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Quasi-Quantum Groups, Knots, Three-Manifolds, and Topological Field Theory

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Abstract. We show how to construct, starting from a quasi-Hopf algebra, or quasi-quantum group, invariants of knots and links. In some cases, these invariants give rise to invariants of the three-manifolds obtained by surgery along these links. This happens for a finite-dimensional quasi-quantum group, whose definition involves a finite group G, and a 3-cocycle ω, which was first studied by Dijkgraaf, Pasquier, and Roche. We treat this example in more detail, and argue that in this case the invariants agree with the partition function of the topological field theory of Dijkgraaf and Witten depending on the same data G, ω.

1. Introduction

It is by now well established that there are deep connections between two-dimensional rational conformal field theories (RCFT), three-dimensional topological field theories (TFT), and quantum groups when q is a root of unity, see e.g. [1–9].

A key element in any attempt at understanding these connections is the fact that both RCFT and quantum groups are sources of topological invariants of knots, links, and three-dimensional manifolds (through the TFT reinterpretation of RCFT). For instance, the invariants of the Hopf link are the elements of the matrix S [1, 2], and consideration of a chain of three circles is the key to proving Verlinde's formula. The construction of invariants of links from the representation theory of quantum groups was developed in [10–12]. In its most general form it appears in [12], where the concept of ribbon Hopf algebras is introduced. Examples of ribbon Hopf algebras are the "usual" quantum groups [13] $U_q\mathscr{G}$, where $\mathscr{G}$ is a semi-simple Lie algebra [7], the double $D(G)$ of a finite group G, and many more are discussed in a recent paper of Kauffman and Radford [14]. To our taste, the above connections are best explained in [12], where a TFT, formalized in

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the sense of Atiyah and Segal [15], is reconstructed from ribbon Hopf algebras of a particular class called modular Hopf algebras by these authors. Roughly speaking, a modular Hopf algebra A is a ribbon Hopf algebra with a finite set of representations which is closed under the tensor product operation, up to representations of quantum dimension zero; $U_q\mathcal{G}$ for q a root of unity [7, 16, 17] and $D(G)$ [18] belong to this class.

In another direction, one may ask how to construct canonically a quantum group, starting from a TFT. Already in the work of Moore and Seiberg [19], it is clear that this problem is analogous to the Tannaka-Krein reconstruction of a group G from a category of vector spaces which at the end, become representations of G. In his work, Majid [20] gives a solution to this problem, showing that the initial data is the category of cobordisms instead of a category of vector spaces. He defines an algebra which has a natural coproduct, the trouble, however, as he points out, is that in general this coproduct Δ will fail to be coassociative, it will be quasi-coassociative:

$$(\mathrm{id}\otimes\Delta)(\Delta(a))=\phi(\Delta\otimes\mathrm{id})(\Delta(a))\phi^{-1}, \tag{1.1}$$

where $\phi\in A\otimes A\otimes A$, and satisfies natural pentagon and hexagon identities (there is also a natural R-matrix). This kind of object, now called quasi-Hopf algebra, was invented by Drinfeld [21] some time before, but with a somewhat different motivation, which we explain below. We would like to mention at this point that the relevance of quasi-Hopf algebras for TFT is clearly shown in the paper of Dijkgraaf, Pasquier, and Roche [22, 23], where they built a very interesting example $D^\omega(G)$, which is a "deformation" of $D(G)$ involving a non-trivial 3-cocycle ω of G, in order to reproduce the fusion rules of the Dijkgraaf-Witten TFT [24, 25] defined with the same data G, ω. Mack and Schomerus [26] have proposed to use quasi-Hopf algebras in RCFT to reproduce the primary field content and fusion rules, e.g. for the Ising model. To achieve this, however, they seem to need to generalize even more the quantum groups, as witnessed by their definition of weak quasi-Hopf algebras.

Drinfeld's motivation, as far as we know, was based on the observation that when one tries to deform the coproduct Δ of a Hopf algebra, setting:

$$\Delta^f(a)=f\Delta(a)f^{-1},$$

with $f\in A\otimes A$ an invertible element, then Δ^f is no longer coassociative, but satisfies (1.1) above, where

$$\phi=f_{23}(\mathrm{id}\otimes\Delta)(f)(\Delta\otimes\mathrm{id})(f^{-1})f_{12}^{-1}.$$

Here and later, f_{ij} means f acting non-trivially in the i^{th} and j^{th} place of $A\otimes A\otimes A$. Now if one defines quasi-Hopf algebras by the property (1.1), one gets a class of objects which is stable under the mapping $\Delta\to\Delta^f$, called "twist by f." This twist takes ϕ into

$$\phi^f=f_{23}(\mathrm{id}\otimes\Delta)(f)\phi(\Delta\otimes\mathrm{id})(f^{-1})f_{12}^{-1}.$$

Twist also preserve the class of quasitriangular quasi-Hopf algebras, which will be defined in Sect. 2.

In this paper we present a natural extension of the constructions of Reshetikhin and Turaev [7, 12] to the case of quasi-Hopf algebras. More precisely, for any ribbon quasi-Hopf algebra we define regular isotopy invariants of coloured ribbon graphs, the colours being finite-dimensional representations. This result is very general and can be applied to a much broader set of algebras and topological setups than those considered later in the paper, for instance to the construction of ambient isotopy invariants of links. We intend to explore some of these questions in a future work. Our motivation for constructing these ribbon invariants was to be able to understand the topological field theory of Dijkgraaf and Witten [25], whose theory was further investigated by Freed and Quinn [27], in the case of a non-trivial cocycle ω, using only the algebra $D^{\omega}(G)$ of [22]. We succeeded in finding a 3-manifold invariant, considering surgery on the ribbon graphs coloured by a representation of $D^{\omega}(G)$, which in the examples that we computed explicitly, coincides with the invariant of [25]. We conjecture that this holds in general. One advantage of our approach for constructing the invariants is that it lends itself to practical computation from a surgery presentation of the manifold, whereas the original definition requires the knowledge of a triangulation, which is generally more difficult to find.

In Sect. 2, we recall the basic definitions from Drinfeld's [21] original papers. In Sect. 3, we give an important theorem on the square of the antipode in quasi-Hopf algebras possessing an R-matrix, generalizing a theorem of Drinfeld [28] for Hopf algebras. In Sect. 4 we define invariants of ribbon graphs, which are framed links (tangles) with some open ends. These invariants are intertwining operators for a ribbon quasi-Hopf algebra. In the particular case of graphs with only closed ribbons (annuli), these invariants are pure numbers and similar to the Reshetikhin-Turaev version of Jones' polynomial. In Sect. 5 we first prove that the algebra $D^{\omega}(G)$ is a ribbon quasi-Hopf algebra, and then we show that it even allows to define invariants of 3-manifolds using surgery. In some simple cases we compute these invariants, checking the properties predicted by our conjecture.

2. Definitions

Let A be an associative algebra over $\mathbb{C}$, with a unit element 1. We say that A is a quasi-bialgebra if there are algebra homomorphisms $\Delta : A \to A \otimes A$, $\varepsilon : A \to \mathbb{C}$ and an invertible element $\phi \in A \otimes A \otimes A$, such that:

$$(\mathrm{id} \otimes \Delta)(\Delta(a)) = \phi(\Delta \otimes \mathrm{id})(\Delta(a))\phi^{-1}, \qquad a \in A, \tag{2.1}$$

$$(\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\phi)(\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\phi) = (1 \otimes \phi)(\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\phi)(\phi \otimes 1), \tag{2.2}$$

$$(\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon) \circ \Delta, \tag{2.3}$$

$$(\mathrm{id} \otimes \varepsilon \otimes \mathrm{id})(\phi) = 1. \tag{2.4}$$

The map Δ is called the coproduct and ε the counit.

Let us briefly recall some of the main consequences of these definitions in the representation theory of A. In this paper we will be dealing only with finite-dimensional representations (π, V) of A, which consist of a finite-dimensional vector space V over $\mathbb{C}$, and a representation $\pi : A \to \mathrm{End}\, V$. We will also use the equivalent definition of an A-module V, and write $a \cdot v$ for $\pi(a)v$, $a \in A$, $v \in V$. Given two such representations (π_1, V_1) and (π_2, V_2) one may construct representations $(\pi_{12}, V_1 \otimes V_2)$ and $(\pi_{21}, V_2 \otimes V_1)$ by setting

$$\pi_{12} = (\pi_1 \otimes \pi_2)\Delta \tag{2.5}$$

and similarly for π_{21}. Suppose we are given three representations (π_i, V_i), $i=1,2,3$. Set

$$\phi^{V_1,V_2,V_3}=(\pi_1\otimes\pi_2\otimes\pi_3)(\phi). \tag{2.6}$$

Then (2.1) says that $\phi^{V_1,V_2,V_3}:(V_1\otimes V_2)\otimes V_3\to V_1\otimes(V_2\otimes V_3)$ is an intertwiner, and therefore, the representations (modules) $(V_1\otimes V_2)\otimes V_3$ and $V_1\otimes(V_2\otimes V_3)$ are equivalent. Now take four representations. The identity (2.2) implies that the diagram

$$\begin{array}{ccccc} ((V_1\otimes V_2)\otimes V_3)\otimes V_4 & \longrightarrow & (V_1\otimes V_2)\otimes(V_3\otimes V_4) & \longrightarrow & V_1\otimes(V_2\otimes(V_3\otimes V_4)) \\ \downarrow & & & & \downarrow \\ (V_1\otimes(V_2\otimes V_3))\otimes V_4 & & \longrightarrow & & V_1\otimes((V_2\otimes V_3)\otimes V_4) \end{array}$$

commutes, where the arrows are $\phi^{V_1\otimes V_2,V_3,V_4}$, $\phi^{V_1,V_2,V_3\otimes V_4}$, etc. This explains the use of the name *pentagon identity* for Eq. (2.2).

Using the counit ε, one obtains a one-dimensional representation of A on $\mathbb{C}$. Then (2.3) means that $V\otimes\mathbb{C}=V=\mathbb{C}\otimes V$ for any A-module V. We will refer to $(\varepsilon,\mathbb{C})$ as the trivial representation. One sees that (2.2) and (2.4) together imply

$$(\varepsilon\otimes\mathrm{id}\otimes\mathrm{id})(\phi)=(\mathrm{id}\otimes\mathrm{id}\otimes\varepsilon)(\phi)=1, \tag{2.7}$$

therefore, in a tensor product of three representations one may forget a trivial factor.

A quasi-bialgebra A is called a quasi-Hopf algebra if there exists an antiautomorphism S of A, i.e. $S(ab)=S(b)S(a)$, and two elements $\alpha,\beta\in A$ such that:

$$\sum_i S(a_i^{(1)})\alpha a_i^{(2)}=\varepsilon(a)\alpha, \quad \sum_i a_i^{(1)}\beta S(a_i^{(2)})=\varepsilon(a)\beta \tag{2.8}$$

for $a\in A$ and $\sum_i a_i^{(1)}\otimes a_i^{(2)}=\Delta(a)$, and

$$\sum_i X_i\beta S(Y_i)\alpha Z_i=1, \quad \text{where} \quad \sum_i X_i\otimes Y_i\otimes Z_i=\phi, \tag{2.9}$$

$$\sum_j S(P_j)\alpha Q_j\beta S(R_j)=1, \quad \text{where} \quad \sum_j P_j\otimes Q_j\otimes R_j=\phi^{-1}. \tag{2.10}$$

We note the following two consequences of the definitions of S,α,β:

$$\varepsilon(\alpha)\varepsilon(\beta)=1, \quad \varepsilon\circ S=\varepsilon. \tag{2.11}$$

The map S is called the antipode. It allows us to define the dual representation (π^*, V^*) of (π, V), where V^* is the dual space, by $\pi^*(a)=(\pi\circ S(a))^t$, the superscript t denoting the transposed map.

In the theory of Hopf algebras, the following relation is well-known: $\Delta(a)=(S\otimes S)(\Delta'\circ S^{-1}(a))$, where $\Delta'=\sigma\circ\Delta$, $\sigma:a\otimes b\mapsto b\otimes a$. Later on we will need the generalization of this, which is due to Drinfeld. Let

$$\sum_j A_j\otimes B_j\otimes C_j\otimes D_j=(\phi\otimes 1)(\Delta\otimes\mathrm{id}\otimes\mathrm{id})(\phi^{-1}), \tag{2.12}$$

$$\gamma=\sum_j S(B_j)\alpha C_j\otimes S(A_j)\alpha D_j, \tag{2.13}$$

$$\sum_i K_i\otimes L_i\otimes M_i\otimes N_i=(\Delta\otimes\mathrm{id}\otimes\mathrm{id})(\phi)(\phi^{-1}\otimes 1), \tag{2.14}$$

$$\delta=\sum_i K_i\beta S(N_i)\otimes L_i\beta S(M_i). \tag{2.15}$$

Then for any $a \in A$,

$$f\Delta(a)f^{-1} = (S \otimes S)(\Delta' \circ S^{-1}(a)), \tag{2.16}$$

where

$$f = \sum_i (S \otimes S)(\Delta'(P_i)) \cdot \gamma \cdot \Delta(Q_i \beta S(R_i)). \tag{2.17}$$

Moreover,

$$\gamma = f\Delta(\alpha), \qquad \delta = \Delta(\beta)f^{-1}. \tag{2.18}$$

In fact, Drinfeld shows that f defines a twist of A, where the modified coproduct is the r.h.s. of (2.16).

A quasi-Hopf algebra is termed *quasitriangular*, if there exists an invertible element $R \in A \otimes A$, such that:

$$\Delta'(a) = R\Delta(a)R^{-1}, \tag{2.19}$$

$$(\Delta \otimes \mathrm{id})(R) = \phi_{312} R_{13} \phi_{132}^{-1} R_{23} \phi, \tag{2.20}$$

$$(\mathrm{id} \otimes \Delta)(R) = \phi_{231}^{-1} R_{13} \phi_{213} R_{12} \phi^{-1}, \tag{2.21}$$

where we have used the following notation: R_{ij} means R acting non-trivially in the i^{th} and j^{th} slot of $A \otimes A \otimes A$. If s denotes a permutation of $\{1,2,3\}$ and $\phi = \sum_i a_i^1 \otimes a_i^2 \otimes a_i^3$ then we set $\phi_{s(1)s(2)s(3)} = \sum_i a_i^{s^{-1}(1)} \otimes a_i^{s^{-1}(2)} \otimes a_i^{s^{-1}(3)}$. From these relations one deduces the quasi-Yang-Baxter equation:

$$R_{12} \phi_{312} R_{13} \phi_{132}^{-1} R_{23} \phi = \phi_{321} R_{23} \phi_{231}^{-1} R_{13} \phi_{213} R_{12}. \tag{2.22}$$

The translation of (2.20) and (2.21) in the language of commutative diagrams leads to hexagons [21]. The following property of R can be derived easily:

$$(\varepsilon \otimes \mathrm{id})R = (\mathrm{id} \otimes \varepsilon)R = 1. \tag{2.23}$$

The most significant consequence of (2.19) in representation theory is that the representations $(\pi_{12}, V_1 \otimes V_2)$ and $(\pi_{21}, V_2 \otimes V_1)$ are equivalent:

$$\pi_{21}(a) = \check{R}_{12} \pi_{12}(a) \check{R}_{12}^{-1}, \tag{2.24}$$

where $\check{R}_{12} : V_1 \otimes V_2 \to V_2 \otimes V_1$ is given by $\check{R}_{12} = P_{12}(\pi_1 \otimes \pi_2)R$ and P_{12} is the operator which permutes the vectors in V_1 and V_2.

3. The Square of the Antipode

Let A be a quasi-Hopf algebra with an R-matrix satisfying (2.19). Generalizing a theorem of Drinfeld for Hopf algebras, we will prove that for any $a \in A$,

$$S^2(a) = uau^{-1}, \tag{3.1}$$

where u is given by the formula:

$$u = \sum_{j,p} S(Q_j \beta S(R_j)) S(b_p) \alpha \, a_p P_j, \tag{3.2}$$

in terms of

$$R = \sum_p a_p \otimes b_p, \qquad \phi^{-1} = \sum_j P_j \otimes Q_j \otimes R_j. \tag{3.3}$$

Let us start by showing that:

$$S^2(a)u = ua\,. \tag{3.4}$$

Set $(\Delta\otimes\mathrm{id})\Delta(a)=\sum_k f_k\otimes g_k\otimes h_k$; using (2.3) and (2.8) one has

$$\sum_k S(f_k)\alpha g_k\otimes h_k=\alpha\otimes a\,, \tag{3.5}$$

and therefore,

$$S^2(a)u=\sum_{j,k,p} S^2(h_k)S(Q_j\beta S(R_j))S(b_p)S(f_k)\alpha g_k a_p P_j\,. \tag{3.6}$$

But (2.19) implies

$$\sum_{k,p} a_p f_k\otimes b_p g_k\otimes h_k=\sum_{k,p} g_k a_p\otimes f_k b_p\otimes h_k\,, \tag{3.7}$$

so that:

$$S^2(a)u=\sum_{j,k,p} S(g_k Q_j\beta S(h_k R_j))S(b_p)\alpha a_p f_k P_j\,. \tag{3.8}$$

Now $(\Delta\otimes\mathrm{id})\Delta(a)\phi^{-1}=\phi^{-1}(\mathrm{id}\otimes\Delta)\Delta(a)$, (2.3) and (2.8) lead to (3.4).

Our next move is to establish the lemma:

$$S(\alpha)u=\sum_p S(b_p)\alpha a_p\,. \tag{3.9}$$

To prove it, one performs in u the substitution

$$\sum_j P_j\otimes Q_j\otimes R_j\otimes 1$$
$$=(\Delta\otimes\mathrm{id}\otimes\mathrm{id})(\phi^{-1})(\mathrm{id}\otimes\mathrm{id}\otimes\Delta)(\phi^{-1})(1\otimes\phi)(\mathrm{id}\otimes\Delta\otimes\mathrm{id})(\phi)$$

and simplifies in several steps the resulting expression for $S(\alpha)u$ by use of (2.4), (2.7), (2.8), and (2.9).

Now (3.9) implies

$$ut=\alpha\,, \tag{3.10}$$

where we set:

$$t=\sum_q S^{-1}(\alpha d_q)c_q\,,\qquad R^{-1}=\sum_q c_q\otimes d_q\,. \tag{3.11}$$

Plugging (3.10) into (2.10) gives

$$1=\sum_j S(P_j)utQ_j\beta S(R_j)=u\sum_j S^{-1}(P_j)tQ_j\beta S(R_j)$$
$$=S^2\Big(\sum_j S^{-1}(P_j)tQ_j\beta S(R_j)\Big)u\,. \tag{3.12}$$

Therefore u, which has both a left and right inverse, is invertible, and $S(u)$, too. This completes the proof of (3.1). Some straightforward corollaries are:

1. $S^2(u)=u$.
2. The element $uS(u)=S(u)u$ is central.
3. $\sum_p S(b_p)\alpha a_p=S(\alpha)u=S(t)S(u)u=S(u)u\sum_q S(c_q)\alpha d_q$.

Notice also that (2.4) and (2.11) ensure $\varepsilon(u)=1$.

The most important consequence of this theorem for representation theory, is that for any quasitriangular quasi-Hopf algebra A, and for any finite-dimensional representation (π, V) of A, the double dual (π^{**}, V^{**}) is equivalent to (π, V), the intertwiner being $\pi(u)$. This means also that the (right) dual (π^*, V^*) is equivalent to the *left dual* representation $({}^*\pi, V^*)$ which is defined [21] by ${}^*\pi(a) = (\pi \circ S^{-1}(a))^t$ for $a \in A$.

4. The Generalized Reshetikhin-Turaev Functor

4.1. Ribbon Quasi-Hopf Algebras

Let A be a quasitriangular quasi-Hopf algebra. We propose the following generalization of the notion of ribbon Hopf algebra of Reshetikhin and Turaev. We say that A is a ribbon quasi-Hopf algebra, if there exists a central element $v \in A$ such that

R1. $v^2 = uS(u)$,

R2. $S(v) = v$,

R3. $\varepsilon(v) = 1$,

R4. $\Delta(uv^{-1}) = f^{-1}((S \otimes S)(f_{21}))(uv^{-1} \otimes uv^{-1})$,

where f is defined in (2.17). We shall comment later on the consequence of these conditions, and give a detailed example of ribbon quasi-Hopf algebra.

4.2. Coloured Ribbon Graphs

A ribbon graph [12] can be defined as a regular projection on a plane of a finite set of oriented ribbons in $\mathbb{R}^3$, i.e. two-dimensional oriented manifolds with boundaries which are the images of non-self-intersecting smooth embeddings $[0,1] \times [0,1] \to \mathbb{R}^2 \times [0,1]$ (open ribbons) or $S^1 \times [0,1] \to \mathbb{R}^2 \times [0,1]$ (annuli). Note that Moebius strips are excluded by this definition so that ribbons have a "white" and a "black" side. The definition of ribbon graphs also assumes that the white side is always facing the observer on the top and bottom of the figure. Furthermore, the extremities of all the open ribbons are vertical and lie in $\mathbb{R}^2 \times \{0,1\}$. Ribbons are also directed, i.e. equipped with an arrow. An example of ribbon graph is shown on Fig. 1.

Two graphs are considered equivalent if and only if they are projections of isotopic ribbons. Here by isotopy we mean a smooth isotopy of $\mathbb{R}^3$ which

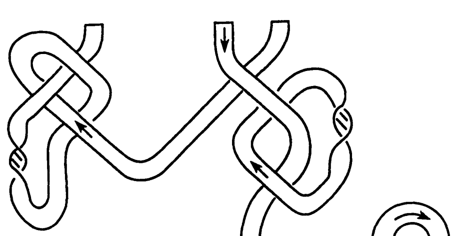

Fig. 1. A ribbon graph

preserves the directions of arrows, the orientation of the graph surface, and keeps the ends of open ribbons fixed. For convenience we will represent pictorially such a ribbon graph as the projection of an oriented link (with possibly open components). This means that we identify the graphs as in Fig. 2.

Now we define *coloured ribbon graphs*, or *c*-graphs for short. Let A be a ribbon quasi-Hopf algebra. Denote by $N(A)_k$ the class of all words (formal non-associative expressions) of the form

$$((((V_1^{\varepsilon_1}\square((V_2^{\varepsilon_2}\square\ldots))\ldots)\square V_k^{\varepsilon_k}))), \tag{4.1}$$

where the k letters V_i are A-modules, $\varepsilon_i=\pm 1$, and $V^1=V$, but V^{-1} is just a symbol, not a module. There is no restriction on the location of parentheses, but we regard two words with the same letters but a different distribution of parentheses as being distinct, e.g. $(V_1\square V_2)\square V_3 \neq V_1\square(V_2\square V_3)$. By definition $N(A)_0$ consists of the single word $\mathbb{C}$, the trivial representation.

A *c*-graph is a ribbon graph equipped with an assignment of two words $w_k\in N(A)_k$, $w_l\in N(A)_l$ to the bottom and top ends of the open ribbons, together with an assignment of an A-module V to each ribbon. (V is then called the colour of the ribbon.) These two assignments must be compatible in the sense that the letters of w_k and w_l corresponding to the ends of an open ribbon must be equal to its colour, and its direction has to be determined by the signs ε_i according to the following rule: if a ribbon end is labeled by a letter $V_i^{\varepsilon_i}$, then it is directed downwards (respectively upwards) if $\varepsilon_i=+1$ (respectively -1). Figure 3 shows an example of *c*-graph.

These definitions can be conveniently organized into a category Grc(A) of *c*-graphs. Its objects are the elements of $N(A)=\bigcup_k N(A)_k$, and the morphisms are the *c*-graphs. For example, the *c*-graph of Fig. 3 is a morphism $V_1\square(V_2\square V_3^{-1})\rightarrow(V_1\square V_3^{-1})\square V_2$. Notice that our convention is that a *c*-graph is a morphism from the bottom to the top. If a *c*-graph has no extremities of open ribbons at the bottom or the top, then it is a morphism to or from $\mathbb{C}$. If it has no open ribbons at all, we say that it is a closed *c*-graph. We stress that the bottom and top objects,

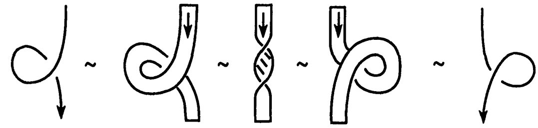

Fig. 2. Representing a ribbon by a single line

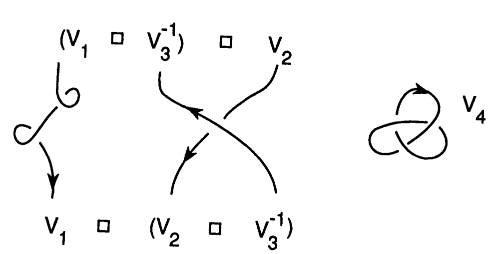

Fig. 3. A *c*-graph

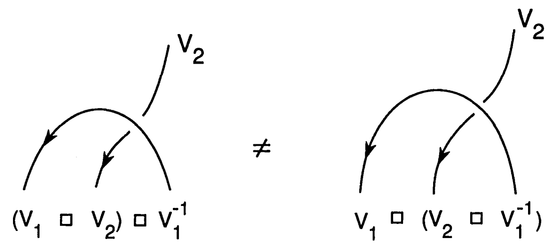

Fig. 4. Two different c-graphs

including the location of parentheses, are essential parts of the definition of a morphism. This is illustrated in Fig. 4.

4.3. The Functor F

Our aim is to define a functor F from Grc(A) to the category Rep(A) of finite-dimensional representations of A, whose objects are finite-dimensional A-modules, and morphisms are intertwiners. If $w \in N(A)$ then $F(w)$ is the A-module obtained by replacing all formal products $\square$ by tensor products $\otimes$, and all occurrences of $V_i^{\varepsilon_i}$ by V_i^* if $\varepsilon_i = -1$. For a c-graph $C: w \to w'$, $F(C)$ is an intertwiner $F(w) \to F(w')$. The image $F(C)$ of a closed c-graph $C: \mathbb{C} \to \mathbb{C}$ is then a pure number, which is the essential ingredient of the invariants of links and 3-manifolds which we construct later. The definition of $F(C)$ is based on the observation that any c-graph C can be built from a few elementary ones by gluing and juxtaposition. These elementary c-graphs $I^\pm$, $X^\pm$, U, D, Φ are shown on Fig. 5.

Let us define more precisely what we mean by gluing and juxtaposition. Suppose that $C: w \to w'$ and $C': w' \to w''$ are two c-graphs. Then by gluing we mean the composition of morphisms in Grc(A), $C' \circ C: w \to w''$, which is obviously defined as in Fig. 6. It is important that the top w' of C is exactly equal to the bottom of C', including the location of parentheses.

Juxtaposition in Grc(A) is a binary operation $\square$. For $w \in N(A)_k$, $w \in N(A)_l$, it is simply $w_k \square w_l \in N(A)_{k+l}$. For c-graphs $C: w \to w'$, $C': x \to x'$, we define $C \square C': w \square x \to w' \square x'$ by placing them side by side, as in Fig. 7.

Observe that in Grc(A) there is a class of c-graphs $\Psi_w^{w'}$, entirely made of vertical lines, and such that w and w' can differ only in the location of parentheses. In Fig. 8 we have displayed the case $w = (V_1 \square (V_2 \square V_3^{-1})) \square V_4$, $w' = (V_1 \square V_2) \square (V_3^{-1} \square V_4)$.

The functor F is required to have the following properties: it is a covariant functor,

$$F(C' \circ C) = F(C') \circ F(C), \tag{4.2}$$

juxtaposition corresponds to tensor products:

$$F(C \square C') = F(C) \otimes F(C'), \tag{4.3}$$

and the Ψ graphs enjoy a "fusion" property, which states that whenever $w, w' \in N(A)_k$ differ only in the location of parentheses, but are such that they have a part $(V_i^{\varepsilon_i} \square V_{i+1}^{\varepsilon_{i+1}}) = w^{(i)}$ in common, then

$$F(\Psi_w^{w'}) = F(\Psi_{w\otimes}^{w'\otimes}), \tag{4.4}$$

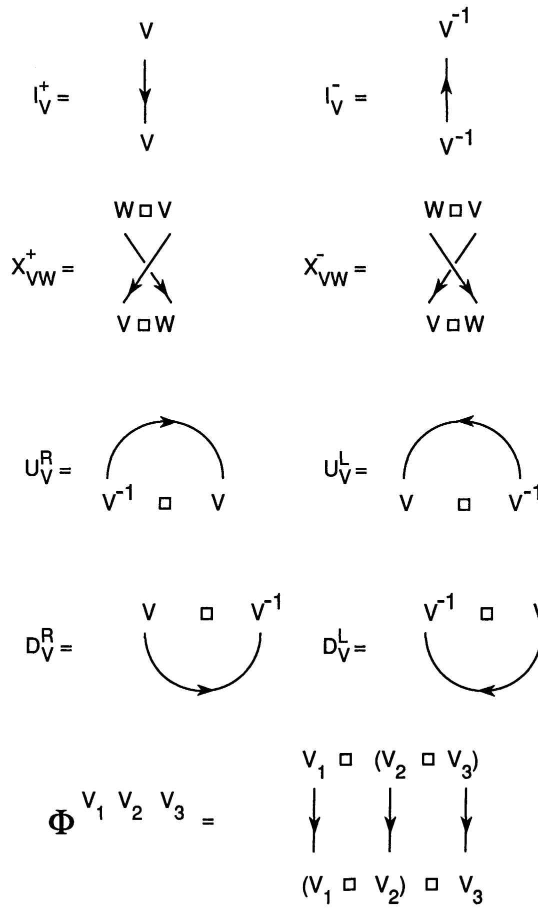

Fig. 5. The elementary *c*-graphs

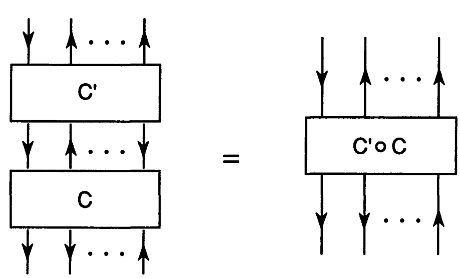

Fig. 6. Gluing

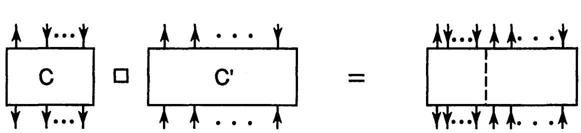

Fig. 7. Juxtaposition

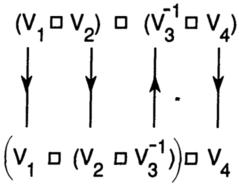

Fig. 8. A Ψ graph

where $w_{\otimes} \in N(A)_{k-1}$ is obtained by replacing $w^{(i)}$ by $F(w^{(i)})$ in w. The functor F is then defined by its values on the elementary graphs of Fig. 5: $I^{\pm}$, $X^{\pm}$, U, D, Φ, as follows:

$$F(I_V^+)=\mathrm{id}_V, \qquad F(I_V^-)=\mathrm{id}_{V^*}, \tag{4.5}$$

$$F(X_{V,W}^+)=\check{R}_{V,W}, \qquad F(X_{V,W}^-)=\check{R}_{W,V}^{-1}, \tag{4.6}$$

$$F(U_V^R)(f\otimes x)=f(\alpha x), \qquad f\in V^*, \qquad x\in V, \tag{4.7}$$

$$F(U_V^L)(x\otimes f)=f(S(\alpha)uv^{-1}x), \tag{4.8}$$

$$F(D_V^R)(1)=\sum_j \beta\cdot e_j\otimes e^j, \tag{4.9}$$

where $\{e_j\}$ is a basis of V, and $\{e^j\}$ the dual basis of V^*,

$$F(D_V^L)(1)=\sum_j e^j\otimes u^{-1}vS(\beta)\cdot e_j, \tag{4.10}$$

$$F(\Phi^{V_1,V_2,V_3})=\phi^{V_1,V_2,V_3}. \tag{4.11}$$

Notice that the r.h.s. of these equations are all intertwiners, as they should be. One has to show that F is well-defined. This means two things: that F preserves all relations coming from isotopy of ribbons, and that the value of F on any c-graph is independent from the choices made in evaluating it, i.e. cutting it into smaller pieces until one reaches a decomposition into elementary graphs. Let us elaborate on this latter point, which is more subtle than in the case of Hopf algebras.

We show first that $F(\Psi_w^{w'})$ is well-defined. In view of the fusion property, it is clear that $F(\Psi_w^{w'})$ is built from ϕ, ϕ^{-1}, and the identity operator. There are several ways to evaluate $F(\Psi_w^{w'})$, however, Mac Lane's "coherence" theorem [29] states that they all give the same result since ϕ satisfies the pentagon identity. The properties of quasi-Hopf algebras involving the counit ε guarantee the well-definedness of the c-graphs containing U or D.

To prove that F depends only on isotopy classes of c-graphs, it is enough to prove that the relations listed on Fig. 9 are preserved [11, 12], for all possible colorings and directions of ribbons. The proof that F preserves relations (a), (b), and (c) is very simple: (a) amounts to Eqs. (2.9) and (2.10), (b) is trivial, and (c) is Eq. (2.22). It can be shown that

$$F(L_V^+)=F(L_V'^+)=\pi(v^{-1}), \tag{4.12}$$

$$F(L_V^-)=F(L_V'^-)=\pi(v), \tag{4.13}$$

where the c-graphs $L_V^{\pm}, L_V'^{\pm}$ are given on Fig. 10. This implies that (d) is also respected. Note that these two equations reflect the fact that the objects we are dealing with are ribbons, see Fig. 2 for a graphical interpretation of (4.12).

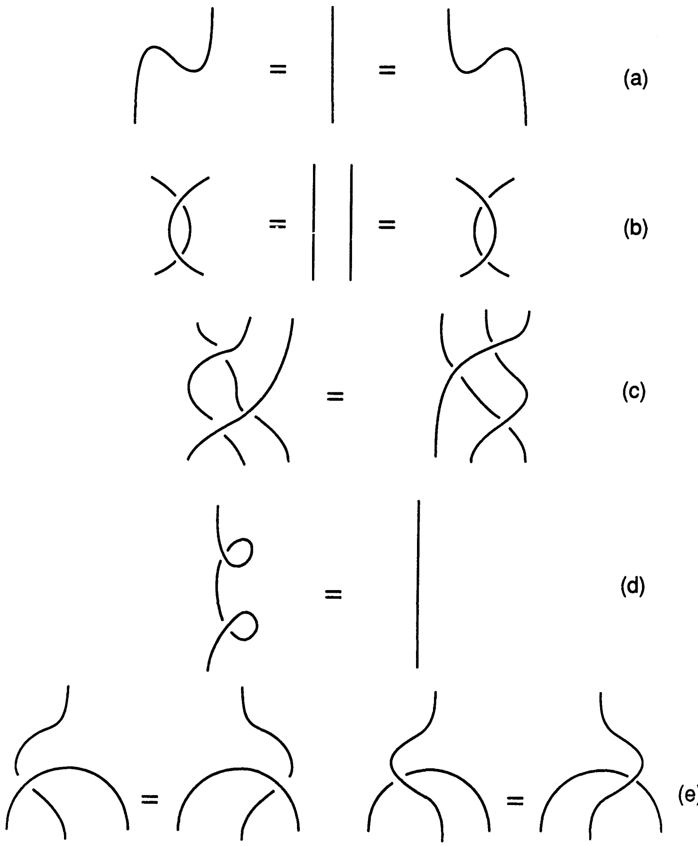

Fig. 9. Isotopy relations

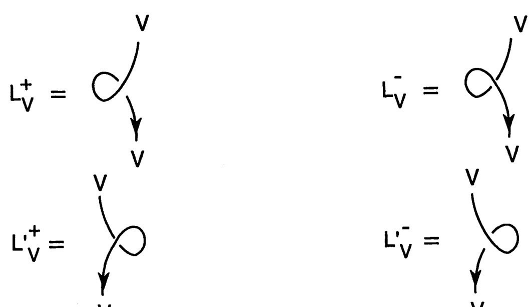

Fig. 10. The four loops

4.4. q-Traces and q-Dimensions

Suppose $C: w \to w$ is a c-graph with the same words on top and bottom, where $w \in N(A)_k$. We define the closure $\hat{C}$ of C by Fig. 11. By construction, $F(C) \in \operatorname{End} F(w)$ is an intertwiner. We put:

$$\operatorname{tr}_q F(C) = \operatorname{tr}_{F(w)}(F(C)\beta S(\alpha)uv^{-1}). \tag{4.14}$$

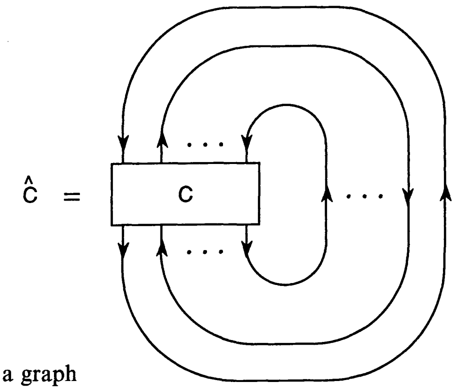

Fig. 11. The closure of a graph

The main properties of this definition are

$$\mathrm{tr}_q(F(C \circ C')) = \mathrm{tr}_q(F(C' \circ C)), \tag{4.15}$$

where C' is also a c-graph $w \to w$, and

$$F(\hat{C}) = \mathrm{tr}_q F(C). \tag{4.16}$$

The proof of (4.16) uses axiom (R4) of ribbon quasi-Hopf algebras, (2.16) and (2.18). Consider first the case $C: V_1 \square V_2 \to V_1 \square V_2$. Let $\Lambda = (\pi_1 \otimes \pi_2 \otimes \pi_2^* \otimes \pi_1^*)(\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\phi)(\phi^{-1} \otimes 1)$. Then

$$\begin{aligned} F(\hat{C}) &= F(U^L_{V_1})(\mathrm{id} \otimes F(U^L_{V_2}) \otimes \mathrm{id})\Lambda^{-1}(F(C) \otimes \mathrm{id}_{V_2^*} \otimes \mathrm{id}_{V_1^*}) \\ &\quad \times \Lambda(\mathrm{id} \otimes F(D^R_{V_2}) \otimes \mathrm{id}) F(D^R_{V_1}) \\ &= \sum_{i,j} \mathrm{tr}_{V_1 \otimes V_2}[(S(\alpha D_i N_j) uv^{-1} A_i \otimes S(\alpha C_i M_j) uv^{-1} B_i) F(C)(K_j \beta \otimes L_j \beta)] \\ &= \mathrm{tr}_{V_1 \otimes V_2}[F(C)(\delta(S \otimes S)(\gamma_{21})(uv^{-1} \otimes uv^{-1})] \\ &= \mathrm{tr}_{V_1 \otimes V_2}[F(C)\Delta(\beta) f^{-1}(S \otimes S)(f_{21}\Delta'(\alpha))(uv^{-1} \otimes uv^{-1})] \\ &= \mathrm{tr}_{V_1 \otimes V_2}[F(C)\Delta(\beta S(\alpha)) f^{-1}(S \otimes S)(f_{21})(uv^{-1} \otimes uv^{-1})] \\ &= \mathrm{tr}_q F(C). \end{aligned} \tag{4.17}$$

The general case follows by induction.

Finally, we define q-dimensions by:

$$\dim_q(V) = \mathrm{tr}_q(\mathrm{id}_V) = \mathrm{tr}_V(\pi(\beta S(\alpha) uv^{-1})). \tag{4.18}$$

Applying (4.16) to the identity graph shows that q-dimensions are multiplicative,

$$\dim_q(V_1 \otimes V_2) = \dim_q(V_1) \cdot \dim_q(V_2). \tag{4.19}$$

Remark. Provided α is invertible, it is possible to give an alternative formulation of (R4), which perhaps will be more appealing to the reader, as it takes exactly the same form as the corresponding axiom for ribbon Hopf algebras. It is based on a computation of $\Delta(u)$: from (3.9), (2.16), and (2.19) one derives:

$$\Delta(u) = f^{-1}(S \otimes S)(\gamma_{21}^{-1} f_{21}) \sum_p (S \otimes S)(\Delta'(b_p))\gamma\Delta(a_p). \tag{4.20}$$

Using the properties of the functor F one can reexpress this as:

$$\Delta(u) = f^{-1}(S \otimes S) f_{21}(u \otimes u)(R_{21}R_{12})^{-1}. \tag{4.21}$$

But since one can also show that

$$(S\otimes S)R=f_{21}Rf^{-1}, \tag{4.22}$$

which implies

$$(S\otimes S)(R_{12}R_{21})=fR_{21}R_{12}f^{-1}, \tag{4.23}$$

the expression for $\Delta(S(u))=f^{-1}(S\otimes S)\Delta'(u)f$ becomes:

$$\Delta(S(u))=(R_{21}R_{12})^{-1}(S(u)\otimes S(u))(S\otimes S)f_{21}^{-1}f. \tag{4.24}$$

This leads to

$$\Delta(S(u)u)=(S(u)u\otimes S(u)u)(R_{21}R_{12})^{-2}, \tag{4.25}$$

in agreement with (R1) and

$$\Delta(v)=(v\otimes v)(R_{21}R_{12})^{-1}. \tag{4.26}$$

This condition is the axiom of ribbon Hopf algebras, which has the same graphical interpretation in the quasi-Hopf case. In other words (4.26) is equivalent to (R4), provided α is invertible.

4.5. *Representations of the Braid Group*

Any representation (π, V) of a quasitriangular quasi-Hopf algebra leads to a representation of the braid group B_n of n strands [26]. The images of the generators b_i, $i=1, \ldots, n-1$ are the following endomorphisms of $(((V\otimes V)\otimes V)\otimes\ldots)\otimes V=V_L^{\otimes n}$ (all left parentheses at the beginning):

$$b_1=\check{R}_{12}, \tag{4.27}$$

$$b_i=\psi_i^{-1}\check{R}_{i,i+1}\psi_i, \quad i>1, \tag{4.28}$$

where

$$\psi_i=\pi^{\otimes n}(\Delta_L^{i-2}(\phi)\otimes 1^{\otimes n-i-1}). \tag{4.29}$$

Here $\check{R}_{i,i+1}$ acts on the i^{th} and $i+1^{\text{th}}$ spaces parenthesed together, Δ_L is defined for any $n\geqq 1$ by

$$\Delta_L(a_1\otimes\ldots\otimes a_n)=\Delta(a_1)\otimes a_2\otimes\ldots\otimes a_n, \tag{4.30}$$

and the notation Δ_L^k stands for $\Delta_L\circ\Delta_L\ldots\Delta_L$ (k times) for $k\geqq 1$, $\Delta_L^0=\text{id}$. For instance, in the case of B_5, $\check{R}_{34}$ is a morphism of $((V\otimes V)\otimes(V\otimes V))\otimes V$ and

$$b_3=\pi^{\otimes 5}((\Delta\otimes\text{id}\otimes\text{id})(\phi^{-1})\otimes 1)\check{R}_{34}\pi^{\otimes 5}((\Delta\otimes\text{id}\otimes\text{id})(\phi)\otimes 1). \tag{4.31}$$

The braid group defining relations:

$$b_ib_j=b_jb_i \quad \text{for} \quad |i-j|\geqq 2, \tag{4.32}$$

$$b_ib_{i+1}b_i=b_{i+1}b_ib_{i+1} \tag{4.33}$$

both come from conservation of isotopy by the functor F, (4.33) being a graphical representation of the quasi-Yang-Baxter equation (2.22). We would like to stress that this result is less obvious than a naive look would suggest, because of the insertions of $\Delta_L^k(\phi)$ operators which ensure the possibility of gluing together the generators contained in a word of the braid group. In other words the properties of F imply identities such as:

$$\Delta_L^{i-1}(\phi)(\Delta_L^{i-2}(\phi^{-1})\otimes 1)=(\mathrm{id}^{\otimes i}\otimes\Delta)\Delta_L^{i-2}(\phi^{-1})(1^{\otimes i-1}\otimes\phi)\Delta_L^{i-2}(\mathrm{id}\otimes\Delta\otimes\mathrm{id})(\phi),$$

$$\Delta_L^{i-2}(\phi)\Delta_L^{j-2}(\phi^{-1})=(\mathrm{id}^{\otimes i-1}\otimes\Delta\otimes\mathrm{id}^{\otimes n-i-1})\Delta_L^{j-3}(\phi^{-1})\Delta_L^{i-2}(\phi),$$

which are consequences of the pentagonal identity, and can be proven directly, although they result from Mac Lane's coherence theorem.

This representation of the braid group depends on the choice of parentheses made in $V_L^{\otimes n}$. However, other choices for tensoring V with itself n times lead to equivalent representations. The above choice allows an easy embedding of B_n into B_{n+1} when adding a strand to the right.

Let us now restrict our attention to the case where (π, V) is an irreducible representation with $\dim_q V \neq 0$. Set

$$\mathscr{T}_n(g)=(\dim_q V)^{-n}\,\mathrm{tr}_{qV_L^{\otimes n}}(g)\,, \tag{4.34}$$

where $g\in B_n$. Due to (4.15), (4.12), and (4.13), $\mathscr{T}_n$ is a Markov trace:

$$\mathscr{T}_n(g_1g_2)=\mathscr{T}_n(g_2g_1)\,, \tag{4.35}$$

$$\mathscr{T}_{n+1}(gb_n^{\pm 1})=\tau_V^{\pm}\mathscr{T}_n(g)\,, \tag{4.36}$$

where $\tau_V^{\pm}=\pi(v^{\mp 1})/\dim_q V$. This trace extends to B_∞, for

$$\mathscr{T}_n(g)=\mathscr{T}_m(g) \quad \text{if} \quad m>n\,, \quad g\in B_n\subset B_m\,. \tag{4.37}$$

From $\mathscr{T}_n$ one can build ambient isotopy invariants of links [30, 31].

5. The Algebra $D^\omega(G)$

In this section, we recall the definition of the quasitriangular quasi-Hopf algebra $D^\omega(G)$ [22, 23]. Then we show that $D^\omega(G)$ is a ribbon quasi-Hopf algebra, and finally, we study the invariants of links all of whose components are coloured by the regular representation of $D^\omega(G)$, showing that they are in fact invariants of the 3-manifolds obtained by surgery on S^3 along those links.

The algebra $D^\omega(G)$ is a quasi-Hopf deformation of $D(G)$, the double of the algebra $\mathscr{F}(G)$ of functions on a finite group G. Its definition involves a 3-cocycle $\omega: G\times G\times G\to U(1)$, which is a normalized cochain, i.e. $\omega(x,y,z)=1$ whenever one (or more) of the three arguments x, y, z is (are) equal to the unit element of G. Recall that by definition, a 3-cocycle ω satisfies:

$$\omega(g,x,y)\omega(gx,y,z)^{-1}\omega(g,xy,z)\omega(g,x,yz)^{-1}\omega(x,y,z)=1\,, \tag{5.1}$$

for any $g,x,y,z\in G$. As a vector space, $D^\omega(G)=\mathscr{F}(G)\otimes\mathbb{C}[G]$, where $\mathbb{C}[G]$ is the group algebra. Its structure will be given in terms of its basis $\underset{h}{{}^{g}\llcorner}=\delta_g\otimes h$, $g,h\in G$. Here $\delta_g(x)=\delta_{g,x}$. To avoid confusion we denote by e the unit element in G, and by $1=\sum_{g\in G}\delta_g$ the unit of $\mathscr{F}(G)$. Sometimes we will use the notation $\underset{g}{{}^{1}\llcorner}=1\otimes g$. The algebra and coalgebra structures in $D^\omega(G)$ are as follows:

$$\underset{x}{{}^{g}\llcorner}\cdot\underset{y}{{}^{h}\llcorner}=\delta_{g,xhx^{-1}}\theta_g(x,y)\,\underset{xy}{{}^{g}\llcorner}\,, \tag{5.2}$$

$$\Delta\left(\underset{h}{{}^{g}\llcorner}\right)=\sum_{xy=g}\gamma_h(x,y)\,\underset{h}{{}^{x}\llcorner}\otimes\underset{h}{{}^{y}\llcorner}\,, \tag{5.3}$$

where $\theta_g(x,y)$ and $\gamma_h(x,y)$ are given by:

$$\theta_g(x,y)=\omega(g,x,y)\omega(x,y,(xy)^{-1}gxy)\omega(x,x^{-1}gx,y)^{-1}, \tag{5.4}$$

$$\gamma_x(g,h)=\omega(g,h,x)\omega(x,x^{-1}gx,x^{-1}hx)\omega(g,x,x^{-1}hx)^{-1}, \tag{5.5}$$

and therefore, $\theta_g(x,y)$ and $\gamma_g(x,y)$ are also equal to one, as soon as one of g,x,y is equal to e. The unit element is ${}^{1}\underset{e}{\llcorner}$. The elements ϕ and R are as follows:

$$\phi=\sum_{g,h,k\in G}\omega(g,h,k)^{-1}\,{}^{g}\underset{e}{\llcorner}\otimes{}^{h}\underset{e}{\llcorner}\otimes{}^{k}\underset{e}{\llcorner}, \tag{5.6}$$

$$R=\sum_{g\in G}{}^{g}\underset{e}{\llcorner}\otimes{}^{1}\underset{g}{\llcorner}. \tag{5.7}$$

The pentagon identity for ϕ is equivalent to the 3-cocycle relation (5.1), and the relations (5.4), (5.5) are equivalent to the quasitriangularity of R, Eqs. (2.20) and (2.21). Using the 3-cocycle relation (5.1), one can check the identities:

$$\theta_g(x,y)\theta_g(xy,z)=\theta_g(x,yz)\theta_{x^{-1}gx}(y,z), \tag{5.8}$$

$$\gamma_x(g,h)\gamma_x(gh,k)\omega(x^{-1}gx,x^{-1}hx,x^{-1}kx)=\gamma_x(h,k)\gamma_x(g,hk)\omega(g,h,k), \tag{5.9}$$

$$\theta_g(x,y)\theta_h(x,y)\gamma_x(g,h)\gamma_y(x^{-1}gx,x^{-1}hx)=\theta_{gh}(x,y)\gamma_{xy}(g,h). \tag{5.10}$$

These relations imply respectively that multiplication is associative, comultiplication is quasicoassociative, and that the coproduct is a morphism of algebras. The counit and the antipode are defined by:

$$\varepsilon\left({}^{g}\underset{h}{\llcorner}\right)=\delta_{g,e}, \tag{5.11}$$

$$S\left({}^{g}\underset{x}{\llcorner}\right)=\theta_{g^{-1}}(x,x^{-1})^{-1}\gamma_x(g,g^{-1})^{-1}\,{}^{x^{-1}g^{-1}x}\underset{x^{-1}}{\llcorner} \tag{5.12}$$

and α,β by:

$$\alpha=1,\quad \beta=\sum_{g\in G}\omega_g\,{}^{g}\underset{e}{\llcorner}, \tag{5.13}$$

where we have set

$$\omega_g=\omega(g,g^{-1},g). \tag{5.14}$$

Note that β is invertible, $\beta^{-1}=\sum_{g\in G}\omega_g^{-1}\,{}^{g}\underset{e}{\llcorner}=S(\beta)$, and also that (5.1) implies:

$$\omega_{g^{-1}}=\omega_g^{-1}. \tag{5.15}$$

From (5.4) and (5.5) one finds:

$$\theta_g(g,g^{-1})=\gamma_g(g^{-1},g)=\theta_g(g^{-1},g)=\gamma_g(g,g^{-1})=\omega_g. \tag{5.16}$$

Now we claim that for any $a\in D^\omega(G)$,

$$S^2(a)=\beta^{-1}a\beta. \tag{5.17}$$

To prove this, one computes explicitly the action of S^2 on the basis ${}^{g}\underset{x}{\llcorner}$ using (5.8), (5.9), and (5.10). An immediate corollary of (5.17) is that $v\in D^\omega(G)$ defined by

$$v=\beta u, \tag{5.18}$$

is central. We now show that v defines a ribbon structure on $D^\omega(G)$. Remark that (5.18) implies that $\mathrm{tr}_q(.)=\mathrm{tr}(.)$ and $\dim_q(.)=\dim(.)\neq 0$. The proof of (R1), (R2), and (R3) consists only of direct computations, and we omit the details. The reader will check that:

$$u=\sum_{g\in G}\omega_g^{-2}\,{}^{g}\llcorner_{g^{-1}}\,, \tag{5.19}$$

$$S(u)=\sum_{g\in G}{}^{g}\llcorner_{g^{-1}}\,, \tag{5.20}$$

$$v=\sum_{g\in G}\omega_g^{-1}\,{}^{g}\llcorner_{g^{-1}}\,, \tag{5.21}$$

from which the equalities $v^2=uS(u)$, $S(v)=v$, and $\varepsilon(v)=1$ follow. It is also easy to compute explicitly:

$$f=\gamma=\sum_{g,h}\omega(g^{-1},g,h)\omega(h^{-1},g^{-1},gh)^{-1}\,{}^{g}\llcorner_{e}\otimes{}^{h}\llcorner_{e}\,, \tag{5.22}$$

$$\delta=\sum_{g,h}\omega_g\omega_h\omega(g,h,h^{-1}g^{-1})\omega(h,h^{-1},g^{-1})^{-1}\,{}^{g}\llcorner_{e}\otimes{}^{h}\llcorner_{e}\,, \tag{5.23}$$

thus (R4) is equivalent to the following identity:

$$\omega_x\omega_y\omega_{xy}^{-1}=\omega(xy,y^{-1},x^{-1})\omega(y^{-1},x^{-1},x)\omega(y^{-1}x^{-1},x,y)^{-1}\omega(x,y,y^{-1})^{-1} \tag{5.24}$$

which is implied by the 3-cocycle relation (5.1).

Remarks. 1. The algebra $D^\omega(G)$ is semisimple, i.e. all representations are completely reducible. The proof of this is parallel to the standard proof that $\mathbb{C}[G]$ is semisimple [27]: let p be a projector on an invariant subspace, and consider

$$p_0=|G|^{-1}\sum_{g,x\in G}\gamma_g(x,x^{-1})S\left({}^{x}\llcorner_{g}\right)p\,{}^{x^{-1}}\llcorner_{g}\,. \tag{5.25}$$

Here $|G|$ is the order of G. Then p_0 is a projector and an intertwiner. Hence the complementary subspace $\mathrm{Ker}\,p_0$ is invariant.

2. The ribbon invariants of closed c-graphs depend only on the cohomology class of ω in $H^3(G,U(1))$. Recall that ω' is equivalent to ω if they differ by a coboundary $\delta\eta$, where $\eta:G\times G\to U(1)$ is a normalized cochain, and

$$\delta\eta(x,y,z)=\eta(y,z)\eta(xy,z)^{-1}\eta(x,yz)\eta(x,y)^{-1}\,. \tag{5.26}$$

Now the element f_η defines a twist of $D^\omega(G)$, where

$$f_\eta=\sum_{g,h\in G}\eta(g,h)\,{}^{g}\llcorner_{e}\otimes{}^{h}\llcorner_{e}\,. \tag{5.27}$$

The twisted algebra is isomorphic to $D^{\omega\delta\eta}(G)$. Since twists preserve equivalence classes of representations, our claim on closed c-graph follows, because their invariants are traces on representations.

In the sequel we shall consider the invariants of c-graphs all of whose ribbons are coloured by the (left) regular representation of $D^\omega(G)$. Let us call those graphs *regular* c-graphs. Recall that the regular representation is the representation on the space $D^\omega(G)$, where the algebra acts by left multiplication. We will show that invariants of closed regular c-graphs are in fact invariants of the 3-manifolds which they define by surgery, and conjecture that these 3-manifolds invariants are equal, up to a normalization factor, to the partition functions of Dijkgraaf and Witten [25]. We will give a number of arguments supporting this conjecture.

As a preliminary step, we give the values of F on the elementary regular c-graphs. We find

$$\check{R}\left({}^{g_1}\underset{x_1}{\llcorner} \otimes {}^{g_2}\underset{x_2}{\llcorner}\right)=\theta_{g_1g_2g_1^{-1}}(g_1,x_2)\,{}^{g_1g_2g_1^{-1}}\underset{g_1x_2}{\llcorner} \otimes {}^{g_1}\underset{x_1}{\llcorner}\,, \tag{5.28}$$

$$\check{R}^{-1}\left({}^{g_1}\underset{x_1}{\llcorner} \otimes {}^{g_2}\underset{x_2}{\llcorner}\right)=\theta_{g_2^{-1}g_1g_2}(g_2^{-1},x_1)\theta_{g_1}(g_2,g_2^{-1})^{-1}\,{}^{g_2}\underset{x_2}{\llcorner} \otimes {}^{g_2^{-1}g_1g_2}\underset{g_2^{-1}x_1}{\llcorner}\,, \tag{5.29}$$

$$F(\Phi)\left({}^{g_1}\underset{x_1}{\llcorner} \otimes {}^{g_2}\underset{x_2}{\llcorner} \otimes {}^{g_3}\underset{x_3}{\llcorner}\right)=\omega(g_1,g_2,g_3)^{-1}\,{}^{g_1}\underset{x_1}{\llcorner} \otimes {}^{g_2}\underset{x_2}{\llcorner} \otimes {}^{g_3}\underset{x_3}{\llcorner}\,, \tag{5.30}$$

$$v\cdot {}^{g}\underset{x}{\llcorner}=\omega(g,g^{-1}x,x^{-1}gx)^{-1}\,{}^{g}\underset{g^{-1}x}{\llcorner}\,, \tag{5.31}$$

$$v^{-1}\cdot {}^{g}\underset{x}{\llcorner}=\omega(g,x,x^{-1}gx)\,{}^{g}\underset{gx}{\llcorner}\,. \tag{5.32}$$

Let $\{\psi_{g,x}\}$ be the dual basis of $\left\{{}^{g}\underset{x}{\llcorner}\right\}$. Then (see Fig. 5):

$$F(U^L_{\text{reg}})\left({}^{g_1}\underset{x_1}{\llcorner} \otimes \psi_{g_2,x_2}\right)=\omega_{g_1}^{-1}\delta_{g_1,g_2}\delta_{x_1,x_2}\,, \tag{5.33}$$

$$F(U^R_{\text{reg}})\left(\psi_{g_1,x_1}\otimes {}^{g_2}\underset{x_2}{\llcorner}\right)=\delta_{g_1,g_2}\delta_{x_1,x_2}\,, \tag{5.34}$$

$$F(D^L_{\text{reg}})(1)=\sum_{g,x}\psi_{g,x}\otimes {}^{g}\underset{x}{\llcorner}\,, \tag{5.35}$$

$$F(D^R_{\text{reg}})(1)=\sum_{g,x}\omega_g\,{}^{g}\underset{x}{\llcorner}\otimes\psi_{g,x}\,. \tag{5.36}$$

To define 3-manifolds invariants we need first to recall the definition of surgery on a link in S^3 [32]. We consider *framed links* (L,f), where $L=L_1\cup L_2\cup\ldots\cup L_n$ is an oriented link in S^3 and $f=(f_1,\ldots,f_n)$ are integers. One can think of (L,f) as being a ribbon graph with an annulus C_i corresponding to each L_i such that the linking number $lk(\partial C_i^+,\partial C_i^-)$ of its two boundary components $\partial C_i^{\pm}$ is equal to f_i. Or one can draw a planar projection of L_i and compute its *writhe* [30]:

$$\sum_{\text{self-crossings } c} w(c)\,, \tag{5.37}$$

where $w(c)$ is defined by the rule: $w(X^{\pm})=\pm 1$, the symbols $X^{\pm}$ being the two crossings of Fig. 5. This quantity is independent of the direction of L_i. By inserting the appropriate number of loops $L^{\pm}$ (Fig. 10) we then adjust the writhe so that it coincides with f_i.

Now we obtain a manifold $M_{L,f}$ from surgery on S^3 as follows: we remove from S^3 a tubular neighbourhood U_i of each L_i. Let μ_i be a meridian on ∂U_i, i.e. a loop which is contractible in U_i, with $lk(\mu_i,L_i)=+1$, and let λ_i be a longitude, i.e. a loop on ∂U_i, which is homologically trivial in S^3-U_i with $lk(\lambda_i,L_i)=0$. Consider a diffeomorphism h of $\bigcup_i \partial U_i$ such that μ_i is mapped to $J_i=\lambda_i+f_i\mu_i$ for each i. Glue U_i with S^3-U_i using h, identifying μ_i on ∂U_i with J_i on $\partial(S^3-U_i)$.

The data (L,f) is called a surgery presentation of the manifold M when M is diffeomorphic to $M_{L,f}$. In fact, every compact 3-manifold M is diffeomorphic to some $M_{L,f}$, in general, there are even many distinct surgery presentations of a given manifold (see below). We claim that

$$\mathscr{F}(M_{L,f})=|G|^{-n}F(C_{L,f})\,, \tag{5.38}$$

where n is the number of components of L, and $C_{L,f}$ is the regular c-graph determined by (L,f), is an invariant of the 3-manifold $M_{L,f}$, i.e. it is independent of the surgery presentation (L,f). To prove this one can appeal to a theorem of Kirby, Fenn, and Rourke [33], which says that $M_{L,f}$ is diffeomorphic to $M_{L',f'}$ if and only if (L,f) and (L',f') are related by a finite sequence of "Kirby moves" (see also Rolfsen [32]). Kirby moves are shown on Fig. 12. The most general move is Fig. 12c, where a part of a framed link, containing p vertical lines intersecting transversally a two-dimensional disc bounded by a circle with framing ± 1, is replaced by p parallel lines forming a composite loop as indicated, or equivalently one performs a full twist on the p lines and the framing of each line changes by ∓ 1. The circle on the left disappears completely, so the number of components of the original link decreases by one. Two important special cases are $p=0$ and $p=1$. When $p=0$ the Kirby move simply consists in removing from the link an unknotted circle Fig. 12a with framing ± 1, which is not linked to the other components. Figure 12b displays the case $p=1$.

It is easy to verify that $\mathscr{F}$ evaluated for the two circles of Fig. 12a is equal to 1, using (5.31), (5.32), and the rules (5.33)–(5.36). This means that $\mathscr{F}(S^3)=1$, as surgery on a circle with framing ± 1 gives back S^3. Notice that this defines also our normalization of $\mathscr{F}$, which is different than the one of [25], where they choose

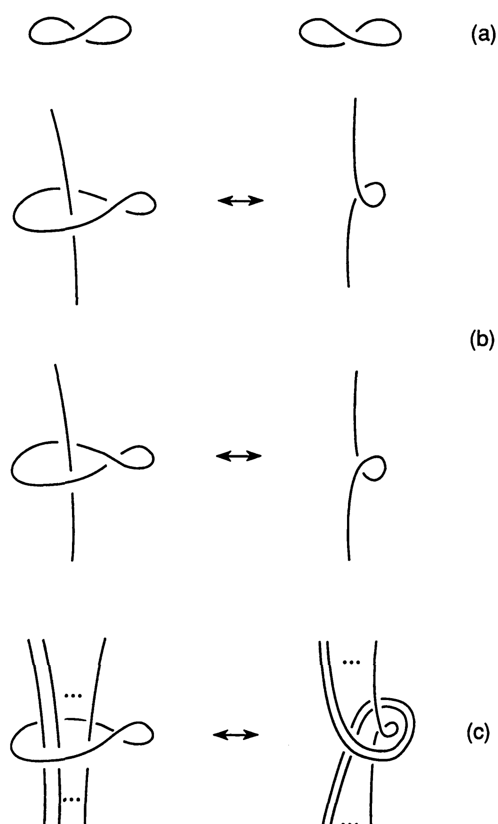

Fig. 12. a Unknotted, unlinked circles with framing ± 1 may be deleted. **b** Example of Kirby move. **c** General Kirby move

instead to normalize the invariant by requiring it to have the value 1 on $S^2 \times S^1$. Our choice, which is the same as in [7], ensures the multiplicativity under connected sums: $\mathscr{F}(M_1 \# M_2) = \mathscr{F}(M_1)\mathscr{F}(M_2)$.

For the proof of invariance of $\mathscr{F}$ under a general Kirby move, we will need the value of $F(C)$ for the c-graph C of Fig. 13, where (π, V) is an arbitrary finite-dimensional representation:

$$F(C)y = \sum_{g,x,h,k} \omega(g^{-1}, g, g^{-1}hg)^{-1}\omega(g^{-1}, h, g)\omega(h, k, g)^{-1}\theta_k(h^{-1}, h)^{-1} \\ \times \pi\left({}^{h}\llcorner_{g}\right) y \otimes r^* \left({}^{k}\llcorner_{h^{-1}}\right) \psi_{g,x} \otimes {}^{g}\llcorner_{x}\,, \tag{5.39}$$

where $y \in V$ and r^* is the dual of the regular representation. The proof that (5.38) is invariant under any Kirby move rests on the following arguments: first we have a very useful graphical interpretation of quasitriangularity, Eqs. (2.20) and (2.21) given by Fig. 14. Of course, we may iterate this identification many times, thereby allowing us to "fuse" an arbitrary number of lines in a crossing, preserving the location of parentheses. Thus the invariant of the regular c-graph on the l.h.s. of Fig. 12c is equal to the invariant of the c-graph on the left of Fig. 12b, but now the line which pierces the disc is coloured by a p-fold tensor product of the regular representation with itself, while the boundary of the disc is coloured by the regular representation. Now for *any* finite-dimensional representation (π, V) colouring the vertical line on the left of Fig. 12b, with the ± 1-framed circle coloured by the regular representation, Eq. (5.39) implies that the value of the corresponding invariant is

$$|G|\pi(v^{\pm 1}). \tag{5.40}$$

Since $\alpha = 1$, we can apply Eq. (4.26) of the remark at the end of Sect. 4.4, whose graphical content is the equality of $F(L_V'^{\mp})$, where V is the p-fold tensor product

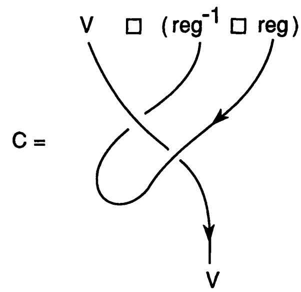

Fig. 13. A c-graph

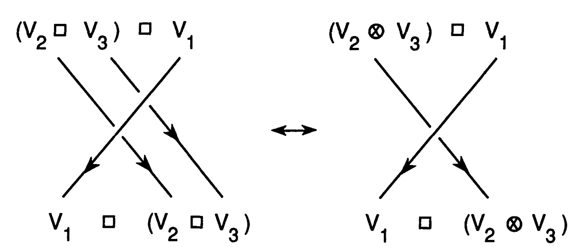

Fig. 14. Quasitriangularity: these two graphs have the same invariants

mentioned before, with the r.h.s. of Fig. 12c. This concludes the proof of the invariance of (5.38).

Note that the regular representation and its dual are equivalent. The reader can check that

$$\psi_{g,x} \mapsto \gamma_x(g^{-1},g)\, {}^{g^{-1}}\llcorner_{x} \tag{5.41}$$

defines an intertwiner. Thus, $\mathscr{F}$ is independent of the directions of the components of the link in the surgery presentation.

Now we state our conjecture, which is that $\mathscr{F}(M)$ is, up to the difference in normalization which we mentioned before, equal to the partition function $Z(M)$ of [25], which has the form:

$$Z(M)=|G|^{-1} \sum_{\varrho \in \mathrm{Hom}(\pi_1 M, G)} \exp(2\pi i \mathscr{A}(\varrho)). \tag{5.42}$$

Here the finite set $\mathrm{Hom}(\pi_1 M, G)$ plays the role of the set of gauge field configurations sectors in this topological "Chern-Simons theory with finite gauge group." The reader should consult [25] for the definition of $\mathscr{A}(\varrho)$. Their paper also contains a combinatorial definition of $Z(M)$, a "state model" formulation in the terminology of Kauffman: take a triangulation of the oriented manifold M, and assign an element of G to each edge, such that the product $g_1 g_2 g_3$ of elements corresponding to a triangle with the induced orientation is equal to the identity. Also identify an edge with positive orientation equipped with $g \in G$ to the same edge with negative orientation, equipped with g^{-1}. Such an assignment is called a state ϱ of the model. The partition function $Z(M)$ will be a sum over the states of the Boltzmann weights of these states. The weight $W(\varrho)=\exp(2\pi i \mathscr{A}(\varrho))$ of a state is

$$W(\varrho)= \prod_{t \in T} W_t, \tag{5.43}$$

where T is the set of all tetrahedra in M, and

$$W_t=\omega(g,h,k) \tag{5.44}$$

for the tetrahedron depicted in Fig. 15. The orientation of M is given by fixing the order of enumeration of the vertices for any tetrahedron to be (a,b,c,d) as in this figure.

Thus the value of $Z(M)$ can be computed from a triangulation of M, whereas $\mathscr{F}(M)$ is computed from a surgery presentation. This is why it is not straightforward to show that the two are equal (up to a constant factor). The general form of $\mathscr{F}(M)$ is

$$\mathscr{F}(M_{L,f})=|G|^{-n} \sum_{g_1,\ldots,g_N,x_1,\ldots,x_N \in G} \left(\prod \delta_{\mathrm{relations},\,e}\right)\Omega(g_1,\ldots,g_N,x_1,\ldots,x_N). \tag{5.45}$$

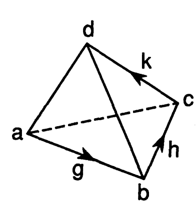

Fig. 15. An oriented tetrahedron with edges labeled by group elements

There is one pair (g_i, x_i) for each minimum in the c-graph $C_{L,f}$ representing (L, f). The relations appearing as δ functions are the image under a homomorphism $\varrho:\pi_1 M \to G$ of a presentation of $\pi_1 M$. Only the g_i, not the x_i, occur in these relations. This comes from the fact that the crossings (5.28), (5.29) in regular c-graphs implement the relations in the Wirtinger presentation of $\pi_1(S^3 - L)$. The additional relations in $\pi_1 M$ resulting from surgery come from the x_i: in fact, in the computation of $F(C_{L,f})$, δ functions appear at each maximum of the graph due to (5.33), (5.34). Relations involving both g_i and x_i are thus produced, from which the x_i, which are only present in the second δ function of (5.33) and (5.34), can be eliminated, at the cost of producing the surgery relations of $\pi_1 M$. The first δ function of (5.33) and (5.34) contributes to the Wirtinger relations. This was noticed independently in [35], where the case of a trivial cocycle ω is discussed. Notice that the phase Ω disappears if the cocycle is trivial, so in this case the preceding argument is the proof that $Z(M)/Z(S^3) = \mathscr{F}(M) = |\mathrm{Hom}(\pi_1 M, G)|$, the number of G-bundles on M (cf. [18] for examples). But when the cocycle ω is non-trivial, the phase Ω is there, coming from the factors θ, γ, ω of the rules for evaluating regular c-graphs. So the precise form of the conjecture is that

$$\Omega(g_1, \ldots, g_N, x_1, \ldots, x_N) = W(\varrho), \tag{5.46}$$

where $\varrho \in \mathrm{Hom}(\pi_1 M, G)$ is defined by the preceding discussion.

In order to check that $\mathscr{F}(M)$ has the correct properties predicted by our conjecture, we have computed its values for the lens spaces $L_{p,1}$ and $L_{pq-1,q} = L_{pq-1,p}$, $p, q \geqq 1$ (see e.g. [32] for the definition and classification of lens spaces). The former is presented by surgery on one unknotted circle with framing p, the latter by surgery on the (framed) Hopf link (two unknotted circles with linking coefficient $+1$) with framings p and q. Here are the results:

$$\mathscr{F}(L_{p,1}) = |G|^{-1} \sum_{g,h} \delta_{g^p,e} \prod_{j=0}^{p-1} \omega(g, g^j h, h^{-1} g h), \tag{5.47}$$

$$\mathscr{F}(L_{pq-1,q}) = |G|^{-2} \sum_{g,h,k} \delta_{g^{pq-1},e} \theta_g(g^{-p}, h) \theta_{g^{-p}}(g, k) \times \prod_{m=1}^{p} \omega(g, g^{-m} h, h^{-1} g h) \prod_{n=0}^{q-1} \omega(g^{-p}, g^{1-np} k, k^{-1} g^{-p} k). \tag{5.48}$$

In general, $\mathscr{F}(M)$ is a complex number. According to [25, 27], $Z(-M) = Z(M)^*$ (complex conjugate), $-M$ being the same manifold with the opposite orientation. Hence $Z(M)$ is real if there exists an orientation-reversing diffeomorphism on M. By the conjecture, $\mathscr{F}(M)$ should have the same properties, and so we checked them for the lens spaces whose invariants are given above; it is easy to show from (5.31) that

$$\mathscr{F}(-L_{p,1}) = |G|^{-1} \sum_{g,h} \delta_{g^p,e} \prod_{j=0}^{p-1} \omega(g, g^j h, h^{-1} g h)^{-1} = \mathscr{F}(L_{p,1})^*. \tag{5.49}$$

It is known that $L_{2,1} = \mathbb{R}P^3 = -\mathbb{R}P^3$. Therefore, (5.47) with $p=2$ should be a real number. A little exercise with the 3-cocycle identity shows that indeed it is real, for any G and ω. Another instructive exercise is to check that the expressions (5.47) and (5.48) are invariant under the substitutions $\omega \mapsto \omega\delta\eta$, see Remark 2 above.

We have also made a direct verification of the conjecture in the case of $L_{p,1}$, by computing $Z(L_{p,1})$ from a triangulation using the state model definition given before. A triangulation of $L_{p,q}$ can be obtained as follows: take p tetrahedra with vertices labeled (a_i, b_i, c_i, d_i) and edges (g_i, h_i, k_i) as in Fig. 15, with $i \in \{1, 2, \ldots, p\}$. First glue together the faces (a_i, b_i, d_i) and $(a_{i+1}, b_{i+1}, c_{i+1})$, then glue together (c_i, d_i, a_i) and $(c_{i+q}, d_{i+q}, b_{i+q})$, for $1 \leqq i \leqq p$, where $p+1$ is identified with 1. This gluing process imposes relations among the group elements (g_i, h_i, k_i) of the edges, leading to a simple expression for $Z(L_{p,q})$ which agrees with (5.47) in the case $q=1$.

For the cyclic group $G = \mathbb{Z}_n$ of order n, there is also an explicit formula [19, 23] for (a representative of) the generator ω of $H^3(\mathbb{Z}_n, U(1))$, which is a cyclic group of order n:

$$\omega(x, y, z) = \exp\left(\frac{2\pi i}{n^2} \bar{z}(\bar{x} + \bar{y} - \overline{x+y})\right), \tag{5.50}$$

where $\bar{x}$ is the representative of x in the set $\{0, 1, \ldots, n-1\}$.

Put $(n, p) = \gcd(n, p)$. It is possible to show that $\mathscr{F}(L_{p,1})$ for $G = \mathbb{Z}_n$ is a Gauss sum:

$$\mathscr{F}(L_{p,1}) = \sum_{g=0}^{(n,p)-1} e^{2i\pi p g^2/(n,p)^2}, \tag{5.51}$$

and one can prove that $Z(L_{p,q})$ is always a Gauss sum for arbitrary q:

$$Z(L_{p,q})/Z(S^3) = \sum_{g=0}^{(n,p)-1} e^{2i\pi p n_{pq} g^2/(n,p)^2}, \tag{5.52}$$

where $n_{pq} \in \{1, \ldots, p-1\}$ is a representative of the multiplicative inverse of q in $\mathbb{Z}_p$. (In the case $G = \mathbb{Z}_2$, $\mathscr{F}(L_{p,1})$ agrees with the expression of $Z(L_{p,1})$ given originally in [25].) The evaluation of these sums is a standard topic in the literature, see e.g. [37]. It would be interesting to study the arithmetic properties of the invariants in general, but for the moment we shall only remark that for any finite group G of order $|G| = N$, and any compact, closed manifold M, $\mathscr{F}(M) \in \mathbb{Q}(q)$, where q is a primitive N^{th} root of unity, since any $\omega \in H^3(G, U(1))$ satisfies $\omega^N = 1$ [38].

Using (5.50) one can compare the invariants of $L_{pq-1,1}$ and $L_{pq-1,p}$ for cyclic groups. (Remember that $\pi_1 L_{p,q} = \mathbb{Z}_p$ for any q.) With the help of a computer program we evaluated the expressions (5.47) and (5.48) in a few cases. We found that $\mathscr{F}(L_{5,1}) = -\mathscr{F}(L_{5,2}) = \sqrt{5}$ for $G = \mathbb{Z}_5$ and $\mathscr{F}(L_{7,1}) = \mathscr{F}(L_{7,2}) = i\sqrt{7}$ for $G = \mathbb{Z}_7$. We plan to present a more detailed computation of the invariants of Lens spaces in a forthcoming publication.

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