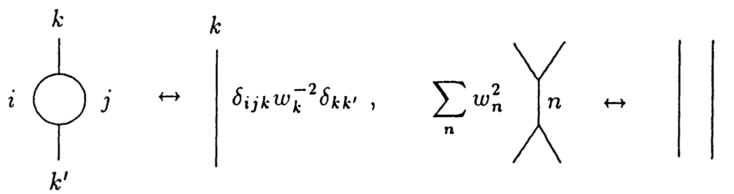
This implies that for the general "0-vertex" Eq. (4.6) is replaced by

$$w(V_0) = w_x^2 f(x)\,. \tag{4.10}$$

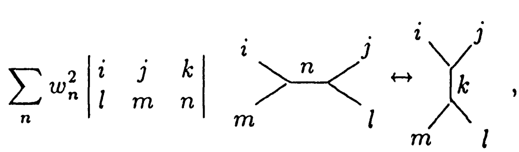
Therefore in Eq. (4.1) the desired factor $Z(|G|_{\underline{x}})$ equals $w_x^2 f(x)$ while the remaining contributions to the state sum give $Z(X) = Z(M)$. This proves the first part, while the second part follows from the fact that $f(x) = 1$ in case $|G|_{\underline{x}}$ is a planar coloured circle S_x^1 (see also Example 3.4).

For later convenience we write some additional formulas for changes of graphs.

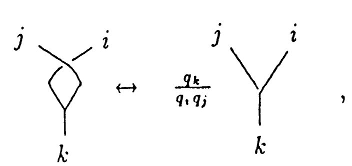
Lemma 4.6. *The following (additive) relations between state sums with the following local form of the coloured graphs hold:*



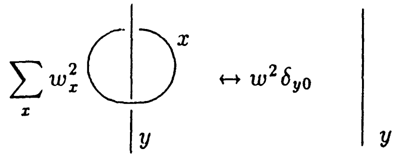
(4.11)



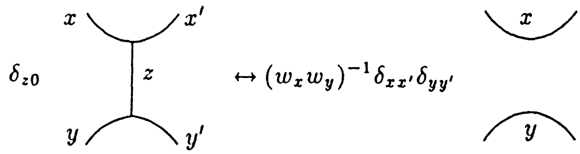
(4.12)



(4.13)



(4.14)



(4.15)

These relations will become important in the next section when we establish cutting rules. The two first relations (4.11) follow from the orthogonality relation (2.7), the Biedenharn-Elliot relation (2.8) and the Racah identity (2.10) imply (4.12) and (4.13), respectively. The equality (4.14) is derived in Appendix A and (4.15) follows from Eqs. (2.6), (3.6), and (4.4). Because of Eqs. (2.9), (2.6), (3.6), and (4.4) this relation remains true if there are additional elementary 4-vertices on the line with colour $z = 0$.

Remark 4.7. Note that if one 1-simplex at an elementary 3-vertex has vanishing colour, then by (2.1) necessarily the colours of the two other 1-simplexes entering this vertex are to be equal in order to have $Z(M, |G|_{\underline{x}}) \neq 0$. This "conservation law" will always be taken into account in what follows.

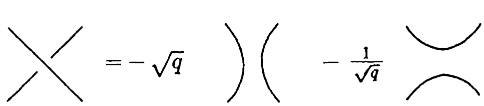
Moreover for the $|G|_{\underline{x}}$ to be considered in the following it turns out that $Z(M, |G|_{\underline{x}})$ is nonvanishing only if one or several lines of the graph have vanishing colour. This motivates the following discussion as a generalization of Eq. (4.15).

Lemma 4.8. *Let $|G|$ be a graph with only elementary vertices v_3 and v_4. For a given fixed colouring $\underline{x}$ of $|G|$ consider the subset $\mathcal{L}_0$ of lines ℓ in $|G|$ with $x(\ell) = 0$. Let $\tilde{G}$ be the 1-dimensional simplicial complex obtained by deleting all the σ^1 which belong to these lines $\ell \in \mathcal{L}_0$. It is esy to see that $|\tilde{G}|$ depends on $|G|_{\underline{x}}$ only and that $|\tilde{G}|$ inherits a colouring $\underline{\tilde{x}}$ leading to a coloured graph $|\tilde{G}|_{\underline{\tilde{x}}}$. Now inspection of* (3.4) *combined with* (2.6) *easily leads us to conclude that*

$$Z(M, |G|_{\underline{x}}) = f(|G|_{\underline{x}}, \mathcal{L}_0) Z(M, |\tilde{G}|_{\underline{\tilde{x}}}), \quad \text{with} \quad f(|G|_{\underline{x}}, \mathcal{L}_0) = \prod_{v_{3,i}} w_{x_i}^{-1} \tag{4.16}$$

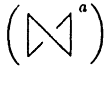
holds. The product is over all 3-vertices with $v_{3,i} \cap \mathcal{L}_0 \neq \emptyset$ and x_i is the colour of the two other lines entering $v_{3,i}$.

Lemma 4.9. *For the fundamental representation of $U_q(sl(2,\mathbf{C}))$ $\delta_{\frac{1}{2}\frac{1}{2}x} = \delta_{x0} + \delta_{x1}$ holds. Therefore when the local colours are $\frac{1}{2}$, one finds by means of Eqs.* (4.1), (2.10), *and* (2.11) *the "skein" relations*

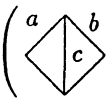


(4.17)

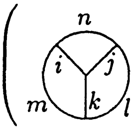
Examples 4.10. i) Let $|G|_a$ be a planar circle with a twist. Its invariant defined by Theorem 4.2 is obtained by the Racah relation (2.10) (for $i = j = a$ and $k = 0$) and Example 3.4,

$Z(|G|_a) = Z$  $= q_a^2 w_a^2.$ (4.18)

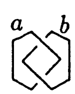
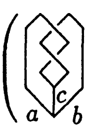
ii) Using the orthogonality relation (2.7) or (4.11) we find

$$Z(|G|_{abc}) = Z\Big($$  $$\Big) = \delta_{abc} \qquad (4.19)$$

if the graph $|G|_{abc}$ is planar.
iii) Relations (4.11) and (4.12) imply that the $6j$-symbol is represented by the following graph invariant:

$$Z(|G|_{ijklmn}) = Z\Big($$  $$\Big) = \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} . \qquad (4.20)$$

iv) Let $|G|_{ab}$ be planar and consist of two circles with linking number one. Its invariant is obtained by Eqs. (2.7), (2.10), and ii)

$$Z(|G|_{ab}) = Z\Big($$  $$\Big) = \sum_c w_c^2 Z\Big($$  $$\Big)$$

$$= \sum_c \frac{w_c^2 q_c^2}{q_a^2 q_b^2} \delta_{abc} = (-1)^{2a+2b}((2a+1)(2b+1))_q . \qquad (4.21)$$

The last equality holds in the context of $U_q(sl(2,\mathbf{C}))$ for the data given by Eq. (2.11). In Appendix A we will discuss the modular properties of the matrices $T_{ab} \propto \delta_{ab} Z(|G|_a)$ and $S_{ab} = wZ(|G|_{ab})$ with the graph invariants of Eqs. (4.18) and (4.21).

5. Cutting Rules

In this section we will introduce some techniques which allow to extend the surgery methods employed in [KMS], where we were able to cut M along a manifold Σ diffeomorphic to S^2, to any Σ. This will in particular enable us to calculate $Z(M)$ for several examples. The main calculations are collected in Lemmas 5.1 and 5.2 below. They make it possible to cut out (solid) cylinders and to introduce (empty) tubes. The following discussion will always make tacit use of the homotopy invariance of $Z(M, |G|_{\underline{x}})$ with respect to the embedding of $|G|_{\underline{x}}$ in ∂M.

Let $C = D^2 \times I$ (D^2 = closed unit disc $\subset \mathbf{R}^2$ with boundary S^1 and $I = [0,1]$ the unit interval) be the cylinder with boundary $(S^1 \times I) \underset{S^1 \times \{0,1\}}{\cup} (D^2 \times \{0,1\})$. For simplicity we will also denote any subset of M homeomorphic to this set by C. Consider now one of the following two situations for $C \subset M$,

$$\begin{aligned} &\text{a) } S^1 \times I \subset \partial M , \qquad (\operatorname{int} D^2) \times \{0,1\} \subset M \backslash \partial M , \\ &\text{b) } D^2 \times \{0,1\} \subset \partial M , \qquad S^1 \times (0,1) \subset M \backslash \partial M . \end{aligned} \qquad (5.1)$$

Intuitively case a) describes a cylinder in M and case b) means that C is a cylinder in M with "top" and "bottom" in ∂M.

In the case a) we consider the cylinder C "cut out" of M giving $\tilde{M}$ with $M = \tilde{M} \underset{D^2\times\{0,1\}}{\cup} C$ and $\partial\tilde{M} = (\partial M \backslash S^1 \times [0,1]) \cup (D^2 \times \{0,1\}))$. Assume now that $|G|_{\underline{x}} \subset \partial M$ is such that only one line ℓ of $|G|_{\underline{x}}$ with colour $x = x(\ell)$ passes through the handle C in the form that $\ell \cap C = \{P\} \times I$ with $P \in S^1$. With these notations we have the following

Lemma 5.1. *The following relation is valid for an arbitrary colouring $\underline{x}$ of $|G|$,*

$$Z(M, |G|_{\underline{x}}) = \delta_{x0} f(|G|_{\underline{x}}, \ell) Z(\tilde{M}, |\tilde{G}|_{\underline{x}}) . \tag{5.2}$$

Here $|\tilde{G}|_{\underline{x}}$ is obtained from $|G|_{\underline{x}}$ by deleting the line ℓ and $f(|G|_{\underline{x}}, \ell)$ is obtained from Eq. (4.16). *In particular $|\tilde{G}|_{\underline{x}}$ is a coloured graph on $\tilde{M}$ making the right-hand side of* (5.2) *well defined (compare Lemma 4.8).*

Proof. Consider the handle $C = D^2 \times [0,1]$ triangulated as in Fig. 8. The contribution to the state sum $Z(M, |G|_{\underline{x}})$ from this piece is

$$\begin{aligned} W = \sum_{a,c,d,f,m,n} & w_a^2 w_c^2 w_d^2 w_f^2 w_m^2 w_n^2 \\ \times & \begin{vmatrix} a & b & c \\ l & f & k \end{vmatrix} \begin{vmatrix} c & d & e \\ m & f & l \end{vmatrix} \begin{vmatrix} f & e & m \\ p & n & o \end{vmatrix} \begin{vmatrix} a & k & f \\ E & B' & A' \end{vmatrix} \\ \times & \begin{vmatrix} f & n & o \\ F & B' & E \end{vmatrix} \begin{vmatrix} n & p & m \\ D & E & F \end{vmatrix} \begin{vmatrix} m & d & l \\ C & E & D \end{vmatrix} \\ \times & \begin{vmatrix} d & c & e \\ B & D & C \end{vmatrix} \begin{vmatrix} b & a & c \\ B & C & A \end{vmatrix} \begin{vmatrix} a & A & B \\ x & B' & A' \end{vmatrix} , \end{aligned}$$

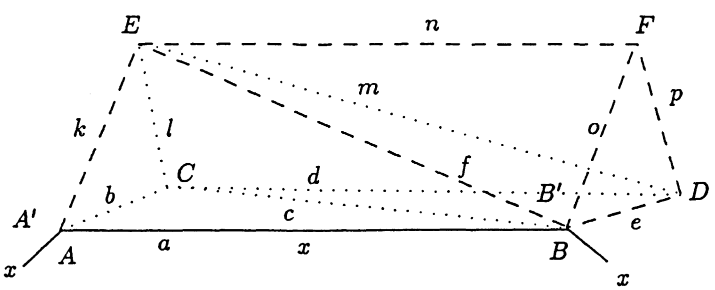


Fig. 8. A triangulated handle with a line of colour x

We use the Biedenharn Elliot relations (2.8) three times in the form

$$\sum_n w_n^2 \begin{vmatrix} f & e & m \\ p & n & o \end{vmatrix} \begin{vmatrix} f & n & o \\ F & B' & E \end{vmatrix} \begin{vmatrix} n & p & m \\ D & E & F \end{vmatrix} = \begin{vmatrix} f & m & e \\ D & B' & E \end{vmatrix} \begin{vmatrix} o & p & e \\ D & B' & F \end{vmatrix} ,$$

$$\sum_m w_m^2 \begin{vmatrix} c & d & e \\ m & f & l \end{vmatrix} \begin{vmatrix} f & m & e \\ D & B' & E \end{vmatrix} \begin{vmatrix} m & d & l \\ C & E & D \end{vmatrix} = \begin{vmatrix} f & l & c \\ C & B' & E \end{vmatrix} \begin{vmatrix} e & d & c \\ C & B' & D \end{vmatrix} ,$$

$$\sum_f w_f^2 \begin{vmatrix} a & b & c \\ l & f & k \end{vmatrix} \begin{vmatrix} a & k & f \\ E & B' & A' \end{vmatrix} \begin{vmatrix} f & l & c \\ C & B' & E \end{vmatrix} = \begin{vmatrix} a & b & c \\ C & B' & A' \end{vmatrix} \begin{vmatrix} k & l & b \\ C & A' & E \end{vmatrix} ,$$

then twice the orthogonality relations (2.7)

$$w_{B'}^2 \sum_d w_d^2 \begin{vmatrix} e & d & c \\ C & B' & D \end{vmatrix} \begin{vmatrix} d & c & e \\ B & D & C \end{vmatrix} = \delta_{B,B'},$$

$$w_{A'}^2 \sum_c w_c^2 \begin{vmatrix} a & b & c \\ C & B & A' \end{vmatrix} \begin{vmatrix} b & a & c \\ B & C & A \end{vmatrix} = \delta_{A,A'},$$

and finally

$$\sum_a w_a^2 \begin{vmatrix} a & A & B \\ x & B & A \end{vmatrix} \propto \delta_{x0},$$

where Eqs. (2.6) and (2.7) have been used. Thus in particular $Z(M, |G|_{\underline{x}})$ vanishes unless $x = 0$. By (4.16) we therefore have

$$Z(M, |G|_{\underline{x}}) = \delta_{x0} f(|G|_{\underline{x}}, \ell) Z(M, |\tilde{G}|_{\underline{x}}). \tag{5.3}$$

On the other hand, by the arguments of [KMS] when the cylinder C is cut the relation

$$Z(M, |\tilde{G}|_{\underline{x}}) = Z(\tilde{M}, |\tilde{G}|_{\underline{x}}) \tag{5.4}$$

is valid. Indeed, this follows by calculating the local factor for $D^2 \times [0,1]$ [now with the line ℓ with colour $x(\ell)$ delected] using the triangulation of Fig. 8. This concludes the proof of Lemma 5.1.

In the case (5.1b) we consider the manifold $\tilde{M} = (M \setminus \operatorname{int} C) \underset{S^1 \times \{0,1\}}{\cup} S^1 \times I$ such that $M = \tilde{M} \underset{S^1 \times I}{\cup} C$. In particular $\partial\tilde{M}$ is obtained by removing $D^2 \times \{0,1\}$ and gluing the tube $T = S^1 \times [0,1]$ along $S^1 \times \{0,1\}$. We will call the operation $M \to \tilde{M}$ the removal of the cylinder C and the introduction of the tube T. We may assume that the coloured graph $|G|_{\underline{x}}$ on ∂M satisfies the condition $|G|_{\underline{x}} \cap (D^2 \times \{0,1\}) = \emptyset$, if necessary by modifying the graph $|G|_{\underline{x}}$ by a homotopy. Thus we may view $|G|_{\underline{x}}$ as a coloured graph on $\partial\tilde{M}$. With these conventions we make the

Definition 5.2. We consider a coloured circle m_x on the tube $T \subset \partial M$ of the form $(S^1 \times \{P\})_x$ $(P \in (0,1))$. Combining $|G|_{\underline{x}}$ and m_x to a coloured disconnected graph, denoted by $|G \cup m|_{\underline{x},x}$ for short, we define

$$Z(\tilde{M}(T_m), |G|_{\underline{x}}) = \sum_x \frac{w_x^2}{w^2} Z(\tilde{M}, |G \cup m|_{\underline{x},x}), \tag{5.5}$$

and say that the tube T is equipped with a meridian m. The following lemma states what happens to the state sum $Z(M, |G|_{\underline{x}})$ if a cylinder C is removed and the tube T is introduced.

Lemma 5.3. *The following relation is valid:*

$$Z(\tilde{M}(T_m), |G|_{\underline{x}}) = \frac{1}{w^2} Z(M, |G|_{\underline{x}}). \tag{5.6}$$

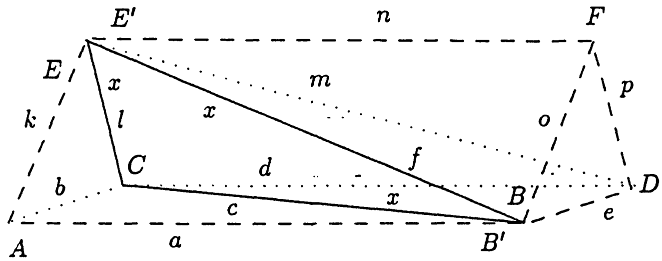


Fig. 9. A triangulated tube with a meridian of colour x

Proof. Let $T = S^1 \times [0,1] \subset \partial\tilde{M}$ be triangulated as in Fig. 9 with the meridian m running along the edges with colours c, f, and l. The contribution to the state sum on the right-hand side of (5.5) from this piece is then given as

$$\begin{aligned} W = {} & \frac{1}{w^2}\sum_x w_x^2 \sum_{B',C',E'} w_{B'}^2 w_{C'}^2 w_{E'}^2 \begin{vmatrix} c & C & B' \\ x & B & C' \end{vmatrix} \begin{vmatrix} f & B' & E \\ x & E' & B \end{vmatrix} \begin{vmatrix} l & E & C \\ x & C' & E' \end{vmatrix} \\ & \times \begin{vmatrix} a & k & f \\ E & B' & A \end{vmatrix} \begin{vmatrix} f & n & o \\ F & B & E' \end{vmatrix} \begin{vmatrix} n & p & m \\ D & E' & F \end{vmatrix} \\ & \times \begin{vmatrix} m & d & l \\ C' & E' & D \end{vmatrix} \begin{vmatrix} d & c & e \\ B & D & C' \end{vmatrix} \begin{vmatrix} b & a & c \\ B' & C & A \end{vmatrix} \\ = {} & \frac{1}{w^2} \begin{vmatrix} a & b & c \\ l & f & k \end{vmatrix} \begin{vmatrix} c & d & e \\ m & f & l \end{vmatrix} \begin{vmatrix} f & e & m \\ p & n & o \end{vmatrix} \begin{vmatrix} k & l & b \\ C & A & E \end{vmatrix} \begin{vmatrix} o & p & e \\ D & B & F \end{vmatrix} . \end{aligned} \tag{5.7}$$

This follows again from the Biedenharn Elliot relations and the orthogonality relations in the form

$$\begin{vmatrix} f & n & o \\ F & B & E' \end{vmatrix} \begin{vmatrix} n & p & m \\ D & E' & F \end{vmatrix} = \sum_{e'} w_{e'}^2 \begin{vmatrix} f & e' & m \\ p & n & o \end{vmatrix} \begin{vmatrix} f & m & e' \\ D & B & E' \end{vmatrix} \begin{vmatrix} o & p & e' \\ D & B & F \end{vmatrix},$$

$$\begin{vmatrix} f & m & e' \\ D & B & E' \end{vmatrix} \begin{vmatrix} m & d & l \\ C' & E' & D \end{vmatrix} = \sum_{c'} w_{c'}^2 \begin{vmatrix} c' & d & e' \\ m & f & l \end{vmatrix} \begin{vmatrix} f & l & c' \\ C' & B & E' \end{vmatrix} \begin{vmatrix} e' & d & c' \\ C' & B & D \end{vmatrix},$$

$$\sum_{E'} w_{E'}^2 \begin{vmatrix} f & l & c' \\ C' & B & E' \end{vmatrix} \begin{vmatrix} f & B' & E \\ x & E' & B \end{vmatrix} \begin{vmatrix} l & E & C \\ x & C' & E' \end{vmatrix} = \begin{vmatrix} f & l & c' \\ C & B' & E \end{vmatrix} \begin{vmatrix} c' & C & B' \\ x & B & C' \end{vmatrix},$$

$$w_{c'}^2 \sum_x w_x^2 \begin{vmatrix} c & C & B' \\ x & B & C' \end{vmatrix} \begin{vmatrix} c' & C & B' \\ x & B & C' \end{vmatrix} = \delta_{cc'},$$

$$w_{e'}^2 \sum_{C'} w_{C'}^2 \begin{vmatrix} d & c & e \\ B & D & C' \end{vmatrix} \begin{vmatrix} e' & d & c \\ C' & B & D \end{vmatrix} = \delta_{ee'},$$

$$\sum_{B'} w_{B'}^2 \begin{vmatrix} a & k & f \\ E & B' & A \end{vmatrix} \begin{vmatrix} b & a & c \\ B' & C & A \end{vmatrix} \begin{vmatrix} f & l & c \\ C & B' & E \end{vmatrix} = \begin{vmatrix} a & b & c \\ l & f & k \end{vmatrix} \begin{vmatrix} k & l & b \\ C & A & E \end{vmatrix}.$$

Now the right-hand side of (5.7) is just the local contribution to the state sum $Z(M, |G|_x)$ for this triangulation concluding the proof of Lemma 5.3.

There are some additional useful formulas for tubes with meridians. In the following we rename $\tilde{M}$ to be M (case 5.1b).

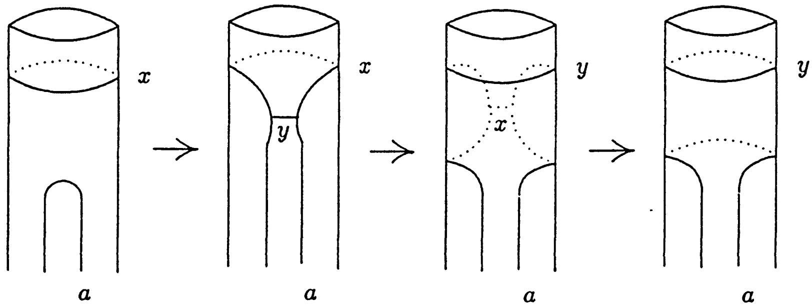


Fig. 10. A nontrivial change of a line in presence of a meridian

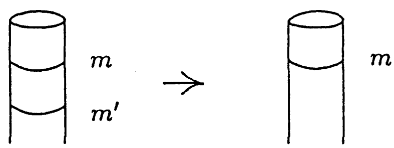


Fig. 11. The projection property of meridians

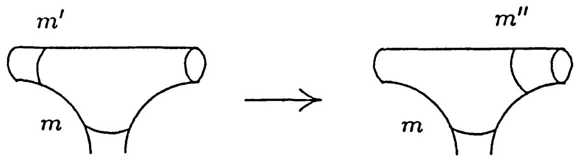


Fig. 12. Shifting meridians at branch points of tubes

Lemma 5.4. *If ∂M contains a tube T with meridian m, then a line l of the coloured graph $|G|_{\underline{x}}$ may be changed nontrivially if $l \cap T \neq \emptyset$ is of the form depicted in Fig. 10 to give*

$$Z(M(T_m), |G|_{\underline{x}}) = Z(M(T_m), |\hat{G}|_{\underline{x}}) . \tag{5.8}$$

Here $|\hat{G}|_{\underline{x}}$ is obtained by replacing a piece of l with colour a, contained in some $S^1 \times \{Q\}$, by its complement in $S^1 \times \{Q\}$. The proof of this lemma is also depicted in Fig. 10, making use of Eq. (2.7) and Theorem 4.1 and where summation over x and y with weight factors w_x^2 and w_y^2 is understood.

Obviously we may generalize Eq. (5.5) by introducing several coloured meridians $m, m', \ldots$ on T (summing over the associated colours $x, x', \ldots$ with weights $w_x^2, w_{x'}^2, \ldots$). By the previous lemma, however, the additional meridians act trivially in the sense of

Corollary 5.5. *Meridians on the same tube act as projections (see Fig. 11), i.e.*

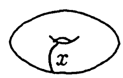
$$Z(M(T_{m,m',\ldots}), |G|_{\underline{x}}) = Z(M(T_m), |G|_{\underline{x}}) . \tag{5.9}$$

Corollary 5.6. *If there is a branching of tubes as depicted in Fig. 12 meridians may be shifted as follows:*

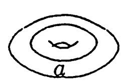
$$Z(M(T_m, T'_{m'}), |G|_{\underline{x}}) = Z(M(T_m, T''_{m''}), |G|_{\underline{x}}) . \tag{5.10}$$

Remark 5.7. So far we have assumed that $|G|_{\underline{x}}$ had support away from the meridians $m, m', \ldots$. However, in Lemma 5.4, Corollaries 5.5 and 5.6, $|G|_{\underline{x}}$ may contain line ℓ_i $(i = 1, \ldots, N)$ where the crossings of each line with the meridians $m, m', \ldots$ are all of the same type ("above" or "below" in the sense of the R-matrix (2.9) or the vertex v_4 in Fig. 3).

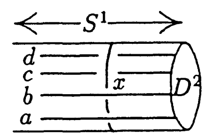
Examples 5.8. i) Let $M = S^1 \times D^2$ be a solid torus ($\partial M = S^1 \times S^1$) and S^1_x a not self crossing circle not contractible in ∂M but contractible in M. By handle cutting (away from S^1_x) due to Lemma 5.1 ($M \to D^3$) we arrive at Example 3.4 and find

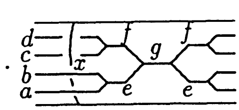
$Z(S^1 \times D^2, S^1_x) = Z\Big($  $\Big) = w_x^2.$ (5.11)

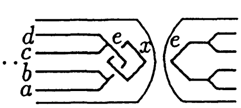
ii) Let $M = S^1 \times D^2$ be a solid torus ($\partial M = S^1 \times S^1$) and S^1_a not contractible in M. By handle cutting due to Lemma 5.1 we now also cut S^1_a and find with the help of Eq. (5.2),

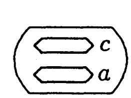
$Z(S^1 \times D^2, S^1_a) = Z\Big($  $\Big) = \delta_{a0}.$ (5.12)

iii) For a later application we consider the more complicated graph on a solid torus depicted in Fig. 13. Let $S^1_a \cup S^1_b \cup S^1_c \cup S^1_d$ be four circles without any crossings not contractible in M and let S^1_x be a not self crossing circle not contractible in ∂M but contractible in M "undercrossing" $S^1_a \cup S^1_b$ and "overcrossing" $S^1_c \cup S^1_d$ in the sense of the R-matrix (2.9) or the elementary vertex v_4 of Fig. 3. We obtain from the orthogonality (4.11), Lemma 5.1, relations (4.14) and (4.15) with $|G|_{abcdx}$ of Fig. 13,

$\sum_x w_x^2 Z(S^1 \times D^2, |G|_{abcdx}) = \sum_x w_x^2 Z\Big($  $\Big)$

$= \sum_{efgx} w_e^2 w_f^2 w_g^2 w_x^2 Z\Big($  $\Big)$

$= \sum_{efgx} w_e^2 w_f^2 w_g^2 w_x^2 \delta_{g0} w_f^{-2} \delta_{ef} Z\Big($  $\Big)$

$= \dfrac{w^2}{w_a^2 w_c^2} \delta_{ab}\delta_{cd} Z\Big($  $\Big)$

$$= \frac{w^2}{w_a^2 w_c^2} \delta_{ab}\delta_{cd} Z(D^3, S^1_a \cup S^1_c) = w^2 \delta_{ab}\delta_{cd}.$$

(5.13)

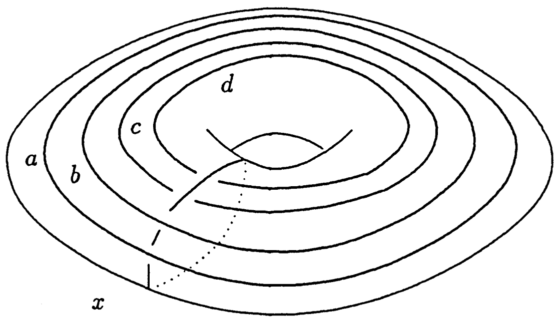


Fig. 13. The graph $|G|_{abcdx} = S^1_a \cup S^1_b \cup S^1_c \cup S^1_d \cup S^1_x$ on a solid torus

6. Coloured Graphs in M

Up to now we have considered graphs $|G|$ embedded in ∂M. In this section we introduce coloured graphs $\mathscr{G}_{\underline{x}}$ in the 3-manifold M.

Definition 6.1. Let M be a compact, oriented 3-manifold. A coloured graph $\mathscr{G}_{\underline{x}}$ in M is given by the following data:
α) A finite 1-dimensional simplicial complex $c(\mathscr{G}_{\underline{x}})$, called the core of $\mathscr{G}_{\underline{x}}$, embedded in $\operatorname{int} M$.
β) An open tubular neighborhood $\mathscr{T}_{\mathscr{G}_{\underline{x}}}$ of $c(\mathscr{G}_{\underline{x}})$ in $\operatorname{int} M$.
γ) A coloured graph $|G|_{\underline{x}}$ in the sense of Sect. 3 on the boundary $\partial\mathscr{T}_{\mathscr{G}_{\underline{x}}}$.

We will always assume that $\mathscr{T}_{\mathscr{G}_{\underline{x}}}$ consists of (connected) components $\mathscr{T}_i$, $(1 \le i \le N)$ each of which is a tubular neighborhood of each component c_i of $c(\mathscr{G}_{\underline{x}})$. $\mathscr{G}_{\underline{x}}$ is called a coloured framed link, which then will be written as $\mathscr{L}_{\underline{x}}$, if each connected component $|c_i|$ of $|c(\mathscr{G}_{\underline{x}})|$ is homeomorphic to a circle S^1 and if $|G|_{\underline{x}} \cap \mathscr{T}_i$ consists of a line ℓ_i (an embedded coloured circle $S^1_{x_i}$) homotopic in $\mathscr{T}_i$ to the core c_i. The line ℓ_i together with the core c_i defines a framing. We write $\mathscr{L}_{\underline{x}} = \bigcup_{i=1}^{N} (\mathscr{L}_i)_{x_i}$ for the resulting canonical decomposition of $\mathscr{L}_{\underline{x}}$. Given $\mathscr{G}_{\underline{x}}$ we denote by $M(\mathscr{G}_{\underline{x}})$ the compact submanifold of M obtained by deleting $\mathscr{T}_{\mathscr{G}_{\underline{x}}}$ from M. By construction $\partial M({}^{G}_{\underline{x}})$ is the disjoint union of $\partial\mathscr{T}_{\mathscr{G}_{\underline{x}}}$ and ∂M. In particular $|G|_{\underline{x}}$ may be viewed as a coloured graph on $\partial M(\mathscr{G}_{\underline{x}})$.

Definition 6.2. The state sum of a compact, oriented 3-manifold M equipped with a coloured graph $\mathscr{G}_{\underline{x}}$ is given as

$$Z(M, \mathscr{G}_{\underline{x}}) = Z(M(\mathscr{G}_{\underline{x}}), |G|_{\underline{x}}), \tag{6.1}$$

where $|G|_{\underline{x}}$ is the graph on $\partial\mathscr{T}_{\mathscr{G}_{\underline{x}}}$ associated to $\mathscr{G}_{\underline{x}}$ by property γ).

In analogy to the discussion in Sect. 5 we may in addition introduce one or several meridians m on any tube T of $\partial\mathscr{T}_{\mathscr{G}_{\underline{x}}}$ which locally looks like $S^1 \times I$ (see Definition 5.2). Again we make the proviso that $|G|_{\underline{x}}$ restricted to such a set T is a (possibly empty) union of coloured straight lines, each of the form $\{P\} \times I$ for some $P \in S^1$.

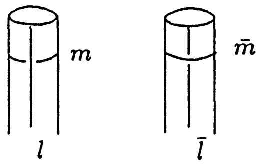


Fig. 14. Left (l) and right ($\bar{l}$) handed lines

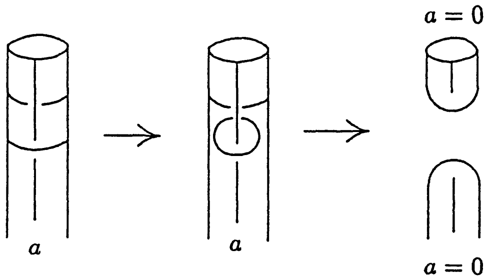


Fig. 15. Left and right handed lines cannot be connected

Definition 6.3. We say such a local line is left handed w.r.t. to a meridian $m = S^1 \times \{Q\}$ ($Q \in (0,1)$) if it "overcrosses" m in the sense of the elementary vertex v_4. It is called right-handed if it "undercrosses" (see Fig. 14). This notion makes sense since $\partial\mathcal{T}_{\mathcal{G}_{\underline{x}}}$ inherits an orientation from the orientation of M.

We may then generalize the state sum (6.1) to

$$Z(M(T_m, T'_{m'}, \ldots), \mathcal{G}_{\underline{x}}) = Z(M(\mathcal{G}_{\underline{x}})(T_m, T'_{m'}, \ldots), |G|_{\underline{x}}) \tag{6.2}$$

which agrees with the state sum (6.1) in case no meridians are present.

Lemma 6.4. *If the local coloured line* $\ell_x = (\{P\} \times I)_x \subset T \subset \partial\mathcal{T}_{\mathcal{G}_{\underline{x}}}$ *in* $G_{\underline{x}}$ *is left-handed* w.r.t. *a meridian* m *and right-handed* w.r.t. *another meridian* $\bar{m}$ *(both living on* T*), then the state sum* (6.2) *vanishes unless the colour* x *of* ℓ_x *vanishes. By the discussion in Sect. 4, we may then delete the whole line in* $|G|_{\underline{x}}$.

We give a graphical presentation of the proof by Fig. 15. Obviously it suffices to consider the case where T only has the meridians m and $\bar{m}$. Here we have used arguments similar to the proof of Lemma 5.4. The claim now follows by relation (4.14) in Lemma 4.6. Using Corollary 5.6 we conclude furthermore

Corollary 6.5. *Left and right handed lines do not "interact," i.e. there is no branching if both left and right lines enter unless the colours are vanishing.*

Using the orthogonality relations (2.7), Eq. (4.14) and Corollaries 5.4 and 5.6 and Remark 5.7 one can prove

Lemma 6.6. *A single coloured local line* ℓ_c *on a set* $T = S^1 \times I \subset \partial\mathcal{T}_{\mathcal{G}_{\underline{x}}}$ *which does not cross any meridian may be decomposed into a right- and left-handed one in the form depicted in Fig. 16 such that*

$$Z(M, \mathcal{G}_{\underline{x}}) = w^{-2} \sum_{a,\bar{b}} w_a^2 w_{\bar{b}}^2 Z(M(T_m, \bar{T}_{\bar{m}}), \hat{\mathcal{G}}_{\underline{x},a,\bar{b}}) \,. \tag{6.3}$$

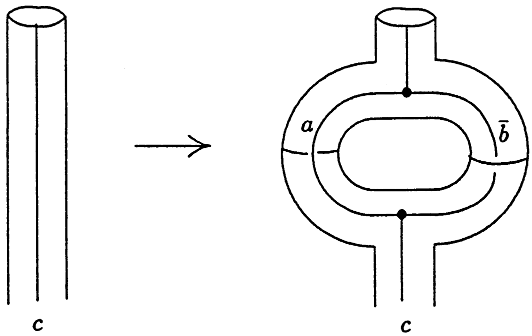


Fig. 16. Decomposition of a line into left and right handed ones

Here $\hat{\mathscr{G}}_{\underline{x},a,\bar{b}}$ is obtained from a local modification of $\mathscr{G}_{\underline{x}}$ as depicted in Fig. 16. Obviously relation (6.3) may be generalized in the sense of (6.2) to the case where meridians are present somewhere else.

In the following we will consider coloured graphs $\mathscr{G}_{\underline{a},\underline{\bar{b}}}$ with left-handed lines of colours $\underline{a}$ and right-handed ones of colours $\underline{\bar{b}}$ only. We write [simplifying the notation of (6.2)]

$$Z(M, \mathscr{G}_{\underline{a},\underline{\bar{b}}}) := Z(M(T_{\underline{m}}, \bar{T}_{\underline{\bar{m}'}}), \mathscr{G}_{\underline{x}}), \tag{6.4}$$

where $\underline{x} = \underline{a} \cup \underline{\bar{b}}$ and the $T_{\underline{m}}(\bar{T}_{\underline{\bar{m}}})$ are the meridians of the left-(right-) handed lines.

The case of a coloured framed link $\mathscr{L}_{\underline{a},\underline{\bar{b}}}$ with N connected components deserves special attentioin. We introduce a meridian on each connected component $\partial\mathscr{T}_i$ of $\mathscr{T}_{\mathscr{L}_{\underline{a},\underline{\bar{b}}}}$.

Theorem 6.7. *The state sums $Z(M, \mathscr{L}_{\underline{a},\underline{\bar{b}}})$ of a coloured framed link $\mathscr{L}_{\underline{a},\underline{\bar{b}}}$ in a 3-manifold yield a representation of the braid group under change of the embedding of $\mathscr{L}_{\underline{a},\underline{\bar{b}}}$ in M. The relative braiding of left-handed and right-handed lines is trivial. The braiding of left-(right-) handed lines is* w.r.t. *the matrix $R(R^{-1})$ defined by* (2.9) *and the vertex v_4 of Eq.* (3.6) *and Fig. 3.*

In particular for the fundamental representation of $U_q(sl(2, \mathbf{C}))$ (i.e. *when the lines involved in the braiding have colours equal to* $1/2$) *one has the skein relations*

$$Z(M, \mathscr{L}_{\underline{a},\underline{\bar{b}}}) = AZ(M, \mathscr{L}'_{\underline{a},\underline{\bar{b}}}) + BZ(M, \mathscr{L}''_{\underline{a},\underline{\bar{b}}}). \tag{6.5}$$

Here the coloured framed links $\mathscr{L}'_{\underline{a},\underline{\bar{b}}}$ and $\mathscr{L}''_{\underline{a},\underline{\bar{b}}}$ coincide with $\mathscr{L}_{\underline{a},\underline{\bar{b}}}$ outside the local braiding region and inside they are depicted by Fig. 17. For two left-handed lines $A = q$, $B = 1 - q^{-2}$ and for two right-handed lines $A = 1/q$, $B = 1 - q^2$,

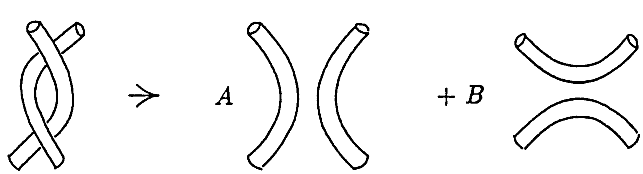


Fig. 17. Skein relation

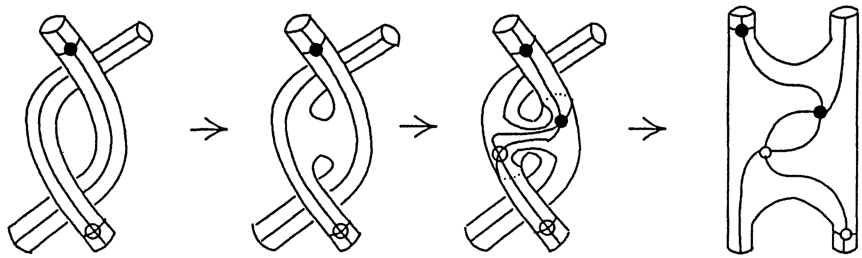
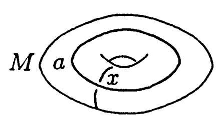


Fig. 18. The braiding of lines $\mathscr{L}$. Here the symbols ✦ and ✧ mean the following. For a left- and right-handed line we have e.g. ✦ = -|- and ✧ = -┬-. For two left- or two right-handed lines we have ✦ = ✧ = -|- and ✦ = ✧ = -┬-, respectively

Proof. Using Lemmas 5.3 and 5.4, Corollaries 5.5 and 5.6 combined with Remark 5.7 we deform $\mathscr{L}_{\underline{a},\underline{\bar{b}}}$ into a $\mathscr{G}_{\underline{a},\underline{\bar{b}}}$ with the same colours $\underline{a}, \underline{\bar{b}}$ as depicted in Fig. 16. The proof is completed by Theorem 4.1 and Eq. (4.17).

Examples 6.8. i) Let $\mathscr{L}_a$ be a simple left-handed loop with colour a and contractible in M, i.e. a link of one component without any nontrivial linking and framing. We have the core $c(\mathscr{L}) \cong S^1$, $\mathscr{T} = S^1 \times D^2$, $\partial\mathscr{T} = S^1 \times S^1$ and the associated graph $|G|_{ax} = S^1_a \times S^1_x$, where S^1_x correspond to the meridian and undercrosses S^1_a. The state sum can be calculated using Lemma 5.1, Example 3.4 [i.e. Eq. (3.9)] and Example 2.1 [i.e. Eq. (2.18)]

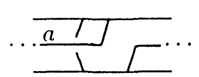
$$Z(M,\mathcal{L}_a) = \sum_x \frac{w_x^2}{w^2} Z\left(M \right.$$



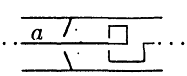
$$\left. \right) = \sum_x \frac{w_x^2}{w^2}\delta_{x0} Z(M \setminus D^3, S^1_a) = Z(M\setminus D^3)\frac{w_a^2}{w^2} = Z(M)w_a^2 . \quad (6.6)$$

For a right-handed simple loop one obtains the same result.
ii) Let $\mathscr{L}^{(n)}_a$ be as in i) but with framing number n [i.e. linking number n of $c(\mathscr{L}_a)$ and S^1_a]. Using Lemma 5.4 and the Racah relation (2.10) one finds

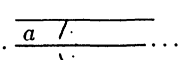
$$Z(M,\mathcal{L}^{(n)}_a) = Z\left(M, \right.$$



$$\left.\right) = Z\left(M,\right.$$



$$\left.\right) = q_a^{2n} Z\left(M,\right.$$



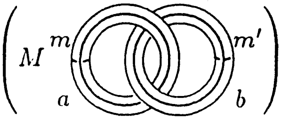
$$\left.\right) = Z(M) q_a^{2n} w_a^2 . \quad (6.7)$$

For a right-handed simple loop with framing number n one obtains

$$Z(M,\mathscr{L}^{(n)}_{\bar{b}}) = Z(M) q_{\bar{b}}^{-2n} w_{\bar{b}}^2 . \quad (6.8)$$

iii) Let $\mathscr{L}_{a,b}$ a two component link (each with zero framing) such that the relative linking number of both components is one. Using the techniques applied in the proof of Theorem 6.7 one finds

$$Z(M,\mathcal{L}_{a,b}) = Z\left(M \right.$$



$$\left.\right) = Z(M) Z(|G|_{ab}) , \quad (6.9)$$

where $|G|_{ab}$ is the planar graph of Example 4.10 iv) and its invariant is given by [Eq. (4.21)]. With Eqs. (A.2) and (A.3) one may write

$$Z(M, \mathscr{L}_{a,b}) = Z(M) w_a^2 w_b^2(a) = Z(M) w S_{ab} \,. \tag{6.10}$$

iv) Let $\mathscr{L}_{a,\bar{b}}$ be as in iii). Since the braiding of a left-handed line relative to a right-handed one is trivial, one obtains

$$Z(M, \mathscr{L}_{a,\bar{b}}) = Z(M) w_a^2 w_{\bar{b}}^2 \,. \tag{6.11}$$

7. Surgery Formulas

In this section we establish the main results of this article. The aim is to derive a surgery formula which generalizes the following relation obtained in [KMS]:

$$Z(M) = \frac{1}{w^2} Z(M_1) Z(M_2) \tag{7.1}$$

for the case

$$M = M_1 \underset{S^2\times\{0\}}{\cup} (S^2 \times I) \underset{S^2\times\{1\}}{\cup} M_2 \,,$$

where $M_1 \cap M_2 = \emptyset$ and where the gluing takes place at components of ∂M_i $(i = 1, 2)$ which look like S^2's. Here M need not be orientable.

The generalization is obtained by replacing S^2 by an arbitrary closed, compact, oriented 2-submanifold Σ in int M. In order to stay close to the case (7.1), we assume that M decomposes into disjoint M_1 and M_2 when we cut along Σ. The case where M stays connected after such a cut may be discussed in the same way with obvious modifications in the resulting formulas.

Below we will introduce so-called canonical coloured graphs $|G|_{\underline{x}}^{\Sigma}$ on Σ. In terms of these coloured graphs we have the

Theorem 7.1. *Let M be a compact oriented 3-manifold and Σ a closed compact 2-submanifold in* int M *such that M may be decomposed in the form*

$$M = M_1 \underset{\Sigma\times\{0\}}{\cup} (\Sigma \times I) \underset{\Sigma\times\{1\}}{\cup} M_2 \,. \tag{7.2}$$

with $M_1 \cap M_2 = \emptyset$, $\partial M_2 = \Sigma$, and $\partial M_1 = \Sigma^*$. Then

$$Z(M) = \sum_{\underline{x}} W_{\underline{x}}^{\Sigma} Z(M_1, |G|_{\underline{x}}^{\Sigma^*}) Z(M_2, |G|_{\underline{x}}^{\Sigma}) \tag{7.3}$$

with a weight factor to be given below for typical examples.

This result provides the alternative triangulation independent formulation of the cobordism analysis of Turaev and Viro as announced in the introduction. To define the canonical graph on Σ, it suffices to consider a connected component Σ^g of genus $g \geq 1$. Indeed, note that for $g = 0$ we already have relation (7.1), such that $|G|_{\underline{x}}^{\Sigma^g}$ by definition is the empty graph for $g = 0$.

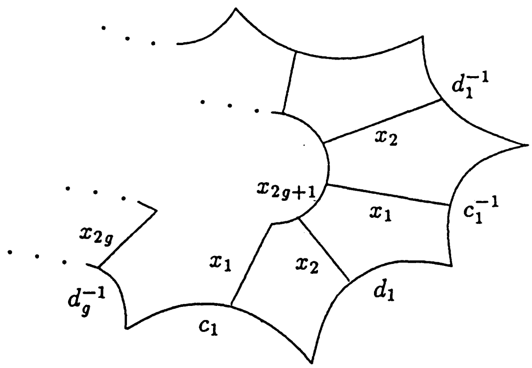


Fig. 19. The canonical graph $|G|_{\underline{x}}^{\Sigma^g}$ with $6g-3$ colours $\underline{x}$

Definition 7.2. Let Σ^g be represented in the standard way by a $4g$-polygon

$$(c_1, d_1, c_1^{-1}, d_1^{-1}, d_1^{-1}, \ldots, c_g^{-1}, d_g^{-1}).$$

The canonical graph $|G|_{\underline{x}}^{\Sigma^g}$ is given by Fig. 19 where $\underline{x} = (x_1, \ldots, x_{6g-3})$. Note that this graph is minimal, i.e. it cannot be reduced nontrivially in the sense of Sect. 4 to a smaller graph having only generalized 3-vertices. Among all such maximally reduced and connected graphs with only (generalized) 3-vertices it is maximal, in the sense that it intersects each element of the homology basis $(c_1, d_1, \ldots, c_g, d_g)$ exactly once. Every other graph with these properties may be obtained from this one by means of "crossing transformation" (called Fierz transformations by physicists) of the form (2.8). The weight factor for Σ^g in Eq. (7.3) is

$$W_{\underline{x}}^{\Sigma^g} = \frac{1}{w^2} \prod_{i=1}^{6g-3} w_{x_i}^2 \,. \tag{7.4}$$

Remark 7.3. Theorem 7.1 may be generalized in the following way to the context where M already contains a coloured graph $\mathscr{G}_{\underline{y}} \subset M$ with the associated graph $G_{\underline{y}}$ as introduced in Sect. 6 [see Eq. (6.1)]. We will assume that Σ intersects $\partial\mathscr{T}_{\mathscr{G}_{\underline{y}}}$ in a union of disjoint discs D^2 enumerated as $D^2_{a_1}, \ldots, D^2_{a_n}, D^2_{\bar{b}_1}, \ldots, D^2_{\bar{b}_{\bar{n}}}$, where the disc $D^2_{a_i}(D^2_{\bar{b}_j})$ is the intersection of Σ^g with that part of $\partial\mathscr{T}_{\mathscr{G}_{\underline{y}}}$ which carries a left-(right-) handed line with colour $a_i(\bar{b}_j) \in \underline{y}$. We define the set

$$\Sigma^g_{\underline{a},\underline{\bar{b}}} = \Sigma^g \backslash D^2_{a_1} \backslash \ldots \backslash D^2_{a_n} \backslash D^2_{\bar{b}_1} \backslash \ldots \backslash D^2_{\bar{b}_{\bar{n}}} \,. \tag{7.5}$$

Also we define the canonical graph $G_{\underline{x}}^{\Sigma^g_{\underline{a},\underline{\bar{b}}}}$ on $\Sigma^g_{\underline{a},\underline{\bar{b}}}$ as depicted in Fig. 20 which extends Fig. 19. Note that both $\Sigma^g_{\underline{a},\underline{\bar{b}}}$ and $G_{\underline{x}}^{\Sigma^g_{\underline{a},\underline{\bar{b}}}}$ are not of the usual form, since they are not closed. However, we replace in Eq. (7.3),

$$\Sigma \to \Sigma^g_{\underline{a},\underline{\bar{b}}} \cup \partial(\mathscr{T}_{\mathscr{G}_{\underline{y}}} \cap M_1)$$

$$|G|_{\underline{x}}^{\Sigma} \to |G|_{\underline{x}}^{\Sigma^g_{\underline{a},\underline{\bar{b}}}} \cup (|G|_{\underline{y}} \cap M_1)$$

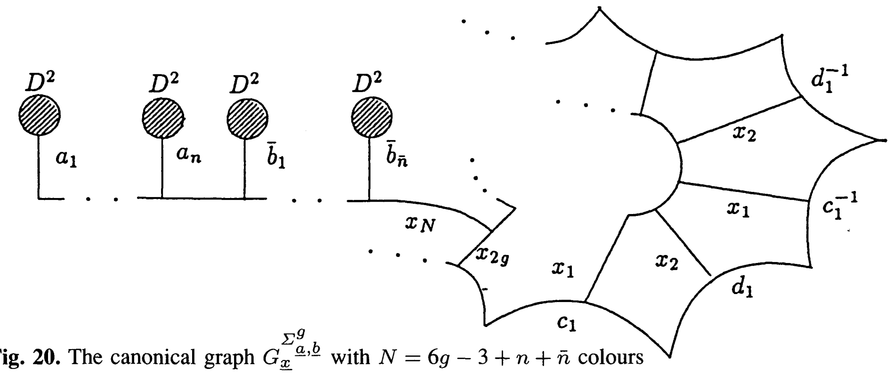


Fig. 20. The canonical graph $G_{\underline{x}}^{\Sigma^g_{\underline{a},\underline{\bar{b}}}}$ with $N = 6g - 3 + n + \bar{n}$ colours

to obtain again a closed 2-manifold as a boundary of a 3-manifold and a closed graph on it (and correspondingly for Σ^* and M_2). Note that e.g. the open line with colour a_1 of $|G|_{\underline{x}}^{\Sigma^g_{\underline{a},\underline{\bar{b}}}}$ is hooked up to the open line with the same colour of $(|G|_{\underline{y}} \cap M_1)$. The weight factor for the case of $\Sigma^g_{\underline{a},\underline{\bar{b}}}$ in Eq. (7.3) is

$$W_{\underline{x}}^{\Sigma^g_{\underline{a},\underline{\bar{b}}}} = w^{2(n+\bar{n}-1)} \prod_{i=1}^{6g-3+n+\bar{n}} w_{x_i}^2 \,. \tag{7.6}$$

The proof of the surgery formula (7.3) and the evaluation of the weight factor (7.4) and its extension (7.6) will be given in Appendix B. It relies mainly on applications of Lemmas 5.1 and 5.3.

The surgery formula (7.3) has an obvious extension to a state sum with an arbitrary coloured graph $|G|_{\underline{y}}$ on ∂M. Of particular interest is the case where Σ is chosen to be ∂M and M_2 a (closed) tubular neighborhood of ∂M in M such that $M_2 \cong \Sigma \times I$. We let M_1 be the closure of the complement of M_2 in M such that $M_1 \cong M$. Then (7.3) generalizes to the relation

$$Z(M, |G|_{\underline{y}}) = \sum_{\underline{x}} W_{\underline{x}}^{\Sigma} Z(\Sigma \times I, |G|_{\underline{y}} \cup |G|_{\underline{x}}^{\Sigma^*}) Z(M, |G|_{\underline{x}}^{\Sigma}) \tag{7.7}$$

valid for $\Sigma = \partial M$.

The notation on the right-hand side of (7.7) is such that $|G|_{\underline{y}}$ is supposed to live on $\Sigma \times \{0\}$ and $G_{\underline{x}}^{\Sigma^*}$ on $\Sigma^* \times \{1\}$. Relation (7.7) in particular proves relation (4.3) with

$$(|G|_{\underline{y}}, |G|_{\underline{x}}^{\Sigma^*}) = Z(\Sigma \times I, |G|_{\underline{y}} \cup |G|_{\underline{x}}^{\Sigma^*}) \,. \tag{7.8}$$

Relations (7.3) and (7.8) suggest the introduction of finite dimensional complex vector spaces V^{Σ} and V^{Σ^*} associated to Σ and Σ^*, respectively. Viewing manifolds and graphs as defining elements in these spaces via

$$\begin{aligned} M_2 &\to v(M_2) \in V^{\Sigma} \quad \text{for} \quad \partial M_2 = \Sigma \,, \\ M_1 &\to v^*(M_1) \in V^{\Sigma^*} \quad \text{for} \quad \partial M_1 = \Sigma^* \,, \end{aligned} \tag{7.9}$$

$$\begin{aligned} |G|_{\underline{x}}^{\Sigma} &\to v^*(\underline{x}) \in V^{\Sigma^*} \quad \text{for} \quad |G|_{\underline{x}}^{\Sigma} \subset \Sigma \,, \\ |G|_{\underline{x}}^{\Sigma^*} &\to v(\underline{x}) \in V^{\Sigma} \quad \text{for} \quad |G|_{\underline{x}}^{\Sigma^*} \subset \Sigma^* \,, \end{aligned} \tag{7.10}$$

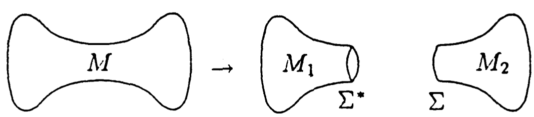
with the pairings (by which V^{Σ^*} is considered to be the dual of V^{Σ})

$$
\begin{aligned}
V^{\Sigma^*} \otimes V^{\Sigma} &\to K\,, \\
v^*(M_1) \otimes v(M_2) &\to (M_1, M_2) = Z(M_1 \cup M_2)\,, \\
v^*(\underline{x}) \otimes (M_2) &\to (\underline{x}, M_2) = Z(M_2, |G|^{\Sigma}_{\underline{x}})\,, \\
v^*(M_1) \otimes v(\underline{x}) &\to (M_1, \underline{x}) = Z(M_1, |G|^{\Sigma^*}_{\underline{x}})\,, \\
v^*(\underline{x}) \otimes v(\underline{x}') &\to (\underline{x}, \underline{x}') = Z(\Sigma \times I, |G|^{\Sigma}_{\underline{x}} \cup |G|^{\Sigma^*}_{\underline{x}'})\,,
\end{aligned}
\tag{7.11}
$$

we obtain a realization of a topological quantum field theory in the sense of [At1] and [At2]. In this notation Eq. (7.3) and (7.7) read for $\partial M_2 = \Sigma$ and $\partial M_1 = \Sigma^*$,

$$Z(M) = (M_1, M_2) = \sum_{\underline{x}} W^{\Sigma}_{\underline{x}}(M_1, \underline{x})\,(\underline{x}, M_2) \tag{7.3$'$}$$

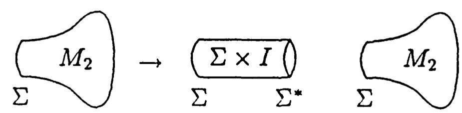
corresponding to the surgery



and

$$(\underline{y}, M_2) = \sum_{\underline{x}} W^{\Sigma}_{\underline{x}}(\underline{y}, \underline{x})\,(\underline{x}, M_2) \tag{7.7$'$}$$

corresponding to the surgery



for the case $|G|_{\underline{y}} = |G|^{\Sigma}_{\underline{y}}$. Obviously, (7.10), (7.11), and (7.7′) may be generalized to arbitrary graphs $|G|_{\underline{y}} \subset \Sigma$. The significance of these equations is that the vectors $v(\underline{x})$ and $v^*(\underline{x})$ form a complete set of vectors in V^{Σ} and V^{Σ^*}, respectively. (They are in general not linear independent since the number of vectors exceeds the dimensions of the spaces, as we will see below. However, we will also construct a basis.) From Eq. (7.7′) we read off that

$$\mathbf{1}^{\Sigma}_{\underline{x},\underline{y}} = (\underline{x}, \underline{y}) \tag{7.12}$$

represents the unit matrix $\mathbf{1}^{\Sigma}$ in the vector space V^{Σ} (and V^{Σ^*}, respectively) in the sense that

$$\mathbf{1}^{\Sigma}_{\underline{x},\underline{y}} = \sum_{\underline{z}} W^{\Sigma}_{\underline{z}} \mathbf{1}^{\Sigma}_{\underline{x},\underline{z}} \mathbf{1}^{\Sigma}_{\underline{z},\underline{y}}\,. \tag{7.13}$$

The dimension d^{Σ} of the vector space V^{Σ} is given by evaluating the trace

$$d^{\Sigma} = Z(\Sigma \times S^1) = \sum_{\underline{x}} W^{\Sigma}_{\underline{x}} \mathbf{1}^{\Sigma}_{\underline{x},\underline{x}}\,. \tag{7.14}$$

Theorem 7.4. *For the $(n+\bar{n})$-fold punctured Riemann surface $\Sigma^g_{\underline{a},\underline{\bar{b}}}$ of genus g, the dimension of the associated vector space is given as*

$$d^{\Sigma^g_{\underline{a},\underline{\bar{b}}}} = \mathrm{tr}(N^{a_1}\dots N^{a_n}(\vec{N}^2)^{g-1})\,\mathrm{tr}(N^{\bar{b}_1}\dots N^{\bar{b}_{\bar{n}}}(\vec{N}^2)^{g-1})\,, \tag{7.15}$$

where the fusion matrices N^a are given as

$$(N^a)_{bc} = \delta_{abc}\,, \tag{7.16}$$

and where $\vec{N}^2$ is the matrix

$$\vec{N}^2 = \sum_a (N^a)^2\,. \tag{7.17}$$

In view of Eq. (1.1) this agrees with Verlindes's formula [Ve] and its generalizations to punctured surfaces (see e.g. [MS, Bo, Th, Wi3]). The proof of Theorem 7.4 will be given in Appendix C. It relies mainly on Theorem 7.1, the disentangling of knots and applying modular formulas derived in Appendix A. Note that in the case of an unpunctured surface Σ and an explicit triangulation of $\Sigma\times S^1$ formula (7.15) gives a nontrivial sum rule for $6j$-symbols.

As promised we now construct a basis in the space V^{Σ} associated to an oriented surface Σ. Again it suffices to consider a connected component of Σ, so we will consider the case $\Sigma=\Sigma^g$ $(g\geq 1)$.

Definition 7.5. Consider the handle body (gefillte pretzel) M_{Σ^g} associated to Σ^g such that $\partial M_{\Sigma^g}=\Sigma^g$. Let $(a_1,\dots,a_g,b_1,\dots,b_g)$ be a canonical homology basis of Σ^g such that the a's are contractible in M_{Σ^g}. The canonical coloured graph $\mathscr{G}^{\Sigma^g}_{\underline{e}}$ is defined as follows. Its core is of the form depicted in Fig. 21, where the circles are homotopic to $(a_1,\dots,a_g)$ in M_{Σ^g}. The tubular neighborhood $\mathscr{T}_{\mathscr{G}^{\Sigma^g}_{\underline{e}}}$ of this core is then a deformation of M_{Σ^g} (it lies in the complement of a tubular neighborhood of Σ^g in M_{Σ^g}). On $\partial\mathscr{T}_{\mathscr{G}_{\underline{e}}}$, which is diffeomorphic to Σ^g, let $|G|_{\underline{e}}$ be the coloured graph with colours $\underline{e}=(e_1,\dots,e_{3g-3})$ for $g>1$ [$\underline{e}=(e_1)$ for $g=1$] as depicted in Fig. 21.

In addition all lines with colours $e_1,\dots,e_{3g-3}$ are assumed to be left-handed, i.e. meridians $T_{\underline{m}}$ ($\underline{m}=(m_1,\dots,m_{3g-3})$) are introduced. Let $\mathscr{G}^{\Sigma^g}_{\underline{\bar{f}}}$ be another canonical graph of the same form, where now the colours $\underline{\bar{f}}=(\bar{f}_1,\dots,\bar{f}_{3g-3})$ are right-handed w.r.t. to meridians $\bar{T}_{\underline{\bar{m}}}$ and such that $\mathscr{T}_{\mathscr{G}^{\Sigma^g}_{\underline{e}}}$ and $\mathscr{T}_{\mathscr{G}^{\Sigma^g}_{\underline{\bar{f}}}}$ are disjoint. Obviously this last condition may be fulfilled. We recall that the relative braiding between these two coloured graphs is trivial. We write for short $\mathscr{G}^{\Sigma^g}_{\underline{e},\underline{\bar{f}}}=\mathscr{G}^{\Sigma^g}_{\underline{e}}\cup\mathscr{G}^{\Sigma^g}_{\underline{\bar{f}}}$. Note that the colours $\underline{e}$ and $\underline{\bar{f}}$ are restricted by the fusion rules i.e. $\delta_{e_1e_1e_2}=\delta_{e_2e_3e_4}=\dots=\delta_{\bar{f}_1\bar{f}_1\bar{f}_2}\dots=1$ holds (see Fig. 21). Therefore the number of colourings of $\mathscr{G}^{\Sigma^g}_{\underline{e},\underline{\bar{f}}}$ is

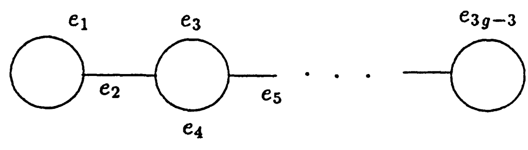


Fig. 21. The core of canonical graph $\mathscr{G}^{\Sigma^g}_{\underline{e}}$ or the graph $|\mathscr{G}|_{\underline{e}}$ associated to $\mathscr{G}^{\Sigma^g}_{\underline{e}}$

given by Eq. (7.15) for the case $n = \bar{n} = 0$,

$$d^{\Sigma^g} = (\mathrm{tr}(\vec{N})^{g-1})^2 . \tag{7.15'}$$

We extend the maps (7.9) by

$$\mathscr{G}_{\underline{e},\underline{\bar{f}}}^{\Sigma^g} \to v(\underline{e},\underline{\bar{f}}) \in V^{\Sigma} \tag{7.18}$$

and the pairings (7.11) (for $\partial M = \Sigma^{g*}$ and $|G|_{\underline{x}}^{\Sigma} \subset \Sigma$) by

$$\begin{aligned} v^*(M) \otimes v(\underline{e},\underline{\bar{f}}) &\to (M, \underline{e},\underline{\bar{f}}) = Z(M \cup M_{\Sigma^g}, \mathscr{G}_{\underline{e},\underline{\bar{f}}}^{\Sigma^g}) , \\ v^*(\underline{x}) \otimes v(\underline{e},\underline{\bar{f}}) &\to (\underline{x}, \underline{e}\underline{\bar{f}}) = Z(M_{\Sigma^g}, \mathscr{G}_{\underline{e},\underline{\bar{f}}}^{\Sigma^g}, |G|_{\underline{x}}^{\Sigma}) \end{aligned} \tag{7.19}$$

and have the

Theorem 7.6. *The handle body M_{Σ^g} equipped with the graph $\mathscr{G}_{\underline{e}}^{\Sigma^g}$ and $\bar{\mathscr{G}}_{\underline{\bar{f}}}^{\Sigma^g}$ defines a vector basis $v(\underline{e},\underline{\bar{f}})$ of the vector space V^{Σ^g} associated to the surface Σ^g. In analogy to relation* (7.9) *the colours $\underline{x}$ describe the "components" of the vectors $v(\underline{e},\underline{\bar{f}})$. In addition there exists a vector basis $v^*(\underline{e},\underline{\bar{f}}) \in V^{\Sigma^{g*}}$ such that the pairing relations*

$$v^*(\underline{e},\underline{\bar{f}}) \otimes v(\underline{e}',\underline{\bar{f}}') \to (\underline{e}\underline{\bar{f}}, \underline{e}'\underline{\bar{f}}') = \delta_{\underline{e}\underline{e}'} \delta_{\underline{\bar{f}}\underline{\bar{f}}'} \tag{7.20}$$

hold and

$$(M_1, M_2) = \sum_{\underline{e},\underline{\bar{f}}} (M_1, \underline{e}\underline{\bar{f}})(\underline{e}\underline{\bar{f}}, M_2) \tag{7.21}$$

is valid for $\partial M_1 = \Sigma^g$ and $\partial M_2 = \Sigma^{g*}$.

Proof. In Appenix D we construct the vectors $v^*(\underline{e},\underline{\bar{f}})$ fulfilling (7.20). The claim follows then by Eq. (7.15′).

Remark 7.7. This theorem may be generalized to the case where Σ^g is replaced by the punctured surface $\Sigma^g_{\underline{a},\underline{\bar{b}}}$. Now $\mathscr{G}_{\underline{e}}^{\Sigma^g}$ has to be replaced by $\mathscr{G}_{\underline{e}}^{\Sigma^g_{\underline{a},\underline{\bar{b}}}}$, obtained from $\mathscr{G}_{\underline{e}}^{\Sigma^g}$ by n tubes starting at $D^2_{a_1}, \ldots, D^2_{a_n}$ and ending on $\partial\mathscr{T}_{\mathscr{G}_{\underline{e}}}$ carrying lines of colour $a_1, \ldots, a_n$ respectively. These lines all end on $|G|_{\underline{e}}$. The construction for $\bar{\mathscr{G}}_{\underline{f}}^{\Sigma^g_{\underline{a},\underline{\bar{b}}}}$ is analogous.

We conclude this section by introducing [for the case $U_q(sl(2,\mathbf{C})$] a hermitian structure on V^{Σ} (Σ an unpunctured Riemann surface) making V^{Σ} a Hilbert space. This structure is analogous to reflection positivity [OS] in euclidean quantum field theories. For simplicity we again consider the case $\Sigma = \Sigma^g$ only. The extension to arbitrary Σ is straightforward.

Theorem 7.8. *For the case $U_q(sl(2,\mathbf{C}))$, there is an antilinear map $\tau^{\Sigma} : V^{\Sigma} \to V^{\Sigma^*}$ such that $(\tau^{\Sigma})^* = \tau^{\Sigma^*}$, $\tau^{\Sigma^*} \circ \tau^{\Sigma} = \mathrm{id}_{V^{\Sigma}}$ and $\tau^{\Sigma} \circ \tau^{\Sigma^*} = \mathrm{id}_{V^{\Sigma^*}}$ with the following properties:*

(i) *The vectors defined by Eqs.* (7.9), (7.10), *and* (7.18) *are mapped as*

$$\tau^{\Sigma} v(M_2) = v^*(M_2^*) \quad \text{for all } M_2 \text{ with } \partial M_2 = \Sigma\,, \tag{7.22a}$$

$$\tau^{\Sigma} v(\underline{x}) = (-1)^{2\sum_j x_j} v^*(\underline{x})\,, \tag{7.22b}$$

$$\tau^{\Sigma} v(\underline{e}, \underline{\bar{f}}) = (-1)^{2\sum_j (e_j+f_j)} v^*(\underline{e}, \underline{\bar{f}})\,. \tag{7.22c}$$

(ii) *The hermitian form*

$$\langle v, v' \rangle = (\tau^{\Sigma} v, v') \quad \text{for} \quad v, v' \in V^{\Sigma} \tag{7.23}$$

on V^{Σ} is positive definite.

Proof. We introduce the basis

$$u(\underline{e}, \underline{\bar{f}}) = (i)^{2\sum_j (e_j+f_j)} v(\underline{e}, \underline{\bar{f}}) \tag{7.24}$$

on V^{Σ} and the dual basis

$$u^*(\underline{e}, \underline{\bar{f}}) = (-i)^{2\sum_j (e_j+f_j)} (\underline{e}, \underline{\bar{f}}) \tag{7.24*}$$

on $V^{\Sigma *}$.

Define τ^{Σ} to be the unique antilinear extension of the map $u(\underline{e}, \underline{\bar{f}}) \mapsto u^*(\underline{e}, \underline{\bar{f}})$ on this basis. Then relation (7.22c) automatically holds and the claim (ii) is a consequence of (7.20). It remains to prove the relations (7.22a, b). They, however, are immediate consequences of the following proposition which is proven in Appendix E.

Proposition 7.9. *The following relations are valid in the $U_q(sl(2, \mathbf{C}))$ case:*

$$(M_1, M_2)^* = (M_2^*, M_1^*)\,, \tag{7.25a}$$

$$(M_1, \underline{x})^* = (-1)^{2\sum_j x_j} (\underline{x}, M_1^*)\,, \tag{7.25b}$$

$$(\underline{x}, \underline{y})^* = (-1)^{2\sum_j (x_j+y_j)} (\underline{y}, \underline{x})\,, \tag{7.25c}$$

$$(\underline{e\bar{f}}, M_2)^* = (-1)^{2\sum_j (e_j+f_j)} (M_2^*, \underline{e\bar{f}})\,, \tag{7.25d}$$

$$(\underline{e\bar{f}}, \underline{x})^* = (-1)^{2\sum_j x_j + 2\sum_j (e_j+f_j)} (\underline{x}, \underline{e\bar{f}})\,, \tag{7.25e}$$

for any M_1 and M_2 with $\partial M_1 = \Sigma^*$, $\partial M_2 = \Sigma$.

Note in this context that the relation

$$W_{\underline{x}}^{\Sigma} = |W_{\underline{x}}^{\Sigma}| (-1)^{2\sum_j x_j} \tag{7.26}$$

is valid. Also this proposition in particular says that the relations

$$Z(M)^* = Z(M^*) = Z(M) \tag{7.27}$$

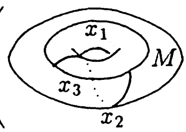
and

$$Z(M^* \underset{\partial M}{\cup} M) \geq 0 \tag{7.28}$$

are valid for all compact oriented (not necessarily closed) 3-manifolds M. In (7.28) we have equality if and only if $v(M) \in V^{\partial M}$ is the zero vector. Note that if the $v(M_2)$ ($\partial M_2 = \Sigma$) span V^{Σ}, then τ^{Σ}, whose existence we just have proven, is uniquely fixed by Eq. (7.22a).

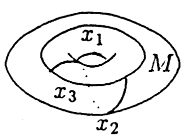
Examples 7.10. i) Let $M_{S^1\times S^1} = S^1 \times D^2$ be the solid torus and $|G|^{S^1\times S^1}_{x_1x_2x_3}$ its canonical graph on $\partial M_{S^1\times S^1} = S^1 \times S^1$. By means of Lemma 5.1 one easily finds (for $e = f = 0$)

$$(\underline{x}, M_{S^1\times S^1}) = (\underline{x}, 00) = Z\left(\quad\right) = \delta_{x_1 0}\delta_{x_2 x_3}. \qquad (7.29)$$



ii) Let $\tilde{M} = S^3\backslash(S^1 \times D^2)$ be the external of a torus in S^3 and $|G|^{(S^1\times S^1)^*}_{x_1x_2x_3}$ its canonical graph on $\partial\tilde{M} = \Sigma^* = (S^1 \times S^1)^*$. By means of Lemma 5.1 we obtain

$$(\tilde{M}, \underline{x}) = Z\left(S^3 \setminus \quad\right) = \delta_{x_2 0}\delta_{x_1 x_3}. \qquad (7.30)$$



Applying the surgery formula (7.3) or (7.3′) we find with $M = M_{S^1\times S^1}$ of i)

$$\begin{aligned}(\tilde{M}, M) &= \sum_{\underline{x}} W^{S^1\times S^1}_{\underline{x}}(\tilde{M}, \underline{x})\,(\underline{x}, M)\\ &= \frac{1}{w^2}\sum_{x_1x_2x_3}\prod_{i=1}^{3} w^2_{x_i}\delta_{x_1 0}\delta_{x_2 0}\delta_{x_3 0} = \frac{1}{w^2} = Z(S^3)\,. \end{aligned} \qquad (7.31)$$

iii) As a generalization of i) let $|G|^{S^1\times S^1}_{x_1x_2x_3}$ and $M_{S^1\times S^1}$ be as in i) but now equipped with its canonical lefthanded graph $\mathscr{G}^{S^1\times S^1}_e \subset M_{S^1\times S^1}$ given by Definition 7.5. Using Lemmas 5.1, 5.3 and Racah's relation (2.10) one finds (for $f = 0$)

$$(\underline{x}, e0) = Z(M_{S^1\times S^1}; \mathscr{G}^{S^1\times S^1}_e, |G|^{S^1\times S^1}_{x_1x_2x_3}) = \frac{q_{x_3}}{q_{x_1}q_{x_2}}\frac{\delta_{x_1 e}}{w^2_{x_1}}\delta_{x_1x_2x_3}\,. \qquad (7.32)$$

iv) As a generalization of ii) let $|G|^{(S^1\times S^1)^*}_{x_1x_2x_3}$ and $\tilde{M}$ be as in ii) but now equipped with a lefthanded graph $\tilde{\mathscr{G}}^{(S^1\times S^1)^*}_{e'}$ (not contractible in $\tilde{M}$). Similar as in iii) we calculate (for $f' = 0$)

$$(\widetilde{e'0}, \underline{x}) = Z(\tilde{M}; \tilde{\mathscr{G}}^{S^1\times S^1}_{e'}, |G|^{(S^1\times S^1)^*}_{x_1x_2x_3}) = \frac{q_{x_3}}{q_{x_1}q_{x_2}}\frac{\delta_{x_2 e'}}{w^2_{x_2}}\delta_{x_1x_2x_3}\,. \qquad (7.33)$$

v) Let $|G|^{(S^1\times S^1)^*}_{x_1x_2x_3}$ and $M_{(S^1\times S^1)^*} = M^*_{S^1\times S^1}$ be as in ii) but with the opposite orientation and equipped with its canonical lefthanded graph $\mathscr{G}^{(S^1\times S^1)^*}_{e'} \subset M_{(S^1\times S^1)}$. Analogously to (7.32) and taking into account the remarks after (3.6) we find (for $f' = 0$)

$$(e'0, \underline{x}) = Z(M^*_{S^1\times S^1}; \mathscr{G}^{(S^1\times S^1)^*}_{e'}, |G|^{(S^1\times S^1)^*}_{x_1x_2x_3}) = \frac{q_{x_1}q_{x_2}}{q_{x_3}}\frac{\delta_{x_1 e'}}{w^2_{x_1}}\delta_{x_1x_2x_3}\,. \qquad (7.34)$$

vi) Combining iii) and iv) we use $\tilde{M} \underset{S^1\times S^1}{\cup} M_{S^1\times S^1} = S^3$ and obtain in agreement with example 6.8 iii) [see Eqs. (6.9) and (6.10)]

$$\widetilde{(e'0, e0)} = Z(S^3, \tilde{\mathscr{G}}_{e'}^{(S^1\times S^1)^*} \cup \mathscr{G}_e^{S^1\times S^1}) = \sum_{\underline{x}} W_{\underline{x}}^{S^1\times S^1}(\tilde{e}', \underline{x})(\underline{x}, e)$$

$$= \frac{1}{w^2} \sum_{x_1x_2x_3} \prod_{i=1}^{3} w_{x_i}^2 \frac{q_{x_3}^2}{q_{x_1}^2 q_{x_2}^2} \frac{\delta_{x_2e'}}{w_{x_2}^2} \frac{\delta_{x_1e}}{w_{x_1}^2} \delta_{x_1x_2x_3} = \frac{1}{w} S_{e'e}, \qquad (7.35)$$

where Eq. (A.1) has been used.

vii) Combining iii) and v) we use $M^*_{S^1\times S^1} \underset{S^1\times S^1}{\cup} M_{S^1\times S^1} = S^1 \times S^2$ and obtain in agreement with Eq. (7.20),

$$(e'0, e0) = Z(S^1 \times S^2, \mathscr{G}_{e'}^{(S^1\times S^1)^*} \cup \mathscr{G}_e^{S^1\times S^1}) = \sum_{\underline{x}} W_{\underline{x}}^{S^1\times S^1}(e', \underline{x})(\underline{x}, e)$$

$$= \frac{1}{w^2} \sum_{x_1x_2x_3} \prod_{i=1}^{3} w_{x_i}^2 \frac{\delta_{x_1e'}}{w_{x_1}^2} \frac{\delta_{x_1e}}{w_{x_1}^2} \delta_{x_1x_2x_3} = \delta_{e'e}\,. \qquad (7.36)$$

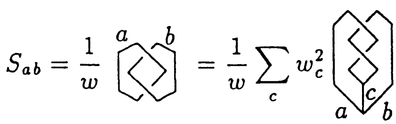
8. Conclusion

This article has shown the richness of structures contained in the combinatorial approach of Turaev and Viro to topological quantum field theories. So far our discussion has basically been limited to the quantum group $U_q(sl(2,\mathbf{C}))$ and to finite groups having only real representations (i.e. real characters) and fusion matrices whose entries are either zero or one (like e.g. the permutation groups S_3 and S_4). By the discussion in [DJN1, DJN2] the Turaev-Viro approach as well as the extension given here may be extended to arbitrary quantum groups or finite groups. Note that for finite groups and the corresponding usual group algebra the Racah relation is essentially trivial leading to an R-matrix which is just the permutation matrix. This implies that finite groups in contrast to quantum groups do not give rise to interesting knot invariants. Also the Reshetikhin-Turaev invariant $\tau(M)$ is trivial in that case [FRS]. On the other hand, every 3 cycle on a finite group as considered in [DPR] actually defines an associator which turns the cocommutative Hopf algebra of the group into a quasitriangular quasi-Hopf algebra [D] (we owe this observation to G. Felder). Our methods also extend to this case and should be related via (1.1) to the discussions in [DW, AC1, AC2, FQ]. The discussion of links and graphs in 3-manifolds and of punctured Riemann surfaces shows that particles may be introduced describing conformally invariant field theories (CFT) on the boundary of the 3-manifold. The appearance of left- and right-handed coloured punctures correspond to left and right handed chiralities. The picture emerging here suggests that CFT describes the asymptotic "free" part of a 3-dim QFT with braid group statistics. This raises the question of finding analogues of the usual Hamilton and LSZ formulation in order to discuss scattering theory in this context.

Acknowledgements. The authors have profited from discussions with W. Müller, F. Nill, S. Novikov, K. H. Rehren, and B. Schroer.

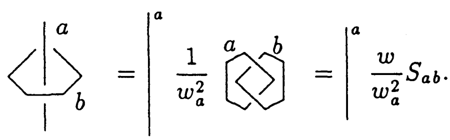
Appendix A

In this appendix we present some usful well known modular formulas. The following equations involving graphs are to be understood as equations of their invariants in the sense of Theorem 4.2 and Eq. (4.1). If there are parts of graphs they are to be interpreted as generalized vertices in the sense of Eq. (4.4). With the conventions used in Sect. 4, let us define the matrix S_{ab} $(a, b \in \mathscr{T})$ with values in K by



$$= \frac{1}{w} \sum_c \frac{w_c^2 q_c^2}{q_a^2 q_b^2} \delta_{abc} = S_{ba}. \tag{A.1}$$

Here we have made use of relation (4.19). By the Wigner-Eckart theorem to be proven below, we also have

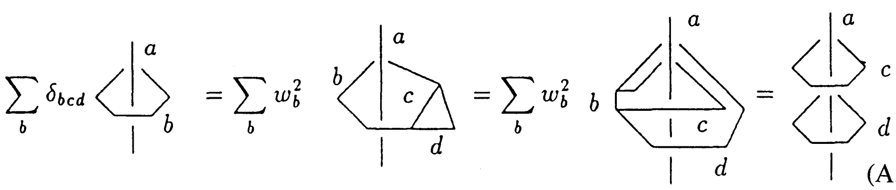


(A.2)

and we define

$$w_b^2(a) = \frac{w}{w_a^2} S_{ab}. \tag{A.3}$$

Now the chain of equalities combined with (A.2)



(A.4)

imply

$$\sum_b w_b^2(a) \delta_{bcd} = w_c^2(a) w_d^2(a) \tag{A.5}$$

for all $a \in \mathscr{T}$. In what follows, we will assume K is an integral domain.

Lemma A.1. *If for given $a \in \mathscr{T}$ there is some $b \in \mathscr{T}$ with $\omega_b^2 \neq \omega_b^2(a)$, then*

$$\sum_b w_b^2 w_b^2(a) = w^2 \delta_{a0}. \tag{A.6}$$

Proof. The proof mimics the orthogonality relation of eigenfunctions for different eigenvalues of the Schrödinger equation. By (2.2) and (A.5) we have

$$(w_b^2 - w_b^2(a)) \sum_c w_c^2 w_c^2(a) = \sum_{c,d} \delta_{bcd} w_d^2 w_c^2(a) - \sum_{c,d} \delta_{bcd} w_c^2 w_d^2(a) = 0$$

for all b and the claim follows.

For the $U_q(sl(2, \mathbf{C})$ case

$$w_b^2(a) = (2b+1)_{-q^{2a+1}} \tag{A.7}$$

such that the assumption made in Lemma A.1 holds for all $a \neq 0$. In the general context one has the following sufficient condition

Lemma A.2. *If* $q_a^2 \neq 1$ *and* $\sum_c q_c^2 w_c^4 \neq 0$ *then there is* $b \in \mathscr{T}$ *with* $w_b^2 \neq w_b^2(a)$.

Proof. Assume the contrary, then

$$q_a^2 w_a^2 q_b^2 w_b^2 = \sum_c w_c^2 q_c^2 \delta_{abc}$$

holds for all b. Multiplying by w_b^2 and summing over b gives

$$q_a^2 w_a^2 \sum_b q_b^2 w_b^4 = \sum_c q_c^2 w_c^4 w_a^2$$

from which we deduce $q_a^2 = 1$ contrary to the assumption, q.e.d.

Note that (A.6) amounts to the statement

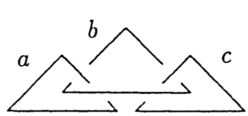
$$\sum_b S_{ab} S_{b0} = \delta_{a0}\,. \tag{A.8}$$

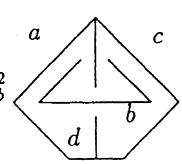
Lemma A.3. *Assume* (A.8) *is valid for all* $a \in \mathscr{T}$. *then*

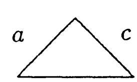
$$\sum_b S_{ab} S_{bc} = \delta_{ac} \tag{A.9}$$

holds for all a *and* c *in* $\mathscr{T}$.

Proof. By (A.8) we have

$$\sum_b w_b^2$$


$$= \sum_d w_d^2 \sum_b w_b^2\,.$$


$$= \sum_b w_b^4 \frac{1}{w_a^2}$$


$$= w^2 \delta_{ac}.$$

On the other hand by (4.2) and (A.2) this quantity also equals

$$\sum_b w_b^2 \bigcirc^a \frac{wS_{ab}}{w_a^2 w_b^2} \bigcirc^b \frac{wS_{bc}}{w_b^2 w_c^2} \bigcirc^c = w^2 \sum_b S_{ab} S_{bc}$$

and the claim follows:

Corollary A.4. *Assume* (A.6) *is valid for all* $a \in \mathcal{T}$ *such that* (A.9) *also holds. Then one has*

$$\delta_{abc} = \sum_d \frac{S_{ad} S_{bd} S_{cd}}{S_{0d}} \tag{A.10}$$

and the fusion matrices N^a *with* $(N^a)_{bc} = \delta_{abc}$ *all commute.*

By methods similar to those used above one can prove the following. Set

$$T_{ab} = \frac{\rho}{w_a^2} \delta_{ab}$$ 

$$= \rho \delta_{ab} q_a^2 \,, \tag{A.11}$$

provided $\varrho = \left(w^{-1} \sum_c w_c^4 q_c^2 \right)^{-1/3} \in K$ exists. Then the relation

$$(TS)^3 = \mathbf{1} \tag{A.12}$$

holds. Relations (A.9) and (A.12) show that S and T form a representation of the modular transformations on a torus with period τ

$$S: \tau \to -1/\tau \,, \qquad T: \tau \to \tau + 1 \,. \tag{A.13}$$

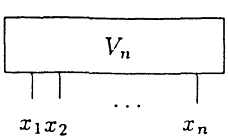
In the case of $U_q(sl(2, \mathrm{C})$ the $w_b^2(a)$ are all real such that S is a real orthogonal matrix. Also ϱ is then of modulus 1 such that T is unitary. In fact, ϱ may then be calculated via Gauss reciprocity theorem to equal $q^{1/2} e^{-i\pi/4}$. In particular, this representation is the same as in [GW].

Lemma A.5. *A Wigner-Eckart type relation holds for any generalized vertex* $V_{n,\underline{x}}$ *in the sense of Sect. 4 (see e.g.* [KR]*)*

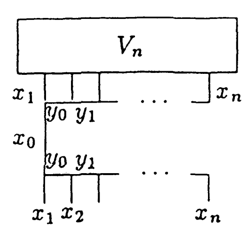
$$w(V_{n,\underline{x}}) = \sum_{\underline{y}} c(\underline{y}) w(v_{n,\underline{x}}(\bar{y})) \,, \tag{A.14}$$

where the $v_{n,\underline{x}}(\underline{y})$ *are called basic vertices (in a "path-basis") and are given by Eq.* (A.15). *For* $n = 2$ *and* 3 *they coincide with the elementary ones of* (3.8).

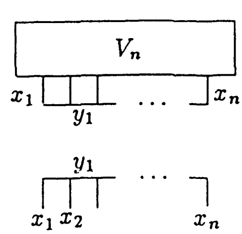
Proof. Using the orthogonality relation (2.7) one obtains



$$= \sum_{x_0, y_0, \underline{y}} w_{x_0}^2 \prod_{i=0}^{n-3} w_{y_i}^2$$

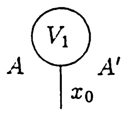


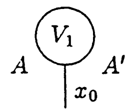
$$= \sum_{\underline{y}} \prod_{i=1}^{n-3} w_{y_i}^2$$



(A.15)

The last equality is a consequence of assumption (2.1). We find



$\propto \delta_{AA'x_0}$ and


$\propto \delta_{AA'}$, (A.16)

where the first proportionality follows from Eq. (2.4) and the second one since A and A' belong to the same sector in the sense of Sect. 3. But because of Eq. (2.1): $\delta_{ab0} = \delta_{ab}$ we obtain

$$\delta_{AA'x_0} \propto \delta_{AA'} \qquad (\forall A, A' \in \mathscr{I}) \Rightarrow x_0 = 0\,. \tag{A.17}$$

Appendix B. Proof of Theorem 7.1

Let M be of the form (7.2). In order to perform surgery along a surface Σ^g of genus g we represent it in a canonical way by a $4g$-polygon $(c_1, d_1, d_1^{-1}, d_1^{-1}, \ldots, d_g^{-1})$. In $\Sigma \times I$ as the intersection point of all cycles c_i, d_i we cut a hole into M of the form D^3 as in Fig. 22a,

$$Z(M) = \frac{1}{w^2}\, Z(M \backslash D^3)\,. \tag{B.1}$$

Along the cycles c_i and d_i we introduce tubes $\partial\mathscr{T}_i$ $(i = 1, \ldots, 2g)$ (by means of Lemma 5.3) with meridians $m_1, \ldots, m_{2g}$ as in Fig. 22b,

$$Z(M) = \frac{1}{w^2} \sum_{\underline{x}} \prod_{i=1}^{2g} w_{x_i}^2\, Z(\tilde{M}, S^1_{x_1} \cup \ldots \cup S^1_{x_{2g}})\,, \tag{B.2}$$

where $\tilde{M} = M \backslash D^3 \backslash \mathscr{T}_1 \backslash \ldots \backslash \mathscr{T}_{2g}$.

As a consequence M_1 and M_2 are now only connected by a cylinder $D^2 \times I$ as depicted in Fig. 22c. The intersections of the meridians with this cylinder are of the form $P_i \times I$ $(i = 1, \ldots, 2g; P_i \in \partial D^2)$. We use the second orthogonality relation of (4.11) $4g - 1$ times (introducing new lines with colours x_0, y_i $(i = 0, \ldots, 4g-3)$ such that only one line $P_0 \times I$ (with colour x_0) intersects the cylinder (see Fig. 22d) to obtain

$$Z(M) = \frac{1}{w^2} \sum_{\underline{x}} \prod_{i=0}^{2g} w_{x_i}^2 \sum_{\underline{y}} \prod_{i=0}^{4g-3} w_{y_i}^2\, Z(\tilde{M}, |G|_{\underline{x},\underline{y}})\,, \tag{B.3}$$

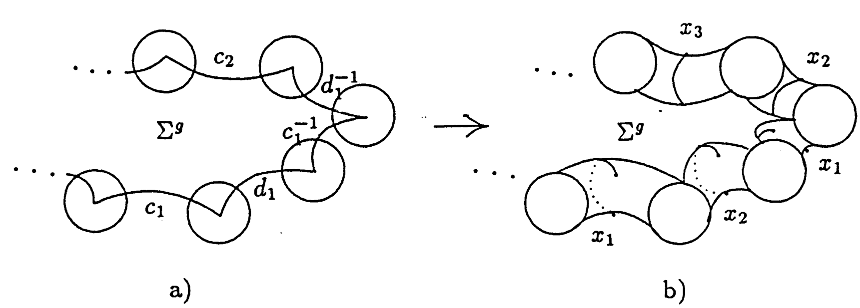


Fig. 22. a Surgery along Σ^g. The "half" meridians with colour x_i drawn in **b** at the c- and d-cycles are connected to the corresponding other "half" ones at c^{-1} and d^{-1}

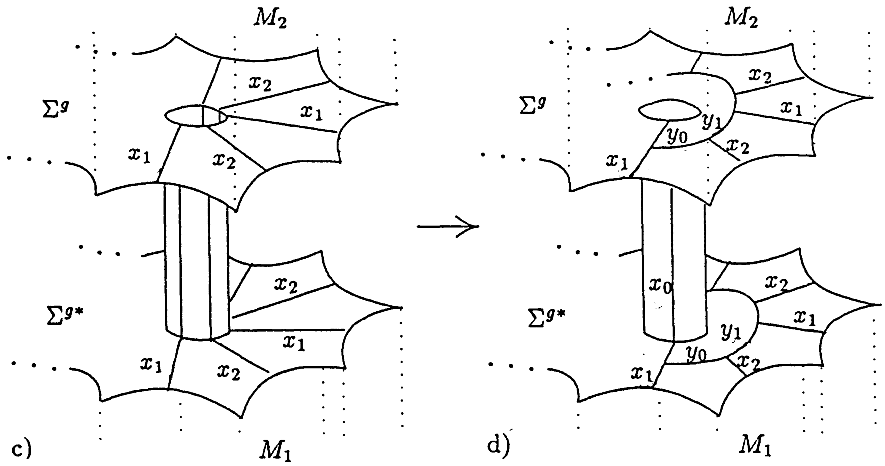


Fig. 22c, d. Surgery along Σ^g

where $|G|_{\underline{x}\underline{y}}$ is the graphic depicted in Fig. 22d. We cut the cylinder by means of Lemma 5.1 to obtain $x_0 = 0$, $y_0 = x_1$ and

$$Z(M) = \frac{1}{w^2} \sum_{\underline{x}} \prod_{i=1}^{6g-3} w_{x_i}^2 Z(M_1, |G|_{\underline{x}}^{\Sigma^{g*}}) Z(M_2, |G|_{\underline{x}}^{\Sigma^g}), \tag{B.4}$$

where we have written $x_{2g+i} = y_i$, $i = 1, \ldots, 4g-3$ and $|G|_{\underline{x}}^{\Sigma^g}$ is given by Fig. 19. Using Lemma 5.3, Corollary 5.5 and 5.6 one can prove the surgery formula for the case that some left- or right-hand lines may cross Σ.

Appendix C. Proof of Theorem 7.4

The dimension of the vector space V^{Σ} associated to a closed oriented 2-manifold Σ is given by

$$d^{\Sigma} = Z(\Sigma \times S^1). \tag{C.1}$$

It suffices to consider the case where $\Sigma = \Sigma^g$ is connected of genus g. Using the surgery formula (7.2) we write (see Fig. 23a)

$$d^{\Sigma^g} = \sum_{\underline{x}} W_{\underline{x}}^{\Sigma^g} Z(\Sigma^g \times I, |G|_{\underline{x}}^{\Sigma^g} \cup |G|_{\underline{x}}^{\Sigma^{g*}}), \tag{C.2}$$

where $G_{\underline{x}}^{\Sigma^g}$ is the canonical graph given by Fig. 19. By means of Lemma 6.4 we may introduce a tube with two meridians over and undercrossing, respectively, a line with colour x_0. The tube connects Σ and Σ^* as in Fig. 23b. Using relation (4.11) we obtain

$$\begin{aligned} d^{\Sigma^g} &= \sum_{u,v} \frac{w_u^2}{w^2} \frac{w_v^2}{w^2} \sum_{\underline{x},\underline{y}} \prod_{i=0}^{2g} w_{x_i}^2 \prod_{i=0}^{4g-3} w_{y_i}^2 Z(H^{2g}, |G|_{\underline{x},\underline{y},u,v}) \\ &= \sum_{u,v} \frac{w_u^2}{w^2} \frac{w_v^2}{w^2} \sum_{\underline{x}} \prod_{i=1}^{2g} w_{x_i}^2 Z(H^{2g}, |G|_{\underline{x},u,v}), \end{aligned} \tag{C.3}$$

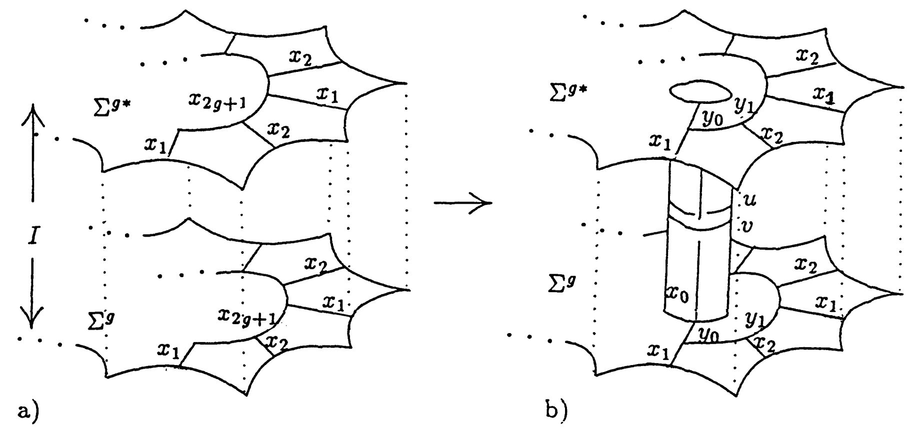


Fig. 23a, b. The state sum $Z(\Sigma^g \times I)$

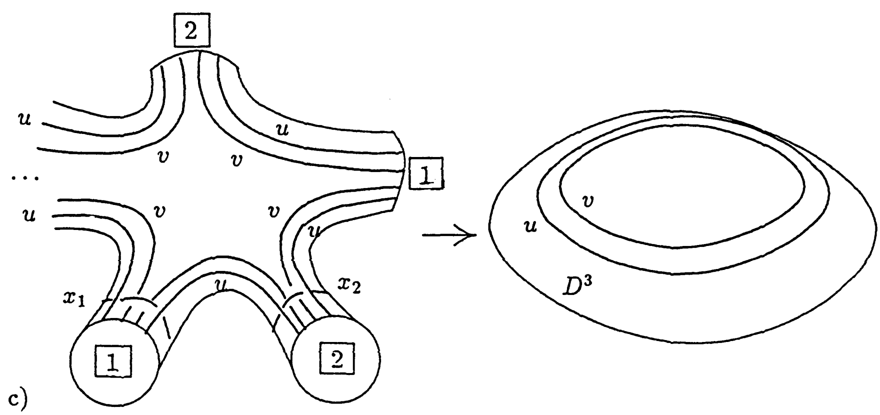


Fig. 23. c The handle body H^{2g} with the graph $|G|_{\underline{x},u,v}$ of Eq. (C.3) (the handles are connected at $1, 2$, etc.); **d** the ball D_3 with the graph $S^1_u \cup S^1_v$ of Eq. (C.4)

where $H^{2g} = (\Sigma^g \times I)\backslash(D^2 \times I)$ is a handle body as depicted in Fig. 23b or Fig. 23c. The $2 - g$ handles may be cut due to Lemma 5.1 and as a generalization of Example 5.8 iii) we find (see Fig. 23d)

$$\begin{aligned} d^{\Sigma^g} &= \sum_{u,v} \frac{w_u^2}{w^2}\frac{w_v^2}{w^2}\left(\frac{w^2}{w_u^2 w_v^2}\right)^{2g} Z(D^3, S^1_u \cup S^1_v) \\ &= \left(\sum_u \left(\frac{w}{w_u^2}\right)^{2g-2}\right)^2 = (\mathrm{tr}(\vec{N}^2)^{g-1})^2 . \end{aligned} \tag{C.4}$$

The last equality follows from (A.10) and (A.9).

For the case $\Sigma = \Sigma^g_{\underline{a},\underline{\bar{b}}}$ ($\underline{a} = (a_1, \ldots, a_n)$, $\underline{\bar{b}} = (\bar{b}_1, \ldots, \bar{b}_{\bar{n}})$) of (7.5) in (C.2) the canonical graph is replaced by $|G|_{\underline{x}}^{\Sigma^g_{\underline{a},\underline{\bar{b}}}}$ of Fig. 20 and Eq. (C.1) by

$$d^{\Sigma^g_{\underline{a},\underline{\bar{b}}}} = Z(\Sigma^g_{\underline{a},\underline{\bar{b}}} \times S^1, \mathscr{G}_{\underline{a},\underline{\bar{b}}}) , \tag{C.5}$$

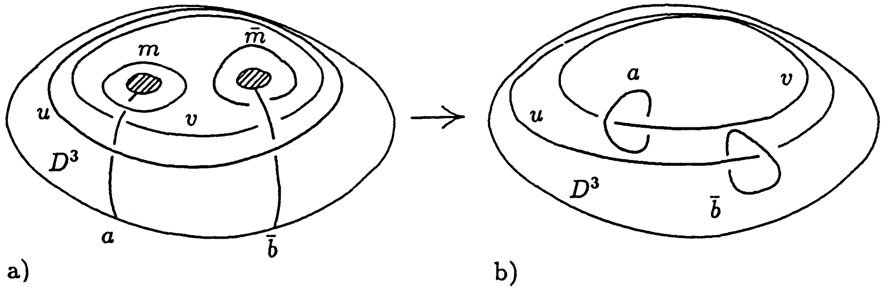


Fig. 24. a The handle body $D^3_{\underline{a},\underline{\bar{b}}}$ with the graph $|G|_{\underline{a},\underline{b},\underline{m},\underline{\bar{m}},u,v}$; **b** the ball D^3 with the graph $G_{\underline{a},\underline{\bar{b}},u,v}$ of Eq. (C.6)

where the tubular neighborhood of $c(\mathscr{G}_{\underline{a},\underline{\bar{b}}})$ is of the form $(D^2_{a_1}\times S^1)\cup\ldots\cup(D^2_{a_n}\times S^1)$. The associated coloured graph $|G|_{\underline{a},\underline{\bar{b}},\underline{m},\underline{\bar{m}}}$ consists of $n+\bar{n}$ loops of the form $(P_{a_1}\times S^1)_{a_1}\cup\ldots\cup(P_{\bar{b}_{\bar{n}}}\times S^1)_{\bar{b}_{\bar{n}}}$ (with $P_{a_i}\in\partial D^2_{a_i}$, $P_{\bar{b}_j}\in\partial D^2_{\bar{b}_j}$) and their meridians $(S^1_{a_1}\times Q)_{m_1}\cup\ldots\cup(S^1_{\bar{b}_{\bar{n}}}\times Q)_{\bar{m}_{\bar{n}}}$ (with $Q\in S^1$). We may proceed as above and in (C.4) D^3 (Fig. 23d) is now replaced by the solid handle body $D^3_{\underline{a},\underline{\bar{b}}}=D^3\backslash D^2_{a_1}\times I\backslash\ldots\backslash D^2_{\bar{b}_{\bar{n}}}\times I$ whose boundary has genus $n+\bar{n}$ (Fig. 24a shows an example for $n=\bar{n}=1$).

The graph $S^1_u\cup S^1_v$ in (C.4) is replaced by a graph $|G|_{\underline{a},\underline{\bar{b}},\underline{m},\underline{\bar{m}},u,v}$ (see Fig. 24a for the case $n=\bar{n}=1$, where the meridians m and $\bar{m}$ have been moved to the ends of their tubes for clarification). A procedure similar to the one used in example 5.8 iii) leads to

$$d^{\Sigma^g}_{\underline{a},\underline{\bar{b}}}=\sum_{u,v}\left(\frac{w^2}{w_u^2w_v^2}\right)^{2g-1}\frac{1}{w^2}\,Z(D^3,|G|_{\underline{a},\underline{\bar{b}},u,v})\,, \qquad \text{(C.7)}$$

where the graph $|G|_{\underline{a},\underline{\bar{b}},u,v}=\left(S^1_v\cup\bigcup_{i=1}^{n}S^1_{a_i}\right)\cup\left(S^1_u\cup\bigcup_{i=1}^{\bar{n}}S^1_{\bar{b}_i}\right)$ is of the form depicted in Fig. 24b (for the case $n=\bar{n}=1$). Using Eq. (A.2) we obtain

$$\begin{aligned} d^{\Sigma^g}_{\underline{a},\underline{\bar{b}}}&=\sum_{u,v}\left(\frac{w^2}{w_u^2w_v^2}\right)^{2g-2}\prod_{i=1}^{n}\frac{S_{a_iv}}{S_{0v}}\prod_{i=1}^{\bar{n}}\frac{S_{\bar{b}_iu}}{S_{0u}}\\ &=\mathrm{tr}(N^{a_1}\ldots N^{a_n}(\vec{N}^2)^{g-1})\,\mathrm{tr}(N^{\bar{b}_1}\ldots N^{\bar{b}_{\bar{n}}}(\vec{N}^2)^{g-1})\,. \end{aligned} \qquad \text{(C.7)}$$

Apendix D. Proof of Theorem 7.6.

Consider in S^3 the external of the handle body M_{Σ^g} of Definition 7.5

$$S^3\backslash M_{\Sigma^g}\,,\quad\text{with}\quad \partial(S^3\backslash M_{\Sigma^g})=\Sigma^{g*}\,, \qquad \text{(D.1)}$$

and the graph $\tilde{\mathscr{G}}^{\Sigma^{g*}}_{\underline{e},\underline{\bar{f}}}$ in int $S^3\backslash M_{\Sigma^g})$ such that the braiding w.r.t. $\mathscr{G}^{\Sigma^g}_{\underline{e'},\underline{\bar{f}'}}$ in int M_{Σ^g} is given by Fig. 25a) for the left-handed part (the right-handed part is defined analogously).

As an extension of Eq. (A.2) we define the modular matrix $S_{\underline{ee'}}$ associated to Σ^g

$$S_{\underline{ee'}}=Z(S^3,\tilde{\mathscr{G}}^{\Sigma^{g*}}_{\underline{e}}\cup\mathscr{G}^{\Sigma^g}_{\underline{e'}})\,. \qquad \text{(D.2)}$$

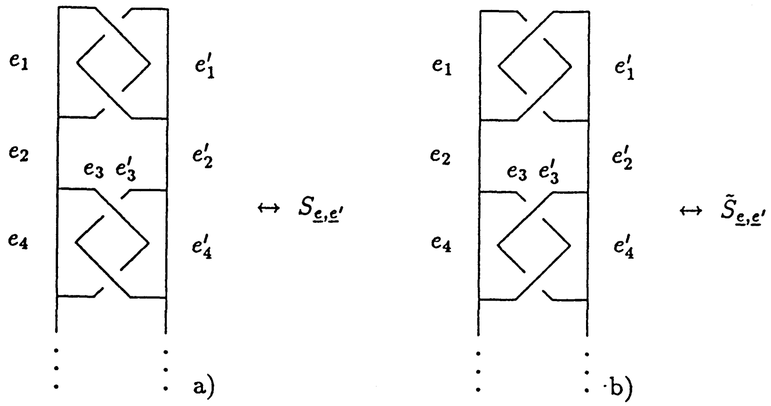


Fig. 25. The modular matrices S and $\tilde{S}$ associated to Σ^g

Analogously to (7.19) we obtain a map

$$\tilde{\mathscr{G}}^{\Sigma^{g*}}_{\underline{e},\underline{\bar{f}}} \to \tilde{v}^*(\underline{e},\underline{\bar{f}}) \in V^{\Sigma^{g*}} \tag{D.3}$$

with the pairing

$$\widetilde{(\underline{e}\underline{\bar{f}},\underline{e}'\underline{\bar{f}}')} = S_{\underline{e}\underline{e}'} S_{\underline{\bar{f}}\underline{\bar{f}}'} \tag{D.4}$$

Let $\tilde{S}$ (the "mirror" of S) be defined analogously to (D.2) by Fig. 25b. Using the Wigner-Eckart like relation (A.15) we find

$$\sum_{\underline{e}'} \tilde{S}_{\underline{e}\underline{e}'} S_{\underline{e}'\underline{e}''} = \delta_{\underline{e}\underline{e}''} \,. \tag{D.5}$$

Hence we have the "orthogonality" relation (7.20) with

$$v^*(\underline{e},\underline{\bar{f}}) = \sum_{\underline{e}',\underline{\bar{f}}'} \tilde{S}_{\underline{e}\underline{e}'} S_{\underline{\bar{f}}\underline{\bar{f}}'} \tilde{v}^*(\underline{e}',\underline{\bar{f}}') \,. \tag{D.6}$$

It is easy to see that $\tilde{S}$ and S are related by "modular shifts" $T^{(i)}$, analogously to Eq. (A.11),

$$T^{(i)}_{\underline{e}\,\underline{e}'} \propto \delta_{\underline{e}\,\underline{e}'} Z\left(\bowtie^{e_i}\right) . \tag{D.7}$$

Appendix E

In this appendix we give a proof of Proposition 7.9. It is an easy consequence of the relations

$$(\underline{x},\underline{e}\underline{\bar{f}}) = Z(M_\Sigma; \mathscr{G}_{\underline{e}\underline{\bar{f}}}, |G|^{\Sigma}_{\underline{x}}) \tag{E.1}$$

and

$$(\underline{e}\underline{\bar{f}},\underline{x}) = Z(M^*_\Sigma; \mathscr{G}^*_{\underline{e}\underline{\bar{f}}}, |G|^{\Sigma^*}_{\underline{x}}) \tag{E.2}$$

and the lemma below. Note that the definition of the vector $v^*(\underline{e}\underline{\bar{f}})$ given by Eqs. (D.3) and (D.6) is in agreement with Eq. (E.2). This follows from the orthogonality relation (7.20) and the fact that

$$Z(M_{\Sigma}^{*} \cup M_{\Sigma}, \mathcal{G}_{\underline{e}\underline{\bar{f}}}^{*} \cup \mathcal{G}_{\underline{e}'\underline{\bar{f}}'}) = \delta_{\underline{e}\underline{e}'}\delta_{\underline{\bar{f}}\underline{\bar{f}}'}\,, \tag{7.20'}$$

where $\mathcal{G}_{\underline{e}\underline{\bar{f}}}^{*} \subset M_{\Sigma}^{*}$ and $\mathcal{G}_{\underline{e}'\underline{\bar{f}}'} \subset M_{\Sigma}$. The derivation of Eq. (7.20′) uses techniques of Sect. 5, especially those applied in example 5.8 iii).

Lemma E. *For any oriented, compact 3 manifold M and any coloured graph $|G|_{\underline{x}}$ on ∂M the following relation is valid in the context of $U_q(sl(2,\mathbf{C}))$:*

$$Z(M, |G|_{\underline{x}})^{*} = \prod_{\substack{\sigma^0 \in G \\ n(\sigma^0)=3}} (-1)^{\sum_{l:\,\partial l \ni \sigma^0} x(l)} Z(M^{*}, |G|_{\underline{x}}^{*})\,. \tag{E.3}$$

Proof. The proof is a hidden cohomology argument and based on the following behaviour of the $6j$-symbols under complex conjugation:

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}^{*} = (-1)^{2i+2j+2k+2l+2m+2n} \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}\,. \tag{E.4}$$

For a given (admissible) edge colouring $\underline{j}$ of a triangulation X of M let

$$\begin{aligned} \operatorname{sign} \underline{j}(\sigma^2) &= (-1)^{\sum_{\sigma^1 \in \partial\sigma^2} j(\sigma^1)} \\ \operatorname{sign} \underline{j}(\sigma^3)\,, &= \prod_{\sigma^2 \in \partial\sigma^3} \operatorname{sign} j(\sigma^2)\,. \end{aligned} \tag{E.5}$$

Then (E.4) implies

$$(6\underline{j})\,(\sigma^3)^{*} = \operatorname{sign} \underline{j}(\sigma^3)\,(6\underline{j})\,(\sigma^3)\,. \tag{E.6}$$

Analogously, if we define $\operatorname{sign}(\underline{j}, \underline{\tilde{J}})\,(\sigma^2)$ $(\sigma^2 \in \partial X)$ by

$$(6\underline{j}, \underline{\tilde{J}})\,(\sigma^2)^{*} = \operatorname{sign}(\underline{j}, \underline{\tilde{J}})\,(\sigma^2)\,(6\underline{j}, \underline{\tilde{J}})\,(\sigma^2)\,, \tag{E.7}$$

then it is easy to see that we can write $\operatorname{sign}(\underline{j}, \underline{\tilde{J}})\,(\sigma^2)$ in the form

$$\operatorname{sign}(\underline{j}, \underline{\tilde{J}})\,(\sigma^2) = \operatorname{sign} \underline{j}(\sigma^2) \prod_{\sigma^1 \in \partial\sigma^2} \operatorname{sign}(\underline{j}, \underline{\tilde{J}})\,(\sigma^1, \sigma^2) \tag{E.8}$$

for suitably defined $\operatorname{sign}(\underline{j}, \underline{\tilde{J}})\,(\sigma^1, \sigma^2)$ $(\sigma^1 \in \partial\sigma^2)$ [see also Eq. (E.10) below for an example]. With these formulas it is easy to see that the relation

$$W(\underline{j}, \underline{\tilde{J}})\,(X)^{*} = \prod_{\substack{\sigma^2 \in \partial X \\ \sigma^1 \in \partial\sigma^2 \cap G(\partial X)}} \operatorname{sign}(\underline{j}, \underline{\tilde{J}})\,(\sigma^1, \sigma^2) W(\underline{j}, \underline{\tilde{J}})\,(X) \tag{E.9}$$

holds. Next we look at (3.4) with the notation of Fig. 2. By (E.4) we have

$$\begin{aligned} W(\underline{x}, \underline{j}, \underline{\tilde{J}})\,(\sigma^1)^{*} &= (-1)^{x+A+A'}(-1)^{x+B+B'} \\ &\quad \times (-1)^{a+A+B}(-1)^{a+B+B'} W(\underline{x}, \underline{j}, \underline{\tilde{J}})\,(\sigma^1) \\ &= \operatorname{sign}(\underline{x}, \underline{\tilde{J}})\,(\sigma_1^0, \sigma^1) \operatorname{sign}(\underline{x}, \underline{\tilde{J}})\,(\sigma_2^0, \sigma^1) \\ &\quad \times \operatorname{sign}(\underline{j}, \underline{\tilde{J}})\,(\sigma^1, \sigma_1^2) \operatorname{sign}(\underline{j}, \underline{\tilde{J}})\,(\sigma^1, \sigma_2^2) W(\underline{x}, \underline{j}, \underline{\tilde{J}})\,(\sigma^1) \end{aligned} \tag{E.10}$$

with the obvious definition of $\operatorname{sign}(\underline{x},\tilde{\underline{J}})(\sigma_i^0,\sigma^1)$ $(i=1,2)$. Here σ_i^2 $(i=1,2)$ are the unique 2-simplexes in ∂X having σ^1 in their boundary.

Finally we look at $w(\underline{x},\tilde{\underline{J}})(\sigma^0)$ $(\sigma^0\in G(\partial X))$ [see (3.6)]. There we have

$$w(\underline{x},\tilde{\underline{J}})(\sigma^0)^* = w(v)^* = w(v^*) \prod_{\sigma^1\in st(\sigma^0)} \operatorname{sign}(\underline{x},\tilde{\underline{J}})(\sigma^0,\sigma^1) \times \begin{cases} 1 & \text{if} \quad n(\sigma^0)=2 \text{ or } 4 \\ (-1)^{\sum_{\sigma^1\in st(\sigma^0)} x(\sigma^1)} & \text{if} \quad n(\sigma^0)=3\,. \end{cases} \tag{E.11}$$

where v^* equals v with the opposite orientation. Then an easy argument shows that

$$W(\underline{j},\tilde{\underline{J}})(X,|G|_{\underline{x}})^* = \prod_{\substack{\sigma^0\in G\\ n(\sigma^0)=3}} (-1)^{\sum_{\sigma^1\in st(\sigma^0)} x(\sigma^1)}\, W(\underline{j},\tilde{\underline{J}})(X^*,|G|_{\underline{x}}^*)\,, \tag{E.12}$$

since all other sign factors $\operatorname{sign}\underline{j}(\sigma^3)$, $\operatorname{sign}(\underline{x},\tilde{\underline{J}})(\sigma^0,\sigma^1)$, and $\operatorname{sign}(\underline{j},\tilde{\underline{J}})(\sigma^1,\sigma^2)$ appear exactly twice. Relation (E.12) obviously proves the lemma.

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