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Diagonalization of the XXZ Hamiltonian by Vertex Operators

Brian Davies[1], **Omar Foda**[2], **Michio Jimbo**[3], **Tetsuji Miwa**[4] **and Atsushi Nakayashiki**[5]

[1] Mathematics Department, the Faculties, The Australian National University, GPO Box 4, Canberra ACT 2601, Australia
[2] Institute for Theoretical Physics, University of Nijmegen, JNL-6525 ED Nijmegen, The Netherlands and Department of Mathematics, University of Melbourne, Parkville, Victoria 3052, Australia
[3] Department of Mathematics, Faculty of Science, Kyoto University, Kyoto 606, Japan
[4] Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606, Japan
[5] The Graduate School of Science and Technology, Kobe University, Rokkodai, Kobe 657, Japan

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Abstract. We diagonalize the anti-ferroelectric XXZ-Hamiltonian directly in the thermodynamic limit, where the model becomes invariant under the action of $U_q(\widehat{\mathfrak{sl}}(2))$. Our method is based on the representation theory of quantum affine algebras, the related vertex operators and KZ equation, and thereby bypasses the usual process of starting from a finite lattice, taking the thermodynamic limit and filling the Dirac sea. From recent results on the algebraic structure of the corner transfer matrix of the model, we obtain the vacuum vector of the Hamiltonian. The rest of the eigenvectors are obtained by applying the vertex operators, which act as particle creation operators in the space of eigenvectors. We check the agreement of our results with those obtained using the Bethe Ansatz in a number of cases, and with others obtained in the scaling limit – the $su(2)$-invariant Thirring model.

0. Introduction

0.1. A Diagonalization Scheme. In this paper we give a new scheme for diagonalizing the 1-dimensional XXZ spin chain

$$H_{XXZ} = -\frac{1}{2}\sum_{k=-\infty}^{\infty} (\sigma^x_{k+1}\sigma^x_k + \sigma^y_{k+1}\sigma^y_k + \Delta\sigma^z_{k+1}\sigma^z_k)\,, \tag{0.1}$$

for $\Delta < -1$, directly in the thermodynamic limit, using the representation theory of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}(2))$: we consider the infinite tensor product

$$W = \cdots \otimes \mathbf{C}^2 \otimes \mathbf{C}^2 \otimes \mathbf{C}^2 \otimes \mathbf{C}^2 \otimes \cdots\,, \tag{0.2}$$

on which H_{XXZ} formally acts, rather than starting with a finite product and subsequently taking the thermodynamic limit. The reason is that the model is $U_q(\widehat{\mathfrak{sl}}(2))$-symmetric only in the thermodynamic limit, and the presence of that symmetry is central to our approach. On the other hand, working directly in that limit makes it difficult to give rigorous analytic proofs, and a number of our statements will be given as conjectures, supported by explicit calculations.

Our scheme is a direct descendant of the recent works [1–4], and Smirnov's picture of the $su(2)$-invariant Thirring model [5], which is the continuum limit of the XXZ model. (See also [6, 7].) More broadly, it originates in many of the developments that took place in the past two decades in the field of integrable models in quantum field theory and statistical mechanics. It is remarkable that in the context of this simple off-critical model, one can recognize many of these developments, and for the rest of this introduction, we wish to recall certain aspects that are relevant to our work.

The peculiarities of this branch of science are twofold: mathematically, its significance is that the systems we are dealing with have infinite degrees of freedom (unlike those in conventional mathematics, in which we manipulate finite degrees of freedom), while physically, it is the fact that they are integrable (unlike most of the physical problems) because of their infinite, and sometimes hidden symmetries. The symmetries serve as the magical word in Arabian Nights that opens the door between the two worlds of infinite and finite degrees of freedom. Thus, through the study of the integrable models we have been discovering many unexpected links between different branches of mathematics such as representation theory, differential equations, combinatorics, topology, algebraic geometry, and so on.

Let us discuss specifically the integrability and the symmetry of lattice systems. For two-dimensional lattice systems, the integrability is commonly understood as the existence of a family of commuting transfer matrices. In the language of one-dimensional quantum chains, it is understood as the existence of infinitely many mutually commuting conservation laws. Let us call them the infinite abelian symmetries. Because they are commuting we can simultaneously diagonalize them, and in most cases the spaces of common eigenvectors are finite-dimensional. In other words, the infinite abelian symmetries reduce the degeneracies of the spectrum from infinite to finite. This is the reason for the solvability. But it is not the whole story. In this introduction, we will discuss other types of symmetries that we will refer to as the non-abelian symmetries and the dynamical symmetries.

0.2. The Bethe Ansatz. The Bethe Ansatz was invented in order to solve the XXX model, then it was applied to the XXZ and many other models. Let us discuss the XXX model specifically. The Bethe Ansatz reduces the diagonalization of the Hamiltonian, a $2^N \times 2^N$ (huge!) matrix, where N is the size of the system, to a system of m coupled algebraic equations, where m is less than or equal to $N/2$. For a finite but large N, this system is very complicated, and far from "solvable." (It is not completely settled even for $m = 2$ [8].) On the other hand, and surprisingly, in the thermodynamic limit ($N \to \infty$) the system of algebraic equations changes into a system of linear integral equations that can be solved. In this way, the ground state and the low-lying excitations have been extensively studied for the XXX model, and in a similar manner for many other models.

In [9, 10] the anti-ferroelectric XXX and XXZ chains are studied in the thermodynamic limit $N = \infty$ and at the physical temperature $T = 0$. We also deal with the same problem in this paper, but by a completely different method. Our diagonalization scheme is totally independent of the. Bethe Ansatz, and goes further: it makes it possible to describe all the eigenvectors by using the powerful tools of representation theory.

0.3. Non-Abelian Symmetries and Particle Picture. In a remarkable paper [11], Faddeev and Takhtajan conjectured the structure of the eigenvectors of the XXX Hamiltonian. Using the Bethe Ansatz, they conclude that all the excitations are composed of elementary ones, called particles, and that the physical space is

$$\mathscr{F} = \left[\sum_{n=0}^{\infty} \int_0^{\pi} \cdots \int_0^{\pi} dk_1 \ldots dk_n \otimes^n \mathbf{C}^2 \right]_{\text{symm}}, \tag{0.3}$$

where $[\ ]_{\text{symm}}$ stands for an appropriate symmetrization. In [11] the summation is only for even n. In our work, we consider both even and odd n.

The issue in [11] is as follows: The XXX model possesses an $\mathfrak{sl}(2)$ symmetry. The Bethe Ansatz produces only the highest weight vectors of this $\mathfrak{sl}(2)$. Therefore, spin-1/2 two-particle states (i.e., the case $n = 2$, $\mathbf{C}^2 \otimes \mathbf{C}^2$), which were correctly observed in [11] in the thermodynamic limit, appear in the finite chain to decompose into spin-1 plus spin-0 particles (i.e., $\mathbf{C}^3 \oplus \mathbf{C}$). In fact, the energy levels of the triplet $\mathbf{C}^3$ and the singlet $\mathbf{C}$ are different for finite lattice and become equal only in the thermodynamic limit. This gives an example of how a system can achieve extra and big symmetries in the infinite volume limit, and shows one of the advantages of a direct diagonalization of the Hamiltonian in the infinite lattice.

Let us briefly consider the particle picture in the XXZ case. By particle picture we mean the structure of common eigenspaces as irreducible modules of the algebra of non-abelian symmetries. Here, we mean by non-abelian symmetries operators that are commuting with the Hamiltonian. As opposed to the abelian symmetries, they are non-abelian, so they change eigenvectors of the Hamiltonian without changing their eigenvalues. The finite XXZ chain has no $\mathfrak{sl}(2)$ symmetry, but as Pasquier and Saleur [12] pointed out, if we add certain boundary terms, then the XXZ Hamiltonian acquires the $U_q(\mathfrak{sl}(2))$ symmetry. But the representations are still highly reducible.

In our approach, instead of adding boundary terms to the finite chain, we consider the infinite chain from the outset, so that the $U_q(\widehat{\mathfrak{sl}}(2))$ symmetry (which is much larger than $U_q(\mathfrak{sl}(2))$) is manifest. As $U_q(\widehat{\mathfrak{sl}}(2))$-module the common eigenspaces of the XXZ and the higher Hamiltonians become the irreducible tensor products $\otimes^n \mathbf{C}^2$ parametrized by n quasi-momentum variables. This is exactly the point of Faddeev and Takhtajan (in the XXX-case), though they did not explain it in these words.

0.4. Dynamical Symmetries and Creation and Annihilation Operators. Since Onsager's solution of the Ising model, a number of alternative methods of solving that model were developed. A method using the infinite abelian symmetries is given, e.g., in Baxter's book [13]. A major difference, when compared with the anti-ferroelectric XXZ model, is that the Ising model has non-degenerate common eigenvectors of the commuting transfer matrices. In other words, the Ising model has no non-abelian symmetries which commute with the Hamiltonian. However we may also consider symmetries in a broader sense: those which do not commute with the Hamiltonian but map eigenvectors to other eigenvectors with different eigenvalues (let us call them the dynamical symmetries). The Ising model has such symmetries: they are generated by the creation and annihilation operators of free fermions.

A natural question arises: what are the dynamical symmetries of the XXZ model? Our answer is that the vertex operators for $U_q(\widehat{\mathfrak{sl}}(2))$ give an appropriate mathematical language for them. We will explain this statement later.

Of course, the lesson of the Ising model is not restricted to the fact that the space of particles is the Fock space $\mathscr{F}$ of free fermions. The most striking fact observed in the Ising model is that the spin operator s_n is completely characterized by its adjoint action on $\mathscr{F}$, i.e., $\mathrm{Ad}\, s_n \in \mathrm{End}(\mathscr{F})$, or on even smaller linear subspace spanned by the free fermions. ($2^N \times 2^N$ reduces to $2N \times 2N$!) This was the key (or the magical word) to the world of the monodromy preserving deformations, the Painlevé transcendents, etc. [14, 15].

Around the same time as the above developments in the Ising model, the theory of the S-matrix was developed by Zamolodchikov and Zamolodchikov, on the basis of the fact that if a field theory has infinite abelian symmetries, then it has a factorized S-matrix. They defined the algebra of the creation and the annihilation operators as the key to the bootstrap program for the determination of the S-matrices. The creation and annihilation operators, which we mentioned above as generators of the dynamical symmetries of the XXZ Hamiltonian, give a lattice realization of the Zamolodchikov algebra [1].

0.5. QISM. The quantum inverse scattering method (QISM), initiated in Leningrad when the city was called by that name, is a large-scale project to understand integrable quantum systems as a whole. Among its many achievements, we recall two that are related to this work. One is, of course, the discovery of quantum groups, or the q-analogue of the universal enveloping algebras, by Drinfeld and Jimbo. The other is Smirnov's work, to which we come later. The reason for the $U_q(\widehat{\mathfrak{sl}}(2))$ symmetry of the XXZ Hamiltonian is simply that the Boltzmann weight of the six-vertex model is the R-matrix for the two-dimensional representation of $U_q(\widehat{\mathfrak{sl}}(2))$ (i.e., $\mathbf{C}^2$ in (0.1)). Also, the existence of the infinite dimensional abelian symmetries is a corollary of this fact. The discovery of quantum groups came as a harvest of QISM, not conversely. All the developments in the earlier days of QISM were made in the absence of quantum groups. Therefore, they are lacking in understanding of the true nature of the symmetries. The QISM project should be further pushed forward on the basis of the developments in the theory of quantum groups.

In the early days of QISM, Smirnov invented a bootstrap program for computing the matrix elements of local fields starting from a factorized S-matrix [5, 16]. He found a system of difference equations which ensures the locality of the field operators, and obtained the form factors (i.e., the matrix elements of the off-shell currents). This is a deep result. We are only beginning to understand its true meaning. Frenkel and Reshetikhin obtained the q-deformed Knizhnik–Zamolodchikov equation, and established that the n-point correlation functions of the vertex operators satisfy this system of q-difference equations [1]. Smirnov's equation was the double Yangian version of that.

As we already mentioned, the creation and the annihilation operators of the XXZ model are given in terms of the vertex operators of $U_q(\widehat{\mathfrak{sl}}(2))$. (They are not equal. See Sect. 7 for a detailed discussion on this point.) The latter are exactly the same as special cases of those considered in [1]. So, their n-point functions satisfy the q-KZ equation. This is the key to the calculation of the energy and the momentum, and the commutation relations of the creation and annihilation operators.

0.6. Conformal Field Theory. We have considered the symmetries of the Ising model and the XXZ model. The importance of such a viewpoint was established in the monumental work of Belavin, Polyakov and Zamolodchikov [17]. To formulate the symmetry picture of conformal field theory in a way that is suitable to our purposes, we will consider the $\mathfrak{sl}(2)$ Wess–Zumino model, following Tsuchiya and Kanie [18]:

(i) The space of states of this model is a direct sum $\mathscr{H}$ of the irreducible highest weight representations of $\widehat{\mathfrak{sl}}(2)$.
(ii) The Hamiltonian of the model is the Virasoro generator L_0 in the Sugawara form. It allows the $\mathfrak{sl}(2)$ symmetry. The total $\widehat{\mathfrak{sl}}(2)$ acts on $\mathscr{H}$ as dynamical symmetries.
(iii) The local fields of this model are given by the vertex operators:

$$\Phi(z)\colon V_z \otimes \mathscr{H} \to \mathscr{H} . \tag{0.4}$$

Here V_z is a level 0 representation of $\widehat{\mathfrak{sl}}(2)$ depending on a parameter z, and Φ intertwines the representations on both sides.
(iv) The vacuum-expectation values of the vertex operators are characterized by the Knizhnik–Zamolodchikov equations, which is a system of linear holonomic differential equations.

Now we can give the definition of the vertex operators for $U_q(\widehat{\mathfrak{sl}}(2))$. They are exactly (0.4) in the context of the quantum affine Lie algebra $U_q(\widehat{\mathfrak{sl}}(2))$. Note, however, that there is a difference between the interpretations of the spectral parameter z in (0.4), in the WZ-model and in our XXZ model. In the former, it is the coordinate variable, and in the latter, it is the momentum variable. The mathematical extension from $q = 1$ to $q \neq 1$ of the vertex operators has a different physical content. (This is already pointed out by Smirnov in a different context [5].)

Conformal field theory has had many achievements in the subject of integrable models. In fact, we should say that it has changed the definition of the game we are playing. However, it has not made progress in all of the subjects studied in earlier days. The reason is obvious: its basic principle, the conformal symmetry, is valid only in massless theories. Zamolodchikov made the first attempt to attack the massive theories from the conformal viewpoint. He proposed to define integrable deformations of conformal field theories by the existence of infinite abelian symmetries embedded in the conformal symmetry [19, 20]. This motivated a reconsideration of the factorized S-matrix theory [21].

0.7. CTM *and Beyond.* Let us return to the XXZ model. The point we wanted to make is the following: The deformation parameter q appears in the XXZ model as the anisotropy parameter $\Delta = (q + q^{-1})/2$. This is the departure from criticality, or simply the mass. So, there is more than a good reason to expect that the quantum affine algebras are relevant in massive integrable models. In fact, many of the recent developments are related to this point. For the continuum theory, we refer the reader to Smirnov's paper [5].

Now let us come to one of the masterpieces of Baxter, the CTM (corner transfer matrix). In [22], after saying that there is no Ising-like reduction from $2^N \times 2^N$ to $2N \times 2N$ in the six or eight vertex models, he wrote

> *A rather ambitious hope is that by examining the CTM's we may stumble on such a group, that the solution of the models may thereby be simplified ...*

A few years later, he succeeded in computing the 1-point functions of the hard-hexagon model and its generalizations, without giving an answer to the original question of his earlier paper. Nevertheless he has given us the magical word – '*Open Sesame*," by inventing the CTM method. When the cave-door opened, there appeared the characters of the affine Lie algebras [23, 24], the theory of crystals [25, 26], the irreducible highest weight representations [3], the q-vertex operators [1, 4] – the passage to the representation theory of the affine quantum groups. And in every case, the use of an infinite system, already assumed in Baxter's original work, is an essential ingredient.

In this paper, we proceed further along this route. We propose that the mathematical picture that bridges (0.2) and (0.3) is

$$\mathrm{End}_{\mathbb{C}}(V(\Lambda_0)) = V(\Lambda_0) \otimes V(\Lambda_0)^* \tag{0.5}$$

(in the even particles sector; the odd particles sector is $\mathrm{Hom}_{\mathbb{C}}(V(\Lambda_0), V(\Lambda_1)) = V(\Lambda_1) \otimes V(\Lambda_0)^*$), where the $V(\Lambda_i)$ are the level 1 highest weight representations of $U_q(\widehat{\mathfrak{sl}}(2))$. By this, the world of the infinite degrees of freedom (i.e., the infinite tensor product) opens to the world of the finite degrees of freedom (i.e., the representation theory of $U_q(\widehat{\mathfrak{sl}}(2))$). The vacuum, the lowest eigenvector of the Hamiltonian, is the identity operator in $\mathrm{End}_{\mathbb{C}}(V(\Lambda_0))$, or the canonical element in $V(\Lambda_0) \otimes V(\Lambda_0)^*$. The particles, as given in (0.3), are created by the vertex operators acting on the left half of $V(\Lambda_i) \otimes V(\Lambda_0)^*$ ($i = 0, 1$). Thus, the space $\mathcal{F}$ (0.3) of the eigenvectors of the Hamiltonian (0.1), that are created by the creation operators upon the vacuum, lies in the level zero $U_q(\widehat{\mathfrak{sl}}(2))$-module (0.5).

Let us summarize our discussion on the symmetries of integrable models in this introduction.

Symmetries	*Ising*	*XXZ* *Thirring*	WZ
Abelian Symmetries	Commuting Transfer Matrices	Infinite Conservation Laws	None
Non-abelian Symmetries	None	$U'_q(\widehat{\mathfrak{sl}}(2))$ The $\mathfrak{sl}(2)$-Yangian	$su(2)$
Dynamical Symmetries	Free Fermions	Vertex Operators	Virasoro $\widehat{su}(2)$
Space of States	$\mathcal{F}$	$\mathcal{F} \subset \mathrm{End}(\mathcal{H}) = \mathcal{H} \otimes \mathcal{H}^*$	$\mathcal{H}$
Local Fields	Clifford group	?	Vertex Operators

0.8. Plan of the Paper. The text is organized as follows. In Sect. 1 we introduce the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}(2))$ as non-abelian symmetries of the XXZ spin chain. In Sect. 2 we describe the embedding of the highest weight modules $V(\Lambda_i)$ into the half-infinite tensor product space $\cdots V \otimes V \otimes V$ given by iterating the vertex operators. Examining the perturbation expansion at $q = 0$ we observe that, choosing the scalar multiple of vertex operators correctly the series defining the embedding would become finite, term by term in powers of q. To ascertain the existence of such a normalization factor we need to know the asymptotics of the n point functions of vertex operators (cf. Sect. 6.8 below) as $n \to \infty$. Though the problem is of its own interest, it is beyond the scope of the present paper. In Sect. 3 we study the decomposition of the level 0 module $V(\Lambda_0) \otimes V(\Lambda_0)^*$ at $q = 0$. We show that its crystal naturally decomposes to "n-particle sectors," each of which can be identified as a certain anti-symmetrization of the affine crystal $\mathrm{Aff}(B)^{\otimes n}$. We study the picture for nonzero q in the next sections. In Sect. 4 we introduce the vacuum vectors as canonical elements of $V(\Lambda_i) \otimes V(\Lambda_i)^*$. In Sect. 5 we formulate the "particle picture" by utilizing a similar but different kind of vertex operators from those used in the embeddings. We argue that in this setting the computation of the one-point function (in the sense of [23], not the staggered polarization [27]) can be done trivially.

Sections 1–5 are devoted to a presentation of our ideas on the problem. In the next two sections we reformulate the problem in the language of representation theory. After reviewing the q-deformed vertex operators following [1] and [4] we give formulas for their two point functions and their commutation relations. The case of general n point functions is discussed in Sect. 6.8. In Sect. 7 we study the vacuum, creation and annihilation operators which are defined in purely representation-theoretic terms. We prove that the vacuum vector has the correct invariance with respect to the translation and energy operators. We then derive the energy-momentum of the creation-annihilation operators using the formulas for two point functions, and show that they coincide with the known results obtained by the Bethe Ansatz method. Section 8 is devoted to a summary and open problems.

In the appendices we collect some data concerning the global crystal base and vertex operators. We give tables for the first few terms of the actions with respect to the global base of $U_q(\widehat{\mathfrak{sl}}(2))$ on $V(\Lambda_0)$ (Appendix 1), of the vertex operators (Appendix 2), the embedding into the infinite tensor product (Appendix 3) and the images of the vacuum, one- and two-particles (Appendix 4). In Appendix 5 we study the $q \to 0$ limit of the Bethe vectors and compare the results with the vertex operator computations.

1. Quantum Affine Symmetries of the XXZ Model

Let $V \simeq \mathbf{C}^2$ be a 2-dimensional vector space, and let σ^x, σ^y, σ^z be the Pauli matrices acting on V. We consider the infinite tensor product W (0.2) and the XXZ Hamiltonian H_{XXZ} that formally acts on W. The action of H_{XXZ} is formal in the sense that it is *a priori* divergent, since we are working directly in the thermodynamic limit, and requires renormalization. We number the tensor components by $k \in \mathbf{Z}$. The limit $k \to \infty$ is to the left, and $k \to -\infty$ to the right. Our aim is to diagonalize the Hamiltonian H_{XXZ}, using the representation theory of quantum affine algebras. The Bethe Ansatz provides us with a method for diagonalizing such Hamiltonians. In this method we start from a finite (periodic or non-periodic) chain, find the eigenvalues and the eigenvectors of the Hamiltonian in a certain form, and take the thermodynamic limit at the end. In this paper we propose another method that diagonalizes H_{XXZ} directly on W. The applicability of our method, at the moment, is limited to the anti-ferroelectric regime $\Delta < -1$. The Bethe Ansatz method in this regime meets with the difficulty of starting from the wrong vacuum, and filling the Dirac sea. Our method is free from this complication.

The main ingredient of our method is the quantum affine symmetries of $U_q(\widehat{\mathfrak{sl}}(2))$. Consider the action of $U'_q(\widehat{\mathfrak{sl}}(2))$ on V;

$$\pi : U'_q(\widehat{\mathfrak{sl}}(2)) \to \mathrm{End}(V)$$

given by

$$\pi(e_0) = \pi(f_1) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$\pi(e_1) = \pi(f_0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$\pi(t_1) = \pi(t_0^{-1}) = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}.$$

We set

$$v_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We relate q to Δ by

$$\Delta = \frac{q + q^{-1}}{2}, \quad -1 < q < 0 .$$

By using the infinite comultiplication $\Delta^{(\infty)}$, we define the action of $U'_q(\widehat{\mathfrak{sl}}(2))$ formally on W:

$$\Delta^{(\infty)}(e_i) = \sum_{k \in \mathbf{Z}} \cdots \otimes t_i \otimes e_i \otimes 1 \cdots, \tag{1.1}$$

$$\Delta^{(\infty)}(f_i) = \sum_{k \in \mathbf{Z}} \cdots \otimes 1 \otimes f_i \otimes t_i^{-1} \cdots, \tag{1.2}$$

$$\Delta^{(\infty)}(t_i) = \cdots \otimes t_i \otimes t_i \otimes t_i \cdots. \tag{1.3}$$

One can check the commutativity

$$[H_{XXZ}, U'_q(\widehat{\mathfrak{sl}}(2))] = 0$$

as discussed in [3]. The major advantage of working in the infinite lattice limit directly is having this infinite symmetry, which inevitably breaks down on a finite lattice. On a finite lattice, one can modify H_{XXZ} in such a way that it possesses the $U_q(\mathfrak{sl}(2))$-symmetries [12], but not the $U_q(\widehat{\mathfrak{sl}}(2))$-symmetries. (Otherwise, the degeneracy of the spectrum on the finite chain would be different.) The difference of the size of the symmetry algebra is crucial. For instance, we can identify the vacuum vector in W as the unique $U'_q(\widehat{\mathfrak{sl}}(2))$-singlet (a vector which generates a 1-dimensional $U'_q(\widehat{\mathfrak{sl}}(2))$-module.) On the other hand, singlets for $U_q(\mathfrak{sl}(2))$ are many.

The full algebra $U_q(\widehat{\mathfrak{sl}}(2))$ is obtained by adding q^d to $U'_q(\widehat{\mathfrak{sl}}(2))$. Following [3] let us identify d with $(H_{\mathrm{CTM}} - S)/2$, where

$$H_{\mathrm{CTM}} = -\frac{q}{1-q^2} \sum_{k \in \mathbf{Z}} k(\sigma^x_{k+1}\sigma^x_k + \sigma^y_{k+1}\sigma^y_k + \Delta\sigma^z_{k+1}\sigma^z_k) \tag{1.4}$$

and $2S = \sum \sigma^z_k$ is the total spin operator. This identification is justified by explicit computations and by checking that

$$[d, e_i] = \delta_{i0}e_i, \quad [d, f_i] = -\delta_{i0}f_i, \quad [d, t_i] = 0 .$$

Consider the shift T of W which induces the outer automorphism of the operator algebra generated by $\sigma^x_k, \sigma^y_k, \sigma^z_k$;

$$T \cdot \sigma^*_k \cdot T^{-1} = \sigma^*_{k-1} .$$

T is a shift to the right by one lattice unit. Because of (1.4), the Hamiltonian (0.1) can be written as

$$\frac{2q}{1-q^2} H_{XXZ} = T^2 \cdot d \cdot T^{-2} - d . \tag{1.5}$$

This is the key to our diagonalization procedure: the point is that the derivation d [3], as well as T can be expressed in the language of representation theory, thus the XXZ Hamiltonian can be expressed in terms of mathematically well-defined objects.

In fact, (1.5) allows us to bypass explicit references to the higher Hamiltonians, obtained by taking higher derivatives of the row-to-row transfer matrix of the six-vertex model. The point is that all higher Hamiltonians can be generated from H_{XXZ}, by taking commutators with d recursively [28]. This means that the essential feature of the integrability (i.e., the existence of the infinite conservation laws), in this model, is incorporated in d.

2. Embedding of the Highest Weight Modules

Let $V(\Lambda_i)$ $(i = 0, 1)$ be the level 1 irreducible highest weight $U_q(\widehat{\mathfrak{sl}}(2))$-module with highest weight Λ_i. We give a conjecture on an embedding of $V(\Lambda_i)$ into the half infinite tensor product

$$W_l = \cdots \otimes V \otimes V$$

by using the vertex operators

$$\Phi\colon V(\Lambda_i) \to V(\Lambda_{1-i}) \otimes V, \quad \Phi(v) = \Phi_+(v) \otimes v_+ + \Phi_-(v) \otimes v_- , \tag{2.1}$$

that satisfies

$$\Phi(xv) = \Delta(x)\Phi(v) \quad \text{for all } x \in U_q(\widehat{\mathfrak{sl}}(2)) \text{ and } v \in V(\Lambda_i) .$$

Precisely speaking, we need a completion of $V(\Lambda_{1-i}) \otimes V$, which we will elaborate in Sect. 6. Also, we do not define W_l (but see the definition $W_l^{(i)}$ below). The subscript l in W_l stands for *left*. It means the left-half-infinite tensor product.

The idea is to consider the composition of the vertex operators;

$$V(\Lambda_0) \xrightarrow{\Phi} V(\Lambda_1) \otimes V \xrightarrow{\Phi \otimes \mathrm{id}} V(\Lambda_0) \otimes V \otimes V \xrightarrow{\Phi \otimes \mathrm{id} \otimes \mathrm{id}} V(\Lambda_1) \otimes V \otimes V \otimes V$$

$$\to \cdots \to W_l \,. \tag{2.2}$$

Our conjecture is that with a proper normalization of (2.1), the composition (2.2) converges to a map

$$\iota\colon V(\Lambda_i) \to W_l$$

satisfying

$$\iota(xv) = \Delta^{(\infty)}(x)\iota(v) \,,$$

and that if v is a weight vector of $V(\Lambda_i)$, then $\iota(v)$ is an eigenvector of H_{CTM} defined by (1.4) with a certain renormalization (see [3]).

Let $\mathscr{P}_l^{(i)}$ be the set of paths which parametrize the affine crystal $B(\Lambda_i)$. (See [2, 29, 30].) We use the convention that the colors $i = 0, 1$ are modulo 2 (e.g., $\mathscr{P}_l^{(i)} = \mathscr{P}_l^{(i+2)}$). A path $p = (p(k))_{k \geqq 1} = (\cdots p(3)p(2)p(1)) \in \mathscr{P}_l^{(i)}$ satisfies $p(k) = (\pm)$ (often identified with ± 1), and $p(2k+i) = (+)$, $p(2k+i+1) = (-)$ for $k \gg 0$. We denote by $|p\rangle$ the vector in W_l corresponding to p;

$$|p\rangle = \cdots \otimes v_{p(3)} \otimes v_{p(2)} \otimes v_{p(1)} \,.$$

The following are called the ground-state paths;

$$\bar{p}_0 = (\cdots + - + -), \quad \bar{p}_1 = (\cdots - + - +) \,.$$

We consider the set of formal infinite linear combinations of paths with coefficients in $\mathbf{Q}(q)$,

$$W_l^{(i)} = \left\{ \sum_{p \in \mathscr{P}_l^{(i)}} c(p)|p\rangle; c(p) \in \mathbf{Q}(q) \right\} .$$

Let $\{G(p); p \in \mathscr{P}_l^{(i)}\}$ be the upper global base of Kashiwara [25] for $V(\Lambda_i)$. Define $c_\pm(r, p)$ $(r, p \in \mathscr{P}_l^{(0)} \sqcup \mathscr{P}_l^{(1)})$ by

$$\Phi_\pm(G(p)) = \sum_r c_\pm(r, p)G(r) \,.$$

If $p \in \mathscr{P}_l^{(i)}$, then $c_\pm(r, p) = 0$ unless $r \in \mathscr{P}_l^{(1-i)}$ and $s(p) = s(r) \pm \frac{1}{2}$. Here $s\colon \mathscr{P}_l^{(0)} \sqcup \mathscr{P}_l^{(1)} \to \frac{1}{2}\mathbf{Z}$ is the spin of paths defined by $s(p) = \frac{1}{2}\lim_{k\to\infty} \sum_{j=1}^{2k+i} p(j)$.

The coefficients $c_\pm(r, p)$ of the vertex operator satisfy the following (see [4]):

$$c_-(\bar{p}_1, \bar{p}_0) = c_+(\bar{p}_0, \bar{p}_1) = 1 \,, \tag{2.3}$$

$$c_\pm(r, p) \equiv 1 \bmod qA \quad \text{if } p(1) = \pm, p(k+1) = r(k)\ \forall k \tag{2.4}$$

$$\equiv 0 \bmod qA \quad \text{otherwise} \,, \tag{2.5}$$

$$\sharp\{r \in \mathscr{P}_l^{(0)} \sqcup \mathscr{P}_l^{(1)}; c_\pm(r, p) \not\equiv 0 \bmod q^N A\} < \infty$$

$$\text{for all } p \in \mathscr{P}_l^{(0)} \sqcup \mathscr{P}_l^{(1)} \text{ and } N \in \mathbf{N} \,. \tag{2.6}$$

Here $A = \{f \in \mathbf{Q}(q);\ q$ has no pole at $q = 0\}$. Equation (2.3) is the normalization and (2.4), (2.5), (2.6) are the key properties which mean the compatibility between the vertex operator and the crystal base.

Define

$$\omega^{(n)}(a, p) = \langle G^*(a_{n+1}) | \Phi_{a(n)} \circ \cdots \circ \Phi_{a(1)} | G(p) \rangle . \tag{2.7}$$

Here $a = (a_{n+1}, a(n), \ldots, a(1))$ and p are paths in $\mathscr{P}_l^{(i)}$, and $\{G^*(p)\}$ is the dual base to $\{G(p)\}$. Because of (2.4–6), for a fixed n, (2.7) is convergent in $\mathbf{Q}[[q]]$.

Conjecture.

(i) *There exists a limit*

$$\omega(a, p) = \lim_{n \to \infty} \frac{\omega^{(n)}(a, p)}{\omega^{(n)}(\bar{p}_i, \bar{p}_i)} \in \mathbf{Z}[[q]] .$$

(ii) $\omega(a, p) \in (\mathbf{Q}(q) \to A)$.

(iii) *Setting*

$$\iota(G(p)) = \sum_{a \in \mathscr{P}_l^{(i)}} \omega(a, p) | a \rangle , \tag{2.8}$$

we have

$$\iota(G(p))|_{q=0} = | p \rangle . \tag{2.9}$$

(iv)

$$\iota(xG(p)) = \sum_{a \in \mathscr{P}_l^{(i)}} \omega(a, p) \Delta^{(\infty)}(x) | a \rangle \tag{2.10}$$

for $x \in U_q'(\widehat{\mathfrak{sl}}(2))$ *and* $p \in \mathscr{P}_l^{(i)}$.

The statements of the conjecture are independent of the normalization of the vertex operator (2.1). With the normalization (2.3) we have (see 6.8 and Appendix 3)

$$\lim_{n \to \infty} \omega^{(n)}(\bar{p}_i, \bar{p}_i)^{1/n} = 1 + q^4 - q^6 + q^8 \bmod q^{10} .$$

Therefore, we also conjecture that this is actually convergent in $\mathbf{Z}[[q]]$. We have no reasonable conjecture what the limit is.

In [3] an embedding $V(\Lambda_0) \hookrightarrow V \otimes V \otimes \cdots$ is constructed by using a different coproduct;

$$\Delta_-(e_i) = e_i \otimes t_i^{-1} + 1 \otimes e_i ,$$

$$\Delta_-(f_i) = f_i \otimes 1 + t_i \otimes f_i ,$$

$$\Delta_-(t_i) = t_i \otimes t_i .$$

To compare these two different embeddings, let us flip right and left in the notation of [3]. (Namely we change the embedding of [3] into the form $V(\Lambda_0) \to \cdots \otimes V \otimes V$.) If we denote the infinite coproduct in [3] after the flip by $\Delta_{FM}^{(\infty)}$, we have

$$\Delta_{FM}^{(\infty)}(f_i) = q\Delta^{(\infty)}(t_i f_i) ,$$

$$\Delta_{FM}^{(\infty)}(e_i) = q\Delta^{(\infty)}(t_i^{-1} e_i) .$$

Therefore these two embeddings differ only by certain power of q. In Appendices 3, 4 we compute $c_\pm(r, p)$ and $\omega(a, p)$ for several cases.

Let us discuss the similarity and the difference of [3] and the present paper. [3] studied the CTM Hamiltonian (1.4) on W_l and identified its eigenvectors with the weight vectors of $V(\Lambda_0)$ embedded in W_l. In this paper we study the XXZ Hamiltonian (0.1) on W and identify its eigenvectors with the weight vectors in certain $U'_q(\widehat{\mathfrak{sl}}(2))$-modules embedded in W. In both cases the main aim is to give an equivalent mathematical picture to the physical content of the problem, which is given by the CTM Hamiltonian or the XXZ Hamiltonian. The mathematical picture presented in [3] is $V(\Lambda_0)$, the level 1 irreducible highest weight $U_q(\widehat{\mathfrak{sl}}(2))$-module. The method employed is the q-perturbation. An obvious implication of [3] to the present problem is to consider

$$V(\Lambda_0) \otimes V(\Lambda_0)^{*a} \tag{2.11}$$

as a mathematical model for the infinite product W. A similar analysis applied to

$$W_r = V \otimes V \otimes \cdots$$

gives the level -1 irreducible lowest weight $U_q(\widehat{\mathfrak{sl}}(2))$-module. The right half of (2.11) is the most appropriate realization of that. Here $V(\Lambda_0)^* = \mathrm{Hom}_{\mathbf{Q}(q)}(V(\Lambda_0), \mathbf{Q}(q))$ endowed with the natural right $U_q(\widehat{\mathfrak{sl}}(2))$-action ρ_R; if $f \in V(\Lambda_0)^*$ then $(\rho_R(x)f)(v) = f(xv)$ for $x \in U_q(\widehat{\mathfrak{sl}}(2))$. Then $V(\Lambda_0)^{*a}$ is the left $U_q(\widehat{\mathfrak{sl}}(2))$-module with the left $U_q(\widehat{\mathfrak{sl}}(2))$ action $\rho_{L,a}$ in terms of the antipode a (see [2], Sect. 5);

$$(\rho_{L,a}(x)f)(v) = f(a(x)v)\,.$$

We will discuss why we choose the antipode to construct the left action a bit later. Now we want to discuss a big difference between [3] and the present paper. The basic tool in [3] was the q-series expansion. Underlying this were two things;

(i) The success of the theory of the affine crystals ([2, 29, 30]).
(ii) The CTM magic, i.e., the discreteness of the spectrum of the CTM Hamiltonian ([13]).

In this paper, we further exploit (i) to analyse the $q = 0$ structure. Note that the XXZ Hamiltonian is already diagonal at $q = 0$, i.e., $\Delta = -\infty$. The eigenvectors are nothing more than paths $p = (p(k))_{k \in \mathbf{Z}}$ with appropriate boundary conditions. The notion of crystal gives a nice combinatorial structure to the set of paths. This is described in detail in Sect. 3. The discreteness of the spectrum is no longer true for the XXZ Hamiltonian. It means the breakdown of the q-expansion as a tool for finding the eigenvectors. The continuum spectrum at $q \neq 0$ degenerates to the discrete spectrum at $q = 0$. Therefore we do not know a priori the correct form of the eigenvectors at $q = 0$ with which we should start our expansions. In [31] the expansion of the vacuum vector (i.e., the lowest eigenvector of H_{XXZ}) was discussed. This was possible because the lowest eigenvalue is not degenerate even at $q = 0$ (or more precisely, it is doubly degenerate, but the two eigenvectors generate two orthogonal sectors). So we could start with

$$|\mathrm{vac}\rangle|_{q=0} = (\cdots + \; - \; + \; - \cdots)\,.$$

Similarly it is not difficult to tell the correct limiting form of the 1 particle state because the 1 particle states must be eigenvectors of T^2 (the shift of two lattice units). Therefore we can start with

$$|u, +\rangle|_{q=0} = \sum_k (\cdots - \; + \; + \; - \cdots) e^{iku} ,$$

$$|u, -\rangle|_{q=0} = \sum_k (\cdots + \; - \; - \; + \cdots) e^{iku} .$$

When we consider an even particle state, e.g., the vacuum, we take

$$p(2k) = (+), \quad p(2k+1) = (-) \quad \text{for } k \to \pm\infty \tag{2.12}$$

as the boundary condition. On the other hand, we take

$$p(2k) = (\mp), \quad p(2k+1) = (\pm) \quad \text{for } k \to \pm\infty \tag{2.13}$$

for an odd particle state. The ordinary approach of the Bethe Ansatz dismisses all the odd particle states, because the cyclic boundary condition on a lattice with even number of sites is discussed. Let us go back to the discussion of the q expansion. If the particle-number is greater than one, there is no a priori choice of the limit $q = 0$. The shift T^2 can fix only the total momentum (at $q = 0$), which is certainly not enough. So we need some new idea other than (i) and (ii) above. The idea is, of course, to use the quantum affine symmetries discussed in Sect. 1. What can one do if one has the symmetries of the Hamiltonian? The answer is to decompose the space of states into the irreducible pieces. This idea works very well if the irreducible decomposition is multiplicity free. As for the XXZ model in the anti-ferroelectric regime, this is not the case if we consider only the abelian symmetries and the non-abelian $U_q(\mathfrak{sl}(2))$-symmetries. But if we consider the $U_q(\widehat{\mathfrak{sl}}(2))$-symmetries this is what we actually get. In the case of the CTM Hamiltonian, the decomposability of the physical space is rather trivial; the space $V(\Lambda_0)$ is already irreducible. This is one big reason why the CTM magic works so nicely. For the XXZ Hamiltonian, we should "decompose" $V(\Lambda_0) \otimes V(\Lambda_0)^{*a}$, (or equivalently $\mathrm{End}_{\mathbf{Q}(q)}(V(\Lambda_0))$) which is a level 0 representation of $U'_q(\widehat{\mathfrak{sl}}(2))$, an object highly reducible and much more interesting than the irreducible modules.

3. Decomposition of Crystals

At $q = 0$ the Hamiltonian (0.1) is diagonal. Apart from the divergence of its eigenvalue, a vector of the form

$$\cdots \otimes v_{p(2)} \otimes v_{p(1)} \otimes v_{p(0)} \otimes v_{p(-1)} \otimes \cdots$$

for an arbitrary map $p\colon \mathbf{Z} \to \{+, -\}$ is an eigenvector of the Hamiltonian at $q = 0$. Among these, we consider only those which satisfy appropriate boundary conditions. We call them paths. The set of paths is equal to the product of two affine crystals, those corresponding to $V(\Lambda_{0,1})$ and $V(\Lambda_0)^{*a}$. (See [2] for affine crystals.) We also consider the $q = 0$ limit of the creation operators ψ_j^*. (We borrow the commutation relation (7.10b) from Sect. 7.) We introduce the algebra generated by ψ_j^*, and show that it is isomorphic to the crystal of the paths. By using this isomorphism we give the $q = 0$ limit of the n-particle eigenvectors.

3.1. Crystal $\mathscr{P}^i$. Recall that the highest weight module $V(\Lambda_i)$ $(i = 0, 1)$ has the crystal which is realized as paths [29] (see below). Let us denote it by $\mathscr{P}^i_{\text{left}}$. Similarly, we have the crystal $\mathscr{P}^i_{\text{right}}$ for $V(\Lambda_i)^{*a}$. The crystal $\mathscr{P}^{0,1}$ of $V(\Lambda_{0,1}) \otimes V(\Lambda_0)^{*a}$ is the tensor product $\mathscr{P}^{0,1}_{\text{left}} \otimes \mathscr{P}^0_{\text{right}}$ in the sense of crystals. As a set it is just a direct product $\mathscr{P}^i = \mathscr{P}^i_{\text{left}} \times \mathscr{P}^0_{\text{right}}$, equipped with the action of the modified Chevalley generators $\tilde{e}_i, \tilde{f}_i$ according to the tensor product rule [25].

Practically the crystal $\mathscr{P}^i$ is described as follows. An element of $\mathscr{P}^i$ is represented as a path extending to both directions:

$$p = (p(k))_{k \in \mathbf{Z}}, \quad p(k) \in \{+, -\},$$
$$p(k) = (-)^k \text{ for } k \ll 0, \quad p(k) = (-)^{k+i} \text{ for } k \gg 0\,. \tag{3.1}$$

The labeling k is from right to left, $k \geqq 1$ (respectively $k \leqq 0$) for $\mathscr{P}^{0,1}_{\text{left}}$ (respectively $\mathscr{P}^0_{\text{right}}$). Alternatively $\mathscr{P}^i$ can be described as the set of arbitrary decreasing sequence of integers $l_1 > \cdots > l_n$ with $n \equiv i \bmod 2$. The two pictures are related via

$$\{k \in \mathbf{Z} \,|\, p(k) = p(k+1)\} = \{l_1, \ldots, l_n\}, \quad l_1 > \cdots > l_n\,.$$

We call $l_1, \ldots, l_n$ "domain walls" of the path p, and write $[[l_1, \ldots, l_n]]$ to represent p. We have the obvious decomposition

$$\mathscr{P}^0 = \bigcup_{n:\,\text{even}} \mathscr{P}(n), \quad \mathscr{P}^1 = \bigcup_{n:\,\text{odd}} \mathscr{P}(n)\,, \tag{3.2}$$

where $\mathscr{P}(n)$ is the set of paths with n domain walls.

There are two types of domain walls: adjacent $(+\ +)$ and $(-\ -)$ pairs. By definition the spin $s(p)$ of a path p is simply $(\sharp(+\ +) - \sharp(-\ -))/2$. The total weight of (3.1) has the form

$$\operatorname{wt} p = s(p)\alpha_1 - h(p)\delta\,.$$

The $h(p) \in \mathbf{Z}$ is given in terms of the energy function $H(\varepsilon, \varepsilon')$,

$$H(+, -) = -1, \quad H(\varepsilon, \varepsilon') = 0 \text{ otherwise}$$

as follows:

$$h(p) = \sum_{k \in \mathbf{Z}} k(H(p(k), p(k+1)) - H(\bar{p}_i(k), \bar{p}_i(k+1)))\,.$$

Here $p \in \mathscr{P}^i$ and $\bar{p}_i(k) = (-)^k$ for $k \leqq 0$, $\bar{p}_i(k) = (-)^{k+i}$ for $k > 0$.

The rules for the action of $\tilde{e}_i, \tilde{f}_i$ may be expressed as

$$\tilde{f}_0\colon (-) \mapsto (+), \quad \tilde{f}_0\colon (-\ +) \mapsto 0\,,$$
$$\tilde{f}_1\colon (+) \mapsto (-), \quad \tilde{f}_1\colon (+\ -) \mapsto 0\,.$$

Their application is as follows (we shall describe $\tilde{f}_0$). $\tilde{f}_0$ has no action on any $(-\ +)$ pair (a singlet). Given a path p we first cut its left tail $\cdots - + - + - +$ and right tail $- + - + - + \cdots$ to make it a finite sequence. We reduce it further by the rule that each time there is an adjacent singlet pair $(-\ +)$ we drop that pair. For example, $(\cdots - + + - - + + - + \cdots)$ reduces to $(+)$. The above procedure is not unique but the result does not depend on the particular choice of it, reflecting the coassociativity of the coproduct. If no spins are left then the path as an entity is a singlet with respect to $\tilde{f}_0$ and it is annihilated. Otherwise we necessarily have a sequence $(\underbrace{+\cdots+}_{k_1}\ \underbrace{-\cdots-}_{k_2})$, and it is mapped to $(\underbrace{+\cdots+}_{k_1+1}\ \underbrace{-\cdots-}_{k_2-1})$. Namely, if $k_2 \neq 0$, one and only one $-$ is changed to $+$ by the action of $\tilde{f}_0$. Note that the configuration of the path (before the reduction of singlets), at the next left and the next right to this $(-)$ is $(+\ -\ -)$. This is changed into $(+\ +\ -)$. Thus $\tilde{f}_0$ shifts a domain wall to the left and cannot cause two domain walls to coalesce. In the above situation $\tilde{e}_0$ produces $(\underbrace{+\cdots+}_{k_1-1}\ \underbrace{-\cdots-}_{k_2+1})$. The rules for $\tilde{e}_1, \tilde{f}_1$ are given by exchanging the roles of $+$ and $-$.

Consider first the case $n = 1$. The paths in $\mathscr{P}(1)$ have spin $\pm 1/2$ according as l is even or odd. They are isomorphic to the crystal $\mathrm{Aff}(B)$ according to Definition 2.2.3 of [2] with non-zero action ($l \in 2\mathbf{Z}$)

$$\tilde{f}_1\colon [[l]] \mapsto [[l+1]],\quad \tilde{f}_0\colon [[l-1]] \mapsto [[l]],\quad l \in 2\mathbf{Z}\,, \tag{3.3}$$

and weights

$$\mathrm{wt}[[l]] = -(l/2)\delta + (1/2)\alpha_1,\quad \mathrm{wt}[[l+1]] = -(l/2)\delta - (1/2)\alpha_1\,.$$

Thus $\mathscr{P}(1)$ is a connected crystal.

Now consider $n = 2$. Corresponding to (3.3) we have (with $l_1, l_2 \in 2\mathbf{Z}, l_1 > l_2$)

$$\tilde{f}_1\colon [[l_1-1, l_2]] \mapsto [[l_1, l_2]] \mapsto [[l_1, l_2+1]] \mapsto 0\,, \tag{3.4a}$$

$$\tilde{f}_0\colon [[l_1, l_2+1]] \mapsto [[l_1+1, l_2+1]] \mapsto [[l_1+1, l_2+2]] \mapsto 0\,, \tag{3.4b}$$

and the weights of these triplets are respectively

$$-\frac{l_1+l_2}{2}\delta + \alpha_1, 0, -\alpha_1 \qquad \text{for (3.4a)}\,,$$

$$-\frac{l_1+l_2+2}{2}\delta + \alpha_0, 0, -\alpha_0 \quad \text{for (3.4b)}\,.$$

So $\mathscr{P}(2)$ has an infinite number of disjoint connected components: each such component may be labeled by the fact that it is generated from the spin-1 path $[[l, 0]]$ ($l \in 2\mathbf{Z}_{\geqq 0} + 1$) by (3.4). $\mathscr{P}(2)$ is only "half" of $\mathrm{Aff}(B) \otimes \mathrm{Aff}(B)$, the latter being labeled by a pair of integers without restriction.

Generally, in $\mathscr{P}(n)$ the maximum spin of a path is $n/2$, and such a path has all $(+\ +)$ walls. Typical is

$$[[l_1, l_2, \ldots, l_n]],\quad l_j - l_{j+1} \in 2\mathbf{Z}_{\geqq 0} + 1 \tag{3.5}$$

with wall labels alternating between even and odd integers and weight

$$\mathrm{wt}[[l_1, l_2, \ldots, l_n]] = -\frac{\left(l_1 + \cdots + l_n + \left[\frac{n}{2}\right]\right)}{2}\delta + \frac{n}{2}\alpha_1\,. \tag{3.6}$$

Here $[n/2]$ denotes integer part. Because of the ordering of the wall labels, there are in general only "$1/n!$ times as many" connected components in $\mathscr{P}(n)$ as in $\mathrm{Aff}(B)^{\otimes n}$. The precise formulation will be given in the next subsection.

3.2. Paths as a Quotient of $\sqcup_{n=0}^{\infty} \mathrm{Aff}(B)^{\otimes n}$. Let $\mathscr{Z}$ be the $\mathbf{Z}$-algebra generated by ψ_j^* $(j \in \mathbf{Z})$ satisfying the relations

$$\psi_j^* \psi_k^* + \psi_k^* \psi_j^* = 0, \quad \psi_{j+1}^* \psi_k^* + \psi_{k+1}^* \psi_j^* = 0\,, \tag{3.7}$$

for all $j, k \in \mathbf{Z}$ such that $j \equiv k \bmod 2$. This algebra arises from the commutation relation (6.20b) of the creation operators expanding them formally in q and z.

$$\varphi_+^*(z) = \sum_{j \in \mathbf{Z}} \psi_{2j}^* z^j + O(q)\,,$$

$$\varphi_-^*(z) = \sum_{j \in \mathbf{Z}} \psi_{2j+1}^* z^j + O(q)\,.$$

Here we assume that the creation operators preserve the crystal lattice, i.e., there is no negative powers in q (see Appendix 4). We have also removed the fractional powers in z from $\varphi_\pm^*$.

Consider the $\mathbf{Z}$-module $\mathbf{Z}\mathscr{P}$ spanned by the set of paths $\mathscr{P}^0 \sqcup \mathscr{P}^1$. We define an action of $\mathscr{Z}$ on $\mathbf{Z}\mathscr{P}$ as follows. Take $[[l_1, \ldots, l_n]] \in \mathscr{P}(n)$. If $n = 0$ we set

$$\psi_j^*[[\ ]] = [[j]]\,.$$

If $j > l_1 + n$, then we define

$$\psi_j^*[[l_1, \ldots, l_n]] = [[j - n, l_1, \ldots, l_n]]\,.$$

If $j = l_1 + n$ or $l_1 + n - 1$, we define

$$\psi_j^*[[l_1, \ldots, l_n]] = 0\,.$$

Finally, if $j < l_1 + n - 1$, then we define

$$\begin{aligned} \psi_j^*[[l_1, \ldots, l_n]] &= -\psi_{l_1+n-1}^* \psi_j^*[[l_2 \ldots l_n]] \quad \text{if } j \equiv l_1 + n - 1 \bmod 2\,, \\ &= -\psi_{l_1+n}^* \psi_{j-1}^*[[l_2 \ldots l_n]] \quad \text{if } j \not\equiv l_1 + n - 1 \bmod 2\,. \end{aligned}$$

From this action we see that the following is a linear base of $\mathscr{Z}$:

$$\mathscr{B} = \sqcup_{n=0}^{\infty} \mathscr{B}(n), \ \mathscr{B}(n) = \{\psi_{j_1}^* \cdots \psi_{j_n}^*; j_1 - n + 1 > j_2 - n + 2 > \cdots > j_n\}\,.$$

There is a bijection ω from $\mathscr{Z}$ to $\mathbf{Z}\mathscr{P}$ which maps $\mathscr{B}(n)$ to $\mathscr{P}(n)$,

$$\begin{aligned} &\omega\colon \mathscr{B}(n) \to \mathscr{P}(n)\,, \\ &\psi_{j_1}^* \cdots \psi_{j_n}^* \mapsto [[j_1 - n + 1, j_2 - n + 2, \ldots, j_n]]\,. \end{aligned} \tag{3.8}$$

This is obtained from the action of $\mathscr{Z}$ on the vector $[[\ ]] \in \mathscr{P}(0)$. (In particular, $\omega(1) = [[\ ]]$.)

We make $\mathscr{B}$ an affine crystal as follows. We define the weight by

$$\mathrm{wt}(\psi_{j_1}^* \cdots \psi_{j_n}^*) = \sum_{k=1}^{n} \mathrm{wt}(\psi_{j_k}^*)\,,$$

$$\mathrm{wt}(\psi_{2j}^*) = -j\delta + \frac{1}{2}\alpha_1, \quad \mathrm{wt}(\psi_{2j+1}^*) = -j\delta - \frac{1}{2}\alpha_1\,.$$

We define the crystal structure on $\mathscr{B}(1)$ by identifying it with $\mathrm{Aff}(B)$ by $\psi_j^* = [[j]]$.

To define the crystal structure on $\mathscr{B}(n)$ $(n > 1)$, let us consider the map

$$\sqcup_{n=0}^{\infty} \mathrm{Aff}(B)^{\otimes n} \to \sqcup_{n=0}^{\infty}(\mathscr{B}(n) \sqcup -\mathscr{B}(n)) \sqcup 0\,,$$

$$\psi_{j_1}^* \otimes \cdots \otimes \psi_{j_n}^* \mapsto \psi_{j_1}^* \cdots \psi_{j_n}^*\,.$$

Starting from the product $\psi_{j_1}^* \cdots \psi_{j_n}^*$ with an arbitrary set of indices $j = (j_1, \ldots, j_n) \in \mathbf{Z}^n$, we can modify it by using the rule (3.7) to $\pm \psi_{j'_1}^* \cdots \psi_{j'_n}^*$, where $j'_k > j'_{k+1} + 1$ for all k, or to 0. This process is compatible with the arrows in the crystal $\sqcup_{n=0}^{\infty}\mathrm{Aff}(B)^{\otimes n}$ in the following sense. Suppose that an index set j changes to j' by applying the rule (3.7) once. Suppose also that an arrow goes from $\psi_{j_1}^* \otimes \cdots \otimes \psi_{j_n}^*$ to $\psi_{\bar{j}_1}^* \otimes \cdots \otimes \psi_{\bar{j}_n}^*$, and $\psi_{j'_1}^* \otimes \cdots \otimes \psi_{j'_n}^*$ to $\psi_{\bar{j}'_1}^* \otimes \cdots \otimes \psi_{\bar{j}'_n}^*$. Then, by case checking one can show that $\bar{j}$ changes to $\bar{j}'$ by applying (3.7) once. Therefore we can induce a crystal structure on $\mathscr{B}$ consistently from $\sqcup_{n=0}^{\infty}\mathrm{Aff}(B)^{\otimes n}$.

The map ω from $\mathscr{B}(n)$ to $\mathscr{P}(n)$ is an isomorphism of crystals, i.e., it is a bijection and it commutes with $\tilde{f}_i$ and $\tilde{e}_i$. This is not an isomorphism of affine crystals, because the affine weights in $\mathscr{B}(n)$ and $\mathscr{P}(n)$ differ by a multiple of δ:

$$\begin{aligned}\mathrm{wt}(\omega(\psi_{j_1}^* \otimes \cdots \otimes \psi_{j_n}^*)) - \mathrm{wt}(\psi_{j_1}^* \otimes \cdots \otimes \psi_{j_n}^*) &= n^2/4 \quad \text{if } n \text{ is even} \\ &= (n^2-1)/4 \quad \text{if } n \text{ is odd.}\end{aligned}$$

3.3. The $q = 0$ Limit of the Eigenvectors. We discussed in Sect. 2 that there is no *a priori* method to tell what the $q = 0$ limit of the eigenvectors of the Hamiltonian is. Here we give a prediction using the map ω (3.8). Note that the definition (3.8) is based on the commutation relation of the creation operators which will be discussed in Sect. 7.

Define

$$\psi_+^*(z) = \sum_{j \in \mathbf{Z}} z^j \psi_{2j}^*, \quad \psi_-^*(z) = \sum_{j \in \mathbf{Z}} z^j \psi_{2j+1}^*\,.$$

We conjecture that the $q = 0$ limit of the n-particle eigenstates (see Sects. 5 and 7) are given by

$$\omega(\psi_{\varepsilon_1}^*(z_1) \cdots \psi_{\varepsilon_n}^*(z_n))\,.$$

The validity of this conjecture is checked for $n = 1, 2$ in Appendix 2. It is also consistent with the Bethe Ansatz calculation for $n = 2h$ and $s = h, h-1$ in Appendix 5.

4. The Vacuum State

The vacuum state is the unique $U'_q(\widehat{\mathfrak{sl}}(2))$-singlet in W under the boundary condition (2.12). To find the vacuum in W means to find a $U'_q(\widehat{\mathfrak{sl}}(2))$-linear map

$$\mathbf{Q}(q) \to V(\Lambda_0) \otimes V(\Lambda_0)^{*a}\,. \tag{4.1}$$

It is given by the canonical element;

$$|\mathrm{vac}\rangle = \sum_k v_k \otimes v_k^*\,. \tag{4.2}$$

Here v_k and v_k^* are dual bases. Since $V(\Lambda_0)$ is infinite dimensional, the above sum is an infinite sum. Therefore we need a completion of $V(\Lambda_0) \otimes V(\Lambda_0)^{*a}$ in order that the vacuum vector actually belong to it. For our purpose the way of completion is not very important. We take the largest one, i.e., the direct product of all the spaces $V(\Lambda_0)_\lambda \otimes V(\Lambda_0)_\mu^{*a}$, where λ and μ runs the weights of $V(\Lambda_0)$ and $V(\Lambda_0)^{*a}$ respectively. We do not use any particular notation for the completion. We just use $V(\Lambda_0) \otimes V(\Lambda_0)^{*a}$ as being completed. It is convenient to use another realization of $V(\Lambda_0) \otimes V(\Lambda_0)^{*a}$. Namely we use the canonical isomorphism

$$V(\Lambda_0) \otimes V(\Lambda_0)^{*a} \simeq \mathrm{End}_{\mathbf{Q}(q)}(V(\Lambda_0)) .$$

(Strictly speaking, the right-hand side is the set of $\mathbf{Q}(q)$-linear homomorphism from $V(\Lambda_0)$ to the completion of $V(\Lambda_0)$.) The $U_q(\widehat{\mathfrak{sl}}(2))$-action on the left-hand side is given by the coproduct. Then the action on the right-hand side is given by the adjoint action which we denote by ad x; suppose that

$$f \in \mathrm{End}_{\mathbf{Q}(q)}(V(\Lambda_0)),\ v \in V(\Lambda_0), \quad x \in U_q(\widehat{\mathfrak{sl}}(2))$$

and

$$\Delta(x) = \sum_k x_k^{(1)} \otimes x_k^{(2)} .$$

Then we have

$$((\mathrm{ad}\, x) f)(v) = \sum_k x_k^{(1)} f(a(x_k^{(2)})v) .$$

If $f = \mathrm{id}$, then $(\mathrm{ad}\, x) f = \varepsilon(x)$ (ε: the counit). Here we used the special choice of the left module structure V^{*a} in (4.1). Therefore id $\in \mathrm{End}_{\mathbf{Q}(q)}(V(\Lambda_0))$ generates a singlet. Or equivalently, the canonical element (4.2) actually realizes the vacuum. In Appendix 4 we give some explicit computation of the embedding of the canonical element in W.

One may raise a question about the translational invariance of the vacuum. Because of the splitting of the whole line into the right and the left pieces, it is not obvious that the definition of the vacuum is independent of the choice of the position of the splitting. Let us distinguish the vacuum in $V(\Lambda_0) \otimes V(\Lambda_0)^{*a}$ and the vacuum in $V(\Lambda_1) \otimes V(\Lambda_1)^{*a}$ by denoting the former by $|\mathrm{vac}\rangle_0$ and the latter by $|\mathrm{vac}\rangle_1$. We want to prove

$$T|\mathrm{vac}\rangle_i = |\mathrm{vac}\rangle_{1-i} \quad i = 0, 1 . \tag{4.3}$$

In order to prove this we need the interpretation of the translation T in our mathematical picture. For this purpose we use the vertex operators

$$\Phi\colon V(\Lambda_i) \to V(\Lambda_{1-i}) \otimes V$$

and

$$\Psi^*\colon V \otimes V(\Lambda_i)^{*a} \to V(\Lambda_{1-i})^{*a} .$$

We introduced Φ in Sect. 2 with the normalization

$$\Phi(u_{\Lambda_0}) = u_{\Lambda_1} \otimes v_- + \cdots, \quad \Phi(u_{\Lambda_1}) = u_{\Lambda_0} \otimes v_+ + \cdots .$$

We define Ψ^* as the intertwiner with the normalization $\Psi^*(v_- \otimes u_{\Lambda_0}^*) = u_{\Lambda_1}^* + \cdots$ and $\Psi^*(v_+ \otimes u_{\Lambda_1}^*) = u_{\Lambda_0}^* + \cdots$, where $u_{\Lambda_i}^*$ is the lowest weight vector of $V(\Lambda_i)^{*a}$ dual to u_{Λ_i}. We define

$$T\colon V(\Lambda_i)\otimes V(\Lambda_i)^{*a}\to V(\Lambda_{1-i})\otimes V(\Lambda_{1-i})^{*a}$$

by the composition

$$V(\Lambda_i)\otimes V(\Lambda_i)^{*a}\xrightarrow{\Phi\otimes \mathrm{id}} V(\Lambda_{1-i})\otimes V\otimes V(\Lambda_i)^{*a}\xrightarrow{\mathrm{id}\otimes\Psi^*} V(\Lambda_{1-i})\otimes V(\Lambda_{1-i})^{*a}$$

up to a constant multiple. This constant is determined in Sect. 7 and the proof of the assertion (4.3) is also given (see (7.4), Proposition 7.1). Once we establish the statement (4.3) then

$$H_{XXZ}|\mathrm{vac}\rangle_i=0 \tag{4.4}$$

follows immediately, because by the definition it is obvious that

$$d|\mathrm{vac}\rangle_i=0\,. \tag{4.5}$$

Before ending this section, we wish to comment on [31], where it was first conjectured, on the basis of direct perturbative calculations up to a low order in q, that the vacuum vector of the XXZ Hamiltonian is also an eigenvector of (1.4). In our algebraic scheme, this follows directly from (4.5). Our scheme also makes it clear that the excited states of the XXZ Hamiltonian are not eigenstates of d.

5. Particle Picture

By a particle we mean a finite-dimensional $U_q'(\widehat{\mathfrak{sl}}(2))$-module consisting of eigenvectors of the XXZ-Hamiltonian.

Starting from the infinite tensor product W, we have argued how the $U_q'(\widehat{\mathfrak{sl}}(2))$-module

$$V(\Lambda_0)\otimes V(\Lambda_0)^{*a}\simeq \mathrm{End}_{\mathbb{Q}(q)}(V(\Lambda_0))$$

is embedded therein, and how the vacuum (i.e., the zero particle state) is understood as a vector of the latter. Now, guided by the crystal decomposition of the even and the odd paths, we reach the following; Modulo statistics to be discussed later, the particle picture for the XXZ-Hamiltonian in the anti-ferroelectric regime is

$$\mathscr{F}=\left[\oplus_{n=0}^{\infty}\int\cdots\int V_{z_n}\otimes\cdots\otimes V_{z_1}du_n\cdots du_1\right]_{\mathrm{symm}}, \tag{5.1}$$

where u_i is a quasi-momentum of the 2-dimensional $U_q'(\widehat{\mathfrak{sl}}(2))$-module $V_{z_k}(z_k=e^{iu_k})$. We call $\mathscr{F}$ the Fock space. (For the meaning of the symmetrization, see (7.10).) The even particle (n: even) are contained in the even sector $V(\Lambda_0)\otimes V(\Lambda_0)^{*a}$, and the odd particles (n: odd) in the odd sector $V(\Lambda_1)\otimes V(\Lambda_0)^{*a}$. To see this we want to find a $U_q'(\widehat{\mathfrak{sl}}(2))$-linear map

$$V_{z_n}\otimes\cdots\otimes V_{z_1}\to V(\Lambda_{0\text{ or }1})\otimes V(\Lambda_0)^{*a}\,,$$

or equivalently, a $U_q'(\widehat{\mathfrak{sl}}(2))$-linear map

$$V_{z_n}\otimes\cdots\otimes V_{z_1}\otimes V(\Lambda_0)\to V(\Lambda_{0\text{ or }1})\,.$$

Therefore, the problem reduces to finding the vertex operators

$$\Phi(z)\colon V_z \otimes V(\Lambda_i) \to \mathrm{V}(\Lambda_{1-i}) \quad (i = 0, 1)\,.$$

The existence and the uniqueness of such vertex operators are given in [DJO] in a general setting. In Appendix 2, we give some explicit calculation of the above vertex operator in terms of the global base of Kashiwara.

Let us argue the physical content of our particle picture. The XXZ-Hamiltonian possesses the infinite hierarchy of the abelian higher order Hamiltonians, and the infinite dimensional non-abelian symmetries of $U'_q(\widehat{\mathfrak{sl}}(2))$. These are the symmetries of the XXZ-Hamiltonian in the strict sense, i.e., they commute with H_{XXZ}. The abelian symmetries do not change the eigenvectors, and the non-abelian symmetries do not change the eigenvalues. The Lorentz boost $e^{\varepsilon d}$ (ε: a scalar parameter) actually changes the energy, but never creates new particles nor annihilates them. Now we introduce the third symmetry of the XXZ-Hamiltonian, the dynamical symmetries which create and annihilate particles.

Define the creation operator

$$\varphi^*_\pm(z)\colon V(\Lambda_i) \otimes V(\Lambda_0)^{*a} \to V(\Lambda_{1-i}) \otimes V(\Lambda_0)^{*a}$$

by

$$\varphi^*_\pm(z)(v \otimes v^*) = \Phi(z)(v_\pm \otimes v) \otimes v^*\,.$$

Then, it is easy to see that $\varphi^*_\pm(z)$ acting on an n-particle state create an $(n+1)$-particle state. In this way, the Fock space $\mathscr{F}$ is embedded in $(V(\Lambda_0) \oplus V(\Lambda_1)) \otimes V(\Lambda_0)^{*a}$.

In Sects. 6 and 7, we give a mathematical treatment of the creation and the annihilation operators. Here, we give rather heuristic discussions on several points.

We argued the embedding of $V(\Lambda_0) \otimes V(\Lambda_0)^{*a}$ to the infinite tensor product. The vectors $G(p)$ in $V(\Lambda_0)$ are expanded in terms of the paths $|p\rangle$, where we consider the latter as vectors in the infinite tensor product. A question arises: Is it possible to expand the paths in terms of the vectors in $V(\Lambda_0)$. The answer is no. The infinite matrix which expresses the transition from $|p\rangle$ to $G(p)$ is of the form $1 + qA_1 + q^2A_2 + \cdots$. If we invert this, we get formally $1 - qA_1 + q^2(A_1^2 - A_2) + \cdots$. But A_1^2 has divergence on the diagonal. So, the transition matrix is not invertible. This means that our definition of the creation operators does not apply to the paths. Suppose we were to apply the creation operator to the bare vacuum $(\cdots + - + - \cdots)$. Since the action of the creation operators changes only the left half, the right half is unchanged. This contradicts the fact that the creation operator satisfies the proper commutation relations with the shift operator (see Sect. 7).

We conjecture that the creation operators preserve the crystal structure in the following sense. The vacuum vector embedded in the infinite tensor product is expanded in power series of q. The conjecture is that the particle states created by the creation operators upon the vacuum also have the same property. This is remarkable because the vertex operators used in the definition of the creation operators do not preserve the crystal lattices. In fact, if they do we have the following contradiction. At $q = 0$ the vacuum considered in $V(\Lambda_0) \otimes V(\Lambda_0)^{*a}$ reduces to $u_{\Lambda_0} \otimes u^*_{\Lambda_0}$, where u_{Λ_0} is the highest weight vector in $V(\Lambda_0)$ and $u^*_{\Lambda_0}$ the dual lowest weight vector in $V(\Lambda_0)^{*a}$ (see Appendix 4). So, if the crystal lattice were preserved by the vertex operators, then to get the $q = 0$ limit of a one-particle state, we only have to apply the vertex operator to u_{Λ_0}. But, this breaks the translational covariance which is expected for the one-particle state. In fact, we will see in Appendix 4, that we have contributions to the q^0-term in the one-particle state from the higher order terms in the vacuum state.

Now we come to a more subtle point. In Appendix 4, we computed 1 and 2 particle states by applying the vertex operators successively. In the computation of

$$\Phi(z_2)(v_- \otimes \Phi(z_1)(v_+ \otimes u_{\Lambda_0}))$$

(which is a part of the computation of $\varphi^*_-(z_2)\varphi^*_+(z_1)|\text{vac}\rangle$), we find terms having poles at $q = 0$. But these terms seem to be summed up to a meromorphic function in z_1, z_2 and q that has no pole (actually has a zero) at $q = 0$. So, even though the vertex operator $\Phi(z)$: $V_z \otimes V(\Lambda_i) \to V(\Lambda_{1-i})$ does not preserve the crystal lattice, the particles created by $\varphi^*_\pm(z)$ may not (and, in fact, do not) have poles at $q = 0$. Namely, the crystal picture survives. The necessity of this summation also tells that the Fourier components of the creation and the annihilation operators are not equal to those of the vertex operators. We will further discuss this in Sect. 7.

Let us mention a few words about the statistics of our particles. The tensor products $V_{z_2} \otimes V_{z_1}$ and $V_{z_1} \otimes V_{z_2}$ are different but isomorphic. They are intertwined by the R-matrix. In fact, the embedded images of these tensor products in $V(\Lambda_0) \otimes V(\Lambda_0)^{*a}$ are equal. This follows from the uniqueness (up to scalar multiple) of the vertex operator $V_{z_1} \otimes V_{z_2} \otimes V(\Lambda_0) \to V(\Lambda_0)$. So the key point is to determine this scalar multiple. We will give the answer to this question in Sect. 6 by using the quantum K–Z equation.

Do particles with spin higher than $\frac{1}{2}$ exist? Let $V^{(j)}$ $(j \in \frac{1}{2}\mathbf{Z})$ be the $(2j+1)$-dimensional $U_q(\mathfrak{sl}(2))$-module, and $V_z^{(j)}$ its $U'_q(\widehat{\mathfrak{sl}}(2))$-extension. If we understand this question as the existence of the vertex operator

$$V_z^{(j)} \otimes V(\Lambda_0) \to V(\Lambda_0) \text{ or } V(\Lambda_1)\,,$$

one can show the non-existence by an argument of crystals. If we treat the problem honestly, we should start from the finite-lattice Bethe Ansatz and examine the thermodynamic limit. There are papers [11] for the XXX-case $(\Delta = -1)$ and [9] for the XXZ-case $(\Delta < -1)$, which are in support of our picture.

Let us consider the dual space V^* of V. The XXZ-Hamiltonian formally acts on the infinite tensor product of V^*, too. So, we can formulate the theory in a completely analogous way by using right modules. In that case, the (even) particle picture will be developed on the tensor product of two right modules $V(\Lambda_0)^* \otimes V(\Lambda_0)^{**a^{-1}}$. Let us, in general, define a natural pairing $\langle\ |\ \rangle$

$$(V(\Lambda')^* \otimes V(\Lambda)^{**a^{-1}}) \otimes (V(\Lambda') \otimes V(\Lambda)^{*a}) \to \mathbf{Q}(q)\,,$$

where Λ and Λ' are arbitrary dominant integral weights. The left $U_q(\widehat{\mathfrak{sl}}(2))$-module $V(\Lambda') \otimes V(\Lambda)^{*a}$ and the right $U_q(\widehat{\mathfrak{sl}}(2))$-module $V(\Lambda')^* \otimes V(\Lambda)^{**a}$ have the canonical pairing $\langle\ ,\ \rangle$ induced from the pairing between $V(\Lambda')$ and $V(\Lambda')^*$ and the pairing between $V(\Lambda)^*$ and $V(\Lambda)^{**}$. This pairing $\langle\ ,\ \rangle$ satisfies

$$\langle fx, g\rangle = \langle f, xg\rangle\,,$$

for $f \in V(\Lambda')^* \otimes V(\Lambda)^{**a}$, $g \in V(\Lambda') \otimes V(\Lambda)^{*a}$ and $x \in U_q(\widehat{\mathfrak{sl}}(2))$. Note that there is an isomorphism of $U_q(\widehat{\mathfrak{sl}}(2))$-modules $V(\Lambda)^{**a^{-1}} \simeq V(\Lambda)^{**a}$ given by

$$(V(\Lambda)^{**a^{-1}})_\lambda \ni v \mapsto q^{-4(\rho,\lambda)} v \in (V(\Lambda)^{**a})_\lambda .$$

See Section 6 for the definition of ρ. We define $\langle \ | \ \rangle$ by setting

$$\langle u' \otimes w' | u \otimes w \rangle = \langle q^{-4(\rho,\, wt(w'))} u' \otimes w', u \otimes w \rangle ,$$

for weight vectors u', w', u, v. If we regard $f \in V(\Lambda')^* \otimes V(\Lambda)^{**a^{-1}}$ (respectively $g \in V(\Lambda') \otimes V(\Lambda)^{*a}$) as an element of $\mathrm{Hom}(V(\Lambda'), V(\Lambda))$ (respectively $\mathrm{Hom}(V(\Lambda), V(\Lambda'))$) then we have

$$\langle f | g \rangle = \mathrm{tr}_{V(\Lambda)}(q^{-4\rho} f \circ g) .$$

Obviously this pairing satisfies

$$\langle fx | g \rangle = \langle f | xg \rangle \quad \text{for } x \in U_q(\widehat{\mathfrak{sl}}(2)) .$$

In particular, we have

$$\langle \mathrm{vac} | \mathrm{vac} \rangle = \mathrm{tr}_{V(\Lambda_0)}(q^{-4\rho}) ,$$

which is the specialized character for $V(\Lambda_0)$.

Getting the character expression for the one-point function of the six-vertex model [2, 23] is a simple corollary of this formula. Define a non-local operator

$$\tau_k = \sum_{j>k} \sigma_j^z$$

on $V(\Lambda_0) \otimes V(\Lambda_0)^{*a}$. We understand the meaning of the infinite sum by the normalization $\tau_0(u_{\Lambda_0} \otimes u^*_{\Lambda_0}) = 0$. The one-point function

$$P(m) = \frac{\langle \mathrm{vac} | \mathrm{Proj}(\tau_0 = m) | \mathrm{vac} \rangle}{\langle \mathrm{vac} | \mathrm{vac} \rangle} \quad (m \in \mathbf{Z})$$

of the six-vertex model is by definition the expectation value of the projection operator to the $\tau_0 = m$ eigenspace. Therefore, it is given by

$$P(m) = \frac{\mathrm{tr}_{V(\Lambda_0)_m} q^{-4\rho}}{\mathrm{tr}_{V(\Lambda_0)} q^{-4\rho}} ,$$

where $V(\Lambda_0)_m$ is the spin m subspace of $V(\Lambda_0)$.

By a similar construction, the dual Fock space $\mathscr{F}^*$ is embedded in $(V(\Lambda_0)^* \oplus V(\Lambda_1)^*) \otimes V(\Lambda_0)^{**a^{-1}}$. The bilinear coupling between $V(\Lambda_i) \otimes V(\Lambda_0)^{*a}$ and $V(\Lambda_i)^* \otimes V(\Lambda_0)^{**a^{-1}}$ induces a non-degenerate coupling between $\mathscr{F}$ and $\mathscr{F}^*$. The n-particle states $V_{z_n} \otimes \cdots \otimes V_{z_1}$ in $\mathscr{F}$ and the m-particle states $V^*_{z'_m} \otimes \cdots \otimes V^*_{z'_1}$ in $\mathscr{F}^*$ are orthogonal unless $n = m$ and $\{z_i\} = \{z'_i\}F$.

Finally, we will consider the annihilation operator. As we defined the creation operator in the frame of left modules, we define the operator which create finite dimensional right $U'_q(\widehat{\mathfrak{sl}}(2))$-modules, in $V(\Lambda_0)^* \otimes V(\Lambda_0)^{**^{a^{-1}}}$. It is given by means of the vertex operator of the form

$$\Phi^*(z)\colon V_z^* \otimes V(\Lambda_i)^* \to V(\Lambda_{1-i})^* .$$

Let $v_\pm^* \in V_z^*$ be the dual elements to $v_\pm$; $\langle v_\varepsilon^*, v_{\varepsilon'}\rangle = \delta_{\varepsilon\varepsilon'}$. Define

$$\varphi_\pm(z)\colon V(\Lambda_i)^* \otimes V(\Lambda_0)^{**^{a^{-1}}} \to V(\Lambda_{1-i})^* \otimes V(\Lambda_0)^{**^{a^{-1}}}$$

by

$$\varphi_\pm(z)(w^* \otimes w) = \Phi^*(z)(v_\pm^* \otimes w^*) \otimes w ,$$

where $w^* \in V(\Lambda_i)^*$ and $w \in V(\Lambda_0)$. We define the annihilation operator $\varphi_\pm(z)$ acting on $\mathscr{F}$ by the dual action of $\varphi_\pm(z)\colon \mathscr{F}^* \to \mathscr{F}^*$. Since $\varphi_\pm(z)$ creates $(n+1)$-particle states from n-particle states in $\mathscr{F}^*$, $\varphi_\pm(z)$ annihilates $(n+1)$-particle states to n-particle states in $\mathscr{F}$.

6. Vertex Operators

So far we have argued how one can treat the elementary excitations in the antiferroelectric XXZ model in the framework of representation theory. The motivations being given, we now turn to the mathematics of vertex operators.

6.1. Notation. Let us restart by fixing the notations. Thus let $P = \mathbf{Z}\Lambda_0 \oplus \mathbf{Z}\Lambda_1 \oplus \mathbf{Z}\delta$ and $P^* = \mathbf{Z}h_0 \oplus \mathbf{Z}h_1 \oplus \mathbf{Z}d$ be the weight lattice of $\widehat{\mathfrak{sl}}(2)$ and its dual lattice respectively. We have $\langle \Lambda_i, h_j\rangle = \delta_{ij}$, $\langle \Lambda_i, d\rangle = 0$, $\langle \delta, h_i\rangle = 0$ and $\langle \delta, d\rangle = 1$. We set $\alpha_1 = 2\Lambda_1 - 2\Lambda_0, \alpha_0 = \delta - \alpha_1$ and $\rho = \Lambda_0 + \Lambda_1$. We normalize the invariant symmetric bilinear form $(\ ,\ )$ on P by $(\alpha_i, \alpha_i) = 1$. Explicitly we have $(\Lambda_i, \Lambda_j) = \delta_{i1}\delta_{j1}/4$, $(\Lambda_i, \delta) = 1/2$ and $(\delta, \delta) = 0$. We shall regard P^* as a subset of P via $(\ ,\ )$, so that $2\alpha_i = h_i$ and $4\rho = h_1 + 4d$.

The quantized affine algebra $U = U_q(\widehat{\mathfrak{sl}}(2))$ is defined on generators e_i, f_i $(i = 0, 1)$, q^h $(h \in P^*)$ over the base field $\mathbf{Q}(q)$. The defining relations are as given in [2, 25], e.g. $[e_i, f_j] = \delta_{ij}(t_i - t_i^{-1})/(q - q^{-1})$, where $t_i = q^{h_i}$. We let $U' = U'_q(\widehat{\mathfrak{sl}}(2))$ denote the subalgebra generated by e_i, f_i and t_i $(i = 0, 1)$. We shall take the coproduct Δ to be

$$\begin{aligned} &\Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i , \\ &\Delta(q^h) = q^h \otimes q^h \quad (h \in P^*) . \end{aligned} \tag{6.1}$$

Accordingly the formula for the antipode a reads

$$a(e_i) = -t_i^{-1}e_i, \quad a(f_i) = -f_it_i, \quad a(q^h) = q^{-h} .$$

6.2. Modules. Given a left U-module M we write its weight space as $M_\nu = \{v \in M \,|\, q^h v = q^{\langle \nu, h\rangle} v \ \ \forall h \in P^*\}$. For $u \in M_\nu$ we write $\mathrm{wt}(u) = \nu$. Suppose $M = \bigoplus_\nu M_\nu$, and let ϕ be an anti-automorphism of U. Then the restricted dual $M^* = \bigoplus_\nu M_\nu^*$ is endowed with a left module structure $M^{*\phi}$ via

$$\langle u, xv\rangle = \langle \phi(x)u, v\rangle \quad \text{for } x \in U, u \in M, v \in M^* .$$

We have $M \simeq (M^{*\phi})^{*\phi^{-1}}$ (canonically). Similar convention is used for right modules and U'-modules. Taking $\phi = a$ we have the canonical identification

$$\mathrm{Hom}_{U'}(L, M \otimes N) = \mathrm{Hom}_{U'}(M^{*a} \otimes L, N)\,,$$

$$\mathrm{Hom}_{U'}(L \otimes N, M) = \mathrm{Hom}_{U'}(L, M \otimes N^{*a}) \tag{6.2}$$

for left modules (for right modules we replace a by a^{-1}).

In what follows we set

$$\lambda = \Lambda_i, \quad \mu = \Lambda_{1-i} \quad \text{for } i = 0 \text{ or } 1. \tag{6.3}$$

As before let $V(\lambda)$ (respectively $V^r(\lambda)$) denote the integrable left (respectively right) highest weight module with highest weight λ. To distinguish the left and right structures we shall use the bra-ket notation $\langle u|, |u\rangle$ for vectors in $V^r(\lambda)$ and $V(\lambda)$, respectively. We fix nonzero highest weight vectors $\langle u_\lambda| \in V^r(\lambda), |u_\lambda\rangle \in V(\lambda)$ once for all. Then there is a unique non-degenerate symmetric bilinear pairing $V^r(\lambda) \times V(\lambda) \to \mathbf{Q}(q)$ such that

$$\langle u_\lambda | u_\lambda\rangle = 1, \quad \langle ux|u'\rangle = \langle u|xu'\rangle \quad \text{for any } \langle u| \in V^r(\lambda), |u'\rangle \in V(\lambda)\,.$$

We shall also consider the simplest two-dimensional U'-module $V = \mathbf{Q}(q)v_+ + \mathbf{Q}(q)v_-$ given by

$$e_1 v_+ = 0, \quad e_1 v_- = v_+, \quad f_1 v_+ = v_-, \quad f_1 v_- = 0, \quad t_1 v_\pm = q^{\pm 1} v_\pm\,,$$

$$e_0 = f_1, \quad f_0 = e_1, \quad t_0 = t_1^{-1} \quad \text{on } V\,.$$

For fixed $m \in \mathbf{Z}$ we equip $V \otimes \mathbf{Q}(q)[z, z^{-1}]$ with a U-module structure by letting

$$e_i \text{ act as } e_i \otimes (zq^m)^{\delta_{i0}}, \ f_i \text{ act as } f_i \otimes (zq^m)^{-\delta_{i0}}\,,$$

$$\text{the weight of } v_\pm \otimes z^n = n\delta \pm (\Lambda_1 - \Lambda_0)\,.$$

The resulting U-module will be denoted by V_{zq^m}. (This conflicts with the notation for weight spaces, but the meaning will be clear from the context.)

Let $v_\pm^*$ be the basis of V^* dual to $v_\pm$: $\langle v_i, v_j^*\rangle = \delta_{ij}$. One checks readily that the following give isomorphisms ("charge conjugation") of U-modules

$$C_\pm : V_{zq^{\mp 2}} \xrightarrow{\sim} V_z^{*a^{\pm 1}}\,,$$

$$C_\pm v_+ = v_-^*, \quad C_\pm v_- = -q^{\pm 1} v_+^*\,. \tag{6.4}$$

6.3. Basic Vertex Operators. By vertex operators (VOs) we will mean the intertwiners of U'-modules of the type

$$\tilde{\Phi}_\lambda^{\mu V}: V(\lambda) \to \hat{V}(\mu) \otimes V\,, \tag{6.5}$$

$$\tilde{\Phi}_\lambda^{V\mu}: V(\lambda) \to V \otimes \hat{V}(\mu)\,. \tag{6.6}$$

Here $\hat{V}(\mu) = \prod_\nu V(\mu)_\nu$ is a completion of $V(\mu)$. Since the weight is preserved modulo δ, for given $v \in V(\lambda)_\nu$ one can write $\tilde{\Phi}_\lambda^{\mu V}\, v = \sum_{n \in \mathbf{Z}} (u_{+,n} \otimes v_+ + u_{-,n} \otimes v_-)$, where $u_{\pm,n} \in V(\mu)_{\nu \mp (\Lambda_1 - \Lambda_0) + n\delta}$. (Because the weights of $V(\mu)$ are bounded from above, $u_{\pm,n} = 0$ for n large enough). Thus one can define the weight components $(\tilde{\Phi}_\lambda^{\mu V})_{\pm,n}$ $(\tilde{\Phi}_\lambda^{V\mu})_{\pm,n}$ by

$$\tilde{\Phi}_\lambda^{\mu V} = \sum_{n \in \mathbf{Z}, \pm} (\tilde{\Phi}_\lambda^{\mu V})_{\pm,n} \otimes v_\pm, \quad \tilde{\Phi}_\lambda^{V\mu} = \sum_{n \in \mathbf{Z}, \pm} v_\pm \otimes (\tilde{\Phi}_\lambda^{V\mu})_{\pm,n},$$

$$(\tilde{\Phi}_\lambda^{\mu V})_{\pm,n}, (\tilde{\Phi}_\lambda^{V\mu})_{\pm,n} \colon V(\lambda)_\nu \to V(\mu)_{\nu \mp (\Lambda_1 - \Lambda_0) + n\delta} . \tag{6.7}$$

We shall fix the normalization as follows:

$$\tilde{\Phi}_\lambda^{\mu V} |u_\lambda\rangle = |u_\mu\rangle \otimes v_\mp + \cdots, \tag{6.8a}$$

$$\tilde{\Phi}_\lambda^{V\mu} |u_\lambda\rangle = v_\mp \otimes |u_\mu\rangle + \cdots. \tag{6.8b}$$

Here v_- (respectively v_+) is chosen for $\lambda = \Lambda_0$ (respectively $\lambda = \Lambda_1$). In (6.8a) $\cdots$ means terms of the form $|u\rangle \otimes v$ with $|u\rangle \notin V(\mu)_\mu$, and similarly for (6.8b). The existence and uniqueness of such VOs are shown in [1, 4].

The VOs can be equivalently formulated as intertwiners of U-modules of the form [1]

$$\Phi_\lambda^{\mu V}(z) = \tilde{\Phi}_\lambda^{\mu V}(z) z^{\Delta_\mu - \Delta_\lambda}, \quad \Phi_\lambda^{V\mu}(z) = \tilde{\Phi}_\lambda^{V\mu}(z) z^{\Delta_\mu - \Delta_\lambda},$$

$$\tilde{\Phi}_\lambda^{\mu V}(z) = \sum (\tilde{\Phi}_\lambda^{\mu V})_{\varepsilon,n} \otimes v_\varepsilon z^{-n} \colon V(\lambda) \to V(\mu) \hat{\otimes} V_z, \tag{6.9a}$$

$$\tilde{\Phi}_\lambda^{V\mu}(z) = \sum v_\varepsilon z^{-n} \otimes (\tilde{\Phi}_\lambda^{V\mu})_{\varepsilon,n} \colon V(\lambda) \to V_z \hat{\otimes} V(\mu) . \tag{6.9b}$$

Here we set $\Delta_\lambda = (\lambda, \lambda + 2\rho)/(k + h^\vee)$, where $k = 1$ is the level and $h^\vee = 2$ is the dual Coxeter number; explicitly $\Delta_{\Lambda_0} = 0$, $\Delta_{\Lambda_1} = 1/4$. The right-hand sides of (6.9a), (6.9b) mean e.g. $V(\mu) \hat{\otimes} V_z = \bigoplus_\nu \prod_\xi V(\mu)_\xi \otimes (V_z)_{\nu - \xi}$. The fractional powers are so designed as to put the q-KZ equation in neater form, see below.

By abuse of notation we let $d \in \mathrm{End}(V(\lambda))$ denote the operator

$$d|u\rangle = \langle d, \nu\rangle |u\rangle \quad |u\rangle \in V(\lambda)_\nu . \tag{6.10}$$

It is easy to see that $[d, (\tilde{\Phi}_\lambda^{\mu V})_{\pm,n}] = n(\tilde{\Phi}_\lambda^{\mu V})_{\pm,n}$ and hence that

$$(d \otimes \mathrm{id}) \Phi_\lambda^{\mu V}(z) - \Phi_\lambda^{\mu V}(z) d = -\left(z \frac{d}{dz} - \Delta_\mu + \Delta_\lambda\right) \Phi_\lambda^{\mu V}(z) .$$

Similar relation holds for $\Phi_\lambda^{V\mu}(z)$.

Remark. In [2, 4] the coproduct of U is chosen to be

$$\Delta_-(e_i) = e_i \otimes t_i^{-1} + 1 \otimes e_i, \quad \Delta_-(f_i) = f_i \otimes 1 + t_i \otimes f_i,$$

$$\Delta_-(q^h) = q^h \otimes q^h \quad (h \in P^*) .$$

The present formulation using the coproduct $\Delta = \Delta_+$ (6.1) is related to the references above as follows.

Let M, N be U-modules such that $\mathrm{wt}(M) \subset \lambda_0 + \sum \mathbf{Z}\alpha_i$, $\mathrm{wt}(N) \subset \mu_0 + \sum \mathbf{Z}\alpha_i$ for some $\lambda_0, \mu_0 \in P$. We define operators β_M, γ_{MN} by

$$\beta_M u = q^{-(\lambda,\lambda) + (\lambda_0,\lambda_0)} u \quad u \in M_\lambda,$$

$$\gamma_{MN} u \otimes v = q^{2(\lambda,\mu) - 2(\lambda_0,\mu_0)} u \otimes v, \quad u \in M_\lambda, \quad v \in N_\mu .$$

Then $\Delta_+(x) = \gamma_{MN} \circ \Delta_-(x) \circ \gamma_{MN}^{-1}$ $(x \in U)$ and $\beta_M \otimes \beta_N = \beta_{M \otimes N} \circ \gamma_{MN} = \gamma_{MN} \circ \beta_{M \otimes N}$. We extend γ_{MN} also to $M \hat{\otimes} N$. It is known [25] that

(i) (L, B) is a lower crystal base of M if and only if $(\beta_M(L), \beta_M(B))$ is an upper crystal base of M.

Suppose M_i have lower crystal bases $(L_i^{\mathrm{low}}, B_i^{\mathrm{low}})$ $(i = 1, 2, 3)$, and set $L_i^{\mathrm{up}} = \beta_{M_i}(L_i^{\mathrm{low}})$, $B_i^{\mathrm{up}} = \beta_{M_i}(B_i^{\mathrm{low}})$. For Φ^{low}: $M_1 \to M_2 \hat{\otimes} M_3$ we put $\Phi^{\mathrm{up}} = \gamma_{M_2 M_3} \circ \Phi^{\mathrm{low}}$. Then we have

(ii) $\Phi^{\mathrm{low}} x = \Delta_-(x) \Phi^{\mathrm{low}}$ if and only if $\Phi^{\mathrm{up}} x = \Delta_+(x) \Phi^{\mathrm{up}}$ $(x \in U)$,
(iii) $\Phi^{\mathrm{up}} \beta_{M_1} = \beta_{M_2} \otimes \beta_{M_3} \Phi^{\mathrm{low}}$. Hence $\Phi^{\mathrm{low}}(L_1^{\mathrm{low}}) \subset L_2^{\mathrm{low}} \hat{\otimes} L_3^{\mathrm{low}}$ if and only if $\Phi^{\mathrm{up}}(L_1^{\mathrm{up}}) \subset L_2^{\mathrm{up}} \hat{\otimes} L_3^{\mathrm{up}}$.

Similar statements are valid for the intertwiners of the type Ψ: $M_1 \otimes M_2 \to M_3$.

6.4. Variants of Vertex Operators. The identification (6.2) along with the isomorphisms (6.4) gives rise to the following variants of intertwiners.

Type I:

$$\tilde{\Phi}_\lambda^{\mu V}: V(\lambda) \to \hat{V}(\mu) \otimes V, \tag{6.11a}$$

$$\tilde{\Phi}_{\mu V}^{\lambda}: V(\mu) \otimes V \to \hat{V}(\lambda), \tag{6.11b}$$

$$\tilde{\Phi}_{V\mu}^{*\lambda}: V \otimes V(\mu)^{*a} \to \hat{V}(\lambda)^{*a}, \tag{6.11c}$$

$$\tilde{\Phi}_\lambda^{*V\mu}: V(\lambda)^{*a} \to V \otimes \hat{V}(\mu)^{*a}. \tag{6.11d}$$

Type II:

$$\tilde{\Phi}_\lambda^{V\mu}: V(\lambda) \to V \otimes \hat{V}(\mu), \tag{6.12a}$$

$$\tilde{\Phi}_{V\mu}^{\lambda}: V \otimes V(\mu) \to \hat{V}(\lambda), \tag{6.12b}$$

$$\tilde{\Phi}_{\mu V}^{*\lambda}: V(\mu)^{*a} \otimes V \to \hat{V}(\lambda)^{*a}, \tag{6.12c}$$

$$\tilde{\Phi}_\lambda^{*\mu V}: V(\lambda)^{*a} \to \hat{V}(\mu)^{*a} \otimes V. \tag{6.12d}$$

That is, (6.11b) is obtained from

$$V(\mu) \to \hat{V}(\lambda) \otimes V^{*a},$$

and (6.11c), (6.11d) are transpose of

$$V(\lambda) \to \hat{V}(\mu) \otimes V^{*a^{-1}}, \quad V(\mu) \otimes V^{*a^{-1}} \to \hat{V}(\lambda)$$

respectively. The case of Type II is similar. We define

$$\Phi_{\lambda V}^{\mu}(z)(u \otimes v) = (\mathrm{id} \otimes \langle v, \rangle) \Phi_\lambda^{\mu V^{*a}}(z) u, \ \Phi_\lambda^{\mu V^{*a}}(z) = (\mathrm{id} \otimes C_+) \Phi_\lambda^{\mu V}(zq^{-2}), \tag{6.13a}$$

$$\Phi_{V\lambda}^{\mu}(z)(v \otimes u) = (\langle v, \rangle \otimes \mathrm{id}) \Phi_\lambda^{V^{*a^{-1}}\mu}(z) u, \ \Phi_\lambda^{V^{*a^{-1}}\mu}(z) = (C_- \otimes \mathrm{id}) \Phi_\lambda^{V\mu}(zq^2). \tag{6.13b}$$

This implies the normalization

$$\Phi_{\lambda V}^{\mu}(z)(|u_\lambda\rangle \otimes v_\pm) = \mp z^{\pm 1/4} q^{1/2} |u_\mu\rangle + \cdots,$$

$$\Phi_{V\lambda}^{\mu}(z)(v_\pm \otimes |u_\lambda\rangle) = \mp z^{\pm 1/4} q^{-1/2} |u_\mu\rangle + \cdots,$$

where the upper (respectively lower) sign is chosen for $\lambda = \Lambda_0$ (respectively $\lambda = \Lambda_1$). For example $z^{-\Delta_\mu + \Delta_\lambda} \Phi_{\lambda V}^{\mu}(z)$ is a $U \otimes \mathbf{Q}(q)[z, z^{-1}]$-linear map

$$V(\lambda) \hat{\otimes} V_z \to \hat{V}(\mu) \otimes \mathbf{Q}(q)[z, z^{-1}] ,$$

where on the right-hand side $x \in U'$ acts as $x \otimes \mathrm{id}$ and q^d acts as $q^d \otimes q^{zd/dz}$, $(q^{zd/dz} f)(z) = f(qz)$.

We have distinguished the two types of intertwiners. The main difference is that the type I operators preserve the crystal lattice (see Sect. 6.7), while the type II operators do not. We shall see explicit examples of the latter phenomenon in Appendix 4.

6.5. Two Point Functions. Frenkel and Reshetikhin [1] showed that the correlation functions of the vertex operators satisfy a q analog of the Knizhnik–Zamolodchikov (q-KZ) equation. For our subsequent discussions we need mostly the case of two point functions for various combinations of VOs. The general case will be discussed in Sect. 6.8.

Let $\Psi(z_1, z_2)$ be one of the following correlations:

$$\langle u_\lambda | \Phi_\mu^{\lambda V_2}(z_2) \Phi_\lambda^{\mu V_1}(z_1) | u_\lambda \rangle , \tag{6.14a}$$

$$\langle u_\lambda | \Phi_\mu^{V_2 \lambda}(z_2) \Phi_\lambda^{\mu V_1}(z_1) | u_\lambda \rangle , \tag{6.14b}$$

$$\langle u_\lambda | \Phi_\mu^{\lambda V_2}(z_2) \Phi_\lambda^{V_1 \mu}(z_1) | u_\lambda \rangle , \tag{6.14c}$$

$$\langle u_\lambda | \Phi_\mu^{V_2 \lambda}(z_2) \Phi_\lambda^{V_1 \mu}(z_1) | u_\lambda \rangle . \tag{6.14d}$$

Clearly

$$\Psi(z_1, z_2) \in V \otimes V \otimes (z_1/z_2)^{\Delta_\mu - \Delta_\lambda} \mathbf{Q}(q)[[z_1/z_2]] . \tag{6.15}$$

The subscripts of V in (6.14a)–(6.14d) indicate the tensor components. In (6.15) the left V is V_1 and the right one is V_2. Thus, for example, if we replace $|u_\lambda\rangle$ in (6.14a) by $f_i |u_\lambda\rangle$ then the result becomes

$$(f_i \otimes 1 + t_i^{-1} \otimes f_i) \langle u_\lambda | \Phi_\mu^{\lambda V_2}(z_2) \Phi_\lambda^{\mu V_1}(z_1) | u_\lambda \rangle \in V \otimes V .$$

(Note that in the discussion of the embedding of $V(\lambda)$ into $V^{\otimes \infty}$ in Sect. 3 and in Sect. 6.8 we use the opposite ordering of the components.) In what follows we introduce the element $q^{h_1/4}$ in U and extend the base field $\mathbf{Q}(q)$ by adding $q^{1/4}$. (We could avoid using fractional powers, but the formulas would become slightly more cumbersome.)

We need also prepare the R matrix. Define $\bar{R}(z)$, $R^+(z)$ by

$$\begin{aligned}
\bar{R}(z) v_\pm \otimes v_\pm &= v_\pm \otimes v_\pm , \\
\bar{R}(z) v_+ \otimes v_- &= \frac{1-z}{1-q^2 z} q v_+ \otimes v_- + \frac{1-q^2}{1-q^2 z} z v_- \otimes v_+ , \\
\bar{R}(z) v_- \otimes v_+ &= \frac{1-q^2}{1-q^2 z} v_+ \otimes v_- + \frac{1-z}{1-q^2 z} q v_- \otimes v_+ .
\end{aligned} \tag{6.16}$$

$$R^+(z) = \rho(z) \bar{R}(z), \quad \rho(z) = q^{-1/2} \frac{(q^2 z)_\infty^2}{(z)_\infty (q^4 z)_\infty} ,$$

where we put

$$(z)_\infty = (z; q^4)_\infty, \quad (z; p)_\infty = \prod_{j=0}^{\infty} (1 - zp^j) .$$

Let further $P \in \mathrm{End}(V \otimes V)$ be $Pv \otimes v' = v' \otimes v$. Then $P\bar{R}(z_1/z_2)$: $V_{z_1} \otimes V_{z_2} \to V_{z_2} \otimes V_{z_1}$ is an intertwiner of U-modules, and $R^+(z_1/z_2)$ is the image of the universal R matrix of U in $\mathrm{End}(V_{z_1} \otimes V_{z_2})$. The scalar factor $\rho(z)$ is determined by the argument in [1].

With these notations the q-KZ equations read as follows:

$$\Psi(q^6 z_1, z_2) = A(z_1, z_2)\Psi(z_1, z_2), \; \Psi(q^6 z_1, q^6 z_2) = (q^{-\phi} \otimes q^{-\phi})\Psi(z_1, z_2) , \quad (6.17)$$

where $\phi = 4\bar{\lambda} + h_1$, $\bar{\Lambda}_0 = 0$, $\bar{\Lambda}_1 = h_1/4$, and

$$\begin{aligned} A(z_1, z_2) &= R^+(q^6 z_1/z_2)(q^{-\phi} \otimes 1) \quad \text{for (6.14a)} , \\ &= (q^{-2\bar{\lambda}} \otimes 1)R^+(q^5 z_1/z_2)(q^{-\phi+2\bar{\lambda}} \otimes 1) \quad \text{for (6.14b)} , \\ &= (q^{2\bar{\lambda}-\phi} \otimes 1)R^+(q z_1/z_2)(q^{-2\bar{\lambda}} \otimes 1) \quad \text{for (6.14c)} , \\ &= (q^{-\phi} \otimes 1)R^+(z_1/z_2) \quad \text{for (6.14d)} . \end{aligned}$$

In the present case the property (6.15) and the normalization (6.8a)–(6.8b) specify the solutions of (6.17) uniquely. We list the answers below.

$\lambda = \Lambda_0$:

$$\begin{aligned} (z_1/z_2)^{-1/4}\Psi(z_1, z_2) &= \frac{(q^6 z_1/z_2)_\infty}{(q^4 z_1/z_2)_\infty}(v_- \otimes v_+ - qv_+ \otimes v_-) \quad \text{for (6.14a)} , \\ &= \frac{(q^5 z_1/z_2)_\infty}{(q^3 z_1/z_2)_\infty}(v_- \otimes v_+ - qv_+ \otimes v_-) \quad \text{for (6.14b)} , \\ &= \frac{(q z_1/z_2)_\infty}{(q^{-1} z_1/z_2)_\infty}(v_- \otimes v_+ - q^{-1}v_+ \otimes v_-) \quad \text{for (6.14c)} , \\ &= \frac{(z_1/z_2)_\infty}{(q^{-2} z_1/z_2)_\infty}(v_- \otimes v_+ - q^{-1}v_+ \otimes v_-) \quad \text{for (6.14d)} . \end{aligned}$$

$\lambda = \Lambda_1$:

$$\begin{aligned} (z_1/z_2)^{1/4}\Psi(z_1, z_2) &= \frac{(q^6 z_1/z_2)_\infty}{(q^4 z_1/z_2)_\infty}(v_+ \otimes v_- - qz_1/z_2 v_- \otimes v_+) \quad \text{for (6.14a)} , \\ &= \frac{(q^5 z_1/z_2)_\infty}{(q^3 z_1/z_2)_\infty}(v_+ \otimes v_- - qz_1/z_2 v_- \otimes v_+) \quad \text{for (6.14b)} , \\ &= \frac{(q z_1/z_2)_\infty}{(q^{-1} z_1/z_2)_\infty}(v_+ \otimes v_- - q^{-1}z_1/z_2 v_- \otimes v_+) \quad \text{for (6.14c)} , \\ &= \frac{(z_1/z_2)_\infty}{(q^{-2} z_1/z_2)_\infty}(v_+ \otimes v_- - q^{-1}z_1/z_2 v_- \otimes v_+) \quad \text{for (6.14d)} . \end{aligned}$$

6.6. Commutation Relations. The general theory of q-difference equations tells [32, 33] that the n point functions of VOs can be continued meromorphically to the entire space $(\mathbf{C}^\times)^n$, apart from overall powers or logarithms in z_i. In our case this is apparent from the explicit formulas. It follows that the same is true of all the matrix elements of compositions of VOs (see the proof of Proposition 6.1 below.)

From the knowledge of the two point functions one can derive the commutation relations of VOs [1]. To write down the relations which will be used later, we need to modify the scalar multiple of the R matrix and define

$$\begin{aligned} &R_{VV}(z) = r_0(z)\bar{R}(z),\ R_{V^*V^*}(z) = (C_- \otimes C_-)R_{VV}(z)(C_- \otimes C_-)^{-1}\,, \\ &R_{VV^*}(z) = (\mathrm{id} \otimes C_-)R_{VV}(zq^{-2})(\mathrm{id} \otimes C_-)^{-1}\,. \end{aligned} \tag{6.18}$$

Here

$$z^{-1/2}r_0(z) = \frac{(z^{-1})_\infty (q^2 z)_\infty}{(z)_\infty (q^2 z^{-1})_\infty} = \frac{\Gamma_{q^4}\left(\frac{1}{2} + \frac{\beta}{2\pi i}\right)\Gamma_{q^4}\left(-\frac{\beta}{2\pi i}\right)}{\Gamma_{q^4}\left(\frac{1}{2} - \frac{\beta}{2\pi i}\right)\Gamma_{q^4}\left(\frac{\beta}{2\pi i}\right)} \tag{6.19}$$

with $z = q^{-2\beta/i\pi}$ and $\Gamma_p(x) = (p;p)_\infty/(p^x;p)_\infty(1-p)^{1-x}$ denoting the q-gamma function.

We have the unitarity and crossing symmetry:

$$\begin{aligned} &R_{VV}(z)PR_{VV}(z^{-1})P = \mathrm{id}\,, \\ &(R_{VV}(z)^{-1})^{t_2} = -R_{VV^*}(z)\,. \end{aligned}$$

Notice that $R_{VV}(z)$, $R_{V^*V^*}(z)$ and $R_{VV^*}(z)$ have no poles in the neighborhood of $|z| = 1$.

In the following theorem, we list the commutation relations of VOs of type (6.12a)–(6.12b). The commutation relations and the holomorphy properties are in the sense of matrix elements. For example (6.20c) below states that for each $\langle u| \in V^r(\lambda)$, $|u'\rangle \in V(\lambda)$ the following hold as meromorphic functions in z_1, z_2 times $(z_1/z_2)^{\pm 1/4}$:

$$\langle u|\Phi_\mu^{V_1\lambda}(z_1)\Phi_\lambda^{V_2^{*a^{-1}}\mu}(z_2)|u'\rangle = R_{VV^*}(z_1/z_2)\langle u|\Phi_\mu^{V_2^{*a^{-1}}\lambda}(z_2)\Phi_\lambda^{V_1\mu}(z_1)|u'\rangle\,. \tag{6.20x}$$

Proposition 6.1.

(i) *The following commutation relation holds*:

$$\Phi_\mu^{V_1\lambda}(z_1)\Phi_\lambda^{V_2\mu}(z_2) = R_{V_1V_2}(z_1/z_2)\Phi_\mu^{V_2\lambda}(z_2)\Phi_\lambda^{V_1\mu}(z_1)\,, \tag{6.20a}$$

$$\Phi_\mu^{V_1^{*a^{-1}}\lambda}(z_1)\Phi_\lambda^{V_2^{*a^{-1}}\mu}(z_2) = R_{V_1^*V_2^*}(z_1/z_2)\Phi_\mu^{V_2^{*a^{-1}}\lambda}(z_2)\Phi_\lambda^{V_1^{*a^{-1}}\mu}(z_1)\,, \tag{6.20b}$$

$$\Phi_\mu^{V_1\lambda}(z_1)\Phi_\lambda^{V_2^{*a^{-1}}\mu}(z_2) = R_{V_1V_2^*}(z_1/z_2)\Phi_\mu^{V_2^{*a^{-1}}\lambda}(z_2)\Phi_\lambda^{V_1\mu}(z_1)\,. \tag{6.20c}$$

(ii) *In the neighborhood of* $|z_1/z_2| = 1$ *both sides of* (6.20a)–(6.20b) *are holomorphic, while* (6.20c) *has a simple pole at* $z_1 = z_2$. *The residue of the latter is given by*

$$\mathrm{Res}_{z=1}\,\Phi_\mu^{V_1\lambda}(z_1)\Phi_\lambda^{V_2^{*a^{-1}}\mu}(z_2)dz = g_\lambda\,\mathrm{id}_{V(\lambda)} \otimes (v_+ \otimes v_+^* + v_- \otimes v_-^*)\,, \tag{6.20d}$$

where $z = z_1/z_2$ *and*

$$g_\lambda = \mp q^{-1/2}\frac{(q^2)_\infty}{(q^4)_\infty} \tag{6.21}$$

with the sign $-$ *(respectively* $+$*) being taken for* $\lambda = \Lambda_0$ *(respectively* Λ_1*).*

Proof. The argument being similar, we concentrate on (6.20c).

If $\langle u| = \langle u_\lambda|$ and $|u'\rangle = |u_\lambda\rangle$ the assertions (i), (ii) follow from the explicit formulas (6.14) and (6.13a)–(6.13b). Suppose $|u'\rangle = |xu''\rangle$ with $x \in U'$. Then the intertwining property of vertex operators entails

$$\begin{aligned} &P\langle u|\Phi_\mu^{V_1\lambda}(z_1)\Phi_\lambda^{V_2^{*a^{-1}}\mu}(z_2)|xu''\rangle \\ &= \sum x_{(1)}\otimes x_{(2)}P\langle ux_{(3)}|\Phi_\mu^{V_1\lambda}(z_1)\Phi_\lambda^{V_2^{*a^{-1}}\mu}(z_2)|u''\rangle \in V_2^{*a^{-1}}\otimes V_1\,. \end{aligned} \tag{6.22}$$

Here $\Delta^{(2)}(x) = \sum x_{(1)}\otimes x_{(2)}\otimes x_{(3)}$. Analogous formula holds for $\langle u| = \langle u''x|$. Since the action of U' on $V_{z_2}^{*a^{-1}}\otimes V_{z_1}$ involves only powers of z_i, the analyticity property follows by induction on the weight of $\langle u|$, $|u'\rangle$. Because $PR(z)$ is an intertwiner the relation (6.20c) is unchanged in the process. To see (ii) note that if we take the residue of (6.22) then the left-hand side reduces to

$$= P\langle ux|\Phi_\mu^{V_1\lambda}(z_1)\Phi_\lambda^{V_2^{*a^{-1}}\mu}(z_2)|u''\rangle\,.$$

This is because the vector $w = v_+^*\otimes v_+ + v_-^*\otimes v_- \in V_z^{*a^{-1}}\otimes V_z$ belongs to the trivial representation. We then obtain by the induction hypothesis $\langle ux|u''\rangle g_\lambda w = \langle u|xu''\rangle g_\lambda w$ as desired. This completes the proof. □

We shall also need the following commutation relations, which can be proved in a similar manner.

$$R_{V_1V_2}(z_1/z_2)\Phi_\mu^{\lambda V_1}(z_1)\Phi_\lambda^{\mu V_2}(z_2) = -\Phi_\mu^{\lambda V_2}(z_2)\Phi_\lambda^{\mu V_1}(z_1)\,, \tag{6.23}$$

$$\Phi_\mu^{V_1\lambda}(z_1)\Phi_\lambda^{\mu V_2}(z_2) = \Phi_\mu^{\lambda V_2}(z_2)\Phi_\lambda^{V_1\mu}(z_1)\times\left(\frac{z_1}{z_2}\right)^{1/2}\frac{\Theta(qz_2/z_1)}{\Theta(qz_1/z_2)}\,, \tag{6.24}$$

$$\Phi_\mu^{V_1^{*a^{-1}}\lambda}(z_1)\Phi_\lambda^{\mu V_2}(z_2) = \Phi_\mu^{V_2}(z_2)\Phi_\lambda^{V_1^{*a^{-1}}\mu}(z_1)\times\left(\frac{z_1}{z_2}\right)^{-1/2}\frac{\Theta(qz_1/z_2)}{\Theta(qz_2/z_1)}\,. \tag{6.25}$$

Here we set

$$\Theta(z) = (z)_\infty(q^4z^{-1})_\infty(q^4)_\infty\,.$$

It is known [4] that the composition $\tilde{\Phi}_{\mu V}^\lambda\circ\tilde{\Phi}_\lambda^{\mu V}$ is convergent in the q-adic topology and is proportional to the identity. The proportionality scalar can be determined from the two point function by setting $z_1 = z_2$. Using the results (6.14) we find

$$\Phi_{\mu V}^\lambda(z)\circ\Phi_\lambda^{\mu V}(z) = -\frac{1}{g_\lambda}\times\mathrm{id}\,, \tag{6.26}$$

where g_λ is given in (6.21).

6.7. Crystal Lattice. Recall that on an integrable U-module one has the operators $\tilde{e}_i^{\mathrm{up}}, \tilde{f}_i^{\mathrm{up}}, \tilde{e}_i^{\mathrm{low}}, \tilde{f}_i^{\mathrm{low}}$ and the notion of the upper/lower crystal lattice [25]. The finite dimensional module V has $L = Av_+ \oplus Av_-$ as a (both upper and lower) crystal lattice.

The integrable highest weight module $V(\lambda)$ has the standard lower/upper crystal lattice $L^{\mathrm{low}}(\lambda)$, $L^{\mathrm{up}}(\lambda)$ given as follows [25]. Let φ denote the anti-automorphism of U given by

$$\varphi(e_i) = f_i, \quad \varphi(f_i) = e_i, \quad \varphi(q^h) = q^h , \tag{6.27}$$

and let (,) be the unique symmetric bilinear form on $V(\lambda)$ such that

$$(|u_\lambda\rangle, |u_\lambda\rangle) = 1, \quad (|xu\rangle, |u'\rangle) = (|u\rangle, |\varphi(x)u'\rangle) . \tag{6.28}$$

Then

$$L^{\mathrm{low}}(\lambda) = \text{the smallest } A\text{-module containing } |u_\lambda\rangle \text{ and stable under } \tilde{f}_i^{\mathrm{low}} ,$$

$$L^{\mathrm{up}}(\lambda) = \{u \in V(\lambda) | (u, L^{\mathrm{low}}(\lambda)) \subset A\} . \tag{6.29}$$

$$L^{\mathrm{low}}(\lambda) = \{u \in V(\lambda) | (u, L^{\mathrm{up}}(\lambda)) \subset A\} . \tag{6.30}$$

Here $A = \{f \in \mathbf{Q}(q) | f \text{ has no pole at } q = 0\}$. We have further

$$L^{\mathrm{up}}(\lambda)_\nu = q^{(\lambda,\lambda)-(\nu,\nu)} L^{\mathrm{low}}(\lambda)_\nu .$$

The crystal lattices of integrable lowest weight modules are defined similarly by replacing $|u_\lambda\rangle$ by the lowest weight vector and $\tilde{f}_i^{\mathrm{low}}$ by $\tilde{e}_i^{\mathrm{low}}$. Let $T_\lambda \in \mathrm{End}(V(\lambda))$ denote the linear map

$$T_\lambda u = q^{(2\rho, \lambda-\nu)} u \quad \text{for } u \in V(\lambda)_\nu .$$

Proposition 6.2. *The upper crystal lattice $L^{*a\,\mathrm{up}}(\lambda)$ of $V(\lambda)^{*a}$ is characterized as*

$$L^{*a\,\mathrm{up}}(\lambda) = \{u \in V(\lambda)^{*a} | \langle u, T_\lambda L^{\mathrm{up}}(\lambda)\rangle \subset A\} , \tag{6.31}$$

where $\langle\ ,\ \rangle$ denotes the canonical pairing.

Proof. Let η' denote the map $V(\lambda) \to V(\lambda)^{*a}$ given by $\langle \eta'(u), v\rangle = (u, v)$, and set

$$\eta(u) = (-1)^{ht(\lambda-\nu)} q^{(\nu,\nu+2\rho)-(\lambda,\lambda+2\rho)} \eta'(u) \quad u \in V(\lambda)_\nu . \tag{6.32}$$

Here for $\xi = m_0\alpha_0 + m_1\alpha_1$ we set $ht(\xi) = m_0 + m_1$. One can verify that

$$\eta(e_i u) = f_i \eta(u), \quad \eta(f_i u) = e_i \eta(u), \quad \eta(q^h u) = q^{-h}\eta(u) .$$

This implies $\eta(\tilde{f}_i^{\mathrm{low}} v) = \tilde{e}_i^{\mathrm{low}} \eta(v)$, $\eta(\tilde{e}_i^{\mathrm{low}} v) = \tilde{f}_i^{\mathrm{low}} \eta(v)$, and hence

$$L^{*a\,\mathrm{low}}(\lambda) = \eta(L^{\mathrm{low}}(\lambda)) .$$

Using the relation $(L^{*a\,\mathrm{up}}(\lambda))_{-\nu} = q^{(\lambda,\lambda)-(\nu,\nu)} (L^{*a\,\mathrm{low}}(\lambda))_{-\nu}$ we find

$$(L^{*a\,\mathrm{up}}(\lambda))_{-\nu} = \eta'(T_\lambda^{-1}(L^{\mathrm{low}}(\lambda))_\nu) .$$

The assertion follows from this and (6.30). $\square$

Hereafter crystal lattice will mean the *upper* crystal lattice at $q = 0$ (we drop the superscript).

A basic property of the VOs (6.11a), (6.11c) is that they preserve the crystal lattice [4]. Fixing the lowest weight vector $|u_\lambda^*\rangle \in V^{*a}(\lambda)$ such that $\langle |u_\lambda\rangle, |u_\lambda^*\rangle\rangle = 1$ we normalize (6.11c) as

$$\tilde{\Phi}_{V\mu}^{*\lambda}(v_\pm \otimes |u_\mu^*\rangle) = |u_\lambda^*\rangle + \cdots . \tag{6.33}$$

Let $\Phi_{\varepsilon,n}$ be the weight components of $\Phi = \tilde{\Phi}_\lambda^{\mu V}$, and similarly let $\tilde{\Phi}_{V\mu}^{*\lambda}(v_\pm \otimes \cdot) = \sum_n \Phi_{\pm n}^*$, $\Phi_{\pm n}^*$: $V^{*a}(\mu)_\nu \to V^{*a}(\lambda)_{\nu \mp \alpha_1/2 + n\delta}$.

Proposition 6.3. *Notations being as above, we have for all ε and n*

(i)

$$\Phi_{\varepsilon n} L(\lambda) \subset L(\mu) , \tag{6.34a}$$

$$\Phi_{\varepsilon n}^* L^{*a}(\mu) \subset L^{*a}(\lambda) . \tag{6.34b}$$

(ii) *For some $s > 0$ we have*

$$\Phi_{\varepsilon n}(L(\lambda)_{\lambda-\xi}) \subset q^{-3n-2(\rho,\xi)-s} L(\mu) , \tag{6.35a}$$

$$\Phi_{\varepsilon n}^*(L^{*a}(\mu)_{-\mu+\xi}) \subset q^{-3n-2(\rho,\xi)-s} L^{*a}(\lambda) . \tag{6.35b}$$

Proof. Assertion (6.34a) is proved in [4]. The argument in [4] shows also that (6.35a) is true if it holds for $\xi = 0$. In Appendix 3 we verify the latter statement explicitly.

From the normalization of $\tilde{\Phi}_{V\mu}^{*\lambda}$ we have the relation

$$\tilde{\Phi}_{V\mu}^{*\lambda} = (\Psi^*)^t, \quad \Psi^* = (\mathrm{id} \otimes C_-)\Phi(q^2) \times (-q) \text{ or } 1$$

according as $\lambda = \Lambda_0$ or Λ_1. We can verify further that

$$(T_\mu^{-1} \otimes \mathrm{id}) \circ \Psi^* \circ T_\lambda = \pm (\Phi_+ \otimes v_-^* - \Phi_- \otimes v_+^*) ,$$

where we put $\Phi = \Phi_+ \otimes v_+ + \Phi_- \otimes v_-$. In view of (6.31) the assertions (6.34b), (6.35b) follow from these relations. □

Thanks to (ii), $\tilde{\Phi}_\lambda^{\mu V}$ is we well defined on the q-adic completion

$$\varprojlim (L(\mu) \otimes L)/(q^n L(\mu) \otimes L) ,$$

i.e., when applied to an infinite sum $u = \sum_n u_n$ such that $u_n \in q^{M_n} L(\lambda)$, $\lim_{n\to\infty} M_n = \infty$, then for any N there are only a finite number of non-zero terms modulo q^N in $\tilde{\Phi}_\lambda^{\mu V} u$. Hence we can iterate the VOs of type (6.11a) finitely many times (we drop the symbols for completion)

$$L(\lambda) \xrightarrow{\Phi_\lambda^{\mu V}} L(\mu) \otimes L \xrightarrow{\Phi_\mu^{\lambda V} \otimes \mathrm{id}} L(\lambda) \otimes L \otimes L \to \cdots . \tag{6.36}$$

As mentioned already (Sect. 2) we conjecture that after suitably normalizing VO the infinite iteration also makes sense.

6.8. n-Point Functions. To examine the behavior of the embedding (6.36) we need to study the n point correlators as n tends to ∞. We give below what is known to us about the general n point functions. In this subsection we consider VOs of the type $\Phi(z) = \Phi_\mu^{\lambda V}(z)$.

Let $n = 2m$ or $2m - 1$, $\lambda = \Lambda_0$ and $\mu = \Lambda_0$ or Λ_1 according to whether n is even or odd. We define the vector $w_n(z) \in V^{\otimes n}$ by

$$\langle u_\lambda | \Phi(z_1) \cdots \Phi(z_n) | u_\mu \rangle$$
$$= \prod_{j=1}^{n} z_j^{(-1)^j/4} \prod_{j=1}^{m-1} (z_{2j-1} z_{2j})^{-m+j} \prod_{i<j} \frac{(q^6 z_j/z_i)_\infty}{(q^4 z_j/z_i)_\infty} \times w_n(z_1, \ldots, z_n) . \tag{6.37}$$

A priori $w_n(z) \prod_{j=1}^{m-1} (z_{2j-1} z_{2j})^{-m+j}$ is then a formal power series in z_n, $z_n/z_{n-1}, \ldots, z_2/z_1$. In general, let Φ: $V(\mu) \to V(\lambda) \otimes W$ be an intertwiner (where W is a finite dimensional module) and set $\Phi u_\mu = u_\lambda \otimes w + \cdots$. Then w must satisfy [4] $e_i^{\langle\lambda, h_i\rangle + 1}\, w = 0$ for all i. In the present case $W = V_{z_1} \otimes \cdots \otimes V_{z_n}$, $\lambda = \Lambda_0$. Because V has a perfect crystal, so does $V^{\otimes n}$ [2]. This implies that up to scalar the linear equations

$$\Delta(e_0)^2 w(z) = 0, \quad \Delta(e_1) w(z) = 0$$

admit at most one solution (recall that the first equation involves the indeterminates z_i), and hence that our $w(z)$ is a rational function in $z_1, \cdots, z_n$ up to an overall scalar factor.

Set $\hat{R}(z) = P\bar{R}(z) \times (1 - q^2 z)/(z - q^2)$ with $R(z)$ defined in (6.16). Then we have the following properties of $w_n(z)$:

$$\hat{R}_{i\,i+1}(z_i/z_{i+1}) w_n(z) = (s_i w_n)(z), \quad i = 1, \ldots, n-1 , \tag{6.38a}$$

where $s_i = (i\; i+1)$, and we set $(sf)(z_1, \ldots, z_n) = f(z_{s^{-1}(1)}, \ldots, z_{s^{-1}(n)})$ for a permutation s ,

$$w_n(q^6 z_1, z_2, \ldots, z_n) = \bar{A}_1 q^{3m-3} P_{1n} \ldots P_{13} P_{12} w_n(z_2, \ldots, z_n, z_1) , \tag{6.38b}$$

where $\bar{A}_1 = -t_1^{-1}$ for $n = 2m$, $= (qt_1^{-1})^{3/2}$ for $n = 2m - 1$,

$$w_{2m}(z_1, \ldots, z_{2m-1}, 0) = w_{2m-1}(z_1, \ldots, z_{2m-1}) \otimes v_-$$
$$- q(f_1 \cdot w_{2m-1}(z_1, \ldots, z_{2m-1})) \otimes v_+ ,$$

$$w_{2m-1}(z_1, \ldots, z_{2m-2}, 0) = w_{2m-2}(z_1, \ldots, z_{2m-2}) \otimes v_+ \times \prod_{j=1}^{2m-2} z_j . \tag{6.38c}$$

Property (6.38a) is a consequence of the commutation relations of VO. Property (6.38b) is a reduced form of the qKZ equation under (6.38a). Finally property (6.38c) follows from the normalization of the VOs (6.8a).

Let us introduce the coefficients $a_n(\varepsilon | z)$ by

$$w_n(z_1, \ldots, z_n) = \sum_{\varepsilon} a_n(\varepsilon | z) v_{\varepsilon_1} \otimes \cdots \otimes v_{\varepsilon_n} .$$

The $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ range over $\varepsilon_i = \pm$ such that the number of $+$'s equals m. Then (6.38a) is rewritten in the form

$$a(\cdots \overset{i}{-} + \cdots | z) = \sigma_i a(\cdots \overset{i}{+} - \cdots | z) ,$$
$$a(\cdots \overset{i}{\varepsilon} \varepsilon \cdots | z) = q\sigma_i a(\cdots \overset{i}{\varepsilon}\, \varepsilon \cdots | z) . \tag{6.39}$$

Here the operator σ_i is given by

$$(\sigma_i^{\pm 1} f)(z) = (q^{-1} z_i - q z_{i+1}) \frac{f(z) - (s_i f)(z)}{z_i - z_{i+1}} - q^{\pm 1} f(z) .$$

One can verify that they obey the Hecke algebra relations (Lusztig [34])

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad \sigma_i \sigma_j = \sigma_j \sigma_i \ (|i-j| > 1), \quad (\sigma_i + q)(\sigma_i - q^{-1}) = 0 .$$

Using (6.39) the coefficients $a(\varepsilon|z)$ are determined uniquely once we know e.g. $a_0(z) = a(\varepsilon|z)$ for $\varepsilon_1 = \cdots = \varepsilon_m = +$, $\varepsilon_{m+1} = \cdots = \varepsilon_n = -$. From computations for small n we expect that $w(z_1, \ldots, z_n)$ is actually a polynomial in z_i and q, and that $a_0(z)$ is given by

$$(-q)^{m(m-1)/2} \prod_{1 \leqq i < j \leqq m} (z_i - q^2 z_j) \prod_{m+1 \leqq i < j \leqq n} (z_i - q^2 z_j) \times \begin{cases} 1 & \text{for } n = 2m, \\ \times z_{m+1} \cdots z_{2m-1} & \text{for } n = 2m-1 . \end{cases} \tag{6.40}$$

The representation of the Hecke algebra generated by this polynomial is irreducible for generic q; at $q = 1$ it specializes to the Specht module [35] of the symmetric group associated with the Young diagram of shape (m, m) or $(m, m-1)$. We hope to be able to extract exact information about the asymptotics of n point functions by analysing these representations.

7. Space of States

7.1. Vacuum, Shift and Energy. In Sect. 1 through Sect. 5 we have emphasized the role of the space $V(\lambda) \otimes V(\lambda')^{*a}$, where $\lambda, \lambda' \in \{\Lambda_0, \Lambda_1\}$. To treat the infinite sums of vectors safely, we introduce its q-adic completion. Define

$$\mathscr{L}_{\lambda,\lambda'} = \varprojlim L(\lambda) \otimes L^{*a}(\lambda')/q^n L(\lambda) \otimes L^{*a}(\lambda') ,$$
$$\mathscr{V}_{\lambda,\lambda'} = \mathbf{Q}((q)) \otimes_{\mathbf{Q}[[q]]} \mathscr{L}_{\lambda,\lambda'} .$$

Here $\mathbf{Q}((q))$ (respectively $\mathbf{Q}[[q]]$) denotes the field (respectively ring) of formal Laurent (respectively power) series in q. The space $\mathscr{V}_{\lambda,\lambda'}$ inherits the left U-module structure of level 0. Let $\mathscr{V}^r_{\lambda,\lambda'}$ denote the right U-module structure on $\mathscr{V}_{\lambda,\lambda'}$ given by $f \cdot x = a^{-1}(x) \cdot f$ ($f \in \mathscr{V}_{\lambda,\lambda'}$, $x \in U$). If we regard an element $f \in \mathscr{V}_{\lambda',\lambda}$ (respectively $g \in \mathscr{V}^r_{\lambda,\lambda'}$) as a linear map $V(\lambda) \to \hat{V}(\lambda')$ (respectively $V(\lambda') \to \hat{V}(\lambda)$), the left (respectively right) U-action is given by the adjoint action:

$$\mathrm{ad}(x) \cdot f = \sum x_{(1)} \circ f \circ a(x_{(2)}) ,$$
$$g \cdot \mathrm{ad}^r(x) = \sum a^{-1}(x_{(2)}) \circ g \circ x_{(1)} ,$$

where $x \in U$. We define the bilinear pairing $\mathscr{V}^r_{\lambda,\lambda'} \times \mathscr{V}_{\lambda',\lambda} \to \mathbf{Q}((q))$ by

$$\langle g | f \rangle = \frac{\mathrm{tr}_{V(\lambda)}(q^{-4\rho} g \circ f)}{\mathrm{tr}_{V(\lambda)}(q^{-4\rho})} . \tag{7.1}$$

It is non-degenerate and enjoys the property

$$\langle g \cdot \mathrm{ad}^r(x) | f \rangle = \langle g | \mathrm{ad}(x) \cdot f \rangle \quad \forall x \in U \,. \tag{7.2}$$

Note that the denominator in (7.1)

$$\mathrm{tr}_{V(\lambda)}(q^{-4\rho}) = \frac{q^{-(4\rho, \lambda)}}{(q^2)_\infty}$$

is the principally specialized character of $V(\lambda)$.

For each ν we take bases $\{|u_i^{(\nu)}\rangle\} \subset L(\lambda)_\nu$, $\{|u_i^{(\nu)*}\rangle\} \subset (L(\lambda)_\nu)^*$ which are dual with respect to the canonical pairing $\langle\ ,\ \rangle$. We call the canonical element

$$|\mathrm{vac}\rangle_\lambda = \sum_{i,\nu} |u_i^{(\nu)}\rangle \otimes |u_i^{(\nu)*}\rangle \in \mathscr{V}_{\lambda,\lambda} \tag{7.3}$$

the vacuum. This corresponds to the identity map $V(\lambda) \to \hat{V}(\lambda)$. Hence it is clear that the vacuum gives rise to the trivial representation over U:

$$\mathrm{ad}(x) \cdot |\mathrm{vac}\rangle_\lambda = \varepsilon(x) |\mathrm{vac}\rangle_\lambda \quad \forall x \in U \,.$$

Since $\langle |u_j^{(\nu)*}\rangle, T_\lambda |u_i^{(\nu)}\rangle\rangle \in q^{(2\rho, \lambda - \nu)} A$ we see from (6.31) that $|\mathrm{vac}\rangle_\lambda$ in fact belongs to $\mathscr{V}_{\lambda,\lambda}$. Define the right vacuum ${}_\lambda\langle \mathrm{vac}| \in \mathscr{V}^r_{\lambda,\lambda}$ analogously. The pairing (7.1) is so normalized that ${}_\lambda\langle \mathrm{vac} | \mathrm{vac}\rangle_\lambda = 1$.

Consider the composition of the maps

$$L(\lambda) \otimes L^{*a}(\lambda') \xrightarrow{\tilde{\Phi}_\lambda^{\mu V} \otimes \mathrm{id}} \hat{L}(\mu) \otimes L \otimes L^{*a}(\lambda') \xrightarrow{\mathrm{id} \otimes \tilde{\Phi}_{V\lambda'}^{*\mu'}} \hat{L}(\mu) \otimes \hat{L}^{*a}(\mu') \,.$$

By Proposition 6.3 it extends to the map $\hat{T}\colon \mathscr{V}_{\lambda,\lambda'} \to \mathscr{V}_{\mu,\mu'}$. We define the translation and energy operators T, H by

$$T = \frac{(q^2)_\infty}{(q^4)_\infty} \hat{T} \,, \tag{7.4}$$

$$H = -\frac{q - q^{-1}}{2} (T^2 \circ d \circ T^{-2} - d) \,, \tag{7.5}$$

where d is given in (6.10).

The following is the first property we expect for the vacuum vector:

Proposition 7.1. *The vacuum vector is invariant under T, H:*

$$T |\mathrm{vac}\rangle_\lambda = |\mathrm{vac}\rangle_\mu, \quad H |\mathrm{vac}\rangle_\lambda = 0 \,.$$

Proof. The second equation is clear from the first and $d|\mathrm{vac}\rangle_\lambda = 0$. Put $\tilde{\Phi}_\lambda^{\mu V} = \sum \Phi_\varepsilon \otimes v_\varepsilon$, $(\tilde{\Phi}_{V\lambda}^{*\mu})^t = \tilde{\Phi}_\mu^{\lambda V^{*a^{-1}}} = \sum \Phi_\varepsilon^* \otimes v_\varepsilon^*$. It suffices to show that

$$\frac{(q^2)_\infty}{(q^4)_\infty} \hat{T} |\mathrm{vac}\rangle_\lambda = \frac{(q^2)_\infty}{(q^4)_\infty} \sum \Phi_\varepsilon |u_i^{(\nu)}\rangle \otimes (\Phi_\varepsilon^*)^t |u_i^{(\nu)*}\rangle$$

coincides with $|\mathrm{vac}\rangle_\mu$ (note that $(\Phi_\varepsilon^*)^t$ sends $V(\lambda)^*$ to $V(\mu)^*$). This is equivalent to showing

$$\sum \Phi_\varepsilon |u_i^{(\nu)}\rangle \langle (\Phi_\varepsilon^*)^t |u_i^{(\nu)*}\rangle, |v\rangle\rangle = \frac{(q^4)_\infty}{(q^2)_\infty} |v\rangle$$

for any $|v\rangle \in V(\mu)$. The left-hand side is

$$= \sum_{\varepsilon} \Phi_\varepsilon (|u_i^{(\nu)}\rangle \langle |u_i^{(\nu)*}\rangle, \Phi_\varepsilon^* |v\rangle\rangle)$$

$$= \sum_{\varepsilon} \Phi_\varepsilon \Phi_\varepsilon^* |v\rangle ,$$

which is the image of $|v\rangle$ under the composition

$$V(\mu) \xrightarrow{\tilde{\Phi}_\mu^{\lambda V^{*a^{-1}}}} V(\lambda) \otimes V^{*a^{-1}} \xrightarrow{\tilde{\Phi}_\lambda^{\mu V} \otimes \mathrm{id}} V(\mu) \otimes V \otimes V^{*a^{-1}} \xrightarrow{\mathrm{id} \otimes \langle\, ,\rangle} V(\mu) .$$

In the same way as in (6.26) we find that the latter is $(q^4)_\infty/(q^2)_\infty \times \mathrm{id}$. □

7.2. Creation and Annihilation Operators. Let us define the components of the VOs of type (6.12a), (6.12b) by

$$\Phi_\lambda^{V\mu}(z) = v_+ \otimes \Phi_+(z) + v_- \otimes \Phi_-(z) ,$$

$$\Phi_\pm^*(z) = \Phi_{V\lambda}^{\mu}(z)(v_\pm \otimes (\cdot)) .$$

Hence from (6.13b) we have the relations

$$\Phi_+^*(z) = -q^{-1}\Phi_-(zq^2), \quad \Phi_-^*(z) = \Phi_+(zq^2) .$$

Using these operators, we would like to define the n-particle states to be

$$(\Phi_{\varepsilon_1}^*(z_1) \cdots \Phi_{\varepsilon_n}^*(z_n) \otimes \mathrm{id})|\mathrm{vac}\rangle_\lambda \tag{7.6}$$

with $\varepsilon_1, \ldots, \varepsilon_n \in \{+, -\}$ and $|z_1| = \cdots = |z_n| = 1$. The n-particle states in the dual space are to be defined analogously, using ${}_\lambda\langle\mathrm{vac}|$ and $\Phi_\varepsilon(z)$ in place of $|\mathrm{vac}\rangle_\lambda$ and $\Phi_\varepsilon^*(z)$ respectively. These are eigenstates of the shift and the energy operators (see 7.2).

Let us examine the meaning of (7.6) more closely by studying the Fourier components

$$\oint \cdots \oint_{|z_1| = \cdots = |z_n| = 1} \frac{dz_1}{2\pi i z_1} \cdots \frac{dz_n}{2\pi i z_n} z_1^{m_1} \cdots z_n^{m_n} (\Phi_{\varepsilon_1}^*(z_1) \cdots \Phi_{\varepsilon_n}^*(z_n) \otimes \mathrm{id})|\mathrm{vac}\rangle_\lambda , \tag{7.7}$$

where the m_i range over all integers. For definiteness we take $n = 2$. Equation (7.7) is an infinite sum over i, ν of the terms

$$\oint\oint_{|z_1|=|z_2|=1} \frac{dz_1}{2\pi i z_1} \frac{dz_2}{2\pi i z_2} z_1^{m_1} z_2^{m_2} (\Phi_{\varepsilon_1}^*(z_1)\Phi_{\varepsilon_2}^*(z_2)|u_i^{(\nu)}\rangle) \otimes |u_i^{(\nu)*}\rangle . \tag{7.8}$$

Now for each $\langle u| \in V^r(\lambda)$, $|u'\rangle \in V(\lambda)$ the function $\langle u|\Phi_{\varepsilon_1}^*(z_1)\Phi_{\varepsilon_2}^*(z_2)|u'\rangle$ is convergent in the domain $|z_1| \gg |z_2|$, $|q| < 1$, and has a meromorphic continuation with respect to $z_1, z_2 \in (\mathbf{C})^\times$ with poles only at $z_1/z_2 = q^{-2}, q^2, q^6, \ldots$ (see Proposition 6.1). Hence the integration in (7.8) is meaningful. Notice that (7.8) is different from applying the weight components $\Phi_{\varepsilon,m}$ of $\Phi_\varepsilon(z)$

$$(\Phi_{\varepsilon_1,m_1}\Phi_{\varepsilon_2,m_2}|u_i^{(\nu)}\rangle) \otimes |u_i^{(\nu)*}\rangle .$$

By the definition the latter is obtained by taking the contour in (7.8) to be $|z_1| \gg |z_2|$, and because of the pole at $|z_1/z_2| = q^{-2}$ the two expressions give different answers.

The type II VOs do not preserve the crystal lattice, so the individual terms (7.8) comprise negative powers of q. However computations suggest (see Appendix 4) that when we sum over i, v the negative powers all disappear, getting thereby a well defined element of $\mathscr{V}_{\lambda,\lambda}$. More generally it can be shown using the q-KZ equation that each matrix element $\langle u|\Phi^*_{\varepsilon_1}(z_1)\cdots\Phi^*_{\varepsilon_n}(z_n)|u'\rangle$, $\langle u|\Phi_{\varepsilon_1}(z_1)\cdots\Phi_{\varepsilon_n}(z_n)|u'\rangle$ ($\langle u|\in V^r(\lambda)$, $|u'\rangle\in V(\lambda)$) is holomorphic on $|z_1| = \cdots = |z_n| = 1$. Fixing n let $\mathscr{F}(n)$, $\mathscr{F}^r(n)$ denote the span of the vectors (7.7) and its dual counterpart, respectively. We expect that

1. $\mathscr{F}(n) \subset \mathscr{V}$, $\mathscr{F}^r(n) \subset \mathscr{V}^r$.
2. With respect to the inner product (7.1) the spaces $\mathscr{F}^r(m)$ and $\mathscr{F}(n)$ with $m \neq n$ are orthogonal.
3. (7.1) restricted to $\mathscr{F}^r(n) \times \mathscr{F}(n)$ is non-degenerate.

Assuming these, we define the physical spaces of states by

$$\mathscr{F} = \bigoplus_{n=0}^{\infty} \mathscr{F}(n), \quad \mathscr{F}^r = \bigoplus_{n=0}^{\infty} \mathscr{F}^r(n) .$$

They are analogous to the usual Fock space of free particles.

Define the creation operator $\varphi^*_\varepsilon(z)$: $\mathscr{F}(n) \to \mathscr{F}(n+1)$ on $\mathscr{F}$, $|z| = 1$, by

$$\varphi^*_\varepsilon(z)(\Phi^*_{\varepsilon_1}(z_1)\cdots\Phi^*_{\varepsilon_n}(z_n)\otimes \mathrm{id})|\mathrm{vac}\rangle_\lambda = (\Phi^*_\varepsilon(z)\Phi^*_{\varepsilon_1}(z_1)\cdots\Phi^*_{\varepsilon_n}(z_n)\otimes \mathrm{id})|\mathrm{vac}\rangle_\lambda . \tag{7.9}$$

Both sides are to be understood in the sense of the Fourier coefficients as in (7.7). The annihilation operator $\varphi_\varepsilon(z)$: $\mathscr{F}(n) \to \mathscr{F}(n-1)$ is defined to be the adjoint of the dual counterpart $\mathscr{F}^r(n) \to \mathscr{F}^r(n+1)$ of (7.9). Thus $\varphi_\varepsilon(z)|\mathrm{vac}\rangle = 0$ by definition, and

$$\begin{aligned} &{}_\lambda\langle \mathrm{vac}|\varphi_{\varepsilon_1}(z_1)\cdots\varphi_{\varepsilon_n}(z_n)\varphi^*_{\varepsilon'_n}(z'_n)\cdots\varphi^*_{\varepsilon'_1}(z'_1)|\mathrm{vac}\rangle_\lambda \\ &\quad = \mathrm{tr}_{V(\lambda)}(q^{-4\rho}\Phi_{\varepsilon_1}(z_1)\cdots\Phi_{\varepsilon_n}(z_n)\Phi^*_{\varepsilon'_n}(z'_n)\cdots\Phi^*_{\varepsilon'_1}(z'_1))/\mathrm{tr}_{V(\lambda)}(q^{-4\rho}) . \end{aligned}$$

Because $\varphi_\pm(z)$ act as identity on the second component of $L(\lambda)\otimes L^{*a}(\lambda')$ the commutation relations (6.20a–6.20c) as operators on $V(\lambda)$ can be readily translated. We have

$$\varphi_{\varepsilon_1}(z_1)\varphi_{\varepsilon_2}(z_2) = \sum \varphi_{\varepsilon'_2}(z_2)\varphi_{\varepsilon'_1}(z_1)(R_{VV}(z_1/z_2))^{\varepsilon_1\varepsilon_2}_{\varepsilon'_1\varepsilon'_2} , \tag{7.10a}$$

$$\varphi^*_{\varepsilon_1}(z_1)\varphi^*_{\varepsilon_2}(z_2) = \sum \varphi^*_{\varepsilon'_2}(z_2)\varphi^*_{\varepsilon'_1}(z_1)(R_{V^*V^*}(z_1/z_2))^{\varepsilon_1\varepsilon_2}_{\varepsilon'_1\varepsilon'_2} , \tag{7.10b}$$

$$\varphi_{\varepsilon_1}(z_1)\varphi^*_{\varepsilon_2}(z_2) = \sum \varphi^*_{\varepsilon'_2}(z_2)\varphi_{\varepsilon'_1}(z_1)(R_{VV^*}(z_1/z_2))^{\varepsilon_1\varepsilon_2}_{\varepsilon'_1\varepsilon'_2} + g_\lambda\delta_{\varepsilon_1\varepsilon_2}\delta(z_1/z_2) . \tag{7.10c}$$

Here the delta function $\delta(z) = \sum_{n\in\mathbf{Z}} z^n$ arises because of the pole of (6.20c) at $z_1 = z_2$. Thus the VOs provide us with a lattice realization of the Zamolodchikov algebra.

Remark. In the limit $q \to 1$, $z = q^{-2\beta/i\pi}$, the R matrix (6.18), (6.19) reduces to the S matrix of the $su(2)$ invariant Thirring model [36]

$$R_{VV}(\beta) = \frac{\Gamma\left(\frac{1}{2} + \frac{\beta}{2\pi i}\right)\Gamma\left(-\frac{\beta}{2\pi i}\right)}{\Gamma\left(\frac{1}{2} - \frac{\beta}{2\pi i}\right)\Gamma\left(\frac{\beta}{2\pi i}\right)} \frac{(\beta \cdot 1 - \pi i P)}{\beta - \pi i} .$$

The commutation relations (7.10a)–(7.10c) become those for the Zamolodchikov operators (up to rescaling the operators $\varphi_\varepsilon(z)$, $\varphi_\varepsilon^*(z)$).

Having set up the mathematical definitions of the creation-annihilation operators we can discuss their transformation properties under the shift and the energy operators.

Proposition 7.2.

$$T\varphi_\pm(z)T^{-1} = \tau(z)\varphi_\pm(z), \qquad T\varphi_\pm^*(z)T^{-1} = \tau(z)^{-1}\varphi_\pm^*(z) , \tag{7.11}$$

$$[H, \varphi_\pm(z)] = -\varepsilon(z)\varphi_\pm(z), \quad [H, \varphi_\pm^*(z)] = \varepsilon(z)\varphi_\pm^*(z) , \tag{7.12}$$

where

$$\tau(z) = z^{-1/2}\frac{\Theta(qz)}{\Theta(qz^{-1})} \quad \varepsilon(z) = -(q - q^{-1})z\frac{d}{dz}\log \tau(z) .$$

Proof. We show (7.11), (7.12) for $\varphi_\pm(z)$. To see (7.11) we need to prove $\Phi_\mu^{V\lambda}(z)\hat{T} = \tau(z)^{-1}\hat{T}\Phi_\lambda^{V\mu}(z)$, where the left- and right-hand sides are the compositions of

$$L(\lambda) \otimes L^{*a}(\lambda) \xrightarrow{\tilde{\Phi}_\lambda^{\mu V} \otimes \mathrm{id}} L(\mu) \otimes L \otimes L^{*a}(\lambda) \xrightarrow{\mathrm{id} \otimes \tilde{\Phi}_{V\lambda}^{*\mu}} L(\mu) \otimes L^{*a}(\mu)$$

$$\xrightarrow{\Phi_\mu^{V\lambda}(z) \otimes \mathrm{id}} L \otimes L(\lambda) \otimes L^{*a}(\mu) \tag{7.13}$$

and

$$L(\lambda) \otimes L^{*a}(\lambda) \xrightarrow{\Phi_\lambda^{V\mu} \otimes \mathrm{id}} L \otimes L(\mu) \otimes L^{*a}(\lambda)$$

$$\xrightarrow{\mathrm{id} \otimes \tilde{\Phi}_\mu^{\lambda V} \otimes \mathrm{id}} L \otimes L(\lambda) \otimes L \otimes L^{*a}(\lambda)$$

$$\xrightarrow{\mathrm{id} \otimes \mathrm{id} \otimes \tilde{\Phi}_{V\lambda}^{*\mu}} L \otimes L(\lambda) \otimes L^{*a}(\mu)$$

respectively. Since the last two maps in (7.13) commute, it suffices to show that

$$\Phi_\mu^{V_1\lambda}(z)\tilde{\Phi}_\lambda^{\mu V_2} = \tau(z)^{-1}\tilde{\Phi}_\mu^{\lambda V_2}\Phi_\lambda^{V_1\mu}(z) .$$

This follows from (6.24) by setting $z_2 = 1$.

Using (7.11) we calculate

$$\begin{aligned} T^2 \circ d \circ T^{-2}\varphi_\pm(z) &= \tau(z)^{-2}T^2 \circ ([d, \varphi_\pm(z)] + \varphi_\pm(z)d) \circ T^{-2} \\ &= -\tau(z)^{-2}T \circ \left(z\frac{d}{dz} - \Delta_\mu + \Delta_\lambda\right)\varphi_\pm(z) \circ T^{-2} \\ &\quad + \varphi_\pm(z)T^2 \circ d \circ T^{-2} . \end{aligned}$$

The first term can be written as

$$-\tau(z)^{-2}\left(z\frac{d}{dz}-\Delta_\mu+\Delta_\lambda\right)(T^2\circ\varphi_\pm(z)\circ T^{-2})$$
$$=-\left(2z\frac{d}{dz}\log\tau(z)\right)\varphi_\pm(z)-\left(z\frac{d}{dz}-\Delta_\mu+\Delta_\lambda\right)\varphi_\pm(z)\,.$$

The relations (7.12) follow from these. □

Let us compare them with the formulas for the energy and momentum of "spin waves" obtained in refs. [10, 37] via the Bethe Ansatz method. The result is given in terms of elliptic functions of nome $-q=e^{-\gamma}$ as ([10], Eqs. (26), (27))

$$\varepsilon(\theta)=\frac{2K}{\pi}\sinh\gamma\,\mathrm{dn}\left(\frac{2K}{\pi}\theta,k\right),\quad p(\theta)=\mathrm{am}\left(\frac{2K}{\pi}\theta,k\right)-\frac{1}{2}\pi\,.$$

Identifying $z=-e^{2i\theta}$ and using the identity ([38], p. 509)

$$e^{-ip(\theta)}=z^{-1/2}\frac{\Theta(qz)}{\Theta(qz^{-1})},\quad z\frac{d}{dz}\log e^{-ip(\theta)}=-\frac{\varepsilon(\theta)}{q-q^{-1}}\,,$$

we see that our (7.11), (7.12) are precisely the same. This gives a strong evidence in favor of our mathematical framework of the particle picture.

8. Discussions

Let us summarize the content of this paper.

1. We conjectured that the embedding [3] of the irreducible highest weight $U_q(\widehat{\mathfrak{sl}}(2))$-module $V(\Lambda_0)$ into the semi-infinite tensor product, is given by the limit of n-point correlation functions of the $U_q(\widehat{\mathfrak{sl}}(2))$ vertex operators as $n\to\infty$. We do not know the exact form of the correct normalization of the vertex operator.
2. We postulated that the mathematical content of the infinite tensor product on which the XXZ Hamiltonian (after a suitable renormalization) is acting, is

$$\mathrm{End}_{\mathbf{C}}(V(\Lambda_0)\oplus V(\Lambda_1))\,.$$

For simplicity, we mainly used the half of this, i.e., $\mathrm{Hom}_{\mathbf{C}}(V(\Lambda_0),V(\Lambda_0)\oplus V(\Lambda_1))$.
3. We made the dictionary between the physical and mathematical pictures, which contains:

(a) The translation and energy operators (7.4), (7.5).
(b) The inner product of states (7.1).
(c) The vacuum states (7.3).
(d) The creation and annihilation operators (7.9).

4. By utilizing the q-deformed Knizhnik–Zamolodchikov equation, we got the following:

(e) The energy and momentum of the creation and annihilation operators (Proposition 7.2).
(f) The commutation relations of the creation and annihilation operators among themselves, i.e., the Zamolodchikov algebra (7.10a)–(7.10c).

(g) A conjectural formula for the n-point correlation functions of the q-deformed vertex operators for arbitrary n (6.37), (6.40). (This is not the same as the n-point correlation functions of the creation and annihilation operators.)

5. We checked the validity of our picture on the following points:

(h) A one-line proof of the character expression for the one-point function (in the sense of [23]) of the six-vertex model (Sect. 5).
(i) Comparison of the energy and momentum of particles with the result obtained by the Bethe Ansatz (Sect. 6.7).
(j) Comparison of the vacuum and two-particle states with the Bethe Ansatz eigenvectors in the anisotropic limit $\Delta = -\infty$ (Appendix 5).
(k) Comparison of the commutation relations of the creation and annihilation operators with the S-matrix of the $su(2)$-Thirring model (Sect. 7.2).

6. The following is a list of unsolved problems (besides the several conjectures stated in the main body of the paper):

- For the XXZ model (and the six-vertex model), we want to know about the form factors of the local operators, the staggered polarization [27] and other correlation functions.

Other models, for which a similar diagonalization scheme might apply, are, especially:

- The vertex models with perfect crystals. The framework of the present paper admits immediate extension to the general case. Reshetikhin [39] proposes that the space of n-particle states in the generalized XXX model with higher spin is a tensor product of two factors, $(\mathbf{C}^2)^{\otimes n}$ and the space of paths of length n of an RSOS model. It would be an interesting problem to examine this picture from our viewpoint.
- The XYZ model, the hard hexagon model [13], Kashiwara–Miwa's model [40, 41], the chiral Potts model [42, 43]. These are massive models, each within different category. No obvious extension of our scheme to any one of them is known.

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Appendix 1. Action of $U_q(\widehat{\mathfrak{sl}}(2))$ on $V(\Lambda_0)$

We will recall some basic properties of the global base of the irreducible highest weight $U_q(\mathfrak{g})$-module $V(\lambda)$, which are given in [25, 44]. Then, we will compute the actions of the Chevalley generators on some vectors of the upper global base in the case $\mathfrak{g} = \widehat{\mathfrak{sl}}(2)$ and $\lambda = \Lambda_0$.

Let $B(\lambda)$ be the crystal for $V(\lambda)$. (See Sect. 2 of [2], especially Definition 2.2.3 and Theorem 2.4.1.) We use the map ε_i, φ_i: $B(\lambda) \to N$. (See Definition 2.2.3 and (2.2.15) in [2].) Set $l_i(b) = \varepsilon_i(b) + \varphi_i(b)$. This is the length of the $\mathfrak{sl}(2)$-string of color i through b. For $b \in B(\lambda)$ we denote by $G^{\mathrm{up}}(b) \in V(\lambda)$ the upper global base vector corresponding to b, and by $G^{\mathrm{low}}(b) \in V(\lambda)$ the lower global base vector corresponding to b. (In [26], G^{up} is written as G (see Sect. 5) and G^{low} is written as G_λ (see Sect. 4).)

We use the following properties.

(1) Duality. Let $(\,,\,)$ be the symmetric bilinear form on $V(\lambda)$ given in Sect. 6.7, (6.27), (6.28). We have

$$(G^{\mathrm{up}}(b), G^{\mathrm{low}}(b')) = \delta_{bb'} \,. \tag{A1.1}$$

Therefore, if

$$e_i G^{\mathrm{up}}(b) = \sum_{b'} c_i(b', b) G^{\mathrm{up}}(b'), \qquad f_i G^{\mathrm{up}}(b) = \sum_{b'} c_i'(b', b) G^{\mathrm{up}}(b') \,,$$

then we have

$$e_i G^{\mathrm{low}}(b) = \sum_{b'} c_i'(b, b') G^{\mathrm{low}}(b'), \quad f_i G^{\mathrm{low}}(b) = \sum_{b'} c_i(b, b') G^{\mathrm{low}}(b') \,.$$

(2) Leading order action. We have

$$e_i G^{\mathrm{up}}(b) = [\varepsilon_i(b)]_i G^{\mathrm{up}}(\tilde{e}_i b) + \sum_{b'} E^i_{bb'} G(b') \,,$$

$$f_i G^{\mathrm{up}}(b) = [\varphi_i(b)]_i G^{\mathrm{up}}(\tilde{f}_i b) + \sum_{b'} F^i_{bb'} G(b') \,,$$

where $E^i_{b'b}$ and $F^i_{b'b}$ are zero if $l_i(b') \geqq l_i(b)$. We also have

$$\lim_{q \to 0} \frac{E^i_{b'b}}{[\varepsilon_i(b)]_i} = \lim_{q \to 0} \frac{F^i_{b'b}}{[\varphi_i(b)]_i} = 0 \,.$$

(3) Positivity.

We have $E^i_{b'b}, F^i_{b'b} \in \bigoplus_{n=0}^{\infty} \mathbf{Z}_{\geqq 0}[n]$, where $[n]$ means $(q^n - q^{-n})/(q - q^{-1})$.

(1) is implicitly given in [25]. (2) follows from Proposition 5.3.1 and (5.3.8–10) in [25]. (3) is proved in [44] (Theorem 11.5).

Now we restrict to the case $\mathfrak{g} = \widehat{\mathfrak{sl}}(2)$ and $\lambda = \Lambda_0$. The crystal $B(\Lambda_0)$ is identified with the set of paths: $p = \{p(k)\}_{k \geqq 1}$ is called a path if and only if $p(k) = (+)$ or $(-)$, and

$$\begin{aligned} p(k) &= (+) \quad \text{if } k \text{ is even and } k \gg 0 \,, \\ &= (-) \quad \text{if } k \text{ is odd and } k \gg 0 \,. \end{aligned}$$

For a path p, we define its signature

$$l = \begin{pmatrix} l_1 \\ \vdots \\ l_m \end{pmatrix}$$

in such a way that

$$l_1 > \cdots > l_m > 0 ,$$
$$p(k+1) = p(k) \quad \text{if and only if } k \in \{ l_1, \ldots, l_m \} .$$

We call m the depth of the corresponding path.

Example. If $p = (\ldots p(4)p(3)p(2)p(1)) = (\cdots + - - + + +)$, then, $l = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$.

The depth of p is 3.

In Tables 1 and 2 below, if l is the signature of p, we will write l to represent $G^{\mathrm{up}}(p)$. It is quite convenient to associate a parity symbol

$$s = \begin{pmatrix} s_1 \\ \vdots \\ s_m \end{pmatrix}$$

Table 1

$e_0 \phi = 0 \qquad e_1 \phi = 0 \qquad f_0 \phi = (1) \boxed{} \qquad f_1 \phi = 0$

$e_0 \boxed{} = (-1) \boxed{\bullet} \qquad e_0 \boxed{\bullet} = 0 \qquad e_1 \boxed{} = 0 \qquad e_1 \boxed{\bullet} = (-1) \boxed{}$

$f_0 \boxed{} = 0 \qquad f_0 \boxed{\bullet} = (1) \boxed{} \qquad f_1 \boxed{} = [2](1) \boxed{\bullet} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{array}{|c|}\hline \\ \hline \bullet \\ \hline \end{array} \qquad f_1 \boxed{\bullet} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{array}{|c|}\hline \bullet \\ \hline \bullet \\ \hline \end{array}$

$e_0 \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array} = [2] \begin{pmatrix} 0 \\ -1 \end{pmatrix} \begin{array}{|c|}\hline \\ \hline \bullet \\ \hline \end{array} + \left\{ \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\} \begin{array}{|c|}\hline \bullet \\ \hline \\ \hline \end{array} \qquad e_0 \begin{array}{|c|}\hline \bullet \\ \hline \\ \hline \end{array} = 0$

$e_0 \begin{array}{|c|}\hline \\ \hline \bullet \\ \hline \end{array} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \begin{array}{|c|}\hline \bullet \\ \hline \bullet \\ \hline \end{array} \qquad e_0 \begin{array}{|c|}\hline \bullet \\ \hline \bullet \\ \hline \end{array} = 0$

$e_1 \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array} = 0 \qquad e_1 \begin{array}{|c|}\hline \bullet \\ \hline \\ \hline \end{array} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array} \qquad e_1 \begin{array}{|c|}\hline \\ \hline \bullet \\ \hline \end{array} = 0$

$e_1 \begin{array}{|c|}\hline \bullet \\ \hline \bullet \\ \hline \end{array} = [2] \begin{pmatrix} 0 \\ -1 \end{pmatrix} \begin{array}{|c|}\hline \bullet \\ \hline \\ \hline \end{array} + \left\{ \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\} \begin{array}{|c|}\hline \\ \hline \bullet \\ \hline \end{array}$

$f_0 \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \end{array} \qquad f_0 \begin{array}{|c|}\hline \bullet \\ \hline \\ \hline \end{array} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{array}{|c|}\hline \bullet \\ \hline \\ \hline \\ \hline \end{array}$

$f_0 \begin{array}{|c|}\hline \\ \hline \bullet \\ \hline \end{array} = [2] \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{array}{|c|}\hline \\ \hline \bullet \\ \hline \\ \hline \end{array}$

$f_0 \begin{array}{|c|}\hline \bullet \\ \hline \bullet \\ \hline \end{array} = [3] \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{array}{|c|}\hline \\ \hline \bullet \\ \hline \end{array} + [2] \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\} \begin{array}{|c|}\hline \bullet \\ \hline \\ \hline \end{array} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{array}{|c|}\hline \bullet \\ \hline \bullet \\ \hline \\ \hline \end{array}$

Table 1. (*Continued*)

$$f_1\begin{array}{|c|}\hline \\\hline \\\hline\end{array} = [2]\begin{pmatrix}1\\0\end{pmatrix}\begin{array}{|c|}\hline \bullet\\\hline \\\hline\end{array} + \left\{\begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}2\\-1\end{pmatrix}\right\}\begin{array}{|c|}\hline \\\hline \bullet\\\hline\end{array} \qquad f_1\begin{array}{|c|}\hline \bullet\\\hline \\\hline\end{array} = \begin{pmatrix}0\\1\end{pmatrix}\begin{array}{|c|}\hline \bullet\\\hline \bullet\\\hline\end{array}$$

$$f_1\begin{array}{|c|}\hline \\\hline \bullet\\\hline\end{array} = 0 \qquad f_1\begin{array}{|c|}\hline \bullet\\\hline \bullet\\\hline\end{array} = 0$$

$$e_0\begin{array}{|c|}\hline \\\hline \\\hline \\\hline\end{array} = [3]\begin{pmatrix}0\\0\\-1\end{pmatrix}\begin{array}{|c|}\hline \\\hline \\\hline \bullet\\\hline\end{array} + [2]\left\{\begin{pmatrix}0\\-1\\0\end{pmatrix} + \begin{pmatrix}2\\-1\\-2\end{pmatrix} + \begin{pmatrix}0\\1\\-2\end{pmatrix}\right\}\begin{array}{|c|}\hline \\\hline \bullet\\\hline \\\hline\end{array}$$
$$+\left\{\begin{pmatrix}-1\\0\\0\end{pmatrix} + \begin{pmatrix}1\\-2\\0\end{pmatrix} + \begin{pmatrix}1\\0\\-2\end{pmatrix}\right\}\begin{array}{|c|}\hline \bullet\\\hline \\\hline \\\hline\end{array} + \begin{pmatrix}2\\-2\\-1\end{pmatrix}\begin{array}{|c|}\hline \\\hline\end{array}$$

$$e_0\begin{array}{|c|}\hline \bullet\\\hline \\\hline \\\hline\end{array} = \begin{pmatrix}0\\0\\-1\end{pmatrix}\begin{array}{|c|}\hline \bullet\\\hline \\\hline \bullet\\\hline\end{array} \qquad e_0\begin{array}{|c|}\hline \\\hline \bullet\\\hline \\\hline\end{array} = \begin{pmatrix}-1\\0\\0\end{pmatrix}\begin{array}{|c|}\hline \bullet\\\hline \bullet\\\hline \\\hline\end{array}$$

$$e_0\begin{array}{|c|}\hline \\\hline \\\hline \bullet\\\hline\end{array} = [2]\begin{pmatrix}0\\-1\\0\end{pmatrix}\begin{array}{|c|}\hline \\\hline \bullet\\\hline \bullet\\\hline\end{array} + \left\{\begin{pmatrix}-1\\0\\0\end{pmatrix} + \begin{pmatrix}1\\-2\\0\end{pmatrix} + \begin{pmatrix}1\\0\\-2\end{pmatrix}\right\}\begin{array}{|c|}\hline \bullet\\\hline \\\hline \bullet\\\hline\end{array}$$

$$e_0\begin{array}{|c|}\hline \bullet\\\hline \bullet\\\hline \\\hline\end{array} = 0 \qquad e_0\begin{array}{|c|}\hline \bullet\\\hline \\\hline \bullet\\\hline\end{array} = 0 \qquad e_0\begin{array}{|c|}\hline \\\hline \bullet\\\hline \bullet\\\hline\end{array} = \begin{pmatrix}-1\\0\\0\end{pmatrix}\begin{array}{|c|}\hline \bullet\\\hline \bullet\\\hline \bullet\\\hline\end{array} \qquad e_0\begin{array}{|c|}\hline \bullet\\\hline \bullet\\\hline \bullet\\\hline\end{array} = 0$$

$$e_1\begin{array}{|c|}\hline \\\hline \\\hline \\\hline\end{array} = 0 \qquad e_1\begin{array}{|c|}\hline \bullet\\\hline \\\hline \\\hline\end{array} = \begin{pmatrix}-1\\0\\0\end{pmatrix}\begin{array}{|c|}\hline \\\hline \\\hline \\\hline\end{array} \qquad e_1\begin{array}{|c|}\hline \\\hline \bullet\\\hline \\\hline\end{array} = 0 \qquad e_1\begin{array}{|c|}\hline \\\hline \\\hline \bullet\\\hline\end{array} = 0$$

$$e_1\begin{array}{|c|}\hline \bullet\\\hline \bullet\\\hline \\\hline\end{array} = [2]\begin{pmatrix}0\\-1\\0\end{pmatrix}\begin{array}{|c|}\hline \bullet\\\hline \\\hline \\\hline\end{array} + \left\{\begin{pmatrix}-1\\0\\0\end{pmatrix} + \begin{pmatrix}1\\-2\\0\end{pmatrix} + \begin{pmatrix}1\\0\\-2\end{pmatrix}\right\}\begin{array}{|c|}\hline \\\hline \bullet\\\hline \\\hline\end{array}$$

$$e_1\begin{array}{|c|}\hline \bullet\\\hline \\\hline \bullet\\\hline\end{array} = \begin{pmatrix}-1\\0\\0\end{pmatrix}\begin{array}{|c|}\hline \\\hline \\\hline \bullet\\\hline\end{array} \qquad e_1\begin{array}{|c|}\hline \\\hline \bullet\\\hline \bullet\\\hline\end{array} = \begin{pmatrix}0\\0\\-1\end{pmatrix}\begin{array}{|c|}\hline \\\hline \bullet\\\hline \\\hline\end{array}$$

$$e_1\begin{array}{|c|}\hline \bullet\\\hline \bullet\\\hline \bullet\\\hline\end{array} = [3]\begin{pmatrix}0\\0\\-1\end{pmatrix}\begin{array}{|c|}\hline \bullet\\\hline \bullet\\\hline \\\hline\end{array} + [2]\left\{\begin{pmatrix}0\\-1\\0\end{pmatrix} + \begin{pmatrix}2\\-1\\-2\end{pmatrix} + \begin{pmatrix}0\\1\\-2\end{pmatrix}\right\}\begin{array}{|c|}\hline \bullet\\\hline \\\hline \bullet\\\hline\end{array}$$
$$+\left\{\begin{pmatrix}-1\\0\\0\end{pmatrix} + \begin{pmatrix}1\\-2\\0\end{pmatrix} + \begin{pmatrix}1\\0\\-2\end{pmatrix}\right\}\begin{array}{|c|}\hline \\\hline \bullet\\\hline \bullet\\\hline\end{array}$$

$$f_0\begin{array}{|c|}\hline \\\hline \\\hline \\\hline\end{array} = 0 \qquad f_0\begin{array}{|c|}\hline \bullet\\\hline \\\hline \\\hline\end{array} = 0 \qquad f_0\begin{array}{|c|}\hline \\\hline \bullet\\\hline \\\hline\end{array} = 0 \qquad f_0\begin{array}{|c|}\hline \\\hline \\\hline \bullet\\\hline\end{array} = \begin{pmatrix}0\\0\\1\end{pmatrix}\begin{array}{|c|}\hline \\\hline \\\hline \\\hline\end{array}$$

Table 1. (*Continued*)

$$f_0 \begin{array}{|c|}\hline \bullet\\\hline \bullet\\\hline \\\hline\end{array} = \begin{pmatrix}1\\0\\0\end{pmatrix} \begin{array}{|c|}\hline \\\hline \bullet\\\hline \\\hline\end{array} \qquad f_0 \begin{array}{|c|}\hline \bullet\\\hline \\\hline \bullet\\\hline\end{array} = \begin{pmatrix}0\\0\\1\end{pmatrix} \begin{array}{|c|}\hline \bullet\\\hline \\\hline \\\hline\end{array}$$

$$f_0 \begin{array}{|c|}\hline \\\hline \bullet\\\hline \bullet\\\hline\end{array} = [2] \begin{pmatrix}0\\1\\0\end{pmatrix} \begin{array}{|c|}\hline \\\hline \\\hline \bullet\\\hline\end{array} + \left\{ \begin{pmatrix}0\\1\\0\end{pmatrix} + \begin{pmatrix}2\\0\\-1\end{pmatrix} + \begin{pmatrix}0\\2\\-1\end{pmatrix} \right\} \begin{array}{|c|}\hline \\\hline \bullet\\\hline \\\hline\end{array}$$

$$f_0 \begin{array}{|c|}\hline \bullet\\\hline \bullet\\\hline \bullet\\\hline\end{array} = [3] \begin{pmatrix}1\\0\\0\end{pmatrix} \begin{array}{|c|}\hline \\\hline \bullet\\\hline \bullet\\\hline\end{array} + [2] \left\{ \begin{pmatrix}0\\1\\0\end{pmatrix} + \begin{pmatrix}2\\-1\\0\end{pmatrix} + \begin{pmatrix}2\\1\\-2\end{pmatrix} \right\} \begin{array}{|c|}\hline \bullet\\\hline \\\hline \bullet\\\hline\end{array}$$
$$+ \left\{ \begin{pmatrix}0\\0\\1\end{pmatrix} + \begin{pmatrix}2\\0\\-1\end{pmatrix} + \begin{pmatrix}0\\2\\-1\end{pmatrix} \right\} \begin{array}{|c|}\hline \bullet\\\hline \bullet\\\hline \\\hline\end{array}$$

$$f_1 \begin{array}{|c|}\hline \\\hline \\\hline \\\hline\end{array} = [4] \begin{pmatrix}1\\0\\0\end{pmatrix} \begin{array}{|c|}\hline \bullet\\\hline \\\hline \\\hline\end{array} + [3] \left\{ \begin{pmatrix}0\\1\\0\end{pmatrix} + \begin{pmatrix}2\\-1\\0\end{pmatrix} + \begin{pmatrix}2\\1\\-2\end{pmatrix} \right\} \begin{array}{|c|}\hline \\\hline \bullet\\\hline \\\hline\end{array}$$
$$+ [2] \left\{ \begin{pmatrix}0\\0\\1\end{pmatrix} + \begin{pmatrix}2\\0\\-1\end{pmatrix} + \begin{pmatrix}0\\2\\-1\end{pmatrix} \right\} \begin{array}{|c|}\hline \\\hline \\\hline \bullet\\\hline\end{array} + \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \begin{array}{|c|}\hline \\\hline \\\hline \\\hline \bullet\\\hline\end{array}$$

$$f_1 \begin{array}{|c|}\hline \bullet\\\hline \\\hline \\\hline\end{array} = [3] \begin{pmatrix}0\\1\\0\end{pmatrix} \begin{array}{|c|}\hline \bullet\\\hline \bullet\\\hline \\\hline\end{array} + [2] \left\{ \begin{pmatrix}0\\0\\1\end{pmatrix} + \begin{pmatrix}2\\0\\-1\end{pmatrix} + \begin{pmatrix}0\\2\\-1\end{pmatrix} \right\} \begin{array}{|c|}\hline \bullet\\\hline \\\hline \bullet\\\hline\end{array}$$
$$+ \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \begin{array}{|c|}\hline \bullet\\\hline \\\hline \\\hline \bullet\\\hline\end{array} + \begin{pmatrix}1\\1\\-1\end{pmatrix} \begin{array}{|c|}\hline \\\hline \bullet\\\hline\end{array}$$

$$f_1 \begin{array}{|c|}\hline \\\hline \bullet\\\hline \\\hline\end{array} = [2] \begin{pmatrix}0\\0\\1\end{pmatrix} \begin{array}{|c|}\hline \\\hline \bullet\\\hline \bullet\\\hline\end{array} + \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \begin{array}{|c|}\hline \\\hline \bullet\\\hline \\\hline \bullet\\\hline\end{array} \qquad f_1 \begin{array}{|c|}\hline \\\hline \\\hline \bullet\\\hline\end{array} = [2] \begin{pmatrix}1\\0\\0\end{pmatrix} \begin{array}{|c|}\hline \bullet\\\hline \\\hline \bullet\\\hline\end{array} + \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \begin{array}{|c|}\hline \\\hline \\\hline \bullet\\\hline \bullet\\\hline\end{array}$$

$$f_1 \begin{array}{|c|}\hline \bullet\\\hline \bullet\\\hline \\\hline\end{array} = [2] \begin{pmatrix}0\\0\\1\end{pmatrix} \begin{array}{|c|}\hline \bullet\\\hline \bullet\\\hline \bullet\\\hline\end{array} + \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \begin{array}{|c|}\hline \bullet\\\hline \bullet\\\hline \\\hline \bullet\\\hline\end{array}$$

$$f_1 \begin{array}{|c|}\hline \bullet\\\hline \\\hline \bullet\\\hline\end{array} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \begin{array}{|c|}\hline \bullet\\\hline \\\hline \bullet\\\hline \bullet\\\hline\end{array} \qquad f_1 \begin{array}{|c|}\hline \\\hline \bullet\\\hline \bullet\\\hline\end{array} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \begin{array}{|c|}\hline \\\hline \bullet\\\hline \bullet\\\hline \bullet\\\hline\end{array} \qquad f_1 \begin{array}{|c|}\hline \bullet\\\hline \bullet\\\hline \bullet\\\hline\end{array} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \begin{array}{|c|}\hline \bullet\\\hline \bullet\\\hline \bullet\\\hline \bullet\\\hline\end{array}$$

Table 2

$$\Phi^\sigma(v_+\otimes\phi)=\sum_{n=0}^{\infty}z^n(2n)\boxed{\bullet}$$

$$\Phi^\sigma(v_-\otimes\phi)=\sum_{n=0}^{\infty}z^n(2n+1)\boxed{}$$

$$\Phi^\sigma\big(v_+\otimes(1)\boxed{}\big)=\sum_{n=1}^{\infty}z^n\binom{2n}{1}\begin{array}{|c|}\hline \bullet\\ \hline \bullet\\ \hline\end{array}$$

$$\Phi^\sigma\big(v_-\otimes(1)\boxed{}\big)=-z^{-1}q^{-1}\phi+\sum_{n=0}^{\infty}z^nq(2n+2)\boxed{\bullet}+\sum_{n=1}^{\infty}z^n\binom{2n+1}{1}\begin{array}{|c|}\hline \\ \hline \bullet\\ \hline\end{array}$$

$$\Phi^\sigma\big(v_+\otimes(2)\boxed{\bullet}\big)=z^{-1}\frac{q^{-3}}{[2]}\phi-(2)\boxed{\bullet}+\sum_{n=0}^{\infty}z^n\frac{q}{[2]}(2n+2)\boxed{\bullet}+\sum_{n=1}^{\infty}z^nq\binom{2n+1}{1}\begin{array}{|c|}\hline \\ \hline \bullet\\ \hline\end{array}$$
$$+\sum_{n=2}^{\infty}z^n\binom{2n}{2}\begin{array}{|c|}\hline \bullet\\ \hline \\ \hline\end{array}$$

$$\Phi^\sigma\big(v_-\otimes(2)\boxed{\bullet}\big)=-z^{-1}\frac{q^{-1}}{[2]}(1)\boxed{}+\sum_{n=0}^{\infty}z^n\frac{q}{[2]}(2n+3)\boxed{}+\sum_{n=1}^{\infty}z^n\binom{2n+1}{2}\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}$$

$$\Phi^\sigma\big(v_+\otimes(3)\boxed{}\big)=-\binom{2}{1}\begin{array}{|c|}\hline \bullet\\ \hline \bullet\\ \hline\end{array}+\sum_{n=0}^{\infty}z^n\frac{q}{[2]}\binom{2n+2}{1}\begin{array}{|c|}\hline \bullet\\ \hline \bullet\\ \hline\end{array}+\sum_{n=2}^{\infty}z^n\binom{2n}{3}\begin{array}{|c|}\hline \bullet\\ \hline \bullet\\ \hline\end{array}$$

$$\Phi^\sigma\big(v_-\otimes(3)\boxed{}\big)=-z^{-2}\frac{q^{-4}}{[2]}\phi-z^{-1}\frac{1}{[2]}(2)\boxed{\bullet}-\binom{3}{1}\begin{array}{|c|}\hline \\ \hline \bullet\\ \hline\end{array}+\sum_{n=0}^{\infty}z^n\frac{q^2}{[2]}(2n+4)\boxed{\bullet}$$
$$+\sum_{n=0}^{\infty}z^n\frac{q}{[2]}\binom{2n+3}{1}\begin{array}{|c|}\hline \\ \hline \bullet\\ \hline\end{array}$$
$$+\sum_{n=1}^{\infty}z^nq\binom{2n+2}{2}\begin{array}{|c|}\hline \bullet\\ \hline \\ \hline\end{array}+\sum_{n=2}^{\infty}z^n\binom{2n+1}{3}\begin{array}{|c|}\hline \\ \hline \bullet\\ \hline\end{array}$$

$$\Phi^\sigma\left(v_+\otimes\binom{2}{1}\begin{array}{|c|}\hline \bullet\\ \hline \bullet\\ \hline\end{array}\right)=z^{-1}q^{-2}(1)\boxed{}-(3)\boxed{}+\sum_{n=1}^{\infty}z^nq^2\binom{2n+1}{2}\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}+\sum_{n=2}^{\infty}z^n\begin{pmatrix}2n\\2\\1\end{pmatrix}\begin{array}{|c|}\hline \bullet\\ \hline \\ \hline \\ \hline\end{array}$$

$$\Phi^\sigma\left(v_-\otimes\binom{2}{1}\begin{array}{|c|}\hline \bullet\\ \hline \bullet\\ \hline\end{array}\right)=\sum_{n=1}^{\infty}z^n\begin{pmatrix}2n+1\\2\\1\end{pmatrix}\begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline\end{array}$$

Table 2. (*Continued*)

$$\Phi^\sigma(v_+\otimes(4)\boxed{\bullet}) = z^{-2}\frac{q^{-6}[3]}{[2]^2([3]-1)}\phi - z^{-1}\frac{q^{-2}}{[2]^2([3]-1)}(2)\boxed{\bullet} - (4)\boxed{\bullet}$$

$$-q\binom{3}{1}\begin{array}{|c|}\hline \\\hline \bullet\\\hline\end{array} - z\binom{4}{2}\begin{array}{|c|}\hline \bullet\\\hline \\\hline\end{array}$$

$$+\sum_{n=0}^{\infty} z^n\frac{q^2[3]}{[2]^2([3]-1)}(2n+4)\boxed{\bullet} + \sum_{n=0}^{\infty} z^n\frac{q^2}{[2]}\binom{2n+3}{1}\begin{array}{|c|}\hline \\\hline \bullet\\\hline\end{array}$$

$$+\sum_{n=1}^{\infty} z^n\frac{q}{[2]}\binom{2n+2}{2}\begin{array}{|c|}\hline \bullet\\\hline \\\hline\end{array}$$

$$+\sum_{n=2}^{\infty} z^n q\binom{2n+1}{3}\begin{array}{|c|}\hline \\\hline \bullet\\\hline\end{array} + \sum_{n=3}^{\infty} z^n\binom{2n}{4}\begin{array}{|c|}\hline \bullet\\\hline \\\hline\end{array}$$

$$\Phi^\sigma(v_-\otimes(4)\boxed{\bullet}) = -z^{-2}\frac{q^{-2}}{[2]^2([3]-1)}(1)\boxed{} - z^{-1}\frac{[3]+q^{-3}[2]}{[2]^2([3]-1)}(3)\boxed{} - \binom{3}{2}\begin{array}{|c|}\hline \\\hline \\\hline\end{array}$$

$$+\sum_{n=0}^{\infty}\frac{z^n q^2[3]}{[2]^2([3]-1)}(2n+5)\boxed{} + \sum_{n=0}^{\infty} z^n\frac{q}{[2]}\binom{2n+3}{2}\begin{array}{|c|}\hline \\\hline \\\hline\end{array}$$

$$+\sum_{n=2}^{\infty} z^n\binom{2n+1}{4}\begin{array}{|c|}\hline \\\hline \\\hline\end{array}$$

$$\Phi^\sigma\left(v_+\otimes\binom{3}{1}\begin{array}{|c|}\hline \\\hline \bullet\\\hline\end{array}\right) = -z^{-2}\frac{-q^{-6}}{[2]([3]-1)}\phi + \frac{z^{-1}q^{-2}[3]}{[2]([3]-1)}(2)\boxed{\bullet} - \sum_{n=0}^{\infty}\frac{z^n q^2}{[2]([3]-1)}(2n+4)\boxed{\bullet}$$

$$+\sum_{n=1}^{\infty} z^n q\binom{2n+2}{2}\begin{array}{|c|}\hline \bullet\\\hline \\\hline\end{array} + \sum_{n=2}^{\infty} z^n\begin{pmatrix}2n\\3\\1\end{pmatrix}\begin{array}{|c|}\hline \bullet\\\hline \bullet\\\hline \\\hline\end{array}$$

$$\Phi^\sigma\left(v_-\otimes\binom{3}{1}\begin{array}{|c|}\hline \\\hline \bullet\\\hline\end{array}\right) = -\frac{z^{-2}q^{-6}}{[2]([3]-1)}(1)\boxed{} + \frac{z^{-1}q^{-2}[3]}{[2]([3]-1)}(3)\boxed{} - \sum_{n=0}^{\infty}\frac{z^n q^2}{[2]([3]-1)}(2n+5)\boxed{}$$

$$+\sum_{n=1}^{\infty} z^n q\begin{pmatrix}2n+2\\2\\1\end{pmatrix}\begin{array}{|c|}\hline \bullet\\\hline \\\hline \\\hline\end{array} + \sum_{n=2}^{\infty} z^n\begin{pmatrix}2n+1\\3\\1\end{pmatrix}\begin{array}{|c|}\hline \\\hline \bullet\\\hline \\\hline\end{array}$$

with a signature l in such a way that

$$s_k = \boxed{} \quad \text{if } l_k \equiv k \bmod 2\,,$$

$$= \boxed{\bullet} \quad \text{if } l_k \not\equiv k \bmod 2\,.$$

Namely, we represent $l = \begin{pmatrix}4\\2\\1\end{pmatrix}$ as

$$\begin{pmatrix}4\\2\\1\end{pmatrix}\begin{array}{|c|}\hline \bullet\\\hline \\\hline \\\hline\end{array}$$

In Table 1 we list the actions of e_i and f_i ($i = 0, 1$) on the upper global base labeled by l such that $m \leqq 3$. The symbol ϕ means the highest weight vector. We abbreviate signatures by the differences to the signature of the operand. For example,

$$f_1 \ \square = [2](1)\boxed{\bullet} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{array}{|c|}\hline \\ \hline \bullet \\ \hline \end{array}$$

really means

$$f_1(l_1)\ \square = [2](l_1+1)\boxed{\bullet} + \begin{pmatrix} l_1 \\ 1 \end{pmatrix} \begin{array}{|c|}\hline \\ \hline \bullet \\ \hline \end{array} \quad \text{for all positive odd } l_1 .$$

We have derived these formulas inductively by using the properties (1–3).

In the right-hand-sides of these formulas, we allow a parity symbol to have length greater than the actual depth of the accompanied signature. Namely, if a signature l is accompanied by a parity symbol of length m, then $l = (l_k)_{1 \leqq k \leqq m}$ can be such that $l_1 > \cdots > l_j > 0 = l_{j+1} = \cdots = l_m$, and it is considered as $(l_k)_{1 \leqq k \leqq j}$. If a signature which breaks even this condition does appear, (e.g., $l_1 = l_2 > 0$), we understand that the corresponding $G^{\mathrm{up}}(p)$ to be zero.

There are some cases such that a parity symbol to have length smaller than the depth of the accompanied signature. In those cases we understand the corresponding terms are zero.

In the left-hand sides of the formulas, we always assume the length of a parity symbol is equal to the depth of the accompanied signature.

Appendix 2. Vertex Operator $V_z \otimes V(\Lambda_0) \to V(\Lambda_1)$

Let $V(\Lambda_i)$ ($i = 0, 1$) be the irreducible highest weight $U_q(\widehat{\mathfrak{sl}}(2))$-module with the highest weight Λ_0, and let V_z be the 2 dimensional U-module given in Sect. 6.2.

Denote by $\hat{V}(\Lambda_i)$ the direct product of the weight spaces of $V(\Lambda_i)$. There is a unique U-linear map

$$\Phi\colon V_z \otimes V(\Lambda_0) \to \hat{V}(\Lambda_1)$$

such that $(\Phi(v_+ \otimes u_{\Lambda_0}), u_{\Lambda_1}) = 1$, where $(\ ,\ )$ is as in Appendix 1. (To be precise Φ is a $U \otimes \mathbf{Q}(q)[z, z^{-1}]$-linear map $V_z \hat{\otimes} V(\Lambda_0) \to \hat{V}(\Lambda_1) \otimes \mathbf{Q}(q)[\![z, z^{-1}]\!]$ (see Sect. 6.4), but we shall not bother writing $\mathbf{Q}(q)[z, z^{-1}]$.) We list

$$\Phi(v_\pm \otimes G^{\mathrm{up}}(p)) = \sum_{p'} c_\pm(p', p) G^{\mathrm{up}}(p')$$

for the first seven paths p whose signatures (see Appendix 1) are ϕ, (1), (2), (3), $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, (4), $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

For convenience, we define $\Phi^\sigma\colon V_z \otimes V(\Lambda_0) \to \hat{V}(\Lambda_0)$ by $\Phi^\sigma = \sigma\Phi$, where σ is the $\mathbf{Q}(q)$-linear map $\hat{V}(\Lambda_0) \to \hat{V}(\Lambda_1)$ induced by the Dynkin diagram automorphism of $U_q(\widehat{\mathfrak{sl}}(2))$ which exchanges the colors 0 and 1. The list in Table 2 is for Φ^σ. The vertex operator $\Phi\colon V_z \otimes V(\Lambda_0) \to \hat{V}(\Lambda_1)$ is given by

$$\Phi(v_\pm \otimes v) = \sigma(\Phi^\sigma(v_\pm \otimes v)) ,$$

and the vertex operator Φ^c: $V_z \otimes V(\Lambda_1) \to V(\Lambda_0)$ is given by

$$\Phi^c(v_+ \otimes v) = z\Phi^\sigma(v_- \otimes \sigma(v)), \quad \Phi^c(v_- \otimes v) = \Phi^\sigma(v_+ \otimes \sigma(v)) \,. \tag{A2.1}$$

The formulas below are obtained inductively by using the result of Appendix 1. For example,

$$\begin{aligned}
\Phi^\sigma(v_- \otimes (1)\boxed{}) &= \Phi^\sigma(v_- \otimes f_0\phi) \\
&= f_1\Phi^\sigma(v_- \otimes \phi) - z^{-1}q^{-1}\Phi^\sigma(v_+ \otimes \phi) \\
&= \sum_{n=0}^{\infty} z^n(2n+1)\left\{[2](1)\boxed{\bullet} + \binom{0}{1}\begin{array}{|c|}\hline \\ \hline \bullet \\ \hline \end{array}\right\} - z^{-1}q^{-1}\sum_{n=0}^{\infty} z^n(2n)\boxed{\bullet} \\
&= -z^{-1}q^{-1}\phi + \sum_{n=0}^{\infty} z^n q(2n+2)\boxed{\bullet} + \sum_{n=1}^{\infty} z^n\binom{2n+1}{1}\begin{array}{|c|}\hline \\ \hline \bullet \\ \hline \end{array} \,.
\end{aligned}$$

Appendix 3. Vertex Operator $V(\Lambda_0) \to V(\Lambda_1) \otimes V$

We follow the notation in Appendix 1 except that now we use Φ for

$$\Phi\colon V(\Lambda_0) \to \hat{V}(\Lambda_1) \otimes V$$

and set

$$\Phi(v) = \Phi_+(v) \otimes v_+ + \Phi_-(v) \otimes v_- \,.$$

We list $\Phi_\pm^\sigma = \sigma\Phi_\pm$ in Table 3, where σ is the Dynkin diagram automorphism. We use the normalization $(\Phi_-(u_{\Lambda_0}), u_{\Lambda_1}) = 1$. Let us prove

$$\Phi_+^\sigma(\phi) = -\sum_{n=0}^{\infty} q^{3n+1}(2n+1)\boxed{}\,,$$

$$\Phi_-^\sigma(\phi) = \sum_{n=0}^{\infty} q^{3n}(2n)\boxed{\bullet}\,, \tag{A3.1}$$

from which the rest of formulas are inductively derived by using

$$\Phi_\pm^\sigma(f_0 v) = q^{\pm 1}f_1\Phi_\pm^\sigma(v) + \delta_{\pm,+}\Phi_-^\sigma(v)\,,$$

$$\Phi_\pm^\sigma(f_1 v) = q^{\mp 1}f_0\Phi_\pm^\sigma(v) + \delta_{\pm,-}\Phi_+^\sigma(v)\,.$$

The proof goes as follows. We have

$$(\Phi_\pm^\sigma(e_0 v), v') = (\Phi_\pm^\sigma(v), f_1 v') + \delta_{\pm,-}(\Phi_+^\sigma(v), t_1 v')\,,$$

$$(\Phi_\pm^\sigma(e_1 v), v') = (\Phi_\pm^\sigma(v), f_0 v') + \delta_{\pm,+}(\Phi_-^\sigma(v), t_0 v')\,. \tag{A3.2}$$

By using these formulas for $v = \phi$, we find that

$$(\Phi_\pm^\sigma(\phi), v') = 0 \quad \text{if } v' \in \operatorname{Im} f_0^2 \cup \operatorname{Im} f_1^2\,.$$

Table 3

$$\Phi^\sigma_+(\phi) = -\sum_{n=0}^{\infty} q^{3n+1}(2n+1)\boxed{} \qquad \Phi^\sigma_-(\phi) = \sum_{n=0}^{\infty} q^{3n}(2n)\boxed{\bullet}$$

$$\Phi^\sigma_+\left((1)\boxed{}\right) = \phi - \sum_{n=0}^{\infty} q^{3n+1}(2n+2)\boxed{\bullet} - \sum_{n=1}^{\infty} q^{3n+2}\binom{2n+1}{1}\begin{array}{|c|}\hline \\ \hline \bullet \\ \hline\end{array}$$

$$\Phi^\sigma_-\left((1)\boxed{}\right) = \sum_{n=1}^{\infty} q^{3n-1}\binom{2n}{1}\begin{array}{|c|}\hline \bullet \\ \hline \bullet \\ \hline\end{array}$$

$$\Phi^\sigma_+\left((2)\boxed{\bullet}\right) = \frac{q^{-1}}{[2]}(1)\boxed{} - \sum_{n=0}^{\infty}\frac{q^{3n}}{[2]}(2n+3)\boxed{} - \sum_{n=1}^{\infty} q^{3n+1}\binom{2n+1}{2}\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}$$

$$\Phi^\sigma_-\left((2)\boxed{\bullet}\right) = \frac{1}{[2]}\phi - \frac{q}{[2]}(2)\boxed{\bullet} + \sum_{n=1}^{\infty}\frac{q^{3n-1}}{[2]}(2n+2)\boxed{\bullet}$$

$$+ \sum_{n=1}^{\infty} q^{3n-1}\binom{2n+1}{1}\begin{array}{|c|}\hline \\ \hline \bullet \\ \hline\end{array} + \sum_{n=2}^{\infty} q^{3n}\binom{2n}{2}\begin{array}{|c|}\hline \bullet \\ \hline \\ \hline\end{array}$$

$$\Phi^\sigma_+\left((3)\boxed{}\right) = \frac{1}{[2]}\phi + \frac{q^{-1}}{[2]}(2)\boxed{\bullet} - \sum_{n=0}^{\infty}\frac{q^{3n}}{[2]}(2n+4)\boxed{\bullet}$$

$$+ \frac{q^3}{[2]}\binom{3}{1}\begin{array}{|c|}\hline \\ \hline \bullet \\ \hline\end{array} - \sum_{n=1}^{\infty}\frac{q^{3n+1}}{[2]}\binom{2n+3}{1}\begin{array}{|c|}\hline \\ \hline \bullet \\ \hline\end{array}$$

$$- \sum_{n=2}^{\infty} q^{3n+2}\binom{2n+1}{3}\begin{array}{|c|}\hline \\ \hline \bullet \\ \hline\end{array} - \sum_{n=1}^{\infty} q^{3n+1}\binom{2n+2}{2}\begin{array}{|c|}\hline \bullet \\ \hline \\ \hline\end{array}$$

$$\Phi^\sigma_-\left((3)\boxed{}\right) = -\frac{1}{[2]}\binom{2}{1}\begin{array}{|c|}\hline \bullet \\ \hline \bullet \\ \hline\end{array} + \sum_{n=1}^{\infty}\frac{q^{3n-2}}{[2]}\binom{2n+2}{1}\begin{array}{|c|}\hline \bullet \\ \hline \bullet \\ \hline\end{array}$$

$$+ \sum_{n=2}^{\infty} q^{3n-1}\binom{2n}{3}\begin{array}{|c|}\hline \bullet \\ \hline \bullet \\ \hline\end{array}$$

$$\Phi^\sigma_+\left(\binom{2}{1}\begin{array}{|c|}\hline \bullet \\ \hline \\ \hline\end{array}\right) = -\sum_{n=1}^{\infty} q^{3n}\begin{pmatrix}2n+1\\2\\1\end{pmatrix}\begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline\end{array}$$

$$\Phi^\sigma_-\left(\binom{2}{1}\begin{array}{|c|}\hline \bullet \\ \hline \\ \hline\end{array}\right) = (1)\boxed{} - q(3)\boxed{} + \sum_{n=1}^{\infty} q^{3n-1}\binom{2n+1}{2}\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}$$

$$+ \sum_{n=2}^{\infty} q^{3n+1}\begin{pmatrix}2n\\2\\1\end{pmatrix}\begin{array}{|c|}\hline \bullet \\ \hline \\ \hline \\ \hline\end{array}$$

Table 3. (*Continued*)

$$\Phi^{\sigma}_{+}((4)\square)=\frac{q^{-3}}{[2]^2([3]-1)}(1)\square+\frac{q^{-4}+q^{-2}+2+q^{2}}{[2]^2([3]-1)}(3)\square$$

$$+\frac{q^2}{[2]}\binom{3}{2}\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}-\sum_{n=0}^{\infty}\frac{q^{3n-1}[3]}{[2]^2([3]-1)}(2n+5)\square$$

$$-\sum_{n=1}^{\infty}\frac{q^{3n}}{[2]}\binom{2n+3}{2}\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}-\sum_{n=2}^{\infty}q^{3n+1}\binom{2n+1}{4}\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}$$

$$\Phi^{\sigma}_{-}((4)\boxdot)=\frac{[3]}{[2]^2([3]-1)}\phi-\frac{q^{-1}}{[2]^2([3]-1)}(2)\boxdot+\frac{1-q[3][2]}{[2]^2([3]-1)}(4)\boxdot$$

$$-\frac{1}{[2]}\binom{3}{1}\begin{array}{|c|}\hline \\ \hline \bullet\\ \hline\end{array}-\frac{q^4}{[2]}\binom{4}{2}\begin{array}{|c|}\hline \bullet\\ \hline \\ \hline\end{array}+\sum_{n=1}^{\infty}\frac{q^{3n-2}[3]}{[2]^2([3]-1)}(2n+4)\boxdot$$

$$+\sum_{n=1}^{\infty}\frac{q^{3n-2}}{[2]}\binom{2n+3}{1}\begin{array}{|c|}\hline \\ \hline \bullet\\ \hline\end{array}+\sum_{n=2}^{\infty}q^{3n-1}\binom{2n+1}{3}\begin{array}{|c|}\hline \\ \hline \bullet\\ \hline\end{array}$$

$$+\sum_{n=2}^{\infty}\frac{q^{3n-1}}{[2]}\binom{2n+2}{2}\begin{array}{|c|}\hline \bullet\\ \hline \\ \hline\end{array}+\sum_{n=3}^{\infty}q^{3n}\binom{2n}{4}\begin{array}{|c|}\hline \bullet\\ \hline \\ \hline\end{array}$$

$$\Phi^{\sigma}_{+}\left(\binom{3}{1}\begin{array}{|c|}\hline \\ \hline \bullet\\ \hline\end{array}\right)=\frac{q}{[2]([3]-1)}(1)\square-\frac{[3]}{[2]([3]-1)}(3)\square$$

$$+\sum_{n=0}^{\infty}\frac{q^{3n-1}}{[2]([3]-1)}(2n+5)\square-\sum_{n=1}^{\infty}q^{3n}\begin{pmatrix}2n+2\\2\\1\end{pmatrix}\begin{array}{|c|}\hline \bullet\\ \hline \\ \hline \\ \hline\end{array}$$

$$-\sum_{n=2}^{\infty}q^{3n+1}\begin{pmatrix}2n+1\\3\\1\end{pmatrix}\begin{array}{|c|}\hline \\ \hline \bullet\\ \hline \\ \hline\end{array}$$

$$\Phi^{\sigma}_{-}\left(\binom{3}{1}\begin{array}{|c|}\hline \\ \hline \bullet\\ \hline\end{array}\right)=-\frac{1}{[2]([3]-1)}\phi+\frac{q^{-1}[3]}{[2]([3]-1)}(2)\boxdot-\frac{q^{-2}}{[2]([3]-1)}(4)\boxdot+q^2\binom{4}{2}\begin{array}{|c|}\hline \bullet\\ \hline \\ \hline\end{array}$$

$$-\sum_{n=1}^{\infty}\frac{q^{3n-2}}{[2]([3]-1)}(2n+4)\boxdot+\sum_{n=2}^{\infty}q^{3n-1}\binom{2n+2}{2}\begin{array}{|c|}\hline \bullet\\ \hline \\ \hline\end{array}$$

$$+\sum_{n=2}^{\infty}q^{3n}\begin{pmatrix}2n\\3\\1\end{pmatrix}\begin{array}{|c|}\hline \bullet\\ \hline \bullet\\ \hline \\ \hline\end{array}$$

In general, the lower global base $G^{\mathrm{low}}(b)$ $(b\in B)$ has the property that $\varepsilon_i(b)\geqq k$ if and only if $G^{\mathrm{low}}(b)\in\operatorname{Im}f_i^{k}$. In our situation, from this follows that

$$(\Phi^{\sigma}_{\pm}(\phi),G^{\mathrm{low}}(p))=0\quad\text{if }\varepsilon_0(p)\geqq 2\text{ or }\varepsilon_1(p)\geqq 2\,. \tag{A3.3}$$

Now, for a given $p\in\mathscr{P}_0$, take any sequence $(i_N,\dots,i_1)$ such that $p=\tilde{f}_{i_N}\dots\tilde{f}_{i_1}\bar{p}_0$. Again, in general, the actions of e_i and f_i on the global base are given by

$$e_i G^{\mathrm{low}}(b) = [\phi_i(b) + 1]_i G^{\mathrm{low}}(\tilde{e}_i b) + \sum_{b'} F^i_{b'b} G^{\mathrm{low}}(b') ,$$

$$f_i G^{\mathrm{low}}(b) = [\varepsilon_i(b) + 1]_i G^{\mathrm{low}}(\tilde{f}_i b) + \sum_{b'} E^i_{b'b} G^{\mathrm{low}}(b') ,$$

where $F^i_{b'b}$ and $E^i_{b'b}$ are the same as those in Appendix 1. In short, the actions create the correction terms (i.e., terms other than the combinatorial ones $G^{\mathrm{low}}(\tilde{e}_i b)$ and $G^{\mathrm{low}}(\tilde{f}_i b)$) of the string length greater than the combinatorial term. Therefore, we conclude that

$$(\Phi^{\sigma}_{\pm}(\phi), G^{\mathrm{low}}(p)) = c(\Phi^{\sigma}_{\pm}(\phi), f_{i_N} \ldots f_{i_1}\phi)$$

with some non-zero $c \in \mathbf{Q}(q)$, and further that

$$(\Phi^{\sigma}_{\pm}(\phi), G^{\mathrm{low}}(p)) \neq 0$$

if and only if

$$\begin{aligned} p &= \tilde{f}_0(\tilde{f}_1\tilde{f}_0)^k\phi \quad \text{for } + \\ &= (\tilde{f}_1\tilde{f}_0)^k\phi \quad \text{for } - . \end{aligned}$$

Finally, by using (A3.2) (with $v = \phi$) recursively, we get (A3.1).

Let us compute $\omega^{(n)}(\bar{p}_i, \bar{p}_i) = \langle G(\phi)| \underbrace{\Phi^{\sigma}_{-} \circ \ldots \circ \Phi^{\sigma}_{-}}_{n} |G(\phi)\rangle$ up to q^8 for an arbitrary n by using Table 3. Let v_i $(i = 1, \ldots, 6)$ be the upper global base corresponding to the following symbols.

v_1	v_2	v_3	v_4	v_5	v_6
ϕ	$(2)\begin{array}{\|c\|}\hline \bullet \\ \hline\end{array}$	$(4)\begin{array}{\|c\|}\hline \bullet \\ \hline\end{array}$	$\binom{3}{1}\begin{array}{\|c\|}\hline \ \\ \hline \bullet \\ \hline\end{array}$	$\binom{5}{1}\begin{array}{\|c\|}\hline \ \\ \hline \bullet \\ \hline\end{array}$	$\binom{4}{2}\begin{array}{\|c\|}\hline \bullet \\ \hline \ \\ \hline\end{array}$

We can extract the following from Table 3.

$$\Phi^{\sigma}_{-}(v_1) = v_1 + q^3 v_2 + q^6 v_3 \bmod q^8 ,$$

$$\Phi^{\sigma}_{-}(v_2) = (q - q^3 + q^5)v_1 + (-q^2 + q^4)v_2 + q^3 v_3 + q^2 v_4 \bmod q^5 ,$$

$$\Phi^{\sigma}_{-}(v_3) = q^2 v_1 - q v_4 \bmod q^2 ,$$

$$\Phi^{\sigma}_{-}(v_4) = -q^3 v_1 + v_2 - q v_3 + q^2 v_6 \bmod q^3 ,$$

$$\Phi^{\sigma}_{-}(v_5) = v_3 \bmod q^1 ,$$

$$\Phi^{\sigma}_{-}(v_6) = v_4 \bmod q^1 .$$

Here "mod q^k" is to be understood for those terms except for v_1. The v_1-terms are given by "mod q^{k+1}." The reason we truncated the expansions at order less than q^8 is as follows. For example, starting from v_1 and applying Φ^{σ}_{-} repeatedly one will get v_2-terms only with power q^3 at the lowest. Then, unless one gets a v_1-term in the next application of Φ^{σ}_{-}, we get at least one more power of q until one does finally reach a v_1-term. Therefore, mod q^5 is enough $(8 - 3 - 1 = 4)$ except for the v_1-term.

Let us denote by $(1, j_1, \ldots, j_k)$ a process

$$v_1 \xrightarrow{\Phi^\sigma_-} v_{j_1} \xrightarrow{\Phi^\sigma_-} \cdots \xrightarrow{\Phi^\sigma_-} v_{j_k} .$$

For example, if $k = 1$ we have two processes (12) and (13) that contribute to the final answer. Along with coefficients, the $k = 1$ process gives rise to $q^3(12) + q^6(13)$. We then proceed for a larger k step by step. Since we consider only up to q^8, this terminates in finite steps. Picking up the k-processes ending at $j_k = 1$, we have

$$\begin{aligned}
&k = 0 \quad (1) , \\
&k = 2 \quad (q^4 - q^6 + q^8)(121) + q^8(131) , \\
&k = 3 \quad (-q^6 + 2q^8)(1221) + q^8(1231) - q^8(1241) , \\
&k = 4 \quad q^8(12221) + (q^6 - q^8)(12421) - q^8(12431) - q^8(13421) , \\
&k = 5 \quad -q^8((122421) + (123421) + (124221)) , \\
&k = 6 \quad q^8((1242421) + (1243421) + (1246421)) .
\end{aligned}$$

Taking combinatorial factors into consideration, we get (for $n \geqq 5$)

$$\begin{aligned}
\omega^{(n)}(\bar{p}_0, \bar{p}_0) &= 1 + (n-1)(q^4 - q^6 + 2q^8) + (n-2)(-q^6 + 2q^8) \\
&\quad + (n-3)(q^6 - 2q^8) + (n-4)(-3q^8) + (n-5)(3q^8) \\
&\quad + \frac{(n-2)(n-3)}{2} q^8 \\
&= 1 + (n-1)q^4 - nq^6 + \frac{n(n-1)}{2} q^8 \bmod q^{10} .
\end{aligned}$$

For smaller values of n, we get

$$\begin{aligned}
\omega^{(2)}(\bar{p}_0, \bar{p}_0) &= 1 + q^4 - q^6 + q^8 \bmod q^{10} , \\
\omega^{(3)}(\bar{p}_0, \bar{p}_0) &= 1 + 2q^4 - 3q^6 + 6q^8 \bmod q^{10} , \\
\omega^{(4)}(\bar{p}_0, \bar{p}_0) &= 1 + 3q^4 - 4q^6 + 9q^8 \bmod q^{10} .
\end{aligned}$$

The results agree with those in 6.8 obtained by solving the q-KZ equation. Similarly, we calculate

$$\begin{aligned}
\omega^{(n)}((\cdots + - + - - +), \bar{p}_0) &= -q + q^3 - nq^5 + 2nq^7 , \\
\omega^{(n)}((\cdots + - + + - -), \bar{p}_0) &= -q + 2q^3 - (n+1)q^5 + (3n+5)q^7 .
\end{aligned}$$

Thus we get

$$\begin{aligned}
\iota((\cdots + - + - - +), \bar{p}_0) &= -q + q^3 - q^5 + q^7 , \\
\iota((\cdots + - + + - -), \bar{p}_0) &= -q + 2q^3 - 4q^5 + 7q^7 ,
\end{aligned}$$

which are in agreement with the result in [3].

Appendix 4. Embedding of the Vacuum, 1 and 2 Particle States

First we give a supplement to Sect. 2 about the embedding $V(\Lambda_0)^{*a} \to V \otimes V \otimes \cdots$. Let us denote by ω the automorphism of the algebra $U_q(\widehat{\mathfrak{sl}}(2))$ given by

$$\omega(e_i) = f_i, \quad \omega(f_i) = e_i, \quad \omega(q^h) = q^{-h}.$$

Let V_1 and V_2 be left $U_q(\widehat{\mathfrak{sl}}(2))$-modules. A $\mathbf{Q}(q)$-linear isomorphism η: $V_1 \to V_2$ is called ω-compatible if and only if

$$\omega(x)\eta(v) = \eta(xv)$$

for $x \in U_q(\widehat{\mathfrak{sl}}(2))$ and $v \in V_1$. The map (6.32) in Sect. 6.7

$$\eta\colon V(\Lambda_0) \to V(\Lambda_0)^{*a}$$

is ω-compatible. The map

$$\eta\colon V \to V \quad v_\pm \mapsto v_\mp$$

is also ω-compatible. Suppose that η_i: $V_i \to V_i'$ $(i = 1, 2, 3)$ are ω-compatible, and that Φ: $V_1 \to V_2 \otimes V_3$ is an intertwiner. Then the map

$$\Phi'\colon V_1' \to V_3' \otimes V_2' \quad \eta_1(v) \mapsto (\eta_3 \otimes \eta_2) \circ P \circ \Phi(v) \tag{A4.1}$$

is also an intertwiner. Here, P is the transposition of the tensor components. For a given path $p \in \mathscr{P}_i$ we set

$$p^* = (-p(1) - p(2) - p(3) \ldots).$$

The set $\mathscr{P}_i^* = \{p^*; p \in \mathscr{P}_i\}$ is naturally the crystal of $V(\Lambda_i)^{*a}$. Let Φ: $V(\Lambda_i) \to V(\Lambda_{1-i}) \otimes V$ be the vertex operator given in Sect. 4. Then

$$\Phi'\colon V(\Lambda_i)^{*a} \to V \otimes V(\Lambda_{1-i})^{*a}$$

is also a vertex operator. From this we can define an embedding

$$\iota'\colon V(\Lambda_0)^{*a} \to V \otimes V \otimes V \otimes \ldots.$$

The embeddings ι and ι' commute with the ω-compatible maps η;

$$\begin{array}{ccc} V(\Lambda_0) & \xrightarrow{\iota} & \ldots \otimes V \otimes V \\ \eta \downarrow & & \eta \downarrow \\ V(\Lambda_0)^{*a} & \xrightarrow{\iota'} & V \otimes V \otimes \ldots \end{array}$$

From (6.32) and (A1.1) we see that the vacuum state $|\mathrm{vac}\rangle$ embedded in W is given by

$$|\mathrm{vac}\rangle = \sum_{\mu} \sum_{p \in (P_0)_\mu} (-1)^{ht(\Lambda_0 - \mu)} q^{|\rho + \Lambda_0|^2 - |\rho + \mu|^2} \iota(G^{\mathrm{up}}(p)) \otimes \eta \circ \iota(G^{\mathrm{low}}(p)).$$

As we have noted in Sect. 2, we have

$$\iota(G^{\mathrm{up}}(p))|_{q=0} = |p\rangle.$$

For $G^{\mathrm{low}}(p)$ we must shift the q power;

$$\iota(q^{|\Lambda_0|^2 - |\mu|^2} G^{\mathrm{low}}(p))|_{q=0} = |p\rangle.$$

Note that

$$(-1)^{ht(\Lambda_0-\mu)}q^{|\rho+\Lambda_0|^2-|\rho+\mu|^2}q^{-|\Lambda_0|^2+|\mu|^2}=(-q)^{ht(\Lambda_0-\mu)}\,.$$

Therefore, the term $(-1)^{ht(\Lambda_0-\mu)}q^{|\rho+\Lambda_0|^2-|\rho+\mu|^2}\iota(G^{\mathrm{up}}(p))\otimes\omega\circ\iota(G^{\mathrm{low}}(p))$ contributes to the sum only with power $q^{ht(\Lambda_0-\mu)}$. Let us compute $|\mathrm{vac}\rangle$ up to order q^3. We need the following five vectors;

$$v_1=G^{\mathrm{up}}(\cdots+-+-)=G^{\mathrm{low}}(\cdots+-+-)\,,$$

$$v_2=G^{\mathrm{up}}(\cdots+-++)=G^{\mathrm{low}}(\cdots+-++)\,,$$

$$v_3=G^{\mathrm{up}}(\cdots+--+)=\frac{1}{[2]}G^{\mathrm{low}}(\cdots+--+)\,,$$

$$v_4=G^{\mathrm{up}}(\cdots++-+)=\frac{1}{[2]}G^{\mathrm{low}}(\cdots++-+)\,,$$

$$v_5=G^{\mathrm{up}}(\cdots+---)=G^{\mathrm{low}}(\cdots+---)\,.$$

We have

$$|\mathrm{vac}\rangle\equiv\sum_{i=1}^{5}(-q)^{m(i)}\iota(v_i)\otimes\omega\circ\iota(v_i)\bmod q^4\,,$$

where $\{m(i)\}_{1\leqq i\leqq 5}=\{0,1,2,3,3\}$.

In the following table we show the coefficients of paths in the expansion of $\iota(v_i)$ $(1\leqq i\leqq 5)$, up to the relevant order for each i. If we write ... as a part of a path, we mean that the indicated part of the path is identical with the ground-state-path $\bar{p}_0$. If we write (...), we mean that the indicated part may be void; otherwise it must not be void. We write $\pm$ or $\mp$, if the indicated column of the path differs from the corresponding column of $\bar{p}_0$. Therefore, for example, $\ldots\pm\ \mp\ldots$ actually represents the paths $(\cdots+-++--)$, $(\cdots+--++-)$, $(\cdots++--+-)$, $(\cdots-++-+-)$, etc., simultaneously.

Path	*Coefficient*
v_1	
$\cdots$	1
$\cdots-\ +$	$-q+q^3$
$\cdots\pm\ \mp\cdots$	$-q+2q^3$
$\cdots\pm\ \mp\ \pm\ \mp(\cdots)$	$2q^2$
$\cdots\pm\ \mp\cdots\pm\ \mp(\cdots)$	q^2
$\cdots\pm\ \pm\ \mp\ \mp(\cdots)$	$-q^3$
$\cdots\pm\ \mp\ \pm\ \mp\ \pm\ \mp(\cdots)$	$-5q^3$
$\cdots\pm\ \mp\cdots\pm\ \mp\ \pm\ \mp(\cdots)$	$-2q^3$
$\cdots\pm\ \mp\ \pm\ \mp\cdots\pm\ \mp(\cdots)$	$-2q^3$
$\cdots\pm\ \mp\cdots\pm\ \mp\cdots\pm\ \mp(\cdots)$	$-q^3$
v_2	
$\cdots+$	$1-q^2$
$\cdots+\ -\ +$	$-2q$
$\cdots\pm\ \mp\cdots+$	$-q$

$\cdots + - + - +$	$5q^2$
$\cdots \pm \mp \pm \mp \cdots +$	$2q^2$
$\cdots \pm \mp \cdots + - +$	$2q^2$
$\cdots \pm \mp \cdots \pm \mp \cdots +$	q^2
$\cdots + + -$	q^2
v_3	
$\cdots - +$	1
$\cdots$	q
$\cdots - + - +$	$-3q$
$\cdots \pm \mp \cdots - +$	$-q$
v_4	
$\cdots + - +$	1
v_5	
$\cdots - -$	1.

From this we get the expansion of $|\mathrm{vac}\rangle$ up to order q^3. The result reads as follows.

Path	*Coefficient*
$\cdots$	1
$\cdots \pm \mp \cdots$	$-q + 2q^2$
$\cdots \pm \mp \pm \mp \cdots$	$2q^2$
$\cdots \pm \mp \cdots \pm \mp \cdots$	q^2
$\cdots \pm \mp \pm \mp \pm \mp \cdots$	$-5q^3$
$\cdots \pm \pm \mp \mp \cdots$	$-q^3$
$\cdots \pm \mp \cdots \pm \mp \pm \mp \cdots$	$-2q^3$
$\cdots \pm \mp \pm \mp \cdots \pm \mp \cdots$	$-2q^3$
$\cdots \pm \mp \cdots \pm \mp \cdots \pm \mp \cdots$	$-q^3$

This agrees with the perturbation expansion (see [31]).

The 1-particle state with the quasi momentum u ($z = e^{iu}$) is an embedding of V_z into $V(\Lambda_1) \otimes V(\Lambda_0)^{*a}$, or further into W. We denote the latter embedding by $\iota_z^{(1)}$ (in general, $\iota_{z_n,\ldots,z_1}^{(n)}$ for the n-particle embedding). The quantum affine symmetry discussed in Sect. 1 assures that the vectors obtained by this embedding are doubly degenerate eigenvectors of H_{XXZ} and its higher order relatives. Because of the property of the dual modules discussed in Sect. 5 of [2], the existence of the embedding is equivalent to the existence of the vertex operator $\Phi(z)$: $V_z \otimes V(\Lambda_0) \to V(\Lambda_1)$.

Let us compute this embedding at $q = 0$ (i.e., only the crystal) by using the data in Table 2. We discuss in details only on the v_+-component. First consider the top term in $\iota_z^{(1)}(v_+)$;

$$\begin{aligned}\Phi(z)(v_+ \otimes u_{\Lambda_0}) \otimes u_{\Lambda_0} &= \sum_{n=0}^{\infty} z^n (2n) \boxed{} \otimes \phi \\ &= (\cdots - + - + \mid + - + - \cdots) \\ &\quad + z(\cdots - + + - \mid + - + - \cdots) \\ &\quad + z^2(\cdots + - + - \mid + - + - \cdots) + \cdots .\end{aligned}$$

So, we get $\sum_{n=0}^{\infty} z^n[[2n]]$ from the top term. Because of the translational covariance, we expect the whole limit $\iota_z^{(1)}(v_+)$ to be $\sum_{n\in\mathbf{Z}} z^n[[2n]]$. In the lowest order in q, the term $(-)^{ht(\Lambda_0-\mu)}q^{|\rho+\Lambda_0|^2-|\rho+\mu|^2}\eta\circ\iota(G^{\mathrm{low}}(p))$ gives rise to $(-q)^{ht(\Lambda_0-\mu)}\omega(|p\rangle)$. Therefore, among 6 more data in Table 2, only $p=(\cdots+--+)$ and $(\cdots-+-+)$ give nonzero contributions at $q=0$, i.e., $z^{-1}[[-2]]$ and $z^{-2}[[-4]]$, respectively. So, it is consistent. Similarly, the v_--component has the limit $\sum_{n\in\mathbf{Z}} z^n[[2n+1]]$.

The two-particle state is $V_{z_2}\otimes V_{z_1}$ embedded in $V(\Lambda_0)\otimes V(\Lambda_0)^{*a}$ or in W. This embedding $\iota_{z_2,z_1}^{(2)}$ is obtained by a successive application of the vertex operators. For example, the spin 1 component is

$$\sum_{p\in\mathscr{P}_0\mu}(-)^{ht(\Lambda_0-\mu)}q^{|\rho+\Lambda_0|^2-|\rho+\mu|^2}\Phi_{z_2}(v_+\otimes\Phi_{z_1}(v_+\otimes G^{\mathrm{up}}(p)))\otimes\eta\circ\iota(G^{\mathrm{low}}(p))\,. \tag{A4.2}$$

Let us use signatures and parity symbols to represent the paths and the upper global base vectors (see Appendix 1). If $p=\phi$, we have

$$\Phi(z_1)(v_+\otimes\phi)\equiv\phi+z_1(2)\square+z_1^2(4)\square+\cdots\in V(\Lambda_1)\,.$$

Then, by using (A2.1),

$$\Phi'(z_2)(v_+\otimes\phi)\equiv z_2(1)\square+z_2^2(3)\square+\cdots\in V(\Lambda_0)\,\mathrm{mod}\,q\,,$$

$$\begin{aligned}\Phi'(z_2)(v_+\otimes z_1(2)\square)\equiv{}&-z_1(1)\square\\&+z_1z_2^2\binom{3}{2}\begin{array}{|c|}\hline\ \\\hline\ \\\hline\end{array}+z_1z_2^3\binom{5}{2}\begin{array}{|c|}\hline\ \\\hline\ \\\hline\end{array}+\cdots\in V(\Lambda_0)\,\mathrm{mod}\,q\,,\end{aligned}$$

$$\begin{aligned}\Phi'(z_2)(v_+\otimes z_1^2(4)\square)\equiv{}&-z_1^2(3)\square-z_1^2z_2\binom{3}{2}\\&+z_1^2z_2^3\binom{5}{4}\begin{array}{|c|}\hline\ \\\hline\ \\\hline\end{array}+z_1^2z_2^4\binom{7}{4}\begin{array}{|c|}\hline\ \\\hline\ \\\hline\end{array}+\cdots\in V(\Lambda_0)\,\mathrm{mod}\,q\,.\end{aligned}$$

Therefore, the contribution to (A4.2) at $q=0$ from $p=\phi$ is

$$\begin{aligned}&z_2[[1,0]]+z_2^2[[3,0]]+\cdots-z_1[[1,0]]+z_1z_2^2[[3,2]]+z_1z_2^3[[5,2]]+\cdots\\&\quad-z_1^2[[3,0]]-z_1^2z_2[[3,2]]+z_1^2z_2^3[[5,4]]+z_1^2z_2^4[[7,4]]+\cdots\,.\end{aligned}$$

So, by the translational covariance, we expect

$$\sum_{m\geqq n}(z_1^nz_2^{m+1}-z_1^{m+1}z_2^n)[[2m+1,2n]] \tag{A4.3}$$

as the whole answer. The data in Table 2 turn out to be consistent with this. In Appendix 5, the comparison of this result with the Bethe Ansatz calculation is given.

Similarly, we obtain

$$\iota_{z_2,z_1}^{(2)}(v_-\otimes v_-)|_{q=0}=\sum_{m\geqq n}(z_1^{n-1}z_2^m-z_1^mz_2^{n-1})[[2m,2n-1]]\,. \tag{A4.4}$$

The case spin is equal to zero, is somewhat tricky because the limit $q=0$ can be taken only after summing up certain series. Let us consider $\iota_{z_2,z_1}^{(2)}(v_-\otimes v_+)$. The first term is $p = \phi$, and we want to compute $\Phi'(z_2)(v_- \otimes \Phi(v_+ \otimes \phi))$, or $\Phi^\sigma(z_2)(v_+ \otimes z_1^n(2n)\boxed{\bullet})$ (see (A2.1) and Table 2). A cumbersome (or rather interesting) feature here is that the terms proportional to ϕ contains negative powers in q. Keeping the whole terms that are proportional to ϕ, and neglecting all other terms that vanishes at $q = 0$, we have

$$\Phi^\sigma(z_2)(v_+ \otimes \phi) = \phi + \sum_{n=1}^{\infty} z_2^n(2n)\boxed{\bullet}\,,$$

$$\Phi^\sigma(z_2)(v_+ \otimes z_1(2)\boxed{\bullet}) = \frac{z_1}{z_2 q^2}\frac{1-q^2}{1-q^4}\phi - z_1(2)\boxed{\bullet}$$
$$+ z_1 \sum_{n=2}^{\infty} z_2^n \binom{2n}{2} \begin{array}{|c|}\hline \bullet \\ \hline \\ \hline\end{array} + \cdots\,,$$

$$\Phi^\sigma(z_2)(v_+ \otimes z_1^2(4)\boxed{\bullet}) = \left(\frac{z_1}{z_2 q^2}\right)^2 \frac{1-q^2}{1-q^4}\cdot\frac{1-q^6}{1-q^8}\phi$$
$$- z_1^2(4)\boxed{\bullet} - z_1^2 z_2 \binom{4}{2} \begin{array}{|c|}\hline \bullet \\ \hline \\ \hline\end{array}$$
$$+ z_1^2 \sum_{n=3}^{\infty} z_2^3 \binom{2n}{4} \begin{array}{|c|}\hline \bullet \\ \hline \\ \hline\end{array} + \cdots\,.$$

Extrapolating, we have

$$1 + \frac{z_1}{z_2 q^2}\frac{1-q^2}{1-q^4} + \left(\frac{z_1}{z_2 q^2}\right)^2 \frac{1-q^2}{1-q^4}\cdot\frac{1-q^6}{1-q^8} + \cdots$$
$$= \frac{q^2(z_2 - z_1)}{z_2 q^2 - z_1}\frac{z_2 - z_1 q^4}{z_2 - z_1 q^2}\frac{z_2 - z_1 q^8}{z_2 - z_1 q^6}\cdots\,.$$

Therefore, after summation the terms proportional to ϕ vanish at $q = 0$. (In fact this is exact as shown in Sect. 6.5.) With this understood, we get

$$\iota^{(2)}_{z_2, z_1}(v_- \otimes v_+)|_{q=0} = \sum_{m>n} (z_1^n z_2^m - z_1^m z_2^n)[[2m, 2n]]\,, \tag{A4.5}$$

and similarly,

$$\iota^{(2)}_{z_2, z_1}(v_+ \otimes v_-)|_{q=0} = \sum_{m>n} (z_1^n z_2^{m+1} - z_1^m z_2^{n+1})[[2m+1, 2n+1]]\,. \tag{A4.6}$$

Appendix 5. Comparison with the Bethe Ansatz States

In this appendix we calculate the Bethe vectors at $q = 0$ and compare them with the results in Appendix 3. We assume the periodic boundary condition and hence that the length N of row to be even. Set $N = 2n$. Following [27] we label the standard basis vectors $v_{\varepsilon_1} \otimes \cdots \otimes v_{\varepsilon_N}$ of $V^{\otimes N}$ by the locations of v_-: $(x_1, \ldots, x_M)$, $1 \leqq x_1 < \cdots < x_M \leqq N$. We denote by $|x\rangle = |x_1, \ldots, x_M\rangle \in V^{\otimes N}$ the corresponding vector with spin $n - M$. It is known that the vector

$$v = \sum f(x_1, \ldots, x_M)|x_1, \ldots, x_M\rangle \tag{A5.1}$$

is an eigenvector of the XXZ Hamiltonian H_{XXZ} if $f(x_1, \ldots, x_M)$ is of the following form:

$$f(x_1, \ldots, x_M) = \sum_{\sigma \in S_M} \prod_{1 \leqq j < k \leqq M} E(u_{\sigma(j)}, u_{\sigma(k)}) \prod_{l=1}^{M} (-F(u_{\sigma(l)}))^{x_l - 1}, \tag{A5.2}$$

where S_M denotes the symmetric group,

$$E(u, v) = \frac{\sin(u - v + i\gamma)}{\sin(u - v)}, \quad F(u) = \frac{\sin(u + \frac{1}{2}i\gamma)}{\sin(u - \frac{1}{2}i\gamma)}$$

$$\Delta = \frac{q + q^{-1}}{2}, \quad q = -e^{-\gamma}, \quad \gamma \in \mathbf{R}_{\geqq 0},$$

and $\{u_j\}_{j=1}^{M}$ is a solution of the following Bethe Ansatz equation (BAE)

$$F(u_k)^{2n} = -\prod_{j=1}^{M} S(u_k, u_j) \quad \text{for } 1 \leqq k \leqq M, \tag{A5.3}$$

where

$$S(u, v) = \frac{\sin(u - v + i\gamma)}{\sin(u - v - i\gamma)}.$$

We calculated the form of Bethe vectors at $q = 0$ in the cases of $2h$-particles with spin h and $h - 1$. Here the particle number is the same as the number of holes in [10] or [9]. As a consequence we find that the Bethe vectors at $q = 0$ coincide with the $q \to 0$ limit of eigenvectors calculated in App. 4. The correspondence are given by

- the ground states (Example 1),
- two particle states with spin 1 (Example 2) ((A4.3)),
- two particle states with spin 0 (Example 3) ((A4.6) and (A4.5)).

(i) The case of $2h$-particles and spin h. Here we consider the case $M = n - h$ for some $h \in \mathbf{Z}_{\geqq 1}$. It can be shown that there is a solution $\{u_j\}_{j=1}^{n-h} = \{\lambda_j\}_{j=1}^{n-h}$ of (A5.3) which has the expansion of the form

$$e^{2i\lambda_j} = \lambda_j^{(0)} \left(1 + \sum_{i=1}^{\infty} \lambda_j^{(i)} q^i\right) \quad \text{for } 1 \leqq j \leqq n - h, \tag{A5.4}$$

where $q = -e^{-\gamma}$. Substituting (A5.4) into (A5.3) and comparing the coefficients of q^0 we have

$$(\lambda_j^{(0)})^{-(n+h)} = (-1)^{n-h+1} \prod_{k=1}^{n-h} \lambda_k^{(0)} \quad \text{for } 1 \leqq j \leqq n - h. \tag{A5.5}$$

Proposition A5.1. *Let $\{r_j\}_{j=1}^{2h} \sqcup \{\mu_j\}_{j=1}^{n-h}$ be a partition of $\{0, 1, \ldots, n + h - 1\}$ such that $r_1 < \cdots < r_{2h}$ and $\mu_1 < \cdots < \mu_{n-h}$. Let θ be a real number which satisfies*

$$\theta \equiv \frac{\pi\left(-2h\mu_1 + \sum_{j=1}^{2h} r_j\right)}{n(n + h)} \mod \frac{\pi}{n}\mathbf{Z}, \tag{A5.6}$$

$$-\pi \leqq \theta < -\pi + \frac{2\pi(\mu_1 + 1)}{n + h}. \tag{A5.7}$$

Set

$$\lambda_1^{(0)} = e^{i\theta}, \quad \lambda_j^{(0)} = \lambda_1^{(0)} a^{\mu_j - \mu_1} \quad 1 \leqq j \leqq n-h\,,$$

where $a = \exp\left(\dfrac{2\pi i}{n+h}\right)$. *Then* $\{\lambda_j^{(0)}\}_{j=1}^{n-h}$ *is a solution of* (A5.5) *and any solution of* (A5.5) *is obtained in this way.*

Remark. For each $\{r_j\}_{j=1}^{2h}$, if n is sufficiently large, there are $2(\mu_1+1)$ θ's which satisfy (A5.6) and (A5.7). More exactly let θ_0 be such a θ that satisfies (A5.6) and $-\pi \leqq \theta < -\pi + \dfrac{\pi}{n}$. Then $\theta_{0,j} = \theta_0 + \dfrac{\pi j}{n}$ $(0 \leqq j \leqq 2\mu_1 + 1)$ satisfies (A5.7) if n is sufficiently large.

By expanding $F(\lambda_j)$ and $E(\lambda_j, \lambda_k)$ in q at $q=0$ we have

$$F(\lambda_j) = -(\lambda_j^{(0)})^{-1}(1 + O(q))\,,$$

$$E(\lambda_j, \lambda_k) = \frac{\lambda_k^{(0)}}{\lambda_j^{(0)} - \lambda_k^{(0)}} q^{-1}(1 + O(q))\,.$$

Substituting these expressions to (A5.2) we have

Proposition A5.2. *The eigenvector v is given by*

$$v = \text{const.} \sum_{\{k\}} (\lambda_1^{(0)} a^{-\mu_1})^{-\sum_{j=1}^{2h} k_j} \cdot \prod_{i=1}^{2h} P_{k_i}(a) \cdot V(k, a)$$

$$\times \sum_{x_1} (-1)^{x_1(n+h-1)} (\lambda_1^{(0)})^{-(n-h)x_1} a^{-x_1(2h\mu_1 + \sum_{j=1}^{2h} r_j)} |x_1, \ldots, x_{n-h}\rangle \bmod q\,,$$

where const. *is an overall factor, and*

$$P_l(a) = \prod_{-(n+h-1) < i < j \leqq 0,\, i, j \neq l} (a^i - a^j)\,,$$

$$V(k, a) = \det(a^{-(n-r_i)k_j})_{1 \leqq i, j \leqq 2h}\,.$$

Here $\{k_j\}_{j=1}^{2h}$ *and* x_1 *varies subject to the following conditions:*

$$0 \geqq k_1 > \cdots > k_{2h} \geqq -(n+h-1)\,,$$

$$1 \leqq x_1 \leqq b_{n-h} + n + h + 1 \ (\leqq 2h+1)\,,$$

$$\{b_j\}_{j=1}^{n-h} = \{-j\}_{j=1}^{n+h-1} \backslash \{k_j\}_{j=1}^{2h}, \quad b_1 > \cdots > b_{n-h}\,.$$

For these data $\{x_2, \ldots, x_{2h}\}$ *are determined by*

$$x_j = x_1 + 2(j-1) + i \quad \text{for} \ -k_i - i + 2 \leqq j \leqq -k_{i+1} - i \ \text{and} \ 0 \leqq i \leqq 2h\,,$$

where we consider $k_0 = 1$ *and* $k_{2h+1} = n - h$.

Lemma. $P_{k+1}(a) = -aP_k(a)$.

Set

$$l_j = 2(k_j + n) - n + j - x_1 \quad \text{for } 1 \leqq j \leqq 2h\,.$$

Now we shall take a limit $n \to \infty$ in such a way that a^{r_j} $(1 \leqq j \leqq 2h)$ are finite and $(-\lambda_1^{(0)})^n$ has a limit. By the lemma, if $k_1 - k_{2h}$ is sufficiently small compared with n, the ratio $P_{k_i}(a)/P_{k_1}(a)$ goes to one in the limit $n \to \infty$. Using this and setting

$$z_j = a^{r_j} \quad (1 \leqq j \leqq 2h)$$

we have in the $n \to \infty$ limit

Proposition A5.3. *Suppose $|l_1 - l_{2h}| \ll \infty$ and $|l_1| \ll \infty$. Then in the limit $q \to 0$ and $n \to \infty$, the coefficients $v(x)$ of $v = \sum v(x)|x\rangle$ is*

$$v(x) = \pm\det(z_i^{k_j})_{1 \leqq i,j \leqq 2h}\left(\prod_{j=1}^{2h} z_j\right)^{\frac{x_1-1}{2}},$$

where $x_1 = 1$ or 2 and the signature is same if x_1 is same.

Example 1 ($h = 0$, Ground states). In this case $a = \exp\left(\frac{2\pi i}{n}\right)$, $\mu_j = j - 1$ $(1 \leqq j \leqq n)$, $\lambda_1^{(0)} = -1$ or $-\exp\left(\frac{\pi i}{n}\right)$ and $\lambda_j^{(0)} = \lambda_1^{(0)} a^{j-1}$ $(1 \leqq j \leqq n)$.

(1) $\lambda_1^{(0)} = -1$

$$v = -|1, 3, 5, \ldots, 2n-1\rangle + |2, 4, 6, \ldots, 2n\rangle \bmod q .$$

(2) $\lambda_1^{(0)} = -\exp\left(\frac{\pi i}{n}\right)$

$$v = \langle 1, 3, 5, \ldots, 2n-1\rangle + |2, 4, 6, \ldots, 2n\rangle \bmod q .$$

Let us write the vector of the form $|x_1, \ldots, x_M\rangle$ by using the chain of $+$ and $-$. Namely, for example, the vectors $|1, 3, 5, \ldots, 2n-1\rangle$ and $|2, 4, 6, \ldots, 2n\rangle$ are

$$\begin{array}{cccccccccc} 1 & 2 & 3 & 4 & \cdots & 2n-3 & 2n-2 & 2n-1 & 2n \\ - & + & - & + & \cdots & - & + & - & + \\ + & - & + & - & \cdots & + & - & + & - . \end{array}$$

Example 2 ($h = 1$, 2-particle states, spin 1). The coefficients of the limit vectors are

$$v(1, x_2, \ldots, x_{n-1}) = \pm(z_1^{k_1} z_2^{k_2} - z_1^{k_2} z_2^{k_1}) ,$$

$$v(2, x_2, \ldots, x_{n-1}) = \pm(z_1^{k_1} z_2^{k_2} - z_1^{k_2} z_2^{k_1})(z_1 z_2)^{\frac{1}{2}} .$$

Here

$$0 \geqq k_1 > k_2 \geqq -n ,$$

and

$$l_1 = 2k_1 + n + 1 - x_1 ,$$

$$l_2 = 2k_2 + n + 2 - x_1 .$$

The numbers l_1 and l_2 have the following pictorial meaning. We shall number the place between $i - 1$ and i by $n + 1 - i$. Then l_1 and l_2 are the numbers between $+$ and $+$ which occurs twice in the above picture.

$$- + - + \cdots - + \underset{l_1}{|} + - \cdots - + \underset{l_2}{|} + - \cdots - + .$$

(ii) The case of $2h$-particles and spin $h - 1$ ($h \geqq 1$). Suppose that the solution of BAE consists of real quasi-momenta and one 2-string ([10]). Let us denote them by $\{\lambda_j\}_{j=1}^{n-h-1}$ and $\{z, w\}$ respectively. Then BAE takes the form

$$F(\lambda_j)^{2n} = -S(\lambda_j, z)S(\lambda_j, w) \prod_{k=1}^{n-h-1} S(\lambda_j, \lambda_k) , \tag{A5.8}$$

$$F(z)^{2n} = S(z, w) \prod_{k=1}^{n-h-1} S(z, \lambda_k) , \tag{A5.9}$$

$$F(w)^{2n} = S(w, z) \prod_{k=1}^{n-h-1} S(w, \lambda_k) . \tag{A5.10}$$

It can be shown that there is a set of solutions of (A5.8)–(A5.10) which has the following expansions at $q = 0$.

$$e^{2i\lambda_j} = \lambda_j^{(0)}\left(1 + \sum_{i=1}^{\infty} \lambda_j^{(i)} q^i\right) \quad \text{for } 1 \leqq j \leqq n-h-1 , \tag{A5.11}$$

$$e^{2iz} = z^{(0)} q\left(1 + z^{(n+h-1)} q^{n+h-1} + \sum_{k=n+h}^{\infty} z^{(k)} q^k\right), \tag{A5.12}$$

$$e^{2iw} = z^{(0)} q^{-1}\left(1 + w^{(n+h-1)} q^{n+h-1} + \sum_{k=n+h}^{\infty} w^{(k)} q^k\right). \tag{A5.13}$$

Substituting these expressions into (A5.8)–(A5.10) we have, at $q = 0$,

$$(\lambda_j^{(0)})^{-n-h+1} = (-1)^{n-h}(z^{(0)})^2 \prod_{k=1}^{n-h-1} \lambda_k^{(0)} , \tag{A5.14}$$

$$(z^{(0)})^{-2(h+1)} = \left(\prod_{k=1}^{n-h-1} \lambda_k^{(0)}\right)^2 . \tag{A5.15}$$

We can solve (A5.14) and (A5.15) easily.

Proposition A5.4. *Let $\{r_j\}_{j=1}^{2h} \sqcup \{\mu_j\}_{j=1}^{n-h-1}$ be a partition of $\{0, 1, \ldots, n+h-2\}$ such that $r_1 < \cdots < r_{2h}$ and $\mu_1 < \cdots < \mu_{n-h-1}$. Let θ_r and θ_c be real numbers which satisfy*

$$\theta_r \equiv \frac{\pi(h-1)(-2h\mu_1 + \sum_{j=1}^{2h} r_j)}{nh(n+h-1)} \bmod \frac{\pi}{nh}\mathbf{Z} , \tag{A5.16}$$

$$-\pi \leqq \theta_r < -\pi + \frac{2\pi(\mu_1 + 1)}{n+h-1} . \tag{A5.17}$$

$$\theta_c \equiv -(n-1)\theta_r + \frac{\pi(-2h\mu_1 + \sum_{j=1}^{2h} r_j)}{n+h-1} \bmod \pi\mathbf{Z} ,$$

$$-\pi \leqq \theta_c < \pi . \tag{A5.18}$$

Set

$$\lambda_1^{(0)} = e^{i\theta_r}, \quad z^{(0)} = e^{i\theta_c}, \quad \lambda_j^{(0)} = \lambda_1^{(0)} a^{\mu_j - \mu_1} \quad 1 \leqq j \leqq n-h-1 ,$$

where $a = \exp\left(\dfrac{2\pi i}{n+h-1}\right)$. Then $\{\lambda_j^{(0)}\}_{j=1}^{n-h-1} \sqcup \{z, w\}$ is a solution of (A5.14) *and* (A5.15). *Any set of solutions of* (A5.14) *and* (A5.15) *is obtained in this way.*

Lemma. *Let $f(x_1, \dots, x_{n-h+1})$ be defined by* (A5.2). *Then*

$f(x_1, \dots, x_{n-h+1}) = f_0(x_1, \dots, x_{n-h+1}) q^{\frac{1}{2}(n-h-1)(n-h-4)} + O(q^{\frac{1}{2}(n-h-1)(n-h-4)})$.

Here $f_0(x_1, \dots, x_{n-h+1}) = 0$ unless $(x_1, \dots, x_{n-h+1})$ is one of the following form.

1. *There exists $p \geqq 1$ which satisfies*

$$x_{p+1} = x_p + 1, \quad x_{i+1} - x_i \geqq 2 \quad \text{if } i \neq p,$$

where $x_{n-h+1} \neq 2n$ if $p = 1$.

2. $x_1 = 1$, $x_2 = 2$, $x_{n-h+1} = 2n$, $x_{i+1} - x_i \geqq 2$ *if* $i \neq 1$.
3. $x_1 = 1$, $x_{n-h} = 2n - 1$, $x_{n-h+1} = 2n$, $x_{i+1} - x_i \geqq 2$ *if* $i \neq n - h$.

Let us calculate $f_0(x_1, \dots, x_{n-h+1})$ in the case of (1) in the lemma. Let us again use the $\pm$ description and denote by $(p(i))_{i=1}^{2n}$ be the $\pm$ chain corresponding to $(x_1, \dots, x_{n-h+1})$.

Lemma. *Suppose that $(p(i))_{i=1}^{2n}$ is of the following form:*

$$p(2m_i - i - 2 + x_1) = p(2m_i - i - 1 + x_1) = + \quad \text{for } 1 \leqq i \leqq l-1,$$
$$p(2m_l - l - 1 + x_1) = p(2m_l - l + x_1) = -,$$
$$p(2m_i - i + x_1) = p(2m_i - i + 1 + x_1) = + \quad \text{for } l+1 \leqq i \leqq 2h,$$
$$2 \leqq m_1 < \cdots < m_{2h} \leqq n + h - 1.$$

Set

$$k_j = -m_j + 1 \quad \text{for } 1 \leqq j \leqq 2h \text{ and } x_1 = 1, 2.$$

Then $p = m_l - l$ and, up to constants independent of $\{x_j\}_{j=1}^{n+h-1}$

$$f_0(x_1, \dots, x_{n-h+1}) = (-1)^{x_1(n+h)} (z^{(0)})^{2p - x_{p+1}} (\lambda_1^{(0)} a^{\mu_1})^{-\sum_{j=1}^{n-h+1} x_j + 2x_{p+1} - 2p}$$
$$a^{x_1 \sum_{j=1}^{2h} r_j} \prod_{i=1}^{2h} P_{k_i}(a) D(r_1, \dots, r_{2h} | k_1, \dots, k_{2h}),$$

where

$$D(r_1, \dots, r_{2h} | k_1, \dots, k_{2h}) = \det(a^{(i-1)(j-1)})_{i \neq k_1, \dots, k_{2h}, j \neq r_1, \dots, r_{2h}}.$$

Let us define $\tilde{k}_j$ $(1 \leqq j \leqq 2h)$ by

$$\tilde{k}_j = n + 2k_j + j - x_1 \quad \text{for } 1 \leqq j \leqq l-1,$$
$$\tilde{k}_l = n + 2k_l + l - 1 - x_1,$$
$$\tilde{k}_j = n + 2k_j + j - 2 - x_1 \quad \text{for } l+1 \leqq j \leqq 2h.$$

By taking the limits $n \to \infty$, $q \to 0$ in a similar manner as in Proposition A5.3 we have

Proposition A5.5. *Suppose $|\tilde{k}_1 - \tilde{k}_{2h}| \ll \infty$ and $|\tilde{k}_1| \ll \infty$. Then in the limit $q \to 0$ and $n \to \infty$, the coefficients $v(x)$ of $v = \sum v(x)|x\rangle$ is*

$$v(x) = \pm \det(z_i^{k_j})_{1 \leqq i, j \leqq 2h} \left(\prod_{j=1}^{2h} z_j \right)^{x_1 - \frac{l + x_1}{2h}},$$

where $x_1 = 1$ or 2 and the signature is same as far as x_1 is same.

Example 3 ($h = 1$, 2-particle states, spin 0). The coefficients of the limit vectors are

$$v(1, x_2, \ldots, x_{n-1}) = \pm(z_1^{\tilde{k}_1} z_2^{\tilde{k}_2} - z_1^{\tilde{k}_2} z_2^{\tilde{k}_1}) \quad \text{for } l = 1\,,$$

$$v(1, x_2, \ldots, x_{n-1}) = \pm(z_1^{\tilde{k}_1} z_2^{\tilde{k}_2} - z_1^{\tilde{k}_2} z_2^{\tilde{k}_1})(z_1 z_2)^{-\frac{1}{2}} \quad \text{for } l = 2\,,$$

$$v(2, x_2, \ldots, x_{n-1}) = \pm(z_1^{\tilde{k}_1} z_2^{\tilde{k}_2} - z_1^{\tilde{k}_2} z_2^{\tilde{k}_1})(z_1 z_2)^{\frac{1}{2}} \quad \text{for } l = 1\,,$$

$$v(2, x_2, \ldots, x_{n-1}) = \pm(z_1^{\tilde{k}_1} z_2^{\tilde{k}_2} - z_1^{\tilde{k}_2} z_2^{\tilde{k}_1}) \quad \text{for } l = 2\,.$$

Here

$$0 \geqq k_1 > k_2 \geqq -n\,,$$

and

$$\tilde{k}_j = 2k_j + n - x_1, \quad \text{for } l = 1\,,$$

$$= 2k_j + n + 1 - x_1, \quad \text{for } l = 2\,.$$

The $\pm$-chain picture are

$$- + - + \cdots + - \underset{\tilde{k}_1}{|} - + \cdots - + \underset{\tilde{k}_2}{|} + - \cdots - +\,,$$

for $l = 1$, $x_1 = 1$ and

$$- + - + \cdots - + \underset{\tilde{k}_1}{|} + - \cdots + - \underset{\tilde{k}_2}{|} - + \cdots - +\,,$$

for $l = 2$, $x_1 = 1$.

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