Commun. Math. Phys. 151, 245–262 (1993)

Communications in
Mathematical
Physics

Topological Field Theory and Rational Curves

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Received October 24, 1991

Abstract. We analyze the quantum field theory corresponding to a string propagating on a Calabi–Yau threefold. This theory naturally leads to the consideration of Witten's topological non-linear σ-model and the structure of rational curves on the Calabi–Yau manifold. We study in detail the case of the world-sheet of the string being mapped to a multiple cover of an isolated rational curve and we show that a natural compactification of the moduli space of such a multiple cover leads to a formula in agreement with a conjecture by Candelas, de la Ossa, Green and Parkes.

1. Introduction

In its most fundamental form, string theory is normally considered as a loop of string propagating through space-time to sweep out a two-dimensional world-sheet. This map of the world-sheet into space-time allows one to "pull back" physics from the familiar space-time around us into a more simple two-dimensional quantum field theory. This model is usually cast in the form of the non-linear σ-model where the non-linearity of this field theory arises from curvature in the target space. Thus it is a simple matter to solve string theory in flat space-time, but more general curved target spaces can only be solved perturbatively assuming the curvature is small in some sense.

When one specializes to the case of requiring space-time supersymmetry one picks out a specific class of allowed target spaces in the above approximation, namely that the target space should have vanishing Ricci curvature and be a complex Kähler manifold. When building realistic models of physics one is naturally led in this situation to considering Calabi–Yau threefolds [1, 2].

This model might have been of interest only to superstring phenomenologists if it weren't for the fact that there is an alternative to this method of solution. That is, one can take an algebraic approach to string theory as a conformal field theory [3, 4]. This process does not involve the approximations required in the non-linear σ-model. The analogue of the Calabi–Yau condition in this case is that the conformal field theory should be an $N = 2$ superconformal field theory. At first it was naturally assumed that this provided a more general class of solutions to the superstring than Calabi–Yau's ever could, but it is now generally believed from evidence such as [5] that these two approaches are in some way equivalent although this equivalence can only be manifest in some regions of the moduli space of the theories considered [6].

This equivalence should be a considerable source of new results as the knowledge to date about non-linear σ-models and $N = 2$ superconformal field theories is quite different. Thus, for example, a great deal of knowledge about the classification of superconformal field theories should be obtained from the classification of Calabi–Yau manifolds which is better understood. Going the other way, the mirror symmetry which is trivial in the language of conformal field theories has led to a new symmetry hitherto unsuspected in algebraic geometry [7, 8]. Combining the knowledge of the non-linear σ-model with the mirror symmetry has also allowed further analysis of the structure of a particular Calabi–Yau and has allowed information about the rational curves on this manifold to be obtained in an elegant way [9]. It is the purpose of this paper to analyze the link between the rational curves on a Calabi–Yau manifold and another quantum field theory closely related to the superconformal field theory, namely the topological quantum field theory.

The topological quantum field theory was first introduced in the context of Yang–Mills theory by Witten [10] to provide a model of Donaldson's diffeomorphism invariants for 4-manifolds [11, 12]. Witten then introduced a topological version of the non-linear σ-model [13] which will be our main object of study in this paper. This will allow us to build diffeomorphism invariant polynomials with integer coefficients for Calabi–Yau manifolds. The integer coefficients are related to the number of rational curves within a certain class (a generalization of degree) when this number is finite.

In Sect. 2 we study the $N = 2$ non-linear σ-model and show how it is related to the topological σ-model. In Sect. 3 we begin the calculation of the integer coefficients reducing it to a problem in algebraic geometry. In Sect. 4 we perform this calculation and finally in Sect. 5 we combine the results.

2. Reduction to Topological Field Theory

Let us consider a map of a Riemann surface Σ into a manifold X

$$\phi \colon \Sigma \to X \,. \tag{1}$$

Putting coordinates u^μ on X and σ^α on Σ we can locally consider this map to be determined by functions $u(\sigma)$ on Σ. The bosonic string action (see for example [2]) then associates the following non-linear σ-model to this:

$$S = t \int d^2\sigma (\eta^{\alpha\beta} g_{\mu\nu} \partial_\alpha u^\mu \partial_\beta u^\nu + \varepsilon^{\alpha\beta} B_{\mu\nu} \partial_\alpha u^\mu \partial_\beta u^\nu) \,, \tag{2}$$

where we have imposed a flat (at least locally) metric $\eta^{\alpha\beta}$ on Σ and $\varepsilon^{\alpha\beta}$ is the antisymmetric tensor. The symmetric field $g_{\mu\nu}$ is a function of u^μ and represents the metric on X and there is an antisymmetric field $B_{\mu\nu}$, also a function of u^μ. The field $B_{\mu\nu}$ does not have as clear a geometric interpretation as does $g_{\mu\nu}$. The real parameter, t, can be thought of as representing the string tension or, equivalently, it can be absorbed into $g_{\mu\nu}$ and $B_{\mu\nu}$ and so used to control the volume of X.

If we now impose $N = 2$ supersymmetry we are naturally led to the condition that X is a complex Kähler manifold [14]. We will assume that this manifold has $h^{2,0} = 0$ so that all closed 2-forms are (1, 1)-forms (up to addition of exact forms). Replacing σ^α with the single complex coordinate z and replacing the real index structure of u^μ with a Kähler index structure we obtain the bosonic part of the action from (2),

$$S = \frac{t}{2} \int \{ g_{i\bar{j}}(\partial u^i \bar{\partial} u^{\bar{j}} + \bar{\partial} u^i \partial u^{\bar{j}}) + iB_{i\bar{j}}(\partial u^i \bar{\partial} u^{\bar{j}} - \bar{\partial} u^i \partial u^{\bar{j}}) \} d^2 z \,. \tag{3}$$

We can split this as follows

$$\begin{aligned} S &= t \int g_{i\bar{j}} \bar{\partial} u^i \partial u^{\bar{j}} d^2 z + \frac{t}{2} \int (g_{i\bar{j}} + iB_{i\bar{j}})(\partial u^i \bar{\partial} u^{\bar{j}} - \bar{\partial} u^i \partial u^{\bar{j}}) d^2 z \\ &= t \int g_{i\bar{j}} \bar{\partial} u^i \partial u^{\bar{j}} d^2 z + \frac{t}{2} \int_{\Sigma} (\phi^* J) \,, \end{aligned} \tag{4}$$

where $\phi^* J$ is the pullback from X to Σ of a (1, 1)-form whose real part is the usual Kähler-form on X.

Let us now add in fermions (i.e., anticommuting operators) χ^i and ρ^i and another bosonic field F^i along with their conjugates, $\chi^{\bar{i}}$, $\rho^{\bar{i}}$ and $F^{\bar{i}}$, to obtain an explicitly $N = 2$ supersymmetric action. The above field theory splits naturally into holomorphic (left-moving) and anti-holomorphic (right-moving) pieces so that we can obtain 2 sets of $N = 2$ supersymmetries generated by the infinitesimal parameters ε_n, $\bar{\varepsilon}_n$. We have

$$\begin{aligned}
&\delta_{\varepsilon_1} u^i = i\varepsilon_1 \chi^i \,, && \delta_{\bar{\varepsilon}_1} u^i = 0 \,, \\
&\delta_{\varepsilon_1} u^{\bar{i}} = 0 \,, && \delta_{\bar{\varepsilon}_1} u^{\bar{i}} = i\bar{\varepsilon}_1 \chi^{\bar{i}} \,, \\
&\delta_{\varepsilon_1} \chi^i = 0 \,, && \delta_{\bar{\varepsilon}_1} \chi^i = 0 \,, \\
&\delta_{\varepsilon_1} \chi^{\bar{i}} = 0 \,, && \delta_{\bar{\varepsilon}_1} \chi^{\bar{i}} = 0 \,, \\
&\delta_{\varepsilon_1} \rho^i = i\varepsilon_1 F^i \,, && \delta_{\bar{\varepsilon}_1} \rho^i = 2\bar{\varepsilon}_1 \bar{\partial} u^i \,, \\
&\delta_{\varepsilon_1} \rho^{\bar{i}} = 2\varepsilon_1 \partial u^{\bar{i}} \,, && \delta_{\bar{\varepsilon}_1} \rho^{\bar{i}} = i\bar{\varepsilon}_1 F^{\bar{i}} \,, \\
&\delta_{\varepsilon_1} F^i = 0 \,, && \delta_{\bar{\varepsilon}_1} F^i = -2\bar{\varepsilon}_1 \bar{\partial} \chi^i \,, \\
&\delta_{\varepsilon_1} F^{\bar{i}} = -2\varepsilon_1 \partial \chi^{\bar{i}} \,, && \delta_{\bar{\varepsilon}_1} F^{\bar{i}} = 0 \,,
\end{aligned} \tag{5}$$

and

$$\begin{aligned}
&\delta_{\varepsilon_2} u^i = i\varepsilon_2 \rho^i \,, && \delta_{\bar{\varepsilon}_2} u^i = 0 \,, \\
&\delta_{\varepsilon_2} u^{\bar{i}} = 0 \,, && \delta_{\bar{\varepsilon}_2} u^{\bar{i}} = i\bar{\varepsilon}_2 \rho^{\bar{i}} \,, \\
&\delta_{\varepsilon_2} \chi^i = -i\varepsilon_2 F^i \,, && \delta_{\bar{\varepsilon}_2} \chi^i = 2\bar{\varepsilon}_2 \partial u^i \,, \\
&\delta_{\varepsilon_2} \chi^{\bar{i}} = 2\varepsilon_2 \bar{\partial} u^{\bar{i}} \,, && \delta_{\bar{\varepsilon}_2} \chi^{\bar{i}} = -i\bar{\varepsilon}_2 F^{\bar{i}} \,, \\
&\delta_{\varepsilon_2} \rho^i = 0 \,, && \delta_{\bar{\varepsilon}_2} \rho^i = 0 \,, \\
&\delta_{\varepsilon_2} \rho^{\bar{i}} = 0 \,, && \delta_{\bar{\varepsilon}_2} \rho^{\bar{i}} = 0 \,, \\
&\delta_{\varepsilon_2} F^i = 0 \,, && \delta_{\bar{\varepsilon}_2} F^i = 2\bar{\varepsilon}_2 \partial \rho^i \,, \\
&\delta_{\varepsilon_2} F^{\bar{i}} = 2\varepsilon_2 \bar{\partial} \rho^{\bar{i}} \,, && \delta_{\bar{\varepsilon}_2} F^{\bar{i}} = 0 \,.
\end{aligned} \tag{6}$$

We can now write down the $N = 2$ supersymmetric form of (4) with the fermions χ and ρ. The fields F^i can be determined through the equations of motion to yield

$$F^i = -\Gamma^i_{jk}\chi^j\rho^k, \quad F^{\bar{\imath}} = -\Gamma^{\bar{\imath}}_{\bar{j}\bar{k}}\chi^{\bar{j}}\rho^{\bar{k}}, \tag{7}$$

where Γ^i_{jk} is the usual connection on a Kähler manifold, $g^{i\bar{m}}\partial_j g_{k\bar{m}}$ (it does not depend on $B_{i\bar{j}}$). Eliminating the F^i fields we have

$$S = t\int\left\{g_{i\bar{j}}\bar{\partial}u^i\partial u^{\bar{j}} - \frac{i}{2}g_{i\bar{j}}\rho^i D\chi^{\bar{j}} - \frac{i}{2}g_{i\bar{j}}\rho^{\bar{j}}\bar{D}\chi^i - \frac{1}{4}R_{i\bar{\imath}j\bar{j}}\chi^i\chi^{\bar{\imath}}\rho^j\rho^{\bar{j}}\right\}d^2z$$

$$+\frac{t}{2}\int_{\Sigma}(\phi^*J)\,. \tag{8}$$

In this equation, D is the covariant derivative, e.g., $\bar{D}\chi^i = \bar{\partial}\chi^i + \Gamma^i_{jk}\bar{\partial}u^j\chi^k$ and $R_{i\bar{\imath}j\bar{j}} = \partial_i\partial_{\bar{j}}g_{j\bar{\imath}} - g_{m\bar{m}}\Gamma^m_{ij}\Gamma^{\bar{m}}_{\bar{\imath}\bar{j}}$. The above action is invariant (the integrand is unchanged up to total derivatives) under (5) and (6) if $g_{i\bar{j}}$ is a Kähler metric and J is a closed 2-form.

Consider the last term in (8) which we shall refer to as S_c. The first thing one should notice is that this term has not changed from (4), that is, it has the rather strange property of being supersymmetric even though it does not explicitly contain any fermions. The S_c term is also a topological invariant, i.e., it depends only on the cohomology class of J and the homology class of the image of Σ in X. Under small continuous variations of the field (u^i, χ^i, ρ^i) S_c does not change. Thus we can say that, in some sense, this term has no bosonic degrees of freedom and no fermionic degrees of freedom. This does not stop this term from being of some importance however.

The first integral, S_t, in (8) does not have this property. That is, it is not topological in the above sense. It is, however, topological in another sense. It is the precise action (except that holomorphic and antiholomorphic maps are interchanged) that appears as a topological non-linear σ-model in [13]. Also the part of the $N = 2$ supersymmetry given by (5) is the BRST symmetry used in that theory. Note that we have to reinterpret the roles of the fields χ and ρ when going between the language of $N = 2$ theories and topological theories. That is, we introduced χ and ρ as world-sheet spinors but to interpret S_t as a topological field theory we consider χ^i to be a 0-form and ρ^i to be a (0, 1)-form on Σ. This reinterpretation is usually referred to as "twisting" the $N = 2$ theory.

For our purposes there are two ingredients in establishing a topological quantum field theory. Firstly one must identify a BRST symmetry, Q. This automatically exists in an $N = 2$ theory and is provided by (5). Secondly, the action is itself BRST-exact (i.e., it is the BRST variation of something). This is true of S_t but not of S_c:

$$\delta_{\varepsilon_1}\delta_{\bar{\varepsilon}_1}\left\{-\frac{t}{4}\int g_{i\bar{j}}\rho^i\rho^{\bar{j}}d^2z\right\} = \varepsilon_1\bar{\varepsilon}_1 S_t\,. \tag{9}$$

A theory with these two properties will have correlation functions which are integers (with the observables in a suitable basis) and does not depend on any continuous parameters in the theory, i.e., the n-point functions are diffeomorphism invariant [10]. Thus our action for an $N = 2$ non-linear σ-model does not have any interesting topological properties as a whole but it splits into two pieces S_c and S_t which each have special properties.

Consider calculating a 3-point function between 3 observables $\mathfrak{A}_a$,

$$f_{abc} = \langle \mathfrak{A}_a \mathfrak{A}_b \mathfrak{A}_c \rangle = \int \mathfrak{A}_a \mathfrak{A}_b \mathfrak{A}_c e^{-S} \mathscr{D}u \mathscr{D}\chi \mathscr{D}\rho \ . \tag{10}$$

Because S_c is invariant under smooth continuous deformations of the fields, it is simplest to consider this path integral as a discrete sum over the homotopy classes of the map ϕ. This is the usual instanton calculation. For classes v we have

$$f_{abc} = \sum_v f_{abc}^{(v)} , \tag{11}$$

where

$$f_{abc}^{(v)} = \int_v \mathfrak{A}_a \mathfrak{A}_b \mathfrak{A}_c e^{-S} \mathscr{D}u \mathscr{D}\chi \mathscr{D}\rho = e^{-S_c(v)} \int_v \mathfrak{A}_a \mathfrak{A}_b \mathfrak{A}_c e^{-S_t} \mathscr{D}u \mathscr{D}\chi \mathscr{D}\rho \ . \tag{12}$$

This shows that the only path integral we need perform is within the topological field theory. Strictly speaking, we have a different stress-energy tensor between the $N = 2$ superconformal field theory and the topological model but this does not affect the calculation for 3-point functions [15].

As usual in instanton calculations, we take the approximation of expanding around the classical solutions of the theory. For topological field theories this is not an approximation at all since we know that changing continuous parameters in the theory such as t in (8) have no effect on the correlation functions. Thus we can take the $t \to \infty$ limit suppressing all but the $S_t = 0$ contributions to the path integral. The whole theory is not invariant however and so one should be careful to realize that we are always assuming that we are in some large t, i.e., large radius limit in the following calculations. The topological action is of the form

$$S_t = t \int d^2 z \{ \| \bar{\partial} u^i \|^2 + \text{fermions} \} , \tag{13}$$

and so we see that the instantons are given by the *holomorphic* maps $u^i(z)$. This leads to three possibilities for the map ϕ:

(1) $\phi(\Sigma)$ is a point in X,
(2) $\phi(\Sigma)$ is an algebraic curve in X,
(3) $\phi(\Sigma)$ is a multiple cover of an algebraic curve in X.

We shall often refer to all of the above cases as n-fold covers of algebraic curves with $n = 0$ for case (1).[1]

Let e_a, $a = 1, \ldots, h^{1,1}$ be a set of integral generators of $H^{1,1}(X)$. We therefore have

$$[J] = 2 \sum_{a=1}^{h^{1,1}} c^a [e_a] , \tag{14}$$

where c^a are complex numbers. It follows that

$$S_c = t \sum_{a=1}^{h^{1,1}} c^a m_a , \tag{15}$$

[1] The target point does not need to be contained in such a curve in X in this case

where m_a are integers given by the homology class of $\phi(\Sigma)$. The case $n = 0$ comprises of all m_a vanishing. Let us introduce a new set of variables

$$q_a = e^{-tc^a}, \tag{16}$$

so that

$$e^{-S_c} = q_1^{m_1} q_2^{m_2} \ldots q_{h^{1,1}}^{m_{h^{1,1}}}. \tag{17}$$

The observables, $\mathfrak{A}_a$, in the topological field theory correspond to cohomology classes on X [13]. The $N = 2$ superconformal field theory has 2 sets of fields that naturally correspond to the (1, 1)-forms and the (2, 1)-forms on X. When an $N = 2$ theory is twisted to form a topological field theory, the process of introducing a BRST symmetry forces us to project out one of these sets of fields. That is, observables must be BRST-closed and this can only be achieved for one of the classes of fields at a time. In this case, we are left with only the (1, 1)-forms as observables. One can twist the $N = 2$ model in a different way however and be left with only the (2, 1)-forms. We will have more to say about this later. We can take, as observables, the generators e_a considered as elements of $H^{1,1}(X)$ or, equivalently by Poincaré duality, as algebraic surfaces within X to form 3-point functions f_{abc} which will have the general form

$$f_{abc} = N^0_{abc} + N^1_{abc} q_1 + N^2_{abc} q_2 + \cdots + N^{\cdots}_{abc} q_1^2 q_2 + \cdots . \tag{18}$$

That is, these f_{abc} functions can be expressed as polynomials in q_i. The coefficients of the polynomial, N^v_{abc}, are the 3-point functions from the topological field theory and, as we will see in the next section, these numbers are intersection numbers of cycles in a moduli space and, in particular, are integers.

Thus we see that to each X, the non-linear σ-model associates a set of polynomials f_{abc} in $h^{1,1}$ variables, q_i. The $t \to \infty$ limit, implicit in calculating f_{abc}, corresponds to limit $q_i \to 0$. If we were to calculate the 3-point functions for the purely topological field theory, i.e., if we drop the S_c term from the action, we would lose the q_i dependence of f_{abc} (as one would expect for a topological field theory). In fact, dropping the S_c term corresponds to setting $q_i = 1$ which in turn is like a "zero radius limit" of (18), that is, we shrink X down to zero so that $J = 0$ in (8). We thus have the rather curious result that the topological field theory (or, at least its 3-point functions) related to the usual string action is obtained by taking the large radius approximation of the non-linear σ-model and then putting the radius equal to zero!

It is important to note that we will not explicitly use any information about the complex structure of X in deriving f_{abc}. These functions depend explicitly only on the cohomology class of J. The f_{abc} are thus local diffeomorphism invariant objects of X although they could conceivably depend on global changes in complex structure in much the same way as Donaldson's polynomials [12].

In the large radius limit, f_{abc} is dominated by the N^0 term in (18). This integer, corresponding to the case where $\phi(\Sigma)$ is a point, is given by the intersection number of the 4-cycles e_a, e_b, e_c within the threefold X [15]. It is useful to think of the functions f_{abc} as generalizations of the intersection form to contain more diffeomorphism invariant information.

Having reduced the functions f_{abc} to polynomials with integer coefficients we will now go on to analyze the precise meaning of these integers as regards the geometry of X.

3. Reduction to Algebraic Geometry

We need to compute

$$N_{abc}^{v} = \int_{v} e_a e_b e_c \exp(-S_t) \mathscr{D}u \mathscr{D}\chi \mathscr{D}\rho \ , \tag{19}$$

where

$$S_t = t \int \left\{ g_{i\bar{j}} \bar{\partial} u^i \partial u^{\bar{j}} - \frac{i}{2} g_{i\bar{j}} \rho^i D \chi^{\bar{j}} - \frac{i}{2} g_{i\bar{j}} \rho^{\bar{j}} \bar{D} \chi^i - \frac{1}{4} R_{i\bar{i}j\bar{j}} \chi^i \chi^{\bar{i}} \rho^j \rho^{\bar{j}} \right\} d^2 z \ . \tag{20}$$

The index structure of the fields χ^i and ρ^i should be interpreted as these fields taking values in $R = \phi^* T_X$, the pullback of the holomorphic tangent bundle on X, with the barred quantities taking values in the pullback of the antiholomorphic tangent bundle. The field u^i however takes values in X itself rather than the tangent bundle. Any small variation, δu^i, can be considered as a deformation of X and so this field does take values in the pullback of the tangent bundle. In particular, it will be important to note that the field $\bar{\partial} u^i$ is an R-valued (0, 1)-form.

The integral (19) was effectively solved some time ago for the case where $\phi(\Sigma)$ is a point [16] and when $\phi(\Sigma)$ is a rational curve [17]. In the latter case the idea is that one assumes the $t \to \infty$ limit and so restricts the integral to the case $\bar{\partial} u^i = 0$, i.e., the holomorphic maps. One interprets the path integral as being over a *superspace* parametrized by (u^i, χ^i, ρ^i). The $\bar{\partial} u^i = 0$ constraint restricts this to the moduli superspace comprising of the moduli space of holomorphic maps in question and the fermion zero modes. To yield the value of (19) one integrates over this moduli superspace.

The BRST invariance gives a grading (a BRST "charge") to the fields in the theory. The charges can be assigned as follows; $q(u^i) = 0, q(\chi^i) = 1, q(\rho^i) = -1$ and $q(e_a) = 1$. The path integration measure also has a BRST charge which comes from the fermion zero modes. Each χ^i zero mode contributes -1 and each ρ^i zero mode contributes $+1$. It is also known that the fermion zero modes correspond to harmonic forms on Σ. Therefore

$$\begin{aligned} \# \text{ of } \chi \text{ zero modes} &= \dim H^0(R) \ , \\ \# \text{ of } \rho \text{ zero modes} &= \dim H^1(R) \ . \end{aligned} \tag{21}$$

The 3-point function (19) vanishes unless the BRST charges cancel. This leads to the constraint

$$\dim H^0(R) - \dim H^1(R) = 3 \ . \tag{22}$$

The left-hand side of this equation is the index of the R-twisted Dolbeault complex on Σ and can easily be computed (see [18]) by the index theorem (or the Hirzebruch–Riemann–Roch theorem):

$$\begin{aligned} \dim H^0(R) - \dim H^1(R) &= \int_{\Sigma} \mathrm{ch}(R) \cdot \mathrm{td}(T_\Sigma) \\ &= (1-g) \dim X - \deg(\phi^* K_X) \ , \end{aligned} \tag{23}$$

where g is the genus of Σ and K_X is the canonical divisor of X. Thus we have the result that for a Calabi–Yau threefold, we will generically have non-zero values for f_{abc}^{v} for all classes of maps v of genus 0 curves Σ into X.

Let us now specialize to the case where X is a Calabi–Yau threefold and we will specify v as being a class of maps from Σ, a genus 0 Riemann surface, to give an n-fold cover of an isolated rational curve, C, in X. We will assume that the curve is of type $(-1, -1)$, i.e.,

$$T_X|_C \cong \mathcal{O}_C(2) \oplus \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)\,. \tag{24}$$

Thus

$$\dim H^0(R) = \dim H^0(\mathcal{O}(2n) \oplus \mathcal{O}(-n) \oplus \mathcal{O}(-n)) = 2n+1 \tag{25}$$

and

$$\begin{aligned}\dim H^1(R) &= \dim H^0(K_\Sigma \otimes R^{-1})\\ &= \dim H^0(\mathcal{O}(-2-2n) \oplus \mathcal{O}(n-2) \oplus \mathcal{O}(n-2)) = 2n-2\,.\end{aligned} \tag{26}$$

If $n = 1$, life is comparatively simple. There are no ρ zero modes and three χ zero modes. Putting these into (19), the three operators for the observables e_a "soak up" the χ zero modes and we can do the integral over the moduli space $Sl(2, \mathbb{C})$ without considering the $\exp(-S_t)$ term that appears in (19). (See [19] for the explicit calculation.) When $n > 1$, things become more messy. We now have ρ zero modes and an equal number of extra χ zero modes. These cannot all be soaked up by the observables. The integral is not zero however since we can expand out the $\exp(-S_t)$ term to introduce some $R_{i\bar{i}j\bar{j}}\chi^i\chi^{\bar{i}}\rho^j\rho^{\bar{j}}$ terms into the action. These terms soak up the extra fermion zero modes that have appeared to yield a generically non-zero answer. Such a calculation will be, in general, rather difficult.

There is a much more elegant approach however exploiting the fact that we are dealing with a topological field theory. A topological field theory is based on an action which has the general form

$$S_t = t\int d^2z\,\{\|s\|^2 + \text{fermions}\}\,, \tag{27}$$

where s is a section of a bundle $\mathcal{W}$ over some moduli space, $\mathcal{M}$, of all field configurations. When calculating a path integral containing the above action, a topological field theory has the property that such an integral will not depend on the parameter t. One can thus take the limit $t \to \infty$ so that the path integral only contains contributions from the zero locus of s. What's more, the fermionic part of the action is arranged so that the determinants arising from the integral over the fermions cause each component of the zero locus to contribute $+1$ or -1 to the path integral (according to orientation). See [10] for an explanation of how this happens. In this way, the path integral computes the *Euler class* of the bundle $\mathcal{W}$. If $\dim\mathcal{W} = \dim\mathcal{M}$, the Euler class is an integer. If $\dim\mathcal{W} < \dim\mathcal{M}$, the Euler class can be thought of as an homology cycle within $\mathcal{M}$. To obtain integers in this case, one can insert cohomology classes (observables) into the path integral which leads to intersection numbers on this cycle [10, 13].

It is instructive to do this calculation for the case of the single cover of an isolated rational curve [15]. The zero section of s corresponds to the moduli space, M, of holomorphic maps of a $\mathbb{P}^1$ into this curve, or equivalently, reparametrizations of $\mathbb{P}^1$. This space is well-known to be $Sl(2, \mathbb{C})$. To do intersection theory, we should really have a compact space, $\bar{M}$, for the instanton moduli space. $Sl(2, \mathbb{C})$

naturally compactifies to $\mathbb{P}^3$. We can choose the coordinates u^i to obtain the map function

$$u^3(z) = \frac{az + b}{cz + d} \tag{28}$$

and $u^1(z) = u^2(z) = 0$. The complex parameters in (28) can be chosen to satisfy $ad - bc = 1$. This space naturally compactifies to $\mathbb{P}^3$ by considering the coordinates $[a, b, c, d]$ as homogeneous. The space that we add in to $Sl(2, \mathbb{C})$ to form $\mathbb{P}^3$ (i.e., the *compactification divisor*) is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and can be thought of as the algebraic variety defined by $ad - bc = 0$.

To calculate (19) we need to represent the e_a as homology classes within the compactified instanton moduli space, $\bar{M}$. This can be achieved as follows [15]. Each e_a is associated with a 4-cycle in X or, by Poincaré duality, by a (1, 1)-form whose support lies in an arbitrary small neighborhood around this cycle. Thus any map which does not take a point of Σ to this cycle gives an arbitrarily small contribution to the path integral. Using the same argument about the fermionic determinants as we used above to obtain the Euler class, we can obtain the following result:

$$N^v_{abc} = \#(L_a \cap L_b \cap L_c) , \tag{29}$$

where L_a is the subspace of $\bar{M}$ given by maps that take a particular point on Σ to a point within the 4-cycle associated to e_a. Which particular point in Σ is chosen for each e_a does not affect the above calculation because of the properties of topological field theories.

For the case we are considering we get a contribution to L_a for each point of intersection between C and e_a. Each such point clearly puts a linear constraint on the homogeneous coordinates $[a, b, c, d]$ and thus gives a hyperplane in $\bar{M}$. Three hyperplanes in $\mathbb{P}^3$ intersect at one point and so we reproduce the result

$$N^{v(n=1)}_{abc} = \#(e_a \cap C) \cdot \#(e_b \cap C) \cdot \#(e_c \cap C) . \tag{30}$$

Now let us try to perform the same calculation for a multiple cover of C. The additional complexity of this problem is caused by ρ^i zero modes. The solution to this problem is provided by much the same method as used in [20] where similar "antighost" zero modes also appeared. To understand more clearly what the relationship between the path integral and the Euler class is, we should consider the approach of [21].

Consider a vector bundle $\mathscr{W} \to \mathscr{M}$. We will refer to $\mathscr{M}$ as the horizontal direction of the bundle and the fibre as the vertical direction. We can consider the cohomology of this space by introducing differential forms with compact support in the vertical direction, i.e., each form vanishes outside a compact region of the fibre. It is a general result of algebraic topology (see, for example [22]) that there is a ($\dim \mathscr{W}$)-form in this class, called the *Thom class* which can be pulled back via the zero section to the Euler class on $\mathscr{M}$. In the case we are considering we have a bundle of infinite dimension. In this situation it is more convenient to introduce the Mathai–Quillen form [23],

$$\omega_s = s^* U , \tag{31}$$

where s is a section of $\mathscr{W}$ and U is a slight variant of the Thom class. One replaces the condition for compact vertical support with the constraint that the form should decay at least as fast as $\exp(-x^2)$ along the fibre. If s is the zero section we effectively reproduce the above argument whereas we can represent the path integral idea if we take s to be some $t \to \infty$ limit of a generic section.

The moduli space $\mathscr{M}$ in our theory is the space of *all* maps $\Sigma \to X$. It is convenient to assume that the maps under consideration have derivatives in L^p; then using the Sobolev norm one can give $\mathscr{M}$ the structure of a Banach manifold with an almost-complex structure [24, 25]. With this structure, the holomorphic tangent space to $\mathscr{M}$ at ϕ is the space $T_{\mathscr{M},\phi} := H^0_{(p)}(\Sigma, \phi^* T_X)$ of L^p-sections of the bundle $R = \phi^* T_X$. These spaces fit together to form the holomorphic tangent bundle $T_{\mathscr{M}} \to \mathscr{M}$, which is a Banach bundle over $\mathscr{M}$.

For any map ϕ: $\Sigma \to X$, the derivative $d\phi$ gives a map between tangent bundles $T_\Sigma \to T_X$, which can be thought of as a map $T_\Sigma \to \phi^* T_X$. If we let $\mathscr{A}^1_\Sigma$ denote the 1-forms on Σ, $d\phi$ can also be regarded a section of $\mathscr{A}^1_\Sigma \otimes \phi^* T_X$ (which would determine a map from T_Σ by evaluating the 1-form on the tangent vector). Taking the (0, 1)-part of $d\phi$, we get $\bar{\partial}\phi$, which can be interpreted as a section of $\mathscr{A}^{0,1}_\Sigma \otimes \phi^* T_X$ over Σ, that is, as an element of the space $\mathscr{W}_\phi := H^0_{(p)}(\Sigma, \mathscr{A}^{0,1}_\Sigma \otimes \phi^* T_X)$. The spaces $\mathscr{W}_\phi$ fit together to form another Banach bundle $\mathscr{W} \to \mathscr{M}$. This bundle comes equipped with the natural section

$$s\colon \phi \mapsto \bar{\partial}\phi\,. \tag{32}$$

This is the section s which appears in the action (27), and whose zero locus M is the space of holomorphic maps.

We can also consider the following more heuristic approach to the bundle $\mathscr{W}$ in terms of local coordinates. The fermions in the path integral can be taken to represent differential forms on $\mathscr{W}$.[2] To be more precise, the "anti-ghost" fermions ρ^i represent a basis of 1-forms in the vertical direction and the "ghost" fermions χ^i represent a basis of 1-forms in the horizontal direction. The following diagram should be born in mind:

$$\begin{array}{ccc} u^i & \xrightarrow{\bar{\partial}} & \bar{\partial}u^i \\ \delta\downarrow & & \downarrow\delta \\ \delta u^i & \xrightarrow{\bar{D}} & \bar{D}\delta u^i \end{array}\,. \tag{33}$$

The bottom row of this diagram can be thought of as 1-forms with δ acting as some kind of de Rham d-operator. The left-hand side can be thought of as the horizontal part of the bundle and the right-hand side the vertical part. The correspondence between δu^i and χ^i is central to topological quantum field theory and is what allows us to represent BRST-observables as cohomology classes in the moduli space [10]. In the vertical direction the fibre should be thought of as $\mathbb{R}^\infty$ and so the cotangent bundle in this direction is isomorphic to the fibre itself. Thus the ρ^i fields can be interpreted as a basis for the fibre. When we perform the path integral (19) we can interpret this as taking the Euler class of the vector bundle whose fibre is spanned by the ρ^i fields. That is, the vector bundle in question is the bundle of (0, 1)-forms on Σ taking values in the pullback of the tangent bundle on X.

[2] In topological Yang–Mills there are three sets of fermions. The extra set arises because of gauge invariance and is not relevant here

The moduli space $\mathscr{M}$ has been closely studied by Gromov [26]. (In fact, Gromov's work provided one of the inspirations for Witten's construction of topological σ-models [13].) We follow the approaches of McDuff [24] and Wolfson [27] to Gromov's ideas. Gromov showed that $\mathscr{M}$ has a natural compactification to a space $\bar{\mathscr{M}}$ which includes maps from "simple cusp-curves" $\Sigma_0 \cup \bigcup_{\alpha=1}^{k} \mathbb{P}^1$ to X as limits of maps from $\Sigma \to X$. In the case at hand (multiple covers of an isolated rational curve C), these maps from cusp-curves can be most easily understood by means of their graphs. A sequence of graphs $\Gamma_{\phi_j} \subset \Sigma \times X$ of maps ϕ_j: $\Sigma \to X$ can converge to a curve in $\Sigma \times X$ which is *not* a graph. Such a limiting graph Γ_ϕ will have one component Σ_0 which is the graph of a map of lower degree (possibly even degree 0), and other $\mathbb{P}^1$-components which map to points in X.

If we extend $\mathscr{W}$ to a bundle $\bar{\mathscr{W}}$ over the compactified space $\bar{\mathscr{M}}$ by continuing to use the spaces $\mathscr{W}_\phi := H^0_{(p)}(\Sigma, \mathscr{A}^{0,1}_\Sigma \otimes \phi^* T_X)$ as fibres even in the case of limiting graphs, then the limiting maps ϕ will still have $\bar{\partial}\phi \in \mathscr{W}_\phi$. Thus, the section s extends to a section $\bar{s}$ of $\bar{\mathscr{W}} \to \bar{\mathscr{M}}$.

We now attempt to calculate the Euler class by pulling back the Mathai-Quillen form by our section $\bar{s}$ of $\bar{\mathscr{W}} \to \bar{\mathscr{M}}$. For this process to work correctly, we are required to take a *generic* section of this bundle. Unfortunately, our section $\bar{s}$ is usually not generic. Fortunately, there is a method [20] which can be used to calculate the Euler class from a non-generic section. (We have formally extended this method from the finite-dimensional to the infinite-dimensional case, ignoring convergence questions.)

If we vary the map ϕ by a displacement $\delta\phi$ (or in local coordinates vary u^i by δu^i as in (33)), we find that $\delta\phi \in T_{\mathscr{M},\phi} = H^0_{(p)}(\Sigma, \phi^* T_X)$. Thus, the variation of $\bar{\partial}\phi$ is given by

$$\bar{D}\delta\phi \in \bar{D} H^0_{(p)}(\Sigma, \phi^* T_X) . \tag{34}$$

If we regard $\bar{D}$ as defining a map of bundles

$$\bar{D} \colon T_{\bar{\mathscr{M}}} \to \bar{\mathscr{W}} \tag{35}$$

(a linearization of the section $\bar{\partial}$), we find that the displacements of $\bar{\partial}\phi$ all lie in the image bundle $\mathscr{W}' = \operatorname{Image}(\bar{D})$.

Following [20] we can use this information to calculate the correct Euler class of $\bar{\mathscr{W}}$. We have the following exact sequence:

$$0 \to \mathscr{W}' \xrightarrow{i} \bar{\mathscr{W}} \to \mathscr{V} \to 0 . \tag{36}$$

The cokernel $\mathscr{V}$ is a bundle whose fibres correspond to $H^1(R)$. This bundle has finite dimension $2n-2$. The zero locus of a generic section of $\bar{\mathscr{W}}$ can be taken as the zero locus of a generic section of $\mathscr{V}$ restricted to the zero locus of a generic section of $\mathscr{W}'$. Translating this into a statement about Euler classes, we can do the path integral (19) by considering the moduli space of holomorphic maps (the Euler class of $\mathscr{W}'$) and including the cohomology class corresponding to the Euler class of $\mathscr{V}$ (the top Chern class of $\mathscr{V}$) in the integrand. That is

$$\begin{aligned} N^v_{abc} &= \int_{M_v} e_a e_b e_c \cdot c_{2n-2}(H^1(R)) \\ &= \#(L_a \cap L_b \cap L_c \cap U) , \end{aligned} \tag{37}$$

where the e_a's are interpreted as (1, 1)-forms on M_v and U is a 6-cycle corresponding to the Poincaré dual of the Euler class of $\mathscr{V}$.

The appearance of the top Chern class in (37) can be viewed as the effect of integrating out the ρ^i zero modes by using the four-fermion term in the action (20). This process will bring powers of the curvature tensor into the action and this should correspond to the usual de Rham representation of Chern classes by powers of the curvature.

We have reduced the problem of finding the Euler class of an infinite dimensional vector bundle, which is best done by path integral techniques, to the problem of finding the Euler class of a finite dimensional vector bundle. This can be done by more conventional methods in algebraic geometry.

4. A Bundle Calculation

Let X be a Calabi–Yau threefold, and let $C \subset X$ be an isolated smooth rational curve such that $T_X|_C \cong \mathcal{O}_C(2) \oplus \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)$. Consider the moduli space

$$M_n(C) = \{\phi\colon \mathbb{P}^1 \to X \mid \phi(\mathbb{P}^1) = C, \deg\phi = n\} \tag{38}$$

of parametrized maps from $\Sigma = \mathbb{P}^1$ to X. For each $\phi \in M_n(C)$, the vector space $H^0(\phi^*(T_X))$ is the tangent space at ϕ to this moduli problem,[3] while the vector space $H^1(\phi^*(T_X))$ is the obstruction space (also at ϕ). The "virtual dimension" of the moduli space is therefore

$$\dim H^0(\phi^*(T_X)) - \dim H^1(\phi^*(T_X)) = \chi(\phi^*(T_X)) = 3\ . \tag{39}$$

Motivated by Gromov's work [26] as described in the previous section, we compactify $M_n(C)$ by using graphs. Associate to each $\phi \in M_n(C)$ the graph

$$\Gamma_\phi \subset \mathbb{P}^1 \times C\ . \tag{40}$$

These graphs fit together into a "universal graph"

$$\Gamma \subset M_n(C) \times \mathbb{P}^1 \times C \tag{41}$$

defined by

$$\Gamma = \{(\phi, t, \phi(t)) \mid \phi \in M_n(C),\ t \in \mathbb{P}^1\}\ . \tag{42}$$

The graphs Γ_ϕ all belong to a common linear system on $\mathbb{P}^1 \times C$. We compactify $M_n(C)$ to $\bar{M}_n(C) \cong \mathbb{P}^{2n+1}$ by including the elements of that linear system which are *not* graphs of maps of degree n. The universal graph then compactifies to the "universal divisor in the linear system"

$$\bar{\Gamma} \subset \bar{M}_n(C) \times \mathbb{P}^1 \times C\ . \tag{43}$$

[3] Since C is isolated in X, the conditions $\phi(\mathbb{P}^1) = C$, $\deg\phi = n$ simply serve to pick out a component of the space of all maps from $\mathbb{P}^1$ to X

This has a very concrete description as follows. Let x, y be homogeneous coordinates on $\mathbb{P}^1$, let s, t be homogeneous coordinates on C, and let $a_0, \ldots, a_n$, $b_0, \ldots, b_n$ be homogeneous coordinates on $\bar{M}_n(C)$. Then the map ϕ corresponding to the point $[a_0, \ldots, a_n, b_0, \ldots, b_n]$ can be described as

$$\frac{s}{t} = \frac{\sum a_i x^i y^{n-i}}{\sum b_i x^i y^{n-i}} \tag{44}$$

and the divisor $\bar{\Gamma}$ in $\bar{M}_n(C) \times \mathbb{P}^1 \times C$ is defined by the equation

$$\sum a_i x^i y^{n-i} t - \sum b_i x^i y^{n-i} s = 0 . \tag{45}$$

As before, each new graph which has been added in this compactification contains a graph of a map of lower degree, together with some $\mathbb{P}^1$'s which map to points in C.

It is a simple matter to explicitly construct the compactification divisor of $\bar{M}_n(C)$. The points in $\bar{M}_n(C)$ which do not correspond to smooth n-fold covers are given by values of $[a_0, \ldots, b_n]$ for which (45) factorizes. That is, the resultant of the two polynomials of (45) vanishes:

$$\begin{vmatrix} a_0 & a_1 & a_2 & \ldots & 0 \\ 0 & a_0 & a_1 & \ldots & 0 \\ & & \vdots & & \\ 0 & 0 & 0 & \ldots & a_n \\ b_0 & b_1 & b_2 & \ldots & 0 \\ & & \vdots & & \\ 0 & 0 & 0 & \ldots & b_n \end{vmatrix} = 0 . \tag{46}$$

Thus, the compactification divisor is a hypersurface of degree $2n$ in $\bar{M}_n(C) \cong \mathbb{P}^{2n+1}$. This shows that $M_n(C)$ is isomorphic to the subspace of $\mathbb{C}^{2n+2}$ given by the constraint that the left side of (46) is equal to 1. This generalizes the $ad - bc = 1$ constraint for the single cover case.

It will be convenient to denote $\mathbb{P}^1 \times C$ by S. Let p: $S \to \mathbb{P}^1$ and q: $S \to C$ be the projection maps; let p_ϕ: $\Gamma_\phi \to \mathbb{P}^1$ and q_ϕ: $\Gamma_\phi \to C$ be the induced maps on the graphs. If $\phi \in M_n(C)$, then p_ϕ establishes a natural isomorphism between Γ_ϕ and $\mathbb{P}^1$.

We regard $T_X|_C$ as a fixed bundle on C. Then for each $\phi \in M_n(C)$, the bundle $\phi^*(T_X|_C)$ is mapped to the bundle $q_\phi^*(T_X|_C)$ under the isomorphism p_ϕ. Thus, the important spaces for us to study are the spaces $H^0(q_\phi^*(T_X|_C))$ and $H^1(q_\phi^*(T_X|_C))$. (As ϕ varies, these spaces will fit together to form bundles over the moduli space $M_n(C)$.) We now need to extend this bundle over $M_n(C)$ to a sheaf over the compactified space $\bar{M}_n(C)$. Unfortunately this is not a unique process. However, given the construction we have used for the compactified moduli space, there is a natural way we can do this. As in the previous section, we choose to extend by using $H^0(q_\phi^*(T_X|_C))$ and $H^1(q_\phi^*(T_X|_C))$ for all values of ϕ, even values for which Γ_ϕ is not a graph.

We now claim that the dimensions of these spaces are independent of ϕ, and that they admit another description. Let $\mathscr{B} = q^*(T_X|_C)$. Then the restriction $\mathscr{B}|_{\Gamma_\phi}$ of

$\mathscr{B}$ to Γ_ϕ coincides with $q_\phi^*(T_X|_C)$. Thus, the spaces we wish to study can be identified with $H^0(\mathscr{B}|_{\Gamma_\phi})$ and $H^1(\mathscr{B}|_{\Gamma_\phi})$. Consider now the restriction sequence

$$0 \to \mathscr{B}(-\Gamma_\phi) \to \mathscr{B} \to \mathscr{B}|_{\Gamma_\phi} \to 0\,. \tag{47}$$

The associated long exact sequence in cohomology is then

$$\begin{aligned} 0 \to H^0(\mathscr{B}(-\Gamma_\phi)) &\to H^0(\mathscr{B}) \to H^0(\mathscr{B}|_{\Gamma_\phi}) \to H^1(\mathscr{B}(-\Gamma_\phi)) \\ \to H^1(\mathscr{B}) &\to H^1(\mathscr{B}|_{\Gamma_\phi}) \to H^2(\mathscr{B}(-\Gamma_\phi)) \to H^2(\mathscr{B}) \to 0\,. \end{aligned} \tag{48}$$

(The last H^2 term is 0 since $\operatorname{supp}(\Gamma_\phi)$ is a curve.)

We will abuse notation a bit and let $\mathbb{P}^1$ and C denote divisor classes on S, as follows: for some fixed points $P \in C$, $Q \in \mathbb{P}^1$ we identify $\mathbb{P}^1 = q^{-1}(P)$, $C = p^{-1}(Q)$. Since Eq. (45) has degree 1 in s and t, and degree n in x and y, each Γ_ϕ belongs to the linear system $|\mathbb{P}^1 + nC|$ on S. Moreover, since $T_X|_C \cong \mathcal{O}_C(2) \oplus \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)$ by assumption, we have

$$\mathscr{B} = q^*(T_X|_C) \cong \mathcal{O}_S(2\mathbb{P}^1) \oplus \mathcal{O}_S(-\mathbb{P}^1) \oplus \mathcal{O}_S(-\mathbb{P}^1)\,. \tag{49}$$

It follows that

$$\mathscr{B}(-\Gamma_\phi) \cong \mathcal{O}_S(\mathbb{P}^1 - nC) \oplus \mathcal{O}_S(-2\mathbb{P}^1 - nC) \oplus \mathcal{O}_S(-2\mathbb{P}^1 - nC)\,. \tag{50}$$

Note also that $K_S = -2\mathbb{P}^1 - 2C$.

We calculate the cohomology of $\mathscr{B}$ as follows. First,

$$H^0(\mathscr{B}) = H^0(\mathcal{O}_S(2\mathbb{P}^1) \oplus \mathcal{O}_S(-\mathbb{P}^1) \oplus \mathcal{O}_S(-\mathbb{P}^1)) \cong H^0(\mathcal{O}_S(2\mathbb{P}^1)) \tag{51}$$

which has dimension 3. Also, by Serre duality,

$$H^2(\mathscr{B})^* \cong H^0(\mathcal{O}_S(-4\mathbb{P}^1 - 2C) \oplus \mathcal{O}_S(-\mathbb{P}^1 - 2C) \oplus \mathcal{O}_S(-\mathbb{P}^1 - 2C)) = \{0\}\,. \tag{52}$$

In addition, since $c_1(\mathscr{B}) = c_2(\mathscr{B}) = 0$, by Riemann–Roch we have $\chi(\mathscr{B}) = 3$. It follows that $H^1(\mathscr{B}) = \{0\}$ as well.

Next we calculate the cohomology of $\mathscr{B}(-\Gamma_\phi)$. It follows directly from Eq. (50) that $H^0(\mathscr{B}(-\Gamma_\phi)) = 0$, while by Serre duality

$$\begin{aligned} H^2(\mathscr{B}(-\Gamma_\phi))^* &\cong H^0(\mathcal{O}_S(-3\mathbb{P}^1 + (n-2)C) \oplus \mathcal{O}_S((n-2)C) \oplus \mathcal{O}_S((n-2)C)) \\ &\cong H^0(\mathcal{O}_S((n-2)C) \oplus H^0(\mathcal{O}_S((n-2)C)\,, \end{aligned}$$

which has dimension $(n-1) + (n-1) = 2n-2$. This time, $c_1(\mathscr{B}(-\Gamma_\phi)) = -3\mathbb{P}^1 - 3nC$ and $c_2(\mathscr{B}(-\Gamma_\phi)) = 6n$. Thus, by Riemann–Roch, $\chi(\mathscr{B}(-\Gamma_\phi)) = 0$. It follows that $H^1(\mathscr{B}(-\Gamma_\phi))$ also has dimension $2n-2$.

Thus we see that the long exact sequence (48) can be shortened to

$$0 \to H^0(\mathscr{B}) \to H^0(\mathscr{B}|_{\Gamma_\phi}) \to H^1(\mathscr{B}(-\Gamma_\phi)) \to 0 \to H^1(\mathscr{B}|_{\Gamma_\phi}) \to H^2(\mathscr{B}(-\Gamma_\phi)) \to 0\,. \tag{53}$$

Moreover, the dimensions of all of these spaces are independent of ϕ, so the sheaf we are after will be locally free.

Now to fit these all together, let $\bar{M} = \bar{M}_n(C)$, and look at the space $Z = \bar{M} \times S$, with projections $\pi: Z \to \bar{M}$, $\rho: Z \to S$. Let $\mathscr{E} = \rho^*(\mathscr{B})$. The restriction sequences (47) fit together into a restriction sequence

$$0 \to \mathscr{E}(-\bar{\Gamma}) \to \mathscr{E} \to \mathscr{E}|_{\bar{\Gamma}} \to 0 \,. \tag{54}$$

For any bundle $\mathscr{G}$ on Z, $R^i\pi_*\mathscr{G}$ denotes the bundle of H^i(fibre, $\mathscr{G}_{\text{fibre}}$)'s on $\bar{M}$. Using the vanishing of cohomologies which we have established above, we see that the long exact sequence associated to (54) takes the form

$$0 \to R^0\pi_*\mathscr{E} \to R^0\pi_*(\mathscr{E}|_{\bar{\Gamma}}) \to R^1\pi_*(\mathscr{E}(-\bar{\Gamma})) \to 0 \to R^1\pi_*(\mathscr{E}|_{\bar{\Gamma}}) \to R^2\pi_*(\mathscr{E}(-\bar{\Gamma})) \to 0 \,. \tag{55}$$

In particular, $R^1\pi_*(\mathscr{E}|_{\bar{\Gamma}})$ is isomorphic to $R^2\pi_*(\mathscr{E}(-\bar{\Gamma}))$, and both of these are locally free of rank $2n-2$.

To identify the bundle $R^2\pi_*(\mathscr{E}(-\bar{\Gamma}))$, we need one additional fact: $\bar{\Gamma} \in |\pi^{-1}(H) + \rho^{-1}(\mathbb{P}^1) + n\rho^{-1}(C)|$, where H is a hyperplane in $\bar{M} \cong \mathbb{P}^{2n+1}$. This holds because the equation for $\bar{\Gamma}$, Eq. (45), has degree 1 in $a_0, \ldots, b_n$, degree 1 in s, t and degree n in x, y.

It follows that $\mathscr{E}(\pi^{-1}(H) - \bar{\Gamma})$ will be a pullback from S. That is, there is some bundle $\mathscr{F}$ on S such that $\mathscr{E}(\pi^{-1}(H) - \bar{\Gamma}) = \rho^*(\mathscr{F})$. Then by the projection formula,

$$R^2\pi_*(\mathscr{E}(-\bar{\Gamma})) \cong \mathscr{O}_{\bar{M}}(-H) \otimes R^2\pi_*(\rho^*(\mathscr{F})) \,. \tag{56}$$

Now $R^2\pi_*(\mathscr{E}(-\bar{\Gamma}))$ and $R^2\pi_*(\rho^*(\mathscr{F}))$ are both locally free of rank $2n-2$, and since Z is the product of $\bar{M}$ and S, $R^2\pi_*(\rho^*(\mathscr{F}))$ must actually be the trivial bundle of that rank. Thus,

$$R^2\pi_*(\mathscr{E}(-\bar{\Gamma})) \cong \mathscr{O}_{\bar{M}}(-H) \otimes (\mathscr{O}_{\bar{M}}{}^{\oplus(2n-2)}) \cong (\mathscr{O}_{\bar{M}}(-H))^{\oplus(2n-2)} \,. \tag{57}$$

The Hodge numbers for $\mathbb{P}^{2n+1}$ satisfy $h^{i,j} = \delta_{i,j}$ and so we have a unique integral generator, I, for $H^{2n-2,2n-2}(\mathbb{P}^{2n+1})$. Equation (57) tells us that

$$c_{2n-2}(R^1\pi_*(\mathscr{E}|_{\bar{\Gamma}})) = (-1)^{2n-2}I = I \,. \tag{58}$$

5. The Result

We are now in a position to complete the computation of (37). The result of the previous section is that U is homological to a sub-$\mathbb{P}^3$ of the moduli space $\mathbb{P}^{2n+1}$. The argument now proceeds in an identical way to the case of $n = 1$. For each point of intersection between C and e_a we obtain a contribution to L_a. The condition that a point of Σ maps into a specific point of X puts a linear constraint on the coordinates $[a_0, \ldots, b_n]$ of $\bar{M}_n(C)$ and thus corresponds to a hyperplane of $\bar{M}_n(C)$. The intersection of 3 hyperplanes and a sub-$\mathbb{P}^3$ within $\mathbb{P}^{2n+1}$ is a point and so we yield a result identical to (30) for any $n > 0$

$$N^v_{abc} = \#(e_a \cap C) \cdot \#(e_b \cap C) \cdot \#(e_c \cap C) \,. \tag{59}$$

As an example, consider the case examined in [9]. We take X to be the algebraic variety given by a quintic constraint in $\mathbb{P}^4$. This manifold has the simplifying feature that $h^{1,1} = 1$. In this case we only have one f-polynomial of the form (18) and this polynomial is a function of only one variable, q.

Let us, for the time being, make the assumption that all rational curves are of the type $(-1, -1)$. In the case of $h^{1,1} = 1$, each rational curve in X has a *degree* which is defined by the intersection number for this curve with the unique integral generator of $H^{1,1}(X)$. For the quintic threefold this generator is the generator inherited from the ambient $\mathbb{P}^4$. Let us compute the contribution to the f-polynomial by an n-fold cover of a degree k curve. As we pullback the Kähler form from X to Σ by this map we multiply the degree of this form on Σ by n and k. It is thus a simple matter to show that $m = nk$ in (15). From (59) we see immediately that the contribution to f is

$$k^3 q^{nk} . \tag{60}$$

Thus

$$f = 5 + \sum_{k=1}^{\infty} \frac{a_k k^3 q^k}{1 - q^k} , \tag{61}$$

where a_k is the number of rational curves of degree k on X. This is precisely the formula used in [9] where the expression (60) was conjectured and justified by the fact that it was the only simple form that would yield integers for a_k.

One of the great virtues of the f-polynomials comes from the mirror property. The f-polynomials here are written in terms of the complexified Kähler form and give information about the couplings between (1, 1)-forms on X. However, they also apply to couplings between (2, 1)-forms on the mirror of X where here the q_i variables would encode information about the complex structure. This is how f was derived in [9]. One key point to note is that the f-polynomials were derived in the large-radius, or small q, limit. When using the f-polynomials in the context of complex structure one must ensure that one is in some "large complex structure" limit [6]. There is a natural construction of this situation using methods of algebraic geometry [28].

The mirror symmetry also came in to play when we derived our topological field theory from the original $N = 2$ superconformal field theory. We chose the BRST symmetry to be generated by (5). If we had chosen this symmetry to be generated by (6) we would have obtained identical results except for the fact that instantons would then have been given by antiholomorphic maps. We could also try to work with a BRST generator given by the right-hand set of equations in (5) and the right-hand set of (6). This would have the result of leaving us with a quite different topological quantum field theory containing only the (2, 1)-forms from the original theory rather than the (1, 1)-sector. The f-polynomials generated would be the ones describing the complex structure deformations. By this method we obtain the general result that the coefficients in these polynomials must always be integers. This fact is not apparent from conventional algebraic geometry.

So far, in our example of the quintic threefold, we assumed that the rational curves were isolated. This is believed to be true for a generic quintic threefold [29]. If we consider the Fermat quintic, i.e., the threefold given by the constraint

$$z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 = 0 \tag{62}$$

in $\mathbb{P}^4$ with homogeneous coordinates $[z_0, \ldots, z_4]$, the set of curves of degree one consists of a continuous family that forms a set of 50 cones [30]. A generic member of this set is of type $(-3, 1)$. One might suspect that the form of f will change in this situation. However, when one computes f from the mirror manifold, no account of the Kähler class of the mirror of X is taken. Thus, on X, f must be invariant under deformations of complex structure $-f$ is a local diffeomorphism invariant. This establishes that if one did the path integral over these families of rational curves in the Fermat case, one should obtain the same f-polynomial as in the generic case of isolated curves. It would be interesting to confirm this explicitly.

Acknowledgements. It is a pleasure to thank S. Katz, C.A. Lütken and E. Witten for sharing invaluable insights into this work. The work of D.R.M. was supported in part by NSF grant DMS-9103827.

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Communicated by S.-T. Yau