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The Parallel Propagator as Basic Variable for Yang–Mills Theory

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Abstract. The parallel propagator (associated with a Yang–Mills connection) taken along all null geodesics from a field point x to null infinity is introduced as a basic variable in Yang–Mills theory. It is shown that the Yang–Mills connection can be reconstructed from this parallel propagator.

The Yang–Mills equations are expressed as an equation for the parallel propagator. This equation can be given as a sum of two parts. The first of these, when set equal to zero on its own, satisfies the Huygens property and is soluble. When the second part is included, the Huygens property is destroyed. This leads to an approximation scheme which at first order is soluble yet already captures much of the non-linearity of Yang–Mills theory.

1. Introduction

In this note we wish to describe an alternate formulation of the standard Yang–Mills (Y–M) equations on Minkowski space, $\mathscr{M}$, in terms of a single matrix (or group) valued function on a six dimensional subspace of the space of paths. We will denote this function by G(path). More specifically this subspace is the space of null geodesics beginning at each point x^a of $\mathscr{M}$ and ending on future null infinity, $\mathscr{I}^+$. A natural parametrization for these paths are the coordinates x^a of the starting point of each path and the (complex) stereographic coordinates $(\zeta, \bar{\zeta})$ which label the sphere of generators of the future light-cone of x^a, where the path ends. A path is thus labeled by $(x^a, \zeta, \bar{\zeta})$. The function $G(x^a, \zeta, \bar{\zeta})$ is to be the parallel propagator, (with the Y–M connection, γ_a), of vectors in the fiber over x^a taken along the $(\zeta, \bar{\zeta})$ generator to the fiber over $\mathscr{I}^+$, i.e.

$$G(x^a, \zeta, \bar{\zeta}) = \mathscr{P} \exp\left(\int_x^\infty \gamma_a dx^a \right). \tag{1.1}$$

We will show that (1.1) can be inverted and thus the connection $\gamma_a(x)$ can be reconstructed (mod choice of gauge) from $G(x^a, \zeta, \bar{\zeta})$. From this fact, the Y–M field equations can be rewritten in terms of the G instead of the γ.

In terms of the new variables, the Yang–Mills equations divide into two parts. The first part is already fully nonlinear, but nevertheless, taken as an equation on its own, satisfies the Huygens principle in the sense that the field at a point depends only on the data (and perhaps one or two derivatives) at the intersection of the light cone of the point and the data surface. The second part destroys the Huygens property. This leads to an approximation scheme which at first order is already fully nonlinear but is nevertheless presumably integrable by virtue of the Huygens property. The solution to the first part can be then inserted into the second part and used to generate successive approximations to the full equations.

We hope that the nonlinearity at first order of this approximation scheme may be of use in ascertaining features of Yang–Mills fields that are not sensitive to the full complexities of the interactions but are nevertheless nonlinear such as, for example, the bound states of the theory.

In Sect. 2 our notation and conventions are introduced, while in Sect. 3 we will derive the inversion equations for (1.1). In Sect. 4 will be devoted to a description of the Y–M equations in terms of G.

2. Notation

For simplicity we will consider a trivial $GL(n, C)$ bundle B over $\mathscr{M}$, with basis vectors $\mathbf{e}_\alpha$ that transform as $\mathbf{e}'_\alpha = \mathscr{G}_\alpha{}^\beta \mathbf{e}_\beta$. Parallel transport is introduced by

$$\nabla_a \mathbf{e}_\alpha = \gamma^\beta_{\alpha a} \mathbf{e}_\beta \tag{2.1}$$

with

$$\gamma^\beta_\alpha = \gamma^\beta_{\alpha a} dx^a$$

the connection one-form. Covariant differentiation of a vector $\mathbf{V} = V^\alpha \mathbf{e}_\alpha$, is then given by

$$\nabla_a V^\alpha = V^\alpha{}_{,a} + V^\beta \gamma^\alpha_{\beta a} .$$

A vector, which is parallel transported along some path P with tangent vector v^a, satisfies the equation

$$v^a \nabla_a V^\alpha \equiv v^a (V^\alpha{}_{,a} + V^\beta \gamma^\alpha_{\beta a}) = 0 . \tag{2.2}$$

The Y–M curvature tensor (suppressing the matrix indices) is defined by

$$F_{ab} = \gamma_{b,a} - \gamma_{a,b} - [\gamma_a, \gamma_b] \tag{2.3}$$

with [,] the matrix commutator.

Under a change of basis we have,

$$\gamma'_a = S_{,a} S^{-1} + S \gamma_a S^{-1} ,$$
$$F'_{ab} = S F_{ab} S^{-1} .$$

The Bianchi identities and the source-free Y–M field equations are respectively

$$\nabla_a F^{*ab} = 0 \tag{2.4}$$

and

$$\nabla_a F^{ab} = 0 \tag{2.5}$$

with the dual field defined by

$$F^*_{ab} = \frac{1}{2}\eta_{abcd}F^{cd}\,,$$

and η the alternating tensor with $\eta_{0123} = -1$.

We now recall several simple features of the light-cone structure on $\mathscr{M}$. Let x^a be an arbitrary point of $\mathscr{M}$; the light cone from x^a, C_x, is given by

$$y^a = x^a + r\ell^a\,,$$

where r is an affine parameter along a null generator of C_x, and ℓ^a is a null vector tangent to the generator, normalized by $\ell_a t^a = 1$ with t^a a unit time-like vector. Since ℓ^a sweeps out the sphere of null directions at x^a, it can be parametrized by the complex stereographic coordinates $(\zeta, \bar{\zeta})$ and be given a convenient representation by

$$\ell^a(\zeta, \bar{\zeta}) = [\sqrt{2}(1 + \zeta\bar{\zeta})]^{-1}([1 + \zeta\bar{\zeta}], [\zeta + \bar{\zeta}], i[\zeta - \bar{\zeta}], [-1 + \zeta\bar{\zeta}])\,. \quad (2.6)$$

Note that an entire null-tetrad, $(\ell^a, n^a, m^a, \bar{m}^a)$, can be obtained from ℓ^a by

$$m^a = \eth\ell^a, \quad \bar{m}^a = \bar{\eth}\ell^a, \quad n^a = \ell^a + \eth\bar{\eth}\ell^a\,. \quad (2.7)$$

One also has

$$\eth m^a = 0, \quad \bar{\eth}\bar{m}^a = 0\,,$$

$$\eth n^a = -m^a, \quad \bar{\eth}n^a = -\bar{m}^a, \quad \eth\bar{m}^a = \bar{\eth}m^a = n^a + \ell^a\,.$$

The operators $\eth$ and $\bar{\eth}$ are angular differential operators defined by

$$\eth\eta = P^{1-s}\partial(P^s\eta)/\partial\zeta, \quad \bar{\eth}\eta = P^{1+s}\partial(P^{-s}\eta)/\partial\bar{\zeta}\,,$$

with $P = 1 + \zeta\bar{\zeta}$ and s (the spin weight) being $(0, 1, -1)$ respectively for $(\ell^a, m^a, \bar{m}^a)$. (These operators are just the standard spin-connection on the sphere, spinors, being in this context the spin-weight, $s = 1/2$ and $-1/2$ functions.)

The connecting vector between two points on neighboring generators $[(\zeta, \bar{\zeta}), (\zeta + d\zeta, \bar{\zeta})]$ and $[(\zeta, \bar{\zeta}), (\zeta, \bar{\zeta} + d\bar{\zeta})]$, of C_x then become respectively

$$\eth y^a = rm^a \quad \text{and} \quad \bar{\eth}y^a = r\bar{m}^a\,.$$

Null infinity, $\mathscr{I}^+$, which is RxS^2 and obtained by passing to the limit $r \to \infty$, is coordinatized by $(u, \zeta, \bar{\zeta})$ with ζ being the (complex) stereo-graphic coordinate on the S^2 factor. Each null geodesic acquires an end-point on $\mathscr{I}^+$; the end-point of that one thru x^a along $\ell^a(\zeta, \bar{\zeta})$ being $(u, \zeta, \bar{\zeta}) = (x^a\ell_a(\zeta, \bar{\zeta}), \zeta, \bar{\zeta})$.

3. The Parallel Propagator

Our parallel propagator $G(x^a, \zeta, \bar{\zeta})$, can be defined in the following fashion:

Abstractly the parallel propagator is the map from the fiber of the Yang–Mills bundle at x^a, to the fiber over the point $(u, \zeta, \bar{\zeta}) = (x^a\ell_a(\zeta, \bar{\zeta}), \zeta, \bar{\zeta})$ at $\mathscr{I}^+$ obtained by parallel propagation along the null geodesic thru x^a with tangent vector $\ell^a(\zeta, \bar{\zeta})$. With respect to a choice of gauge (i.e. a frame for the Yang–Mills bundle) the parallel propagator G is represented by the non-singular matrix denoted by

$$G_\alpha{}^\lambda(x^a, \zeta, \bar{\zeta})\,,$$

where the regular Greek indices index the Yang–Mills frame at x^a and the underlined indices index the frame at $\mathscr{I}^+$.

Clearly the inverse, G^{-1}, is the parallel transport operator from $\mathscr{I}^+$ to x^a along $\ell^a(\zeta, \bar{\zeta})$ and hence G satisfies the equation

$$\ell^a \nabla_a G_{\alpha}^{\underline{\lambda}} \equiv \ell^a (G_{\alpha,a}^{\underline{\lambda}} - \gamma_{\alpha a}^{\beta} G_{\beta}^{\underline{\lambda}}) = 0 \tag{3.1}$$

or

$$DG - \gamma_a \ell^a G = 0 \quad \text{with} \quad D = \ell^a \partial / \partial x^a = \partial / \partial r \, .$$

This can be written in integral form as a path-ordered exponential integral

$$G(x^a, \zeta, \bar{\zeta}) = \mathscr{P}\exp\left(\int_0^\infty \gamma_a \ell^a dr \right) . \tag{3.2}$$

(Note that owing to the conformal invariance of the Yang–Mills equations, our formalism remains essentially unchanged, if we choose a finite null-cone instead of $\mathscr{I}^+$.)

We thus see that knowledge of the connection γ leads to the $G(x^a, \zeta, \bar{\zeta})$ via the path ordered integral, Eq. (3.2). We now use Eq. (3.1) to recover the connection itself from $G(x^a, \zeta, \bar{\zeta})$, i.e., to invert the path integral formula.

We consider the G for two infinitesimally close points on the same null generator, i.e., $G(x^a, \zeta, \bar{\zeta})$ and $G(x^a + \Delta r \ell^a, \zeta, \bar{\zeta})$; it is clear, that from their difference and from their definitions that

$$DG = \gamma_a \ell^a G$$

or

$$\gamma_a \ell^a = DG\, G^{-1} \, , \tag{3.3}$$

one of the components of γ. The other components can be obtained by applying the $\eth$ and $\bar{\eth}$ operators [see (2.7)] to (3.3), yielding

$$\gamma_a m^a = \ell^a \eth (G_a G^{-1}) + m^a (G_a G^{-1}) \, ,$$

$$\gamma_a \bar{m}^a = \ell^a \bar{\eth} (G_a G^{-1}) + \bar{m}^a (G_a G^{-1}) \, ,$$

$$\gamma_a n^a = \ell^a \eth\bar{\eth} (G_a G^{-1}) + \bar{m}^a \eth (G_a G^{-1}) + m^a \bar{\eth} (G_a G^{-1}) + n^a (G_a G^{-1}) \, ,$$

or

$$\gamma_a = G_a G^{-1} - m_a \ell^b \bar{\eth} (G_b G^{-1}) - \bar{m}_a \ell^b \eth (G_b G^{-1}) + \ell_a k \tag{3.4}$$

with

$$k = [m^b \bar{\eth} (G_b G^{-1}) + \bar{m}^a \eth (G_a G^{-1}) + \ell^b \eth\bar{\eth} (G_b G^{-1})] \, .$$

Equation (3.4) is the reconstruction of the connection from knowledge of G.

Note the important point that if one constructs a new G by

$$G'(x^a, \zeta, \bar{\zeta}) = g(x^a) G(x^a, \zeta, \bar{\zeta}) \, , \tag{3.5}$$

where $g(x^a)$ is an *arbitrary* matrix function of x^a, the connection obtained from the G' is just the gauge transform of the connection obtained from G. We can thus consider the transformation (3.5) as a gauge transformation.

Also note that if an *arbitrary* $G(x^a, \zeta, \bar{\zeta})$ (i.e. a $G(x^a, \zeta, \bar{\zeta})$ that was not necessarily a parallel propagator) were to be inserted into (3.4) the resulting expansion for γ would depend on both x^a and $(\zeta, \bar{\zeta})$ and hence would not be a connection; a proper connection *would depend only* on x^a. It is easy to see from (3.1) and the equations following (2.7) that both a necessary and sufficient condition for γ to depend only on x^a is

$$\eth^2(DG\,G^{-1}) = 0 \quad \text{and c.c.} \tag{3.6}$$

We shall refer to these as the auxiliary conditions on G. (Under the assumption that G is a global and regular function on the $(\zeta, \bar{\zeta})$ sphere, one or the other of Eqs. (3.6) are sufficient as, for example, the second equation implies that $\bar{\eth}\gamma_a = 0$ so γ_a is global and holomorphic and therefore independent of ζ by virtue of a generalization of Liouville's theorem. Alternatively the real equation $\bar{\eth}^2\eth^2(DG\,G^{-1}) = 0$ is sufficient, by use of the maximum principle for the Laplacian on the sphere, since this quantity is the only nontrivial component of $\bar{\eth}\eth\gamma_a$.)

We will return later to the issue of the $(\zeta, \bar{\zeta})$ dependence of γ.

4. The Field Equations

We now consider the question of reexpressing the conventional Y–M equations in terms of G. At the first level this is quite simple; one expresses the connection in terms of the G, i.e. Eq. (3.4), which is then substituted into the Y–M equations. After some manipulation, (using $\nabla_b F^{ab} = 0 \Leftrightarrow \ell_a \nabla_b F^{ab} = 0$ together with the auxiliary conditions) one obtains the following equation;

$$D^3(\bar{\eth}J) + D[D^2J, \bar{J}] + 2[D^2J, D\bar{J}] = 0 \tag{4.1}$$

or

$$D^3(\bar{\eth}J) + [D^3J, \bar{J}] + 3[D^2J, D\bar{J}] = 0\,,$$

where

$$J = G^{-1}\eth G, \quad \bar{J} = G^{-1}\bar{\eth}G\,.$$

This equation, for $G(x^a, \zeta, \bar{\zeta})$, together with Eq. (3.6), is equivalent to the full vacuum Y–M equations. (Note that by the identity $\bar{\eth}J - \eth\bar{J} + [\bar{J}, J] = 0$ (4.1) is real.) One can actually go a significant step further and perform the radial integrals (i.e. eliminating the D derivatives) and obtain

$$\bar{\eth}J - \int_0^\infty dr\{[D^2J, \bar{J}]r - [D^2J, D\bar{J}]r^2\} = \mathscr{C}\,, \tag{4.2}$$

where $\mathscr{C}$ involves "constants" of integration; i.e., $D^3\mathscr{C} = 0$. By use of the known asymptotic behavior of the Y–M field in the neighborhood of $\mathscr{I}^+$, the "constant" of integration $\mathscr{C}$ can be expressed in terms of the free characteristic data given on $\mathscr{I}^+$. By a lengthy argument [1,2] one has that

$$\mathscr{C} = -(\bar{\eth}A + \eth\bar{A}) - [A, \bar{A}] + 2\int_u^\infty du[\dot{\bar{A}}, A]\,, \tag{4.3}$$

where A is an arbitrary function on $\mathscr{I}^+$, i.e., $A = A(u, \zeta, \bar{\zeta})$ but, where in this expression for $\mathscr{C}$, the value of u is restricted to the intersection of C_x (the light-cone from x^a) with $\mathscr{I}^+$. (Despite appearances, $\mathscr{C}$ is real.) This intersection is given by [3]

$$u = x^a \ell_a(\zeta, \bar{\zeta}) \tag{4.4}$$

so that

$$A = A(x^a \ell_a(\zeta, \bar{\zeta}), \zeta, \bar{\zeta})\,.$$

We thus finally have

$$\begin{aligned} &\bar{\eth} J - \int_0^\infty dr\{[D^2 J, \bar{J}]r - [D^2 J, D\bar{J}]r^2\} \\ &= -(\bar{\eth} A + \eth \bar{A}) - [A, \bar{A}] + 2\int_u^\infty du[\dot{\bar{A}}, A]\,, \end{aligned} \tag{4.5a}$$

with

$$J = G^{-1}\eth G, \quad \bar{J} = G^{-1}\bar{\eth} G\,,$$

a matrix valued space-time scalar equation for the determination of $G(x^a, \zeta, \bar{\zeta})$ given the free data $A = A(x^a \ell_a(\zeta, \bar{\zeta}), \zeta, \bar{\zeta})$. Again, (4.5) despite appearances, is real and can be rewritten in a manifestly real form as

$$\begin{aligned} &\bar{\eth} J + \eth \bar{J} + \int_0^\infty dr(\{[DJ, \bar{J}] + \text{c.c.}\} + \{[D^2 J, D\bar{J}] + \text{c.c.}\}r^2) \\ &\qquad = -2(\bar{\eth} A + \eth \bar{A}) + 2\int_u^\infty du([\dot{\bar{A}}, A] + \text{c.c.})\,, \end{aligned} \tag{4.5b}$$

with again

$$J = G^{-1}\eth G, \quad \bar{J} = G^{-1}\bar{\eth} G\,.$$

5. An Approximation Procedure

Though Eqs. (4.5) appear to be quite formidable, a relatively simple perturbation procedure is available so that a solution procedure exists for the determination of the G at every order of the calculation.

This procedure starts with taking (4.5a) and ignoring the (radial) integral terms on the left-hand side at first order, and solving for J. One uses for this solution, the unique Green's function for the operators $\bar{\eth}$. To proceed further we must then solve the equation $\eth G = JG$ for G with the "first order" J. This equation, while not quite a quadrature, requires the global solution of linear equations on the sphere and is a standard ingredient of one of the solution procedures for the self-dual Yang–Mills equations [5] (in which context J is just a function of $x^a \ell_a$ and $(\zeta, \bar{\zeta})$ and the equation is referred to as the Sparling equation). With this solution for G we can substitute the expressions for J and $\bar{J} = G^{-1}\bar{\eth} G$ into the (radial) integral expressions of (4.5a) and run through the procedure again to determine the next approximation and so on.

When (3.4) has been used to reconstruct γ_a it will not in general be real or independent of $(\zeta, \bar{\zeta})$ except in the limit as the order of the approximation goes to infinity. We can however evaluate it at $(\zeta, \bar{\zeta}) = 0$ in the gauge $G(\zeta = 0, \bar{\zeta} = 0) = 1$ and take its real part; this will converge to the required solution (as will the "raw" γ_a).

If we were to start with (4.5b) rather than (4.5a), we would have to solve the harmonic map equations with source in order to pursue the approximation scheme. We however do not know if a solution procedure exists for these equations.

When the integral terms on the LHS of (4.5a) are ignored, the equations satisfy the Huygens property in the sense that the solution G at x^a only depends on the RHS of the equation, i.e., the data, at the intersection of the light cone of x^a with the data surface, null infinity.

We thus see that our reformulation of Y–M can be considered as a generalized D'Adhemar formulation of Y–M. The perturbation solution is analogous to a series of Penrose's Zig-Zag integrals [6]. It is the presence of the integral terms in (4.5a) that prevents the Y–M equations from satisfying Huygens principle, namely that data propagates along characteristic surfaces.

Remark 1. Note that J and $\bar{J}$ are invariant under the gauge transformation (3.5), i.e., $G'(x^a, \zeta, \bar{\zeta}) = g(x^a)G(x^a, \zeta, \bar{\zeta})$, and hence the solutions G, to (4.5), *are not unique* and are given (at least) up to a gauge transformation. That there is no other freedom in the solution is difficult to show rigorously; nevertheless a perturbative argument clearly indicates that the solutions are unique up to these gauge transformations. If this can be made rigorous then we have *the important result* namely; Eq. (4.5), which was derived from the Y–M equations, is completely equivalent to the Y–M equations. $G(x^a, \zeta, \bar{\zeta})$ is then the Y–M parallel propagator and automatically yields a connection (3.4). It would thus follow that the auxiliary condition (3.6) is then automatically satisfied.

(In [7] further arguments are given for the derivability of the auxiliary condition (3.6) (in weaker form) directly from Eq. (4.1) and its integral (4.5). This weaker form also requires globality on the sphere in order to imply condition (3.6) in total. It is perhaps also worth mentioning that (3.6) follows directly from (4.5) and globality in linearized theory, and so one would expect it to follow at least for "small" solutions of the full Yang–Mills equations.)

Remark 2. In the special case of an Abelian gauge theory (i.e. Maxwell theory), $G(x^a, \zeta, \bar{\zeta})$ is a scalar which can be written as

$$\log G = F(x^a, \zeta, \bar{\zeta}) .$$

The field equation (4.5), equivalent in this case to Maxwell's vacuum equations, then becomes

$$\bar{\eth}\eth F = -(\bar{\eth} A + \eth \bar{A}) .$$

Since there is a simple unique Green's function [4] for the $\bar{\eth}\eth$ operator, it can be immediately integrated for F, which when substituted into the equation for the connection yields the well-known D'Adhemar solution of Maxwell's equations, i.e., the integral form of the solution of Maxwell's equations obtained from characteristic data. The Sparling Equation version of the self (or anti-self)-dual Yang–Mills equations [5] are also special cases of (4.5).

Remark 3. A similar programme can be executed in the context of GR using light-cone cuts, etc. See [7, 8] for a discussion in a similar spirit.

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