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$U(1) \times SU(2)$-Gauge Invariance of Non-Relativistic Quantum Mechanics, and Generalized Hall Effects

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Abstract. We show that the non-relativistic quantum mechanics of particles with spin coupled to an electromagnetic field has a natural $U(1) \times SU(2)$ gauge invariance. Ward identities reflecting this gauge invariance combined with an assumption of incompressibility of a system of such particles in an appropriate external field and for suitable values of the particle density permit us to determine the form of the effective action of the system as a functional of small fluctuations in the electromagnetic field, in the large-distance-, adiabatic limit. In this limit, the action is found to have a universal form. We present explicit results for two-dimensional, incompressible electron fluids and apply them to derive the equations of linear response theory, describing a variety of generalized Hall effects. Sum rules for the Hall conductivities, magnetic susceptibilities and other quantities of physical interest are found.

1. Introduction and Summary of Main Results

In this paper we study the physics of two-dimensional (2d) electronic and magnetic systems, e.g., of heterojunctures or 2d chiral spin liquids. Such systems are described in theories of the quantized Hall effect or of layered superconductors.

A basic recent observation is that the large-scale, low-frequency physics of incompressible electron fluids exhibits *universal features*. Incompressibility is understood as the absence of dissipative processes. Experimentally, it corresponds to a vanishing longitudinal resistance, i.e., $R_L = 0$. For incompressible electron fluids one can identify interesting physical quantities, such as the Hall conductivity or the quantum numbers of excitations above the groundstate, which only depend on the large-scale, low-frequency properties of the system and which can therefore be predicted precisely *without* detailed knowledge of the microscopic dynamics. A related notion of universality is familiar from the theory of critical phenomena accompanying continuous phase transitions. The idea that incompressible quantum fluids exhibit universal large-scale, low-frequency behaviour plays an important role in the analysis of the quantum Hall effect reported in [1–3].

A heterojuncture is an essentially two-dimensional gas of electrons. If a strong, transverse magnetic field is turned on then, for sufficiently small electron density, the spins of the electrons in states of low energy are aligned in the direction of the magnetic field and can therefore be ignored. In this approximation, the electron can be described as a non-relativistic scalar fermion. Using the fact that in $2+1$ dimensions the quantum mechanical electric current can be derived from a vector potential [4, 2], one can show that the large-scale, low-frequency physics of such a system is described by an abelian Chern-Simons gauge theory, the gauge fields being the vector potentials of the electric current [2]. Insisting on the property that, among the physical excitations of the Chern-Simons theory, there be excitations with the quantum numbers of the electron or hole (charge $\mp e$, Fermi statistics) one finds that the set of possible values of σ_H is discrete, that the odd-denominator rule holds, and that the system exhibits, in general, fractionally charged excitations with fractional statistics, (depending on the value of σ_H).

It has become clear that, for certain values of σ_H, *spin effects* $\left(\text{possibly for } \sigma_H = \frac{e^2}{h}\frac{2}{4l+1},\, l = 1,2,\ldots\right)$ or effects of approximate internal symmetries – *"isospin effects"* – $\left(\text{e.g., for } \sigma_H = \frac{e^2}{h}\frac{5}{2}\right)$ may play an important role [5–7, 2, 8]. The present paper is motivated, in part, by a desire to understand the significance of spin- and internal degrees of freedom in the physics of 2d electron fluids. We propose to determine the effective theory describing the large-scale, low-frequency properties of 2d electron fluids, taking into account spin- and internal degrees of freedom. From the effective theory we then derive the basic equations of *linear response theory* and *current sum rules* which describe a variety of well known and less well known effects, including *generalized Hall effects*.

Our derivation of the effective theory in the adiabatic and scaling limit relies on two *basic observations*:

(a) The general observation that systems of non-relativistic electrons have a *local* $U(1) \times SU(2)$ symmetry, i.e., a $U(1) \times SU(2)$-*gauge invariance*.

(b) The observation that if a system of non-relativistic electrons is *incompressible* (i.e., exhibits a positive energy gap above the groundstate energy) then its effective action, as a functional of external electromagnetic fields, is local, and its general form is *computable* in the adiabatic and scaling limit. This is a manifestation of *universality*.

In computing the general form of the effective action, $U(1) \times SU(2)$ *Ward identities* will turn out to play a crucial role, (but the specific form of the microscopic dynamics is unimportant).

In the following, we intend to make observations (a) and (b) more precise and to describe some physical consequences, in particular a *Hall effect for spin currents* and *quantization of magnetic susceptibility*. A detailed discussion of observation (a) can be found in Sect. 2. Our main idea is to treat the dynamics of systems of non-relativistic matter in a geometrical way. The $U(1) \times SU(2)$-gauge invariance of such systems is related to a natural notion of parallel transport, or, equivalently, of covariant differentiation of non-relativistic spinors; (it pays to view such spinors as "sections of a fibre bundle" with $U(1) \times SU(2)$ as structure group). Consider, for example, two-component Pauli spinors, ψ. The covariant derivatives acting on Pauli spinors are given by

$$D_\mu = \partial_\mu + ia_\mu + i\sum_{A=1}^{3} w_{\mu A}\sigma_A\,, \tag{1.1}$$

for $\mu = 0$ (time), 1, 2, (3) (space), where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the usual Pauli matrices. A *geometrically natural* form of an action functional for a system of non-relativistic electrons confined to a region Ω of two- (or three-) dimensional space, in the formalism of second quantization, is given by the following "generally covariant" expression:

$$S_\Omega(\psi^*, \psi; a, w) = \int_{\mathbb{R}\times\Omega} dt d^2\underline{x} \left[i\hbar c \psi^* D_0 \psi + \frac{\hbar^2}{2m} (D_l \psi)^* (D^l \psi) \right] - \int_{\mathbb{R}} dt H_I(t) , \tag{1.2}$$

where ψ^* (the creation operator) is the adjoint of the Pauli spinor ψ (the annihilation operator), and the interaction term $H_I(t)$ is a $U(1) \times SU(2)$-*gauge invariant* functional of ψ^* and ψ. The $U(1) \times SU(2)$-gauge transformations of the gauge fields a and w and of the Pauli spinor ψ are as follows:

$$U(1): a_\mu \mapsto a_\mu + \partial_\mu \chi , \qquad \psi \mapsto e^{-i\chi} \psi , \tag{1.3}$$

where χ is an arbitrary, real-valued function on space-time $\mathbb{R} \times \Omega$, and

$$SU(2): w_\mu \equiv i \sum_{A=1}^{3} w_{\mu A} \sigma_A \mapsto g w_\mu g^{-1} + g \partial_\mu g^{-1} , \qquad \psi \mapsto g\psi , \tag{1.4}$$

where g is an arbitrary $SU(2)$-valued function on $\mathbb{R} \times \Omega$. It follows from Eqs. (1.1), (1.3), and (1.4) that the action S_Ω given in (1.2) is $U(1) \times SU(2)$-gauge invariant.

We must ask whether the equations of motion obtained by varying the action S_Ω with respect to the dynamical fields ψ and ψ^* are related to the Pauli equation for systems of interacting, non-relativistic electrons in an external electromagnetic field, found in standard text books of quantum mechanics [9–11]? This question is answered in Sect. 2. For a certain *natural choice* of $U(1) \times SU(2)$-*gauge*, one finds that the components of the $U(1)$- and $SU(2)$-gauge potentials can be expressed in terms of the electromagnetic vector potentials, A_μ, and the electric and magnetic fields $\vec{E}$ and $\vec{B}$ as follows:

$$a_\mu = \frac{e}{\hbar c} A_\mu , \tag{1.5}$$

$$w_{0A} = -\frac{\mu_e}{2c} B_A , \quad \text{and} \quad w_{lA} = -\frac{\mu_e}{4c} \varepsilon_{lAB} E_B , \tag{1.6}$$

$A, B = 1, 2, 3$, $\vec{B} = (B_1, B_2, B_3)$, $\vec{E} = (E_1, E_2, E_3)$ [see Eqs. (2.23) and (2.24) of Sect. 2)], and $\mu_e \approx -\frac{e}{m_0 c}$ is the magnetic moment of the electron. With identifications (1.5) and (1.6), the equations of motion derived from the action S_Ω given in (1.2) reduce to the usual Pauli equations for ψ and its adjoint ψ^*, including the *Zeeman term* and conventional *spin-orbit couplings*, as well as some additional terms of higher order in $1/m$ that also appear in an expansion of the Dirac equation according to the Foldy-Wouthuysen scheme.

This explains where observation (a) comes from. For more details see Sect. 2.

Observation (b), concerning consequences of *incompressibility* of a system, is substantiated in Sect. 3. To show that a given system is incompressible, e.g. in the sense that connected Green functions of quantum mechanical currents have "good" cluster properties – the form of the incompressibility assumption required in Sect. 3 – is a difficult analytical problem of many-body theory. We shall not solve this problem in the present paper. Rather, we shall elucidate the physical properties of incompressible systems, *assuming* incompressibility and using $U(1) \times SU(2)$-gauge invariance. We shall derive the behaviour of such systems on large distance scales and in the adiabatic limit, for short, in the *scaling limit*, and find that it is *universal*. For that purpose, we develop a *linear response theory*: Assuming that the system under the influence of a certain background magnetic field $\vec{B}_c = (0, 0, B_c)$ is incompressible, we study its response to small fluctuations in the external electromagnetic field, $(\vec{\tilde{E}}, \vec{\tilde{B}})$. In order to find the basic equations of linear response theory, we attempt to determine the general form of the effective action, $S^*_\Omega(a_c, w_c; \tilde{a}, \tilde{w})$, in the scaling limit. Here the potentials a_c, w_c describe the background field $\vec{B}_c$, while $\tilde{a}$ and $\tilde{w}$ describe the fluctuation field $(\vec{\tilde{E}}, \vec{\tilde{B}})$ and are calculated from formulas (1.5) and (1.6), (in a specific choice of gauge). The total gauge potentials are given by $a = a_c + \tilde{a}$, $w = w_c + \tilde{w}$. With the help of observation (a) ($U(1) \times SU(2)$-gauge invariance) and assuming incompressibility, the most general form of $S^*_\Omega(a_c, w_c; \tilde{a}, \tilde{w})$ for a *two-dimensional* electron fluid confined to some domain Ω in the $x - y$ plane is found to be given by the following expression, *independently* of what the specific microscopic dynamics of the system is, (universality!):

$$
\begin{aligned}
-\frac{1}{\hbar} S^*_\Omega(a_c, w_c; \tilde{a}, \tilde{w}) = & \int_{M_3} (*j_c) \wedge \tilde{a} + \int_{M_3} (*m_3) \wedge \tilde{w}_3 \\
& + \frac{\sigma}{4\pi} \int_{M_3} \tilde{a} \wedge d\tilde{a} + \frac{\chi}{2\pi} \int_{M_3} \tilde{a} \wedge d\tilde{w}_3 + \frac{\sigma_s}{4\pi} \int_{M_3} \tilde{w}_3 \wedge d\tilde{w}_3 \\
& + \frac{k}{4\pi} \int_{M_3} \mathrm{tr}\left(w \wedge dw + \frac{2}{3} w \wedge w \wedge w \right) \\
& + \sum_{A=1}^{2} \int_{M_3} \tau_1^{\mu\nu} \tilde{w}_{\mu A} \tilde{w}_{\nu A} d^3\xi + \sum_{A,B=1}^{2} \int_{M_3} \tau_2^{\mu\nu} \varepsilon_{AB} \tilde{w}_{\mu A} \tilde{w}_{\nu B} d^3\xi \\
& + \sum_{A,B,C=1}^{3}{}' \int_{M_3} \eta^{\mu\nu\varrho}_{ABC} \tilde{w}_{\mu A} \tilde{w}_{\nu B} \tilde{w}_{\varrho C} d^3\xi + \text{b.t.}
\end{aligned}
\tag{1.7}
$$

This form holds in an $SU(2)$-gauge where $w_{c,\mu A} = -\delta_{\mu 0}\delta_{A3} \frac{\mu_e}{2c} B_{c,3}$. Each term in (1.7) is explained in great detail in Sect. 3. Here we just note that the coefficients σ, χ, σ_s and k are constants, and the functions j_c^μ, m_3^μ, $\tau_\alpha^{\mu\nu}$ and $\eta^{\mu\nu\varrho}_{ABC}$ are the "scaling limits" of certain current green functions. Depending on the physical situation studied, further restrictions on these functions follow; see our analysis in Sect. 4. Once we have found S^*_Ω, it is a matter of functionally differentiating S^*_Ω with respect to the gauge fields $\tilde{a}$ and $\tilde{w}$, using that

$$
\langle j^\mu(\xi) \rangle_{a,w} = \frac{\delta S^*_\Omega}{\delta \tilde{a}_\mu(\xi)} (a_c, w_c; \tilde{a}, \tilde{w}), \ \ldots, \tag{1.8}
$$

where j^μ is the electric current operator, and ξ a rescaled space-time point, in order to find the linear response equations of an incompressible system in the scaling limit.

Denoting the electric charge density in physical units by ϱ and the electric current in physical units by $\mathscr{J}^i$, $i = 1, 2$, we derive from (1.7) and (1.8),

$$\langle \varrho(\xi) \rangle_{\vec{E},\vec{B}} = \varrho_c(\xi) + \frac{\sigma_H}{c} \tilde{B}_3(\xi) - \ell_0 \underline{\nabla} \cdot \underline{\tilde{E}}(\xi) + \cdots , \tag{1.9}$$

and

$$\begin{aligned} \langle \mathscr{J}^i(\xi) \rangle_{\vec{E},\vec{B}} = {} & \mathscr{J}_c^i(\xi) + \sigma_H \varepsilon^{ij} \tilde{E}_j(\xi) + \ell_0 \delta^{ij} \partial_\tau \tilde{E}_j(\xi) \\ & + c \ell_1 \varepsilon^{ij} \partial_j \tilde{B}_3(\xi) + \cdots , \qquad i = 1, 2 . \end{aligned} \tag{1.10}$$

Here $\varrho_c = \frac{e}{c} j_c^0$ describes the background charge density of the system and $\mathscr{J}_c^i = e j_c^i$ a possible persistent current circulating in the system. Moreover, $\vec{\tilde{E}} = \vec{E}$ is the electric field (we have set $\vec{E}_c = 0$) and $\vec{B} = \vec{B}_c + \vec{\tilde{B}}$ the magnetic field. [For precise definitions see Eqs. (4.8) and (2.13)]. The second terms on the right-hand side of (1.9) and (1.10) describe the *Hall effect* for the electric charge density and current. One finds that

$$\sigma_H = \sigma \frac{e^2}{h} , \tag{1.11}$$

where σ is the coefficient of the third term in S_Ω^*. The remaining terms on the right-hand side describe effects of the *spin degrees of freedom* of electrons and are here discussed systematically for the first time; see Sect. 4, in particular Eqs. (4.14) and (4.16).

Besides the electric current density $\mathscr{J}^\mu$ one can define spin current densities $\mathscr{S}_A^\mu$, $A = 1, 2, 3$, in a natural way; see Eq. (2.14). By differentiating S_Ω^* with respect to the components, $w_{\mu A}$, of the $SU(2)$-gauge potential, using an analogue of (1.8), we discover a *Hall effect* for the *spin current*.

For example, for the expectation value of the 3-component, $\mathscr{S}_3^\mu$, of the spin current density we find the equations

$$\langle \mathscr{S}_3^0(\xi) \rangle_{\vec{E},\vec{B}} = M^0(\xi) + (\sigma_{H1}^{\text{spin}} - \sigma_{H2}^{\text{spin}}) \frac{1}{2c} \underline{\nabla} \cdot \underline{\tilde{E}}(\xi) + \chi_\perp \frac{1}{\mu_e} \tilde{B}_3(\xi) + \cdots , \tag{1.12}$$

and, for $i = 1, 2$,

$$\begin{aligned} \langle \mathscr{S}_3^i(\xi) \rangle_{\vec{E},\vec{B}} = {} & M^i(\xi) + (\sigma_{H1}^{\text{spin}} - \sigma_{H2}^{\text{spin}}) \varepsilon^{ij} \partial_j \tilde{B}_3(\xi) + \sigma_{H1}^{\text{spin}} \varepsilon^{ij} \partial_j B_{c,3}(\xi) \\ & - (\sigma_{H1}^{\text{spin}} - \sigma_{H2}^{\text{spin}}) \delta^{ij} \frac{1}{2c} \partial_\tau \tilde{E}_j(\xi) + \chi_\perp \frac{c}{\mu_e} \varepsilon^{ij} \tilde{E}_j(\xi) + \cdots . \end{aligned} \tag{1.13}$$

Here M^0 is proportional to the *magnetization* of the system when $\vec{\tilde{E}} = \vec{\tilde{B}} = 0$, and M^i is a possible persistent spin current. Furthermore, in terms of the coefficients of the effective action S_Ω^*, one finds that

$$\sigma_{H1}^{\text{spin}} = \frac{k}{2\pi} \frac{\hbar}{2} \mu_e , \qquad \sigma_{H2}^{\text{spin}} = \frac{\sigma_s}{4\pi} \frac{\hbar}{2} \mu_e , \tag{1.14}$$

and

$$\chi_\perp = \frac{\chi}{4\pi} \frac{\mu_e e}{c} . \tag{1.15}$$

We note that $\mathscr{J}^\mu$ and $\mathscr{S}_3^\mu$ are *conserved currents*, while the other components $\mathscr{S}_A^\mu$, $A = 1, 2$, are *not* conserved, for our choice of the background gauge potential $\vec{w}_0 =$

$-\frac{\mu_e}{2c}\vec{B}_c$, $\vec{w}_l = 0$, $\vec{B}_c = (0,0,B_c)$. These components are, however, covariantly conserved. Their expectation values obey equations analogous to (1.12) and (1.13) describing a Hall effect for spin currents (see Sect. 4).

So far, we have not said anything about the possible values of the constants σ_H, $\sigma_{H1}^{\text{spin}}$, $\sigma_{H2}^{\text{spin}}$ and $\chi_\perp$ appearing in the equations above. This is the subject of a forthcoming paper by the authors, where we shall show that all of them belong to certain discrete sets. This "quantization" is a consequence of the following observation: If we consider the transformation properties of S_Ω^* under $U(1) \times SU(2)$-gauge transformations *not* vanishing on the boundary of Ω we realize that, actually, S_Ω^* is gauge-invariant *only* up to boundary terms! Since non-relativistic quantum mechanics is fully $U(1) \times SU(2)$-gauge invariant, violations of $U(1) \times SU(2)$-gauge invariance due to bulk terms of S_Ω^* must be compensated by corresponding violations of gauge invariance due to *anomalous boundary terms* in S_Ω^*. These anomalous boundary terms are uniquely determined by the Chern-Simons bulk terms in S_Ω^*. They correspond to *chiral electric* and *spin currents*, coupled to the gauge potentials $\tilde{a}$ and $\tilde{w}$, respectively, which circulate around the system and are localized near its boundary. Chiral electric currents circulating around the boundary of quantum Hall systems were orignally found by Halperin [12] in a simple quantum mechanical analysis. His observation triggered much of the recent work on bounday excitations in quantum Hall fluids. The derivation of boundary terms in S_Ω^* and of the associated chiral boundary currents is deferred to a separate paper; but see [13, 2, 3, 14]. The study of the algebras of chiral boundary currents and of their representation theory (along with some input from the physics of quantum Hall fluids) will provide us with fairly precise information on the possible, discrete values of the basic constants σ_H, $\sigma_{H1}^{\text{spin}}$, etc.; see also [13, 2, 3]. Put differently, the results in the papers quoted above imply that if σ_H, $\sigma_{H1}^{\text{spin}}$, $\sigma_{H2}^{\text{spin}}$ and $\chi_\perp$ do *not* belong to certain discrete sets, the corresponding quantum Hall fluid *cannot* be incompressible. Some basic ideas concerning these matters are sketched at the end of Sect. 3 and in Sect. 4.

From the form (1.7) of the effective action S_Ω^* one can derive a variety of sum rules for current Green functions. These sum rules enable us to express the coefficients $\sigma, \chi, \sigma_s, \ldots$ of the different terms in S_Ω^* in terms of integrals over connected current Green functions of the system.

While the classical Hall effect for the electric current is standard knowledge, generalized classical Hall effects, e.g. Eq. (1.9), or the Hall effects for spin currents, do not appear to have been discussed in the literature; but see [15]. Surprisingly, one could have predicted them starting from *classical* physics. In order to illustrate this point, let us consider a system of classical point particles with charge q and a magnetic moment $\vec{m}$ which move in the $x-y$ plane under the influence of a time-independent, but *inhomogeneous* magnetic field $\vec{B}(\underline{x})$, $\underline{x} = (x,y) \in \mathbb{R}^2$; but $\vec{E}(\underline{x}) = 0$. The energy, $\mathscr{E}$, of such a particle, located at the point $\underline{x}$, in the field $\vec{B}$ is given by

$$\mathscr{E}(\underline{x}) = -(\vec{m} \cdot \vec{B})(\underline{x}). \tag{1.16}$$

It therefore experiences a force, $\vec{F}$, given by

$$\vec{F}(\underline{x}) = \text{grad}(\vec{m} \cdot \vec{B})(\underline{x}). \tag{1.17}$$

In a stationary state, this force must be balanced by the *Lorentz force*, $\vec{F}_L$, exerted on the particle, moving in the field $\vec{B}(\underline{x})$ with velocity $\vec{v}(\underline{x})$, which is given by

$$\vec{F}_L(\underline{x}) = \frac{q}{c}\,\vec{v}(\underline{x}) \wedge \vec{B}(\underline{x})\,. \tag{1.18}$$

For definiteness, let us choose $\vec{B}(\underline{x}) = (0, 0, B(\underline{x}))$, with a nowhere vanishing $B(\underline{x})$. We assume the standard gyromagnetic relation $\vec{m} = \mu\vec{S}$, where $\vec{S}$ is the spin of the particles. Equating (1.17) to (1.18) we find that

$$\mu S_3 \partial_i B(\underline{x}) = \frac{q}{c}\,\varepsilon_{ij} v^j(\underline{x}) B(\underline{x})\,, \qquad i, j = 1, 2\,. \tag{1.19}$$

If we consider spin-1/2 particles then $S_3 = \pm\frac{\hbar}{2}$. We denote by $\varrho_{\uparrow,\downarrow}(\underline{x})$ the density of particles with $S_3 = \pm\frac{\hbar}{2}$. We also define the current densities in the $x - y$ plane

$$i^l_{\uparrow,\downarrow}(\underline{x}) = \varrho_{\uparrow,\downarrow}(\underline{x}) v^l(\underline{x})\,, \qquad l = 1, 2\,.$$

From (1.19) we then conclude that

$$i^l_{\uparrow,\downarrow}(\underline{x}) = \mp\frac{\mu\hbar c}{2q}\,\frac{\varrho_{\uparrow,\downarrow}(\underline{x})}{B(\underline{x})}\,\varepsilon^{lj}\partial_j B(\underline{x})\,. \tag{1.20}$$

This implies a Hall effect for the total electric current density,

$$\begin{aligned}\mathscr{J}^l(\underline{x}) &\equiv q(i^l_\uparrow + i^l_\downarrow)(\underline{x}) \\ &= -\frac{\mu\hbar c}{2B(\underline{x})}(\varrho_\uparrow - \varrho_\downarrow)(\underline{x})\varepsilon^{lj}\partial_j B(\underline{x})\,.\end{aligned} \tag{1.21}$$

Note that the right-hand side of (1.21) corresponds to the last term in (1.10).

Furthermore, we also find a Hall effect for the 3-component of the total spin current density, namely

$$\begin{aligned}\mathscr{S}^l_3 &\equiv \frac{\hbar}{2}(i^l_\uparrow - i^l_\downarrow)(\underline{x}) \\ &= -\frac{\mu\hbar^2 c}{4qB(\underline{x})}(\varrho_\uparrow + \varrho_\downarrow)(\underline{x})\varepsilon^{lj}\partial_j B(\underline{x})\,.\end{aligned} \tag{1.22}$$

Thus, besides the standard Hall effect for the electric current, as described in Eq. (1.10), quantum mechanics *and* classical physics predict several generalizations of the Hall effect, see Eqs. (1.9), (1.12), (1.13), (1.21), and (1.22), which do not appear to have been described in the literature. Of course, only quantum mechanics enables us to understand the quantization of the coefficients σ, χ, σ_s and k. This topic and extensions of our analysis to other systems, including $(3+1)$-dimensional ones, will be discussed in separate papers.

It would be interesting to test Eqs. (1.9), (1.12), (1.13), ... and the quantization of χ, σ_3 and k (k is predicted to be an *integer*) experimentally. We hope that, before long, such experiments might be possible.

2. Symmetries and Currents in Two-Dimensional Electronic and Magnetic Systems

In this section, we consider 2d electronic and magnetic systems, for example inversion layers or films of ^3He, confined to some surface Ω. We propose to analyze the symmetries and currents of such systems in geometrical terms. For simplicity, we choose Ω to be a connected domain contained in the $x-y$ plane of physical space $\mathbb{R}^3$. A typical example of such a system is a 2d gas of electrons moving in some background. The negative electric charge of the electrons is compensated by the positive charge of ions located on some array of sites in Ω or by a positive background charge density distributed uniformly on Ω, (as in the jellium model). Another example of a physical system we would like to understand is a rotating liquid of neutral atoms or molecules with an electric or magnetic dipole moment pinned to a two-dimensional background and subject to an inhomogeneous electric or magnetic field.

We shall be interested in the properties of such systems at very low temperatures and on large distance and time scales. We shall therefore neglect the dynamics of the background (e.g. of the positive ions, or of the fluctuations of the background charge density).

The basic idea underlying our paper is to analyze such systems by analyzing their symmetry properties and the currents associated to their symmetries. We propose to construct an effective theory of currents, such as the electromagnetic current j, the spin current $\vec{s}$, and possibly further neutral currents, i, sometimes associated with internal symmetries of the system. We attempt to describe the main features of the system by studying the response of the system to coupling those currents to "conjugate" external gauge fields. Clearly, the gauge field to which the electromagnetic current j couples (i.e., the gauge field conjugate to j) is the electromagnetic vector potential A. We must ask what the physical meaning of the gauge potential is to which the spin current $\vec{s}$ couples, more generally of gauge potentials conjugate to further currents i of the system? We start by giving a "geometrial" answer to this question and then provide the physical interpretation.

We describe the electronic degrees of freedom by second-quantized, two-component Pauli spinor fields, ψ and ψ^*. Mathematically, Pauli spinor fields should be viewed as operator-valued sections of a complex vector bundle, Σ, with base space $M_3 = \mathbb{R} \times \Omega$, the $(2+1)$-dimensional space-time of the system, fibre $\mathbb{C}^2$ and structure group $G = U(1) \times SU(2)$. The group $SU(2)$ acts on Pauli spinors in the fundamental representation, the action of $U(1)$ on Pauli spinors is diagonal (phase transformations). The bundle Σ is associated to a principal $U(1) \times SU(2)$-bundle, P, with base space M_3. We shall not dwell on these mathematical notions, since the bundles Σ and P are trivial, i.e.,

$$\Sigma = M_3 \times \mathbb{C}^2 \,, \qquad P = M_3 \times (U(1) \times SU(2)) \,,$$

if we choose Ω to be a connected domain in the $x-y$ plane. (However if Ω were chosen to be the two-dimensional sphere or torus[1], or a more general compact surface, then the bundles Σ and P are non-trivial in general, and the use of a somewhat mathematical jargon would not be a luxury. See e.g. [16, 17] for some background on fibre bundles.)

[1] This choice is frequent in numerical studies of such systems

As announced, we use the language of second quantization. An electron creation operator, $\psi^*(\underset{\sim}{x})$, $\underset{\sim}{x} \in \Omega$, is given by the Pauli spinor

$$\psi^*(\underset{\sim}{x}) = (\psi_\uparrow^*(\underset{\sim}{x}), \psi_\downarrow^*(\underset{\sim}{x})), \tag{2.1}$$

where $\psi_\uparrow^*(\underset{\sim}{x})$ creates an electron at the point $\underset{\sim}{x} = (x^1, x^2) \in \Omega$, whose spin is polarized in the direction of the positive z-axis ("spin up"), and $\psi_\downarrow^*(\underset{\sim}{x})$ creates an electron at $\underset{\sim}{x}$ with spin polarized in the negative z-direction ("spin down"). The electron annihilation operator,

$$\psi(\underset{\sim}{x}) = \begin{pmatrix} \psi_\uparrow(\underset{\sim}{x}) \\ \psi_\downarrow(\underset{\sim}{x}) \end{pmatrix} \tag{2.2}$$

is defined to be the adjoint of $\psi^*(\underset{\sim}{x})$ on the usual Fock space of spin-1/2 fermions. (If the bundle Σ were non-trivial, i.e., for domains Ω with non-trivial topological properties, the definition of creation- and annihilation operators would be more complicated because the introduction of a spin structure on Ω is not as immediate as above; see e.g. [18] for a short summary of notions.) The Fermi statistics of electrons corresponds to canonical anti-commutation relations

$$\{\psi_\alpha^\#(\underset{\sim}{x}), \psi_\beta^\#(\underset{\sim}{y})\} = 0\,, \qquad \{\psi_\alpha(\underset{\sim}{x}), \psi_\beta^*(\underset{\sim}{y})\} = \frac{1}{\sqrt{g(\underset{\sim}{x})}}\, \delta_{\alpha\beta}\delta(\underset{\sim}{x} - \underset{\sim}{y})\,, \tag{2.3}$$

where $\psi^\# = \psi$ or ψ^*, $\alpha, \beta = \uparrow$ or $\downarrow$, $\{A, B\} = AB + BA$, and where $g(\underset{\sim}{x})^{-1/2}\delta(\underset{\sim}{x} - \underset{\sim}{y})$ is the Dirac δ-function on Ω in an arbitrary metric, $g_{kl}(\underset{\sim}{x})$, on Ω, with $g(\underset{\sim}{x})$ the determinant of $g_{kl}(\underset{\sim}{x})$.

In order to formulate dynamical laws for 2d system of electrons, we need to introduce a notion of parallel displacement and covariant differentiation of Pauli spinors. Thus, we must specify some gauge potentials. (In more mathematical jargon, we must equip the principal bundle P with a $U(1) \times SU(2)$-connection.) These gauge potentials, or connections, are given by

$$\omega = (a, w)\,, \tag{2.4}$$

where the electromagnetic vector potential

$$a(x) = \sum_{\mu=0}^{2} a_\mu(x)\, dx^\mu \tag{2.5}$$

is a $U(1)$-gauge potential (i.e., an $\mathbb{R}$-valued 1-form) on space-time M_3. Here $x = (x^0, \underset{\sim}{x})$ is a space-time point, with $x^0 = ct$, where t is time, c is the velocity of light, and $\underset{\sim}{x} \in \Omega$. The components a_μ are real functions on M_3. Furthermore,

$$w(x) = \sum_{\mu=0}^{2} w_\mu(x)\, dx^\mu\,, \tag{2.6}$$

with

$$w_\mu(x) = i\vec{w}_\mu(x) \cdot \vec{\sigma} = i \sum_{A=1}^{3} w_{\mu A}(x)\sigma_A\,, \tag{2.7}$$

is an $SU(2)$-gauge potential (i.e., an $su(2)$-valued 1-form) on M_3. Here $w_{\mu A}(x)$ is a real function on M_3, and $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the three Pauli matrices. In physical units,

$$a_\mu(x) = \frac{e}{\hbar c}\, A_\mu(x)\,, \tag{2.8}$$

and

$$\vec{w}_\mu(x) = \frac{\mu_e}{2c}\, \vec{W}_\mu(x)\,, \tag{2.9}$$

where $-e$ is the electric charge of the electron, $\hbar$ is Planck's constant, $\mu_e \approx -\dfrac{e}{m_0 c}$ is (up to a factor $\hbar/2$) the magnetic moment of the electron, with m_0 the mass of the electron in empty space, A_μ is the usual electromagnetic vector potential, and $\vec{W}_\mu = (W_{\mu 1}, W_{\mu 2}, W_{\mu 3})$ is an $SU(2)$-gauge potential, whose physical meaning is yet to be elucidated. Defining the spin operator, $\vec{S}$, by

$$\vec{S} = \frac{\hbar}{2}\,\vec{\sigma} \tag{2.10}$$

we have that

$$w_\mu(x) = i\vec{w}_\mu(x)\cdot\vec{\sigma} = \frac{i\mu_e}{\hbar c}\, \vec{W}_\mu(x)\cdot\vec{S}\,. \tag{2.7$'$}$$

Covariant differentiation of Pauli spinors is defined by

$$\begin{aligned} D_\mu \equiv D_\mu^\omega &= \nabla_\mu + ia_\mu(x) + w_\mu(x) \\ &= \nabla_\mu + \frac{ie}{\hbar c}\, A_\mu(x) + \frac{i\mu_e}{\hbar c}\, \vec{W}_\mu(x)\cdot\vec{S}\,, \end{aligned} \tag{2.11}$$

where ∇_μ is the (Riemannian) covariant derivative, acting on Pauli spinors as follows:

Let $g_{kl}(x) = g_{kl}(x^0, \underline{x})$ be the metric on Ω at time $t = \dfrac{x^0}{c}$. We assume that $g_{kl}(x^0,\underline{x})$ is positive-definite for all $(x^0,\underline{x}) \in M_3$. By $g^{kl}(x^0,\underline{x})$ we denote the inverse matrix of $g_{kl}(x^0,\underline{x})$, and $g(x^0,\underline{x})$ is the determinant of $g_{kl}(x^0,\underline{x})$. Typically, $g_{kl}(x^0,\underline{x}) = \delta_{kl}$, the standard Euclidian metric, but more general metrics arise in the description of systems with defects (disclinations) and/or off-diagonal disorder. Next, we introduce two vector fields $e^l_A(x)$, $A = 1,2$, with the property that, with $e_{Ak}(x) = g_{kl}(x)e^l_A(x)$,

$$g_{kl}(x) = \sum_{A=1}^{2} e_{Ak}(x)\, e_{Al}(x)\,.$$

(Actually there is a third vector field implicit in our discussion, namely the unit vector field along the positive z-axis, allowing for the definition of "spin up/down.") The action of ∇_μ on Pauli spinors is then defined by

$$\nabla_0 = \frac{\partial}{\partial x^0}\,, \quad \text{and} \quad \nabla_k = \frac{\partial}{\partial x^k} + \gamma_k(x)\,, \qquad k = 1,2\,,$$

with

$$\gamma_k(x) = \sum_{A,B,l=1}^{2} e_{Al}(x)\,(\nabla_k e^l_B)\,(x)\,\frac{1}{4}\,[\sigma_A, \sigma_B]\,. \tag{2.12}$$

Here the action of ∇_k on the vector field $e^l_A(x)$ is the usual covariant differentiation with respect to the metric $g_{kl}(x)$,

$$(\nabla_k e^l_A)\,(x) = \frac{\partial}{\partial x^k}\, e^l_A(x) + \Gamma^l_{kj}(x)\, e^j_A(x)\,,$$

Γ^l_{kj} being the Christoffel symbols of $g_{kl}(x)$. If $g_{kl}(x) = \delta_{kl}$, we have that $\nabla_\mu = \frac{\partial}{\partial x^\mu}$. In general, the derivative ∇_μ allows to incorporate the effect of defects or disorder on the spin degrees of freedom of the electrons in the 2d system. A similar construction (of a non-relativistic spin structure) may be necessary if topologically non-trivial domains Ω are considered.

We may now define the *electromagnetic current* density operator, $j(x)$, by

$$\begin{aligned} j_0(x) &= \psi^*(x)\psi(x)\,, \\ j_k(x) &= -\frac{i\hbar}{2mc}\left[(D_k\psi)^*(x)\psi(x) - \psi^*(x)(D_k\psi)(x)\right], \end{aligned} \tag{2.13}$$

and the *spin current* density operator, $\vec{s}(x)$, by

$$\begin{aligned} \vec{s}_0(x) &= \psi^*(x)\vec{\sigma}\psi(x)\,, \\ \vec{s}_k(x) &= -\frac{i\hbar}{2mc}\left[(D_k\psi)^*(x)\vec{\sigma}\psi(x) - \psi^*(x)\vec{\sigma}(D_k\psi)(x)\right], \end{aligned} \tag{2.14}$$

with $x = (x^0, \underset{\sim}{x}) = (ct, \underset{\sim}{x}) \in M_3$, $k = 1, 2$. (Here we have chosen the currents to be pure densities, omitting factors e, $\frac{\hbar}{2}$ and c from the standard definitions of the physical currents, which will be given and discussed in Sect. 4) In Eqs. (2.13) and (2.14), $\psi^*(x)$, and $\psi(x)$ are the time-dependent electron creation- and annihilation operators in the Heisenberg picture.

In order to define the operators $\psi^*(x)$ and $\psi(x)$ in the Heisenberg picture and to understand the physical meaning of the $SU(2)$-gauge potential $\vec{W}_\mu(x)$, we now must specify the general form of the dynamics of the system. In this section, we use the Hamiltonian formalism. Thus we have to define the Hamiltonian of the system. (In the next section, we shall work in the Lagrangian formalism and introduce path integrals.)

From a geometrical point of view ("general covariance"), one is led to consider the following time-dependent Hamiltonians, $H(t)$, (in second quantized form):

$$H(t) = H_0(A_\mu(x^0,\cdot), \vec{W}_\mu(x^0,\cdot), g_{kl}(x^0,\cdot)) + H_I(t)\,, \tag{2.15}$$

where

$$\begin{aligned} H_0(t) &= H_0(A_\mu(x^0,\cdot), \vec{W}_\mu(x^0,\cdot), g_{kl}(x^0,\cdot)) \\ &= \frac{\hbar^2}{2m}\int_\Omega g^{kl}(x^0,\cdot\underset{\sim}{x})\,(D_k\psi)^*(\underset{\sim}{x})\,(D_l\psi)(\underset{\sim}{x})\sqrt{g(x^0,\underset{\sim}{x})}\,d^2\underset{\sim}{x} \\ &\quad + \int_\Omega \left(e j_0(\underset{\sim}{x})A_0(x^0,\underset{\sim}{x}) + \mu_e\,\frac{\hbar}{2}\,\vec{s}_0(\underset{\sim}{x})\cdot\vec{W}_0(x^0,\underset{\sim}{x})\right)\sqrt{g(x^0,\underset{\sim}{x})}\,d^2x\,, \end{aligned} \tag{2.16}$$

and where m is the effective mass of the electron, and H_I describes electron-electron interactions and the interactions of electrons with the background; H_I does *not* depend on A_μ and $\vec{W}_\mu$ and is gauge-invariant. For example,

$$H_I(t) = \int_\Omega \sqrt{g(x^0, \underline{x})}\, d^2\underline{x}\, \psi^*(\underline{x}) v(x^0, \underline{x}) \psi(\underline{x})$$
$$+ \frac{1}{2} \int_\Omega \sqrt{g(x^0, \underline{x})}\, d^2\underline{x} \int_\Omega \sqrt{g(x^0, \underline{x})}\, d^2\underline{y}\, \psi^*(\underline{x}) \psi^*(\underline{y}) V(\underline{x} - \underline{y}) \psi(\underline{y}) \psi(\underline{x}),$$

where v is a one-body potential, and V is a (possibly screened) Coulomb potential.

The dynamics of the creation- and annihilation operators, ψ^* and ψ, in the Heisenberg picture is given by

$$i\hbar \frac{\partial}{\partial t} \psi^\#(t, \underline{x}) = -\,[H(t), \psi^\#(t, \underline{x})]\,. \tag{2.17}$$

Setting $H(t) = H_0(t) + H_I(t)$ and using (2.16), we find that (2.17) implies the equation

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \psi = {} & \frac{1}{2m} g^{kl} \left(\frac{\hbar}{i} \nabla_k + \frac{e}{c} A_k + \frac{\mu_e}{c} \vec{W}_k \cdot \vec{S} \right) \\ & \times \left(\frac{\hbar}{i} \nabla_l + \frac{e}{c} A_l + \frac{\mu_e}{c} \vec{W}_l \cdot \vec{S} \right) \psi \\ & + \mu_e \vec{W}_0 \cdot \vec{S} \psi + e A_0 \psi - [H_I(t), \psi]\,, \end{aligned} \tag{2.18}$$

and similarly for ψ^*. Here ∇_k is the covariant derivative in the k-direction with respect to the metric g_{kl}, as discussed above; in particular, $\nabla_k = \frac{\partial}{\partial x^k}$, for $g_{kl} = \delta_{kl}$. All the fields in (2.18) are evaluated at $x = (ct, \underline{x})$. We set

$$\pi_k = \frac{\hbar}{i} \nabla_k + \frac{e}{c} A_k \tag{2.19}$$

and expand the product in the first term on the right-hand side of (2.18). Then Eq. (2.18) becomes

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \psi = {} & \frac{1}{2m} g^{kl} \pi_k \pi_l \psi + \frac{\mu_e}{2mc} (\vec{S} \cdot \vec{W}^k) \pi_k \psi \\ & + \frac{\mu_e}{2mc} \pi_k (\vec{S} \cdot \vec{W}^k) \psi + \frac{\mu_e^2}{2mc^2} (\vec{S} \cdot \vec{W}_k)(\vec{S} \cdot \vec{W}^k) \psi \\ & + \mu_e (\vec{S} \cdot \vec{W}_0) \psi + e A_0 \psi - [H_I(t), \psi]\,, \end{aligned} \tag{2.20}$$

where $\vec{W}^k = g^{kl} \vec{W}_l$.

In order to find the physical meaning of the $SU(2)$-gauge potential $\vec{W}_\mu$, we compare Eq. (2.20) with the usual Pauli equation (see e.g. [9–11]), which can be derived from the Dirac equation by expanding in powers of $\frac{1}{m}$, according to the Foldy-Wouthuysen (FW) scheme:

$$i\hbar \frac{\partial}{\partial t} \psi = \frac{1}{2m} g^{kl} \pi_k \pi_l \psi - \mu_e \vec{S} \cdot \vec{B} \psi - e\phi\psi + h_{s-o}\psi - [H_I(t), \psi]\,, \tag{2.21}$$

where the spin-orbit term h_{s-o} is given by

$$\begin{aligned} h_{s-o}\psi &= -\frac{\mu_e}{4mc}\,[\pi_k(\vec{S}\wedge\vec{E})^k + (\vec{S}\wedge\vec{E})^k\pi_k]\,\psi \\ &= \frac{i\hbar\mu_e}{4mc}\left[\frac{1}{\sqrt{g}}\,\partial_k(\sqrt{g}\,g^{kl}\varepsilon_{lAB}S_AE_B\psi) + g^{kl}\varepsilon_{lAB}S_AE_B\partial_k\psi\right] \\ &\quad - \frac{\mu_e e}{2mc^2}\,g^{kl}A_k\varepsilon_{lAB}S_AE_B\psi\,. \end{aligned} \tag{2.22}$$

In (2.21) and (2.22), $\vec{E} = (E_1, E_2, E_3)$ is the external electric field and $\vec{B} = (B_1, B_2, B_3)$ the external magnetic field, ε_{lAB} is the sign of the permutation (lAB) of $(1\,2\,3)$, with $l = 1, 2$ and $A, B = 1, 2, 3$. Furthermore, $\vec{B} = \vec{\nabla}\wedge\vec{A}$, where $\vec{A} = (A_1, A_2, A_3)$ is the three-dimensional electromagnetic vector potential, and by the choice of our units in (2.8), $\underline{A} = (A_1, A_2)$ is the spatial part of the $U(1)$-gauge potential. Finally, ϕ is the external electrostatic potential, and $\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}$, where $\vec{\nabla} = \left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right)$ is the gradient, (while, in (2.18), (2.19), ∇_k denotes covariant differentiation in the k-direction with respect to g_{kl}). We now compare Eq. (2.20) with Eq. (2.21), with h_{s-o} as in (2.22). This suggests to make the following identifications:

$$A_0(x) = -\,\phi(x)\,, \tag{2.23}$$

and

$$\vec{W}_0(x) = -\,\vec{B}(x)\,, \qquad W_{lA}(x) = -\,\frac{1}{2}\,\varepsilon_{lAB}E_B(x)\,. \tag{2.24}$$

The term $\frac{\mu_e^2}{2mc^2}\,(\vec{S}\cdot\vec{W}^k)\,(\vec{S}\cdot\vec{W}_k)\,\psi$ on the right-hand side of Eq. (2.20) is a relativistic correction of order m^{-3} which is missing in Eq. (2.22). Its presence in (2.20) is a direct consequence of $SU(2)$-gauge invariance of the theory. Using the relation

$$S_AS_B = \frac{\hbar^2}{4}\,\delta_{AB} + i\,\frac{\hbar}{2}\,\varepsilon_{ABC}S_C\,,$$

and (2.24), this term can be rewritten as

$$\begin{aligned} &\frac{\hbar^2\mu_e^2}{8mc^2}\int_\Omega g^{kl}W_{kA}W_{lA}\psi^*\psi\,\sqrt{g}\,d^2\underline{x} \\ &= \frac{\hbar^2\mu_e^2}{32mc^2}\int_\Omega g^{kl}\varepsilon_{kAB}\varepsilon_{lAC}E_BE_C\psi^*\psi\,\sqrt{g}\,d^2\underline{x}\,. \end{aligned} \tag{2.25}$$

We note that the "geometric" Eq. (2.20), with the ientifications (2.23) and (2.24), contains *all* terms one finds in a systematic FW-expansion of the Dirac equation up to order m^{-2}, with the exception of the rest energy term of $\mathcal{O}(m)$, and the so-called Darwin term of $\mathcal{O}(m^{-2})$! The term (2.25) can be found among the $\mathcal{O}(m^{-3})$ terms, but only up to a factor 1/2.

It should be emphasized that identifications (2.24) refer to a special choice of $SU(2)$-gauge and are invalidated by changing the gauge; [the gauge potentials $\vec{W}_\mu(x)$ are, of course, *not* $SU(2)$-gauge-invariant]. For example, the effect of an external electromagnetic field, with $\vec{B}(x) = \vec{B}(t)$ independent of (x^1, x^2), and $\vec{E}(x) = 0$, on the *spin degrees of freedom* can be gauged away by an $SU(2)$-gauge transformation ${}^g w = gwg^{-1} + gdg^{-1}$, with $g(x) = T\left[\exp \frac{-i\mu}{\hbar} \int_{t_0}^{t} \vec{B}(\tau)\cdot\vec{S}d\tau\right]$. For $\vec{B}(t) = b(t)\vec{B}_0$, this is related to Larmor's theorem. Similarly, the effects of certain electric fields on the spin degrees of freedom can be gauged away; (e.g., for $\vec{E}(x) = (e(x_2), 0, 0)$). Note that, in the gauge in which (2.24) holds, the $SU(2)$-gauge potentials $\vec{W}_\mu$ are *physically observable* quantities.

Finally, we observe that, as Eq. (2.21) shows, the metric g_{kl} describes disclinations or more general off-diagonal disorder in the system.

In our derivation of Eqs. (2.18), (2.20) from expression (2.15) and (2.16) for the Hamiltonian $H(t)$ we have been careless about *boundary terms* arising from integrations by part on Ω when the boundary $\partial\Omega$ of Ω is non-empty. (For $\Omega = \mathbb{R}^2$, or if $\partial\Omega$ is empty, there are, of course, no boundary terms.) The dynamics of 2d electronic systems near $\partial\Omega$ is very interesting and revealing, and boundary terms play an important role. This will be discussed in [19]; see also [13, 2, 14]. In this paper we focus our attention on *bulk effects* and therefore neglect boundary terms.

In the remainder of this section, we study properties of the electromagnetic- and spin current, j and $\vec{s}$. Multiplying (2.20) by $\psi^*\sqrt{g}$ from the left and the adjoint equation by $\sqrt{g}\psi$ from the right, subtracting one equation from the other one, and rearranging terms, we find that

$$-\frac{1}{\sqrt{g}}\left[\psi^*\sqrt{g}(\partial_t\psi) + (\partial_t\psi^*)\sqrt{g}\psi\right] = \frac{1}{\sqrt{g}}\partial_k(\sqrt{g}\,g^{kl}j_l) \tag{2.26}$$

which yields the continuity equation for the electromgnetic current, provided $g_{kl}(x)$ is time-independent. Defining

$$(\eta_{\mu\nu}(x)) = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & & \\ 0 & \multicolumn{2}{c}{-(g_{kl}(x))} \end{array}\right), \tag{2.27}$$

with $g_{kl}(x)$ independent of $x^0 = ct$, we can write the continuity equation as

$$\nabla_\mu j^\mu = \frac{1}{\sqrt{|\eta|}}\partial_\mu(\sqrt{|\eta|}\,\eta^{\mu\nu}j_\mu) = 0\,. \tag{2.28}$$

It will turn out to be useful to rewrite this equation in terms of differential forms: We define a 1-form j by setting

$$j(x) = \sum_{\mu=0}^{2} j_\mu(x)\,dx^\mu\,, \tag{2.29}$$

and the dual 2-form $\mathscr{J}$ by

$$\mathscr{J}(x) = \sum_{\mu<\nu}\mathscr{J}_{\mu\nu}(x)\,dx^\mu \wedge dx^\nu\,,$$

with

$$\mathscr{J}_{\mu\nu}(x) = |\eta(x)|^{1/2}\varepsilon_{\mu\nu\varrho}\eta^{\varrho\sigma}(x)j_\sigma(x)\,. \tag{2.30}$$

A short-hand rewriting of (2.30) is

$$\mathscr{J} = *j\,, \tag{2.31}$$

where $*$ is the so-called Hodge $*$-operation. Then Eq. (2.28) can be rewritten as

$$d\mathscr{J} = 0\,, \tag{2.32}$$

where d denotes exterior differentiation. Equation (2.32) holds, provided $g_{kl}(x)$, and hence $\eta_{\mu\nu}(x)$, is time-independent. (See e.g. [16] for some basic facts in the theory of differential forms.)

Let us suppose that the external magnetic field $\vec{B}$ has a constant direction, i.e.,

$$\vec{B}(x) = b(x)\vec{v}_0\,, \quad \text{and} \quad E''(x) = (\vec{E}(x)\cdot\vec{B}_0)\,|\vec{B}_0|^{-1} = 0\,. \tag{2.33}$$

Let $s'' = (\vec{s}\cdot\vec{B}_0)\,|\vec{B}_0|^{-1}$ denote the component of the spin current parallel to the magnetic field. An argument similar to the one just sketched for the electromagnnetic current then shows that s''^{μ} is a conserved current, i.e.,

$$\nabla_\mu s''^{\mu} = 0\,. \tag{2.34}$$

If there is no electromagnetic field, at all, then all components of the spin current *in an appropriate gauge* are conserved, i.e.,

$$\nabla_\mu \vec{s}^{\mu} = 0\,, \tag{2.35}$$

or

$$d\vec{\mathscr{S}} = 0\,, \tag{2.36}$$

where $\vec{\mathscr{S}} = *\vec{s}$ is the Hodge dual of $\vec{s}$.

It should be emphasized that the spin current $\vec{s}$ is *not* $SU(2)$-gauge invariant, and the answer to the question whether it has conserved components, s_A, depends on our choice of gauge. This should not come as a surprize: Matter currents coupling to non-abelian gauge fields are not gauge-invariant and, since the gauge field itself carries non-abelian "charge", do not generate conserved charges. The answer to the question whether some components of such currents satisfy a continuity equation, for a specific choice of an external gauge field configuration, depends on the choice of gauge. Generically, such currents do *not* have any conserved components. However, if, in the example of the spin current, we consider an external electromagnetic field configuration whose effect on the spin degrees of freedom can be gauged away by an $SU(2)$-gauge transformation then, *in the gauge where* $\vec{W}_\mu$ *vanishes*, all components of the spin current are conserved, of course. In this case, the components of $\vec{s}$ generate a "non-relativistic" $SU(2)$-current algebra; see e.g. [20].

Let i be some conserved current of the system, and let $\mathscr{I}$ be the 2-form dual to i. (Our main examples are $i = j$, or $i = s''$ if the direction of $\vec{B}(x)$ is constant and $E''(x) = 0$.) Then

$$d\mathscr{I} = 0\,. \tag{2.37}$$

In $2+1$ space-time dimension, Eq. (2.37) can be solved, locally, by introducing a vector potential $\mathscr{A}$ for i, with

$$\mathscr{I} = d\mathscr{A}\,, \quad \text{or} \quad i = *d\mathscr{A}\,. \tag{2.38}$$

If Ω is a connected domain in the $x-y$ plane with a non-empty boundary then (2.38) holds globally. Clearly, $\mathscr{A}$ is determined by i only up to scalar functions, i.e., $\mathscr{A}$

and $\mathscr{A} + d\chi$ correspond to the same current i. Thus, $\mathscr{A}$ is an abelian gauge field. An effective theory of conserved currents in $2+1$ dimensions is, therefore, an *abelian gauge theory*. This remark provides an explanation of why there is such an intimate connection between two-dimensional many-body theory and gauge theory. It is very useful in the theory of the quantized Hall effect; see [2].

The generalization of the main findings of this section to include couplings of some additional "neutral currents" of the 2d system to conjugate external gauge fields, abelian or non-abelian, is straightforward. Such generalizations may be relevant for the analysis of systems with several current-conducting bands. For such systems, one typically introduces several species of creation- and annihilation operators, ψ_a^*, ψ_a, $a = 1, \dots, n$, connected to each other by an approximate *internal symmetry* which may be gauged.

More importantly, as the reader will have noticed, the formalism developed in this section can be applied equally well to systems in three-dimensional physical space as it does to the two-dimensional systems considered above, [21].

3. The Effective Gauge Field Action in the Scaling Limit

The purpose of this section is to study the partition (or generating) function, $\mathscr{Z}(A, \vec{W})$, of a two-dimensional system of electrons or of other kinds of particles with spin 1/2 which carry a magnetic moment, (e.g. ^3He-atoms in a thin film), coupled to an arbitrary electromagnetic vector potential A_μ and an arbitrary $SU(2)$-gauge potential $\vec{W}_\mu$. We are interested in deriving the form of the effective action, $\ln \mathscr{Z}(A, \vec{W})$, at large distance and time scales, (more precisely, in the "scaling limit").

In order to write down an explicit expression for the partition function $\mathscr{Z}(A, \vec{W})$ and exhibit its gauge invariance, it is convenient to work in the Lagrangian formalism and use Feynman path integrals. (Alternatively, one can use the quantum mechanical propagator of the system coupled to A, $\vec{W}$ and define $\mathscr{Z}(A, \vec{W})$ as the expectation value of the propagator in the groundstate of the system computed for the fields A_c and $\vec{W}_c$, where $A_c = \lim_{t\to\pm\infty} A(ct, \underline{x})$, $\vec{W}_c = \lim_{t\to\pm\infty} \vec{W}(ct, \underline{x})$ are fixed; see [2].)

The starting point of the Lagrangian formalism is to note that the Pauli equation (2.18) and its adjoint are the Euler-Lagrange equations corresponding to the action function $S_\Omega(\psi^*, \psi; A, \vec{W})$ given by

$$S_\Omega(\psi^*, \psi; A, \vec{W}) = i\hbar \int_{M_3} \psi^* \frac{\partial \psi}{\partial t} \sqrt{|\eta|}\, d^3x - \int_{\mathbb{R}} dt H(t)\,, \tag{3.1}$$

where $H(t)$ is the Hamiltonian introduced in (2.15), (2.16), and where space-time M_3 is given by $\mathbb{R} \times \Omega$, with a metric $\eta_{\mu\nu}$ as given in (2.27).

More explicitly, S_Ω is given by

$$S_\Omega(\psi^*, \psi; A, \vec{W}) = \int_{M_3} \sqrt{|\eta|}\, d^3x \left[i\hbar\psi^* D_0\psi + \frac{\hbar^2}{2mc} \eta^{kl} (D_k\psi)^* (D_l\psi) \right] - \int_{\mathbb{R}} dt H_I(t)\,, \tag{3.2}$$

where D_μ denotes the covariant derivative defined in (2.11), for $\mu = 0, 1, 2$; ($x^0 = ct$). We observe that, since $H_I(t)$ is assumed to be manifestly gauge-invariant (and independent of A and $\vec{W}$), the action S_Ω is *invariant* under arbitrary $U(1)$- and $SU(2)$-gauge transformations, including time-dependent ones!

Quantization of the system with Feynman path integrals leads to the following formula for the partition function $\mathscr{Z}_\Omega$:

$$\mathscr{Z}_\Omega(A, \vec{W}) = \int \mathscr{D}\psi^* \mathscr{D}\psi \, e^{iS_\Omega(\psi^*,\psi;A,\vec{W})/\hbar}, \tag{3.3}$$

where ψ and ψ^* are interpreted as anti-commuting c-numbers, corresponding to Fermi statistics. (It is necessary to specify suitable "boundary conditions" on $\psi^\#(x^0, \underline{x})$, for $x^0 \to \pm\infty$ and $\underline{x} \to \partial\Omega$. This is discussed in [22], and we shall permit ourselves to be careless about this point, in the present paper – in spite of its importance.)

Passing to the Euclidean region (corresponding to analytic continuation in the time variable from the real to the imaginary axis) and replacing A_0 by iA_0 and $\vec{W}_0$ by $i\vec{W}_0$, one obtains the Euclidean version of Eq. (3.3):

$$\mathscr{Z}_\Omega^E(A, \vec{W}) = \int \mathscr{D}\bar{\psi}\mathscr{D}\psi \, e^{-S_\Omega^E(\bar{\psi},\psi;A,\vec{W})/\hbar}, \tag{3.3'}$$

where the Euclidean action S_Ω^E is defined by a formula similat to (3.2) (obtained by replacing x^0 by ix^0, A_0 by iA_0, etc.). The variables ψ and $\bar{\psi}$ are independent Grassmann variables. In order to describe a system at positive temperature T, one compactifies the imginary-time axis to a circle of circumference $\beta = 1/k_B T$ and imposes anti-periodic boundary conditions, (corresponding to the KMS condition). In this paper, though, we study the *large-scale physics* of systems at *zero temperature*, $T = 0$.

The significance of $\ln \mathscr{Z}_\Omega(A, \vec{W})$ (or $\ln \mathscr{Z}_\Omega^E(A, \vec{W})$) is that it is the generating functional of the *connected time-ordered Green functions* (or Euclidean Green functions) of the electromagnetic- and the spin currents. Let us first work in "mathematical" (rather than physical) units: As in (2.8), (2.9) we set $a_\mu = \frac{e}{\hbar c} A_\mu$, $w_\mu = i\vec{w}_\mu \cdot \vec{\sigma} = \frac{i\mu_e}{\hbar c} \vec{W}_\mu \cdot \vec{S}$, and similarly for the currents. Then, at non-coinciding arguments,

$$\left\langle T \left[\prod_{i=1}^{n} j^{\mu_i}(x_i) \prod_{l=1}^{m} s_{A_l}^{\nu_l}(y_l) \right] \right\rangle_{a,w}^{c} = i^{n+m} \prod_{i=1}^{n} \frac{\delta}{\delta a_{\mu_i}(x_i)} \prod_{l=1}^{m} \frac{\delta}{\delta w_{\nu_l A_l}(y_l)} \ln \mathscr{Z}_\Omega(a, w), \tag{3.4}$$

where $\langle (\cdot) \rangle_{a,w}^c$ denotes the connected "vacuum expectation functional" in an external gauge field configuration, (a, w).

In order for Eq. (3.4) to be meaningful for arbitrary n and m, we need to assume that $\ln \mathscr{Z}_\Omega(a, w)$ is smooth in a and w, at least in the neighbourhood of suitably chosen background gauge fields a_c and w_c. (In this section we shall require four derivatives in a neighbourhood of a_c, w_c and defer a more subtle discussion of the differentiability properties of $\ln \mathscr{Z}_\Omega(a, w)$ to the end of Sect. 4.)

We define the effective gauge field action, S_Ω^{eff}, by

$$S_\Omega^{\text{eff}}(a, w) = \frac{\hbar}{i} \ln \mathscr{Z}_\Omega(a, w). \tag{3.5}$$

It is our aim, in this section, to determine the form of S_Ω^{eff} on large distance and time scales, i.e., to calculate the *scaling limit* of S_Ω^{eff}. For this purpose we define scale transformations of the field variables. We set

$$x = (x^0, \underline{x}) = \lambda(\xi^0, \underline{\xi}) \equiv \lambda\xi\,, \qquad 1 \leq \lambda < \infty\,, \tag{3.6}$$

with $\underline{x} \in \lambda\Omega$, i.e., $\underline{\xi} \in \Omega$. The parameter λ is the scale parameter; the domain $\Omega \subseteq \mathbb{R}^2$ is kept fixed. We assume, for simplicity, that the rescaled metric, $g_{kl}^{(\lambda)}(x) \equiv g_{kl}(\lambda\xi)$, averaged with arbitrary test functions, converges to the Euclidean metric, δ_{kl}. By (3.6),

$$\frac{\partial}{\partial x^\mu} = \lambda^{-1}\,\frac{\partial}{\partial \xi^\mu}\,. \tag{3.7}$$

In order for the covariant derivatives D_μ to have the right behaviour under scale transformations, we *have* to define the scaled gauge potentials as follows:

$$\begin{aligned} a_\mu^{(\lambda)}(x) &= a_{c,\mu}(x) + \tilde{a}_\mu^{(\lambda)}(x) \\ &= \lambda^{-1}[a_{c,\mu}(\xi;\lambda) + \tilde{a}_\mu(\xi)] \end{aligned} \tag{3.8a}$$

and

$$\begin{aligned} w_\mu^{(\lambda)}(x) &= w_{c,\mu}(x) + \tilde{w}_\mu^{(\lambda)}(x) \\ &= \lambda^{-1}[w_{c,\mu}(\xi;\lambda) + \tilde{w}_\mu(\xi)] \end{aligned} \tag{3.8b}$$

where a_c and w_c are external background potentials $\tilde{a}$ and $\tilde{w}$ denote "fluctuation potentials", and $x = \lambda\xi$. More precisely, $\tilde{a}_\mu(\xi)$ and $\tilde{w}_\mu(\xi)$ are taken to be fixed functions on $M_3 = \mathbb{R} \times \Omega$. This means that, considering larger and larger systems, the strength of the fluctuation potentials $\tilde{a}_\mu^{(\lambda)}(x)$ and $\tilde{w}_\mu^{(\lambda)}(x)$ decreases as λ^{-1}, or if we focus on the corresponding rescaled systems on Ω the fluctuation potentials $\tilde{a}_\mu(\xi)$ and $\tilde{w}_\mu(\xi)$ remain fixed. The background potentials, however, are assumed to be fixed in the physical systems on $\lambda\Omega$, i.e., $a_{c,\mu}(x)$ and $w_{c,\mu}(x)$ are kept fixed, for all λ, with $x \in \lambda M_3$. From the point of view of the corresponding rescaled systems on Ω this means that the strength of the background potentials $a_{c,\mu}(\xi;\lambda)$ and $w_{c,\mu}(\xi;\lambda)$ increases linearly in λ. Finally, for example by inspection of the rescaled Pauli equation, we note that, for consistency of the scaling limit, the mass and magnetic moment of the electron must have the following behavior: If we keep the mass, m, and the magnetic moment, μ_e, fixed in physical units then in the rescaled systems on Ω, we must have $m(\lambda) = \lambda m$ and $\mu_e(\lambda) = \dfrac{\mu_e}{\lambda}$.

The scaling limit of the effective gauge-field action is defined by

$$S_\Omega^*(a, w) \equiv \text{“}\lim_{\lambda\to\infty}\text{”}\, S_{\lambda\Omega}^{\text{eff}}(a^{(\lambda)}, w^{(\lambda)})\,. \tag{3.9}$$

Here, and in all subsequent formulas, "$\lim\limits_{\lambda\to\infty}$" is *not* to be understood as an actual limit; rather this symbol indicates that we explore asymptotic behaviour, as λ becomes large, in the form of Laurent series in λ.

Next, we attempt to calculate $S_\Omega^*(a, w)$. For pedagogical reasons it is useful to first consider a situation where $\mu_e = 0$, so that $w \equiv 0$. We examine the Taylor expansion of $S_{\lambda\Omega}^{\text{eff}}(a)$ around a fixed external background potential a_c, setting $a = a_c + \tilde{a}$, where $\tilde{a}$ denotes a small fluctuation vector potential. In general, $S_{\lambda\Omega}^{\text{eff}}(a)$ is a very complicated functional of a, and it is quite impossible to compute S_Ω^* explicitly. The point is that, for *"incompressible" systems*, our task simplifies drastically. Let us suppose that the electromagnetic background potential a_c is such that the corresponding electromagnetic field, da_c, is time-independent. We can then choose a gauge such that the Hamiltonian of the system is time-independent (provided that H_I is time-independent). In this situation, an "incompressible" system is one whose Hamiltonian H has a spectrum with a *positive energy gap* above the ground state energy (or pure point spectrum in a small interval above the groundstate energy). More precisely, we shall assume that in the given background potential a_c, connected Green functions of the electromagnetic current j have *strong cluster properties* (converging to 0 more rapidly that d^{-4} when the distance d between two arguments tends to ∞).

Besides the basic assumption just described and referred to as *"incompressibility of the system"*, in what follows we shall assume that $S^{\text{eff}}_{\lambda\Omega}(a)$ is four times continuously (Fréchet) differentiable in a on some Schwartz-space neighbourhood of a_c. Furthermore, we shall make important use of the $U(1)$*-gauge invariance*, related to the *conservation* of the *electromagnetic current.*

We begin our analysis of $S^*_{\Omega}(a)$ by expanding $S^{\text{eff}}_{\lambda\Omega}(a)$ to third order in $\tilde{a} = a - a_c$, with a forth-order remainder term.

$$
\begin{aligned}
S^{\text{eff}}_{\lambda\Omega}(a) &= S^{\text{eff}}_{\lambda\Omega}(a_c) + \sum_{n=1}^{3} \frac{1}{n!} \int\limits_{(\lambda M_3)^n} \frac{\delta^n S^{\text{eff}}_{\lambda\Omega}}{\delta a_{\mu_1}(x_1)\dots\delta a_{\mu_n}(x_n)}(a_c) \\
&\quad \times \tilde{a}_{\mu_1}(x_1)\dots\tilde{a}_{\mu_n}(x_n) dv(x_1)\dots dv(x_n) \\
&\quad + \frac{1}{4!} \int\limits_{(\lambda M_3)^4} \frac{\delta^4 S^{\text{eff}}_{\lambda\Omega}}{\delta a_{\mu_1}(x_1)\dots\delta a_{\mu_4}(x_4)}(a_c+\alpha) \\
&\quad \times \tilde{a}_{\mu_1}(x_1)\dots\tilde{a}_{\mu_4}(x_4) dv(x_1)\dots dv(x_4) \\
&= S^{\text{eff}}_{\lambda\Omega}(a_c) - i\hbar \sum_{n=1}^{3} \frac{(-i)^n}{n!} \int\limits_{(\lambda M_3)^n} \langle T[j^{\mu_1}(x_1)\dots j^{\mu_n}(x_n)]\rangle^c_{a_c} \\
&\quad \times \tilde{a}_{\mu_1}(x_1)\dots\tilde{a}_{\mu_n}(x_n) dv(x_1)\dots dv(x_n) \\
&\quad - \frac{i\hbar}{4!} \int\limits_{(\lambda M_3)^4} \langle T[j^{\mu_1}(x_1)\dots j^{\mu_4}(x_4)\rangle^c_{a_c+\alpha} \\
&\quad \times \tilde{a}_{\mu_1}(x_1)\dots\tilde{a}_{\mu_4}(x_4) dv(x_1)\dots dv(x_4)\,,
\end{aligned} \tag{3.10}
$$

where $dv(x) = \sqrt{|\eta(x)|}\, d^3x$. The remainder term is evaluated in a background field $a_c + \alpha$, with $\alpha = \theta\tilde{a}$, for some $0 < \theta < 1$.

Next, we study the behaviour of the different terms on the right-hand side of (3.10) in the scaling limit [see (3.6)–(3.-9)]. We begin by stating, in (1) to (4) below, in a more explicit form the basic assumption of incompressibility, and of the other principles mentioned above.

(1) Since we have assumed incompressibility of the system in the form of strong cluster properties of connected current Green functions, we conclude that

$$
\frac{(-i)^{n+1}\lambda^{2n}}{n!} \langle T[j^{\mu_1}(\lambda\xi_1)\dots j^{\mu_n}(\lambda\xi_n)]\rangle^c_{a_c} \to \varphi^{\mu_1\dots\mu_n}(\xi_1, \dots, \xi_n)\,, \tag{3.11}
$$

as $\lambda \to \infty$, where $\varphi^{\mu_1 \dots \mu_n}$ is a *local* distribution, i.e.,

$$\operatorname{supp} \varphi^{\mu_1 \dots \mu_n} = \{\xi_1, \dots, \xi_n : \xi_1 = \dots = \xi_n\}. \tag{3.12}$$

(Note that the current j has scaling dimension 2 which matches with the factor λ^2 per current insertion in (3.11). That the current j must have dimension 2 is a consequence of the fact that $\int\limits_{t=\text{const}} j^0(t, \underline{x})\, d^2\underline{x}$ is a *dimensionless* conserved charge!)

(2) Only *relevant* and *marginal* terms contribute to the action S^*_Ω in the scaling limit, i.e., if the leading term of

$$\lambda^{2n} \int \langle T[j^{\mu_1}(\lambda\xi_1) \dots j^{\mu_n}(\lambda\xi_n)]\rangle^c_{a_c} \tilde{a}_{\mu_1}(\xi_1) \dots \tilde{a}_{\mu_n}(\xi_n)\, d^3\xi_1 \dots d^3\xi_n \tag{3.13}$$

is of order λ^{-D} for $\lambda \to \infty$ with $D > 0$, then this term will not be displayed in S^*_Ω. By (1),

$$D \geq n - 3. \tag{3.14}$$

(This can be seen by first replacing λ by $\lambda\lambda'$ in (3.11), and letting $\lambda' \nearrow \infty$. Next, we recall that in $2+1$ dimensions any local distribution can be written as a sum of derivatives of a product of 3-dimensional δ-functions, which then tells us that $\lambda^{2n}\varphi^{\mu_1 \dots \mu_n}(\lambda\xi_1, \dots, \lambda\xi_n)$ scales as λ^{-D} with $D = -2n + 3(n-1) + a \geq n - 3$, a being the number of derivatives present.) By (3.14) there are no terms of order $n \geq 4$ in $\tilde{a}$ contributing to S^*_Ω. This explains why, in (3.10), we have expanded $S^{\text{eff}}_{\lambda\Omega}$ only to *fourth order* in $\tilde{a}$.

(3) $U(1)$-gauge invariance and current conservation: In $U(1)$-gauge theory – much in contrast to non-abelian gauge theories – the space of vector potentials (on a trivial $U(1)$-bundle) is a real *vector space*: If $a_1, \dots, a_m$ are arbitrary vector potentials then an arbitrary linear combination

$$a = \sum_{l=1}^{m} \lambda_l a_l\,, \qquad \lambda_l \in \mathbb{R}\,,$$

is again a vector potential (provided M_3 does not contain any non-contractible, two-dimensional compact surfaces – an assumption which is true for our choice of Ω). Moreover, the vector potential a', defined by

$$a' = \sum_{l=1}^{m} \lambda_l (a_l + d\chi_l)$$

is gauge-equivalent to a, for arbitrary functions $\chi_1, \dots, \chi_m$. Therefore, for an *arbitrary* background vector potential a_c, the effective action $S^{\text{eff}}_{\lambda\Omega}(\tilde{a})$, where $\tilde{a} = a - a_c$ is the "fluctuation potential", must be a *gauge-invariant functional* of $\tilde{a}$, i.e.,

$$S^{\text{eff}}_{\lambda\Omega}(\tilde{a} + d\tilde{\chi}; a_c + d\chi) = S^{\text{eff}}_{\lambda\Omega}(\tilde{a}; a_c)\,. \tag{3.15}$$

This identity implies that

$$\nabla_{\mu_l} \varphi^{\mu_1 \dots \mu_l \dots \mu_n}(\xi_1, \dots, \xi_l, \dots, \xi_n) = 0\,, \tag{3.15'}$$

for every $l = 1, \dots, n$, and here $\nabla_\mu = \dfrac{1}{\sqrt{|\eta|}}\, \partial_\mu(\sqrt{|\eta|}\,\cdot)$.

Note that Eq. (3.15) implies conservation of the electromagnetic current in the strong form that

$$\nabla_{\mu_l}\langle T[j^{\mu_1}(x_1)\dots j^{\mu_l}(x_l)\dots j^{\mu_n}(x_n)]\rangle^c_{a_c}=0\,, \tag{3.15''}$$

for all $l=1,\dots,n$, as a distribution on $M_3^{\times n}$. Equation (3.15″) is stronger than the statement that j^μ is a conserved current, since, in general, the latter statement only implies that (3.15″) holds as long as $x_l\neq x_j$, for $j=1,\dots,n, j\neq l$.

Thanks to (3.15), (3.15″), the equation

$$j=*d\mathscr{A}\,,$$

for some *operator-valued vector potential* $\mathscr{A}$, [see Eq. (2.38)] holds, in fact, as an *operator equation*, without any restrictions – as used in [2].

(4) We are studying systems *without* relativistic invariance. hence $\varphi^{\mu_1\cdots\mu_n}$ has no reason to be a Lorentz-invariant distribution. However, if the background potential a_c is such that the field da_c is invariant under rotations in the $x-y$ plane (e.g., a_c is the vector potential of a constant magnetic field $\vec{b}=(0,0,b_3)$ in the z-direction) then $\varphi^{\mu_1\cdots\mu_n}$ might be expected to be a rotation – ($S0(2)$-)invariant distribution, provided Ω is a rotation ivnariant domain in the $x-y$ plane. However, we should warn that, in general, rotation invariance may be broken spontaneously. In the limit where $\Omega\nearrow\mathbb{R}^2$, $\varphi^{\mu_1\cdots\mu_n}$ might be expected to be translation-invariant, provided da_c is translation-invariant. This would be the case at positive temperature, $T>0$, by Mermin-Wagner theorem [23, 24], but may fail when $T=0$.

We are now ready to display all terms possibly contributing to S^*_Ω explicitly.

(i) The term in S^*_Ω of first order in $\tilde{a}$ is relevant ($D=-2$), and takes the form

$$\int_{M_3} j^\mu_c(\xi)\tilde{a}_\mu(\xi)d^3\xi\,, \tag{3.16}$$

where by current conservation [see (3), above],

$$\partial_\mu j^\mu_c(\xi)=0\,. \tag{3.17}$$

Since time-translation invariance is not broken, for a time-independent background field da_c, j^μ_c will be independent of ξ^0. Hence (3.17) implies that

$$\underline{\nabla}\cdot\underline{j}_c(\xi)=0\,, \tag{3.18}$$

i.e., $\underline{j}_c(\xi)=$ "$\lim\limits_{\lambda\to\infty}$" $\lambda^2\langle\underline{j}(\lambda\xi)\rangle_{a_c}$ describes a persistent, divergence-free (super-) current circulating in the system. furthermore, $j^0_c(\xi)\equiv\varrho(\xi)$ describes a time-independent background charge density. Thus, if there are no persistent, electric (super-) currents circulating in the system then

$$j^\mu_c(\xi)=\delta^{\mu 0}\varrho(\underline{\xi})\,. \tag{3.19}$$

(ii) The term of second order in $\tilde{a}$ contributing to S^*_Ω is determined as follows. By (1) and (3), ($U(1)$-gauge invariance, or strong current conservation),

$$\varphi^{\mu\nu}(\xi,\eta)=\alpha\varepsilon^{\mu\nu\varrho}\partial_\varrho\delta(\xi-\eta)+\varphi^{\mu\nu}_I(\xi,\eta)\,, \tag{3.20}$$

where α is a constant, and the distribution φ_I consists of second or higher derivatives of the δ-function, (up to distributions localized on the boundary, ∂M_3, of space-time).

The dimensions D [see (3.13)] of the terms corresponding to $\varphi_I^{\mu\nu}$, e.g., the Maxwell term, are strictly positive. Hence they do not occur in S_Ω^*. In conclusion, the term of order 2 in $\tilde{a}$ contributing to S_Ω^* is given by

$$\alpha \int_{M_3} \varepsilon^{\mu\varrho\nu} \tilde{a}_\mu(\xi) (\partial_\varrho \tilde{a}_\nu)(\xi) d^3\xi \,, \tag{3.21}$$

i.e., the *Chern-Simons term* which is *marginal*. In the language of differential forms, (3.21) reads

$$\alpha \int_{M_3} \tilde{a} \wedge d\tilde{a} \,. \tag{3.22}$$

This term is $U(1)$-gauge invariant *only* up to boundary terms!

The unique terms of second order in $\tilde{a}$ and of dimension $D = 1$ are the Maxwell terms

$$\frac{1}{4} \sum_{\mu,\nu=0}^{2} g_{\mu\nu} \int f_{\mu\nu}^2(\xi) d^3\xi \,, \tag{3.23}$$

where $f_{\mu\nu} = \partial_\mu \tilde{a}_\nu - \partial_\nu \tilde{a}_\mu$. Rotation invariance would imply that $g_{j0} = g_{0j} = g^{(0)}$, for $j = 1, 2$, and $g_{12} = g_{21} = g^{(1)}$, but $g^{(0)}$ and $g^{(1)}$ may have different values.

(iii) By principles (1) and (3), $\varphi^{\mu_1\mu_2\mu_3}$ must be a distribution which is a sum of *derivatives* of products of two δ-functions. (There are no local distributions $\varphi^{\mu_1\mu_2\mu_3}$ which are compatible with $U(1)$-gauge invariance and which are measures.) Thus the dimension D of terms of third order in $\tilde{a}$ contributing to $S_{\lambda\Omega}^{\text{eff}}$ is *strictly positive*. Hence S_Ω^* does not contain any third-order terms in $\tilde{a}$. Moreover, as remarked above, all terms of order ≥ 4 in $\tilde{a}$ have dimension $D > 1$ and therefore do not appear in S_Ω^*.

In conclusion $\left(\text{we include a factor } -\frac{1}{\hbar} \text{ for later convenience; see Sect. 4}\right)$

$$-\frac{1}{\hbar} S_\Omega^*(\tilde{a}) = \int_{M_3} j_c^\mu(\xi) \tilde{a}_\mu(\xi) d^3\xi + \alpha \int_{M_3} \varepsilon^{\mu\varrho\nu} \tilde{a}_\mu(\xi) (\partial_\varrho \tilde{a}_\nu)(\xi) d^3\xi$$
$$+ \text{boundary terms (b.t.)} \,. \tag{3.24}$$

In differential-form notation,

$$\begin{aligned} -\frac{1}{\hbar} S_\Omega^*(\tilde{a}) &\equiv -\frac{1}{\hbar} S_\Omega^*(\tilde{a}; a_c) \\ &= \int_{M_3} (*j_c) \wedge \tilde{a} + \alpha \int_{M_3} \tilde{a} \wedge d\tilde{a} + \text{b.t.} \\ &= \int_{M_3} (*j_c) \wedge a + \alpha \int_{M_3} a \wedge da - 2\alpha \int_{M_3} a \wedge da_c + \text{const.} + \text{b.t.} \,, \end{aligned} \tag{3.24$'$}$$

where $a = a_c + \tilde{a}$ is the total vector potential. This formula shows that $S_\Omega^*(a)$ is $U(1)$-gauge-invariant, up to boundary terms which will be studied in detail in [19].

Next, we extend our analysis to systems with $\mu_e \neq 0$, $w \neq 0$, and determine the general form of $S_\Omega^*(a, w)$. This is a variation on the theme just discussed. The only complication encountered is that we cannot use current conservation for $\vec{s}_\mu = (s_{\mu 1}, s_{\mu 2}, s_{\mu 3})$ in the strong from of (3.15″) and (3.15′), *even* if the background $SU(2)$-gauge potential w_c vanishes. If w_c does not vanish, some or all components of $\vec{s}_\mu$ are *not* conserved currents, at all. We set

$$a = a_c + \tilde{a}\,, \qquad w = w_c + \tilde{w}\,, \tag{3.25}$$

and we choose $w_c = i\vec{w}_c \cdot \vec{\sigma}$ to correspond to an external magnetic field $\vec{B}_c(x)$ in the z-direction, i.e.,

$$w_{c,03}(x) = -\frac{\mu_e}{2c} B_{c,3}(x)\,, \tag{3.26}$$

with all other components of w_c vanishing. This corresponds to a standard experimental situation in two-dimensional condensed matter physics. In (3.25) and in the following we suppress the parameter λ for the gauge fields, leaving it to the context to specify whether we are working with respect to the physical systems on $\lambda\Omega$ or the corresponding rescaled systems on Ω; see (3.8). [Here a remark on the use of the term "external gauge field" seems to be appropriate: For the system consisting only of the electrons confined to the two-dimensional region Ω, every gauge field is an external one, and in this way the term is used throughout this paper. However, from the point of view of an experimentalist, the total fields a and w are composed out of a_c and w_c which he can impose on the experimental sample from outside, and out of the small fluctuations $\tilde{a}$ and $\tilde{w}$, respectively, which he can impose only partially from outside, the other part possibly being a property of the background of the experimental sample containing the electrons.] We observe that if in (3.26) $B_{c,3}$ is independent of $\underline{x}$ then w_c is a pure gauge, but the $SU(2)$-gauge transformation gauging away w_c is not localized and, therefore, would change the boundary conditions at ∂M_3 and, in general, at $t = \pm\infty$.

Again, we assume that $S^{\text{eff}}_{\lambda\Omega}(a, w)$ is four times continuously (Fréchet) differentiable in a and w on some suitable neighbourhood of a_c, w_c. The Taylor expansion of $S^{\text{eff}}_{\lambda\Omega}(a, w)$ around (a_c, w_c) to fourth order in $\tilde{a}$, $\tilde{w}$ contains all terms present in (3.10) – whose scaling limits we have already determined – terms analogous to those in (3.10), but with j^μ replaced by a component, s^μ_A, of the spin current and $\tilde{a}_\mu$ replaced by $\tilde{w}_{\mu A}$ and, finally, it contains mixed terms corresponding to the Green functions $\langle T[js]\rangle^c$, $\langle T[jjs]\rangle^c$ and $\langle T[jss]\rangle^c$ which we need to discuss.

Let us start by analyzing the pure $SU(2)$-terms corresponding to the Green functions $\langle s\rangle$, $\langle T[ss]\rangle^c$ and $\langle T[sss]\rangle^c$ in a background field (a_c, w_c). Again, we use principles (1) through (4) above, viz. incompressibility in the form of strong cluster properties, power counting, $U(1)_{\text{spin}}$-gauge invariance (corresponding to local rotations around the 3-axis in spin space) for the gauge potential $\tilde{w}_{\mu 3}$, assuming that w_c satisfies (3.26). This entails conditions analogous to (3.15′), (3.15″) for Green functions *only* depending on s^μ_3 (and j^μ). When applicable we shall also use rotation invariance. In addition, we make use of

(5) full $SU(2)$-gauge invariance of the theory; see (3.2).

We should emphasize that for our choice of w_c, $s^\mu_A(x)$ will *not* be conserved, for $A = 1, 2$, see Sect. 2, so that (3) cannot be used, *except* for Green functionly *only* involving s^μ_3 and j^μ.

(i) The term of first order in $\tilde{w}$ occurring in $S^*_\Omega(a, w)$ comes from the one-point function

$$m^\mu_A(x) = \langle s^\mu_A(x)\rangle_{a_c, w_c}\,. \tag{3.27}$$

If in the scaling limit there are no persistent spin (super-) currents circulating in the system then

$$\text{“}\lim_{\lambda\to\infty}\text{''}\,\lambda^2\langle\vec{s}^{\mu}(\lambda\xi)\rangle_{a_c,w_c} \equiv \vec{m}^{\mu}(\xi) = \delta^{\mu 0}\vec{m}^{0}(\xi)\,. \tag{3.28}$$

If w_c corresponds to a magnetic field in the z-direction, as assumed, then

$$\vec{m}^{\mu} = (0,0,m_3^{\mu})\,, \tag{3.29}$$

where m_3^0 is proportional to the *magnetization* of the system.

If $w_c \to 0$, then, in two dimensions,

$$m_3^0 \to 0\,, \quad \text{for} \quad T > 0\,,$$

by the Mermin-Wagner theorem [23], but at zero temperature, $T = 0$, there could be *spontaneous magnetization.* (In three space dimensions, m_0^3 could be non-vanishing, for $w_c \to 0$, even when $T > 0$.)

It is important to note that $\vec{m}^{\mu}(\xi)$ is not $SU(2)$-gauge invariant: $\vec{m}^{\mu}(\xi) \equiv \vec{m}^{\mu}(\xi; w_c)$ depends on the choice of gauge in which we describe w_c. If ${}^g w_c = g w_c g^{-1} + g dg^{-1}$, where g is an $SU(2)$-gauge transformation, then, at least formally, a change of variables in the path integrals (3.3) or (3.3′) shows that

$$\begin{aligned}\vec{m}^{\mu}(x; {}^g w_c) &= \langle\vec{s}^{\mu}(x)\rangle_{a_c,{}^g w_c} = \langle R(g(x))\,\vec{s}^{\mu}(x)\rangle_{a_c,w_c}\\ &= R(g(x))\,\vec{m}^{\mu}(x; w_c)\,,\end{aligned} \tag{3.30}$$

where $R(g)$ is the $S0(3)$-rotation corresponding to an element g of $SU(2)$ in the adjoint representation. More generally,

$$\begin{aligned}\langle T[s_{A_1}^{\mu_1}(x_1)\dots s_{A_n}^{\mu_n}(x_n)]\rangle_{a_c,w_c}^c = \sum_{B_1\dots B_n} & R_{A_1B_1}(g(x_1))\dots R_{A_nB_n}(g(x_n))\\ &\times \langle T[s_{B_1}^{\mu_1}(x_1)\dots s_{B_n}^{\mu_n}(x_n)]\rangle_{a_c,w_c}^c\,.\end{aligned} \tag{3.31}$$

Furthermore, we note that the “fluctuation field,” $\tilde{w} = w - w_c$, transforms under $SU(2)$-gauge transformations also according to the adjoint representation, $R(g)$, i.e.

$${}^g\tilde{w} = g\tilde{w}g^{-1}\,, \quad \text{or} \quad \overrightarrow{{}^g\tilde{w}} = R(g)\vec{\tilde{w}}\,, \tag{3.32}$$

since

$${}^g w = g w g^{-1} + g dg^{-1}\,, \quad \text{and} \quad {}^g w_c = g w_c g^{-1} + g dg^{-1}\,,$$

so that the inhomogeneous term cancels in (3.32).

By (3.30) and (3.32), the contribution

$$\int_{M_3} m_3^{\mu}(\xi)\,\tilde{w}_{\mu 3}(\xi)\,d^3\xi = \int_{M_3} (*m_3) \wedge \tilde{w}_3 \tag{3.33}$$

to $S_{\Omega}^*(a, w)$ is compatible with $SU(2)$-gauge invariance. It is a relevant term of scaling dimension $D = -2$, and we finally note tht by $U(1)_{\text{spin}}$-gauge invariance (fully explained in the next paragraph) we just have that

$$\partial_{\mu} m_3^{\mu}(\xi) = 0\,. \tag{3.34}$$

(ii) Next, we discuss the terms of second and third order in $\tilde{w}$ contributing to $S_{\Omega}^*(a, w)$. Since we have chosen $w_{c,A} = \delta_{A3} w_{c,3} = \delta_{A3} w_{c,\mu 3} dx^{\mu}$ to have only a non-zero 3-component, we know from Sect. 2 that the 3-component of the spin current $\vec{s}^{\mu}$ is conserved. Note that in the gauge where (3.26) holds, s_3^{μ} is conserved also if there is, in addition to $\vec{B}_c$, a non-vanishing electric fluctuation field $\vec{\tilde{E}}$ in the $x-y$ plane, i.e., besides $w_{c,03}$ and $\tilde{w}_{03}$, also $\tilde{w}_{k3}$, for $k=1,2$, can be non-vanishing and s_3^{μ} is still conserved. Thus if we temporarily restrict the fluctuation field $\tilde{w}$ to be of the form

$$\tilde{w}_A = \delta_{A3}\tilde{w}_3 \qquad (3.26')$$

then the theory is a $U(1)_{\text{spin}}$-gauge theory, where the $U(1)_{\text{spin}}$-gauge transformations correspond to local rotations in spin space around the 3-axis and act on $\tilde{w}_3$ by $\tilde{w}_3 \to \tilde{w}_3 + d\chi$, where $2\chi(x)$ is the angle of rotation. In this case, our problem of determining the second and third order terms in $\tilde{w}_3$ contributing to S_{Ω}^{*} is *identical* to the one already solved for the electromagnetic vector potential $\tilde{a}$. The solution is that if we require Eqs. (3.26) and (3.26′) to hold the second-order term in $S_{\Omega}^{*}(\tilde{a}=0,\tilde{w})$ has the form

$$\int_{M_3} \tilde{w}_3 \wedge d\tilde{w}_3\,, \qquad (3.35)$$

and there is no third-order term.

Now, we should remember that as a functional of the *total* gauge potential $w = w_c + \tilde{w}$, $S_{\Omega}^{*}(a,w)$ must be $SU(2)$-gauge invariant; [principle (5)]. In the scaling limit we write the total $SU(2)$-gauge potential w as

$$w_{\mu A}(\xi) = w_{c,\mu A}(\xi) + \tilde{w}_{\mu A}(\xi)\,,$$

with the definition of

$$w_{c,\mu A}(\xi) \equiv \text{“}\lim_{\lambda\to\infty}\text{”}\,\lambda w_{c,\mu A}(\lambda\xi) = \text{“}\lim_{\lambda\to\infty}\text{”}\,w_{c,\mu A}(\xi;\lambda)\,; \qquad (3.36)$$

see (3.8). Moreover, if w is restricted to gauge potentials of the form $w = w_c + i\tilde{w}_3\cdot\sigma_3$, with w_c as in (3.26) then the second order term in $\tilde{w}_3$ must reduce to one proportional to (3.35). Finally, terms of order $n \geq 4$ in $\tilde{w}$ are irrelevant by power counting, i.e., have $D > 0$, and are therefore absent from S_{Ω}^{*}; [principle (2)]. The terms of dimension $D = 0$ and -1 containing second- *and* third order contributions in $\tilde{w}$ and having all the properties required above are the Chern-Simons term

$$\frac{k}{4\pi}\int_{M_3} \operatorname{tr}\left(w\wedge dw + \frac{2}{3}\,w\wedge w\wedge w\right), \qquad (3.37)$$

for some constant k, and a term of the form

$$\int_{M_3} (\beta_3\tilde{w}_3)\wedge d(\beta_3\tilde{w}_3) + \text{b.t.} \qquad (3.38)$$

In the gauge where w_c is given by (3.26), $\beta_3 \equiv \beta_3(w_c)$ is a constant depending on w_c, because (3.38) has to be $U(1)_{\text{spin}}$-gauge invariant (up to boundary terms). Furthermore under $SU(2)$-gauge transformations $\beta_3(w_c)$ transforms like the 3-component of $\vec{s}^{\mu}$, that is according to

$$\delta_{A3}\beta_3(w_c) \to \beta_A(\xi;{}^{g}w_c) = R_{A3}(g(\xi))\,\beta_3(w_c)\,, \qquad (3.39)$$

for $A = 1,2,3$. This ensures $SU(2)$-gauge invariance of (3.38), taking into account (3.32).

(iii) We shall realize, however, that there can also be relevant ($D < 0$) terms of order 2 and further marginal ($D = 0$) terms of order 3 in $\tilde{w}$ contributing to $S^*_\Omega(\tilde{a}, \tilde{w})$. The reason is that the "fluctuation potential" $\tilde{w}$ and the spin current $\vec{s}^\mu$ both transform under the adjoint representation of $SU(2)$-gauge transformations, and equations like (3.15′) and (3.15″) are not true for Green functions of spin currents other than Green functions *only* involving j and s_3.

Besides the distribution $\varepsilon^{\mu\nu\varrho}\left[\frac{k}{4\pi}\,\delta_{AB} + \beta_3\delta_{A3}\beta_3\delta_{B3}\right]\partial_\varrho\delta(\xi-\eta)$ [see (3.37) and (3.38)], the scaling limit of the Green function $\langle T[s^\mu_A(x)s^\nu_B(y)]\rangle^c_{a_c,w_c}$ can yield a term $\tau^{\mu\nu}_{AB}(\xi)\delta(\xi-\eta)$, where $\tau^{\mu\nu}_{AB}(\xi) \equiv \tau^{\mu\nu}_{AB}(\xi; w_c)$ depends on w_c. By (3.31) and (3.32), the term

$$\sum_{A,B=1}^{3} \int_{M_3} \tau^{\mu\nu}_{AB}(\xi; w_c)\,\tilde{w}_{\mu A}(\xi)\,\tilde{w}_{\nu B}(\xi)\,d^3\xi \tag{3.40}$$

is consistent with $SU(2)$-gauge invariance, since $\tau^{\mu\nu}_{AB}$ transforms under $SU(2)$-gauge transformations according to the representation $R(g)\otimes R(g)$, just like $\tilde{w}_{\mu A}\tilde{w}_{\nu B}$. The coefficient $\tau^{\mu\nu}_{33}$ must vanish, for $w_{c,\mu A} = \delta_{A3}w_{c,\mu 3}$ as in (3.26), since, for this choice of w_c, the current $s^\mu_3(\xi)$ is conserved. Moreover, *global* $U(1)_{\text{spin}}$-invariance under rotations around the 3-axis in spin space implies that

$$\tau^{\mu\nu}_{A3} = \tau^{\mu\nu}_{3A} = 0\,, \quad \text{for} \quad A = 1,2\,, \tag{3.41}$$

and

$$\tau^{\mu\nu}_{AB} = \tau^{\mu\nu}_1\delta_{AB} + \tau^{\mu\nu}_2\varepsilon_{AB}\,, \quad \text{for} \quad A,B = 1,2\,. \tag{3.42}$$

The obvious symmetry of $\tau^{\mu\nu}_{AB}$ under exchanging (μA) with (νB) implies that $\tau^{\mu\nu}_1$ is symmetric and $\tau^{\mu\nu}_2$ is antisymmetric in μ and ν. Could the global $U(1)_{\text{spin}}$-invariance [for w_c as in (3.26)] be spontaneously broken? For $T > 0$, this is ruled out by the Mermin-Wanger theorem. For $T = 0$, spontaneous $U(1)_{\text{spin}}$-breaking might appear in some of the $\tau^{\mu\nu}_{AB}$. But then the system would have a Goldstone boson. As a consequence, connected spin-current Green functions would have slow (power-law) fall off, and hence our basic hypothesis of incompresibility would not hold.

If the system displays invariance under rotations of the $x-y$ plane in the scaling limit then the coefficients $\tau^{\mu\nu}_\alpha$ in (3.42) satisfy

$$\tau^{i0}_\alpha = \tau^{0j}_\alpha = 0\,, \quad \text{for} \quad i,j = 1,2\,, \tag{3.43}$$

and τ^{ij}_α must be invariant under rotations in the $x-y$ plane, so that

$$\tau^{ij}_\alpha = \tau_{\alpha 1}\delta^{ij} + \tau_{\alpha 2}\varepsilon^{ij}\,, \quad \text{for} \quad \alpha = 1,2\,. \tag{3.44}$$

Finally the symmetry of τ^{ij}_1 and the antisymmetry of τ^{ij}_2 imply that τ_{12} and τ_{21} vanish. Hence there are only three independent coefficients

$$\tau^{(0)} \equiv \tau^{00}_1\,, \qquad \tau^{(1)} \equiv \tau_{11}\,, \quad \text{and} \quad \tau^{(2)} \equiv \tau_{22}\,. \tag{3.45}$$

In conclusion, under our hypotheses, in particular the assumption that $w_{c,\mu A} = \delta_{A3}w_{c,\mu 3}$ there might be a term of scaling dimension $D = -1$ in the effective action $S^*_\Omega(\tilde{a}, \tilde{w})$ which generally takes the form

$$\sum_{A=1}^{2} \int_{M_3} \tau^{\mu\nu}_1(\xi)\tilde{w}_{\mu A}(\xi)\tilde{w}_{\nu A}(\xi) + \sum_{A,B=1}^{2} \int_{M_3} \tau^{\mu\nu}_2(\xi)\varepsilon_{AB}\tilde{w}_{\mu A}(\xi)\tilde{w}_{\nu B}(\xi)\,. \tag{3.46}$$

More specially, in the case where rotation invariance is displayed by the system in the scaling limit, the term is given by

$$\int_{M_3} \left\{ \tau^{(0)}(\xi) \sum_{A=1}^{2} \tilde{w}_{0A}(\xi)\tilde{w}_{0A}(\xi) + \tau^{(1)}(\xi) \sum_{i,A=1}^{2} \tilde{w}_{iA}(\xi)\tilde{w}_{iA}(\xi) + 2\tau^{(2)}(\xi)\left(\tilde{w}_{11}(\xi)\tilde{w}_{22}(\xi) - \tilde{w}_{12}(\xi)\tilde{w}_{21}(\xi)\right) \right\} d^3\xi \,. \tag{3.46$'$}$$

Further interesting constraints on the coefficients $\tau_1^{\mu\nu}$ and $\tau_2^{\mu\nu}$, $\tau^{(0)}$, $\tau^{(1)}$, and $\tau^{(2)}$ respectively, will be found in Sect. 4.

(iv) There can also be a third order term of the form

$$\sum_{A,B,C=1}^{3} \int_{M_3} \eta_{ABC}^{\mu\nu\varrho}(\xi)\,\tilde{w}_{\mu A}(\xi)\,\tilde{w}_{\nu B}(\xi)\,\tilde{w}_{\varrho C}(\xi)\,d^3\xi \,, \tag{3.47}$$

a term which is marginal, in $S_\Omega^*(\tilde{a},\tilde{w})$. The tensor $\eta_{ABC}^{\mu\nu\varrho}(\xi) \equiv \eta_{ABC}^{\mu\nu\varrho}(\xi; w_c)$ is computed from the scaling limit of $\langle T[s_A^\mu(x_1)\, s_B^\nu(x_2)\, s_C^\varrho(x_3)]\rangle_{a_c,w_c}^c$. It is obviously symmetric under arbitrary permutations of (μA), (νB) and (ϱC). The term with $A = B = C = 3$ is irrelevant by principle (3), i.e., by invariance under $U(1)_{\text{spin}}$-gauge transformations corresponding to local rotations around the 3-axis in spin space. Under general $SU(2)$-gauge transformations, $\eta_{ABC}^{\mu\nu\varrho}$ transforms according to the representation $R \otimes R \otimes R$; more precisely

$$\eta_{ABC}^{\mu\nu\varrho}(\xi; {}^g w_c) = \sum_{D,E,F} R_{AD}(g(\xi))\, R_{BE}(g(\xi))\, R_{CF}(g(\xi))\, \eta_{DEF}^{\mu\nu\varrho}(\xi; w_c)\,;$$

see (3.31). This implies consistency with gauge invariance of (3.47).

If $w_{c,\mu A} = \delta_{A3} w_{c,\mu 3}$, as assumed, rotations around the 3-axis in spin space form a global, unbroken $U(1)_{\text{spin}}$-symmetry. Then the only terms that are possible, apart from ones arising by permutations of $(\mu 3)\,(\nu A)\,(\varrho B)$, are of the form

$$\eta_{3BC}^{\mu\nu\varrho} = \eta_1^{\mu\nu\varrho}\delta_{BC} + \eta_2^{\mu\nu\varrho}\varepsilon_{BC}\,, \quad \text{with} \quad B, C = 1, 2\,, \tag{3.48}$$

where $\eta_1^{\mu\nu\varrho}$ is symmetric in $\nu\varrho$, and $\eta_2^{\mu\nu\varrho}$ is anti-symmetric in $\nu\varrho$. Let us also assume that, in the scaling limit, $\eta_{ABC}^{\mu\nu\varrho}(\xi) = \eta_{ABC}^{\mu\nu\varrho}(|\xi|)$ is invariant under rotations in the $x-y$ plane. Rotation invariance then yields further restrictions on $\eta_1^{\mu\nu\varrho}$ and $\eta_2^{\mu\nu\varrho}$ which, for example, permit us to decompose $\eta_1^{\mu\nu\varrho}$ into a sum of six terms with independent relative coefficients, and similarly for $\eta_2^{\mu\nu\varrho}$. These decompositions are of little use and are therefore omitted.

(v) Finally, we discuss mixed terms depending on both kinds of gauge potentials, $\tilde{a}$ and $\tilde{w}$. The terms proportional to $\langle T[jjs]\rangle^c$ and to $\langle T[jss]\rangle^c$ in $S_{\lambda\Omega}^{\text{eff}}(\tilde{a},\tilde{w})$ have dimension $D \geq 1$, as follows from $U(1)$-gauge invariance in $\tilde{a}$ and principle (3). Therefore, they disappear in the scaling limit. We are thus left with the possibility of a mixed second order term in $S_{\lambda\Omega}^{\text{eff}}(\tilde{a},\tilde{w})$ proportional to

$$\langle T[j^\mu(x)\, s_A^\nu(y)]\rangle_{a_c,w_c}^c \,. \tag{3.49}$$

If $w_{c,\mu A} = \delta_{A3} w_{c,\mu 3}$, as assumed [see (3.26)], then invariance under global rotations around the 3-axis in spin space implies that (3.49) vanishes unless $A = 3$. Furthermore, for $A = 3$, we can apply principle (3), i.e., $U(1)$-gauge invariance under local phase transformations and $U(1)_{\text{spin}}$-gauge invariance under local rotations around the 3-axis in spin space, to conclude that

$$\partial_\mu \langle T[j^\mu(x) s_3^\nu(y)]\rangle^c_{a_c,w_c} = \partial_\nu \langle T[j^\mu(x) s_3^\nu(y)]\rangle^c_{a_c,w_c} = 0\,, \tag{3.50}$$

as distributions. Thus, in the scaling limit, $\langle T[j^\mu(x) s_3^\nu(y)]\rangle^c_{a_c,w_c}$ approaches the distribution

$$2\gamma_3 \varepsilon^{\mu\nu\varrho}\partial_\varrho \delta(\xi - \eta)\,, \tag{3.51}$$

where, for w_c as in (3.26), $\gamma_3 \equiv \gamma_3(w_c)$ is a constant depending on w_c. Under $SU(2)$-gauge transformations $\gamma_3(w_c)$ transforms like $\beta_3(w_c)$, given in (3.39). In conclusion then, for $w_{c,\mu A} = \delta_{A3} w_{c,\mu 3}$, as assumed, there can be a marginal mixed term

$$2 \int\limits_{M_3} \varepsilon^{\mu\nu\varrho} \tilde{a}_\mu \partial_\nu (\gamma_3 \tilde{w}_{\varrho 3}) d^3\xi + \text{b.t.} = \int\limits_{M_3} d\tilde{a} \wedge (\gamma_3 \tilde{w}_3) + \int\limits_{M_3} \tilde{a} \wedge d(\gamma_3 \tilde{w}_3) + \text{b.t.} \tag{3.52}$$

contributing to $S^*_\Omega(\tilde{a}, \tilde{w})$. For $\gamma_3 \neq 0$, this term is $SU(2)$-gauge invariant but $U(1)$-gauge invariant *only up to boundary terms*, (terms localized on ∂M_3).

Even in the *classical* theory of the Hall effect, a term proportional to (3.52) is present, in general; see the discussion at the end of Sect. 1.

We have now completed our task of determining, in the scaling limit, the most general form of the effective action of a two-dimensional incompressible electronic system in an external electromagnetic field. In an $SU(2)$-gauge where w_c satisfies (3.26) the result reads as follows:

$$\begin{aligned}
-\frac{1}{\hbar} S^*_\Omega(\tilde{a}, \tilde{w}) &\equiv -\frac{1}{\hbar} S^*_\Omega(\tilde{a}, \tilde{w}; a_c, w_c) \\
&= \int\limits_{M_3} (*j_c) \wedge \tilde{a} + \alpha \int\limits_{M_3} \tilde{a} \wedge d\tilde{a} \\
&\quad + \int\limits_{M_3} (*m_3) \wedge \tilde{w}_3 + \frac{k}{4\pi} \int\limits_{M_3} \operatorname{tr}\left(w \wedge dw + \frac{2}{3} w \wedge w \wedge w \right) \\
&\quad + \int\limits_{M_3} (\beta_3 \tilde{w}_3) \wedge d(\beta_3 \tilde{w}) \\
&\quad + \sum_{A=1}^{2} \int\limits_{M_3} \tau_1^{\mu\nu} \tilde{w}_{\mu A} \tilde{w}_{\nu A} d^3\xi + \sum_{A,B=1}^{2} \int\limits_{M_3} \tau_2^{\mu\nu} \varepsilon_{AB} \tilde{w}_{\mu A} \tilde{w}_{\nu B} d^3\xi \\
&\quad + {\sum_{A,B,C=1}^{3}}' \int\limits_{M_3} \eta^{\mu\nu\varrho}_{ABC} \tilde{w}_{\mu A} \tilde{w}_{\nu B} \tilde{w}_{\varrho C} d^3\xi \\
&\quad + \int\limits_{M_3} d\tilde{a} \wedge (\gamma_3 \tilde{w}_3) + \int\limits_{M_3} \tilde{a} \wedge d(\gamma_3 \tilde{w}_3) + \text{b.t.}\,,
\end{aligned} \tag{3.53}$$

where $w = w_c + \tilde{w}$ [see (3.36)], and b.t. denotes a collection of boundary terms localized on ∂M_3. In the η-term the primed sum means that there is no contribution if two or more of the indices A, B, and C simultaneously equal 3. The first two terms on the right-hand side of (3.53) have been identified in (3.24) [see also (3.17)–(3.22)], the third term in (3.33), the fourth and fifth term in (3.37) and (3.38), respectively, the sixth and seventh term in (3.40)–(3.42), the eight terms in (3.47), and the last two terms in (3.52).

Note that (3.53) is valid without assuming rotation invariance of the system in the scaling limit. If the latter is true, however, further restrictions on the form of the τ- and η-terms can be taken into account, see e.g. (3.46′). Depending on the application, it is appropriate to collect the second, fifth and the last two terms and rewrite them either in the form

$$\frac{\sigma}{4\pi}\int\limits_{M_3} \tilde{a}\wedge d\tilde{a} + \frac{\sigma_s}{4\pi}\int\limits_{M_3} \tilde{w}_3 \wedge d\tilde{w}_3 + \frac{\chi}{2\pi}\int\limits_{M_3} \tilde{a}\wedge d\tilde{w}_3 + \text{b.t.}\,, \tag{3.53′}$$

or equivalently as

$$\frac{\sigma^{(1)}}{4\pi}\int\limits_{M_3} \tilde{a}\wedge d\tilde{a} + \frac{\sigma^{(2)}}{4\pi}\int\limits_{M_3} (\tilde{a}+\delta\tilde{w}_3)\wedge d(\tilde{a}+\delta\tilde{w}_3)\,. \tag{3.53″}$$

In a gauge where w_c satisfies (3.26) the new constants (depending on w_c) are defined by

$$\sigma = 4\pi\alpha\,, \qquad \sigma_s = 4\pi\beta_3^2 \quad \text{and} \quad \chi = 4\pi\gamma_3\,, \tag{3.54}$$

or

$$\sigma^{(1)} = 4\pi\left[\alpha - \frac{\gamma_3^2}{\beta_3^2}\right], \qquad \sigma^{(2)} = 4\pi\,\frac{\gamma_3^2}{\beta_3^2} \quad \text{and} \quad \delta = \frac{\beta_3^2}{\gamma_3}\,, \tag{3.55}$$

if $\beta_3 \neq 0 \neq \gamma_3$. The limiting cases of vanishing β_3 and/or γ_3 can be treted by imposing suitable conditions on $\sigma^{(1)}+\sigma^{(2)}$, $\sigma^{(2)}\delta$ and $\sigma^{(2)}\delta^2$ at the end of a particular discussion.

The form (3.53) of the effective action $S^*_\Omega(\tilde{a},\tilde{w})$ has been gained by successively constructing terms which are invariant (up to boundary terms) under $\tilde{a} \to \tilde{a} + d\chi$, $w \to gwg^{-1} + gdg^{-1}$, $\tilde{w} \to g\tilde{w}g^{-1}$, and $\tilde{w}_3 \to \tilde{w}_3 + d\chi_s$ corresponding to $U(1)$-, $SU(2)$- and $U(1)_{\text{spin}}$-gauge invariance, respectively. The $U(1)_{\text{spin}}$-gauge invariance is a result of the particular choice of w_c as in (3.26). While at first sight one might think to have made exhaustive use of gauge invariance in the construction of $S^*_\Omega(\tilde{a},\tilde{w})$, there is an important observation yet to be made: Recalling the definition of the scaling limit in (3.6)–(3.9) we emphasize that (with respect to the scaled systems on $\lambda\Omega$) $\tilde{w}^{(\lambda)}_\mu(x)$ scales with λ^{-1} while $w_{c,\mu}(x)$ remains fixed. Therefore we expect $SU(2)$-gauge transformations, $w_{c,\mu}(x) + \tilde{w}^{(\lambda)}_\mu(x) = w^{(\lambda)}_\mu(x) \to g(x)w^{(\lambda)}_\mu(x)g^{-1}(x) + (g\partial_\mu g^{-1})(x)$, to mix terms of different scaling dimensions in $S^{\text{eff}}_{\lambda\Omega}(a^{(\lambda)}, w^{(\lambda)})$. Clearly $(g\partial_\mu g^{-1})(x)$ scales with λ^{-1}.

This then leads to further restrictions on the coefficients of the terms in (3.53). In order to be more explicit we recall a standard argument from non-abelian gauge theory. Writing $g(x) = e^{-\Lambda(x)}$ with $\Lambda(x) = i\vec{\Lambda}(x)\cdot\vec{\sigma} \in su(2)$, we denote by δ_Λ an infinitesimal $SU(2)$-gauge transformation,

$$\delta_\Lambda w^{(\lambda)}_{\mu A}(x) = i[\partial_\mu \Lambda_A(x) - 2\varepsilon_{ABC}w_{\mu B}(x)\Lambda_C(x)] \equiv i(D_\mu\Lambda)_A(x)\,. \tag{3.56}$$

$SU(2)$-gauge invariance of $S^{\text{eff}}_{\lambda\Omega}(a^{(\lambda)}, w^{(\lambda)})$ can then be expressed by

$$0 = \delta_A S^{\text{eff}}_{\lambda\Omega}(a^{(\lambda)}, w^{(\lambda)}) = \int\limits_{\lambda M_3} d^3x \, \frac{\delta S^{\text{eff}}_{\lambda\Omega}}{\delta w^{(\lambda)}_{\mu A}(x)} (a^{(\lambda)}, w^{(\lambda)}) \, \delta_A w^{(\lambda)}_{\mu A}(x)$$

$$= -i\hbar \int\limits_{\lambda M_3} d^3x \langle s^\mu_A(x)\rangle_{a,w} (D_\mu \Lambda)_A(x) \, , \tag{3.57}$$

where (3.4) has been used for the last equality. In (3.57), integrating by parts leads to the "covariant conservation" of the spin current

$$\begin{aligned} 0 &= (D_\mu \langle s^\mu \rangle_{a,w})_A(x) \\ &\equiv \partial_\mu \langle s^\mu_A(x)\rangle_{a,w} - 2\varepsilon_{ABC} w^{(\lambda)}_{\mu B}(x) \, \langle s^\mu_C(x)\rangle_{a,w} \, . \end{aligned} \tag{3.58}$$

In a slightly different form this reads

$$\partial_\mu \langle \vec{s}^\mu(x)\rangle_{a,w} = 2\vec{w}_{c,\mu}(x) \wedge \langle \vec{s}^\mu(x)\rangle_{a,w} + 2\vec{\tilde{w}}^{(\lambda)}_\mu(x) \wedge \langle \vec{s}^\mu(x)\rangle_{a,w} \, , \tag{3.59}$$

which makes evident the mixing of terms with different scaling dimensions if we notice that $\langle \vec{s}^\mu(x)\rangle_{a,w}$ is expanded in powers of λ (given by varying $S^{\text{eff}}_{\lambda\Omega}$ with respect to $\tilde{w}^{(\lambda)}_{\mu A}$). In Sect. 4 we will discuss the implications of (3.59) for the coefficients in (3.53) depending on different physical settings that might be considered. Finally we mention that (3.57) is valid of course only if there are no anomalies. That the theory is *anomaly-free* seems to be ensured by the fact that Eq. (3.58) or (3.59) can be derived purely from microscopic quantum mechanics [i.e., by a straightforward but somewhat lengthy calculation, only using the definition of the spin current (2.14) and the Pauli equation (2.20) or (2.21)]. A similar discussion with respect to $U(1)$-gauge invariance just leads to the continuity equation of the electromagnetic current $\langle j^\mu(x)\rangle_{a,w}$ which has already been taken into account in (3.53).

It follows from the definition of the effective gauge field action given in (3.5) that

$$\mathscr{Z}_{\lambda\Omega}(a^{(\lambda)}, w^{(\lambda)}) \underset{\lambda\to\infty}{\sim} \exp \frac{i}{\hbar} S^*_\Omega(\tilde{a}, \tilde{w}; a_c, w_c) \, ; \tag{3.60}$$

see (3.5)–(3.9). From the absence of gauge anomalies we know that $\mathscr{Z}_\Omega(a, w)$ is $U(1) \times SU(2)$-gauge invariant. This has some very important implications which we now briefly discuss; but see also [13, 2, 3, 19].

(1) Let us consider a system in infinite space, $M_3 = \mathbb{R} \times \Omega$, with $\Omega = \mathbb{R}^2$. We impose the boundary conditions that the gauge potentials a and w tend to pure gauges at infinity, i.e., $a(\xi) \to d\chi(\xi)$, $w(\xi) \to (gdg^{-1})(\xi)$, as $|\xi| \to \infty$. By general covariance of the Chern-Simons terms, we may then compactify M_3 to the 3-sphere S^3. Since $SU(2)$ is the 3-sphere, as well, there exist $SU(2)$-gauge transformations with non-trivial winding number, i.e., of non-zero degree. Let $g^{(n)}$ denote such a gauge transformation of degree n. Let us consider the factor

$$z_k(w) = \exp - \frac{ik}{4\pi} \int\limits_{S^3} \text{tr}\left(w \wedge dw + \frac{2}{3} \, w \wedge w \wedge w \right) \ldots \tag{3.61}$$

contributing to the partition function $\mathscr{Z}_{\lambda\mathbb{R}^2}(a^{(\lambda)}, w^{(\lambda)})$; see (3.53), (3.60). it is well known (see e.g. [25]) that

$$z_k({}^{g^{(n)}}w) = z_k(w) \exp(2\pi i k n) \, , \tag{3.62}$$

for arbitrary $n \in \mathbb{Z}$. There is no other term in $S^*_{\mathbb{R}^2}(\tilde{a}, \tilde{w})$ cancelling the factor $\exp(2\pi i k n)$. Thus the gauge invariance of the complete partition function yields the famous constraint

$$k \in \mathbb{Z}\,. \tag{3.63}$$

It will turn out that k is essentially the "Hall conductivity" for the spin current, and (3.63) establishes its *quantization.*

(2) Next, let us consider a system on a space-time of the form $M_3 = \mathbb{R} \times \Omega$, with $\partial\Omega$ non-empty, as usual. Then the three Chern-Simons terms

(a)
$$\frac{k}{4\pi} \int_{M_3} \mathrm{tr}\left(w \wedge dw + \frac{2}{3} w \wedge w \wedge w \right), \tag{3.64}$$

(b)
$$\frac{\sigma^{(1)}}{4\pi} \int_{M_3} \tilde{a} \wedge d\tilde{a}\,, \tag{3.65}$$

and

(c)
$$\frac{\sigma^{(2)}}{4\pi} \int_{M_3} (\tilde{a} + \delta\tilde{w}_3) \wedge d(\tilde{a} + \delta\tilde{w}_3) \tag{3.66}$$

in S^*_Ω are *not* invariant under gauge transformations not vanishing at the boundary of M_3, i.e., they are *"anomalous."* Term (a) displays the two-dimensional chiral $SU(2)$-anomaly, terms (b) and (c) the two-dimensional chiral $U(1)$-anomaly. Since the partition function $\mathscr{Z}_\Omega(a, w)$ is fully gauge-invariant, these anomalies must be cancelled by additional terms in $S^*_\Omega(\tilde{a}, \tilde{w})$ localized on the boundary of space-time M_3, i.e., by terms among those denoted by "b.t." in (3.53). The structure of these additional terms is well known [26]. They are the generating functions of the connected Green functions of chiral $SU(2)$-[term (a)] and chiral $U(1)$-[term (b)+term (c)] currents which are localized on ∂M_3 and form Kac-Moody algebras, see. e.g. [27]. These Kac-Moody algebras of chiral edge currents and their representations provide extremely interesting information on the physics of two-dimensional electronic systems [13, 2, 3]. A detailed analysis of anomaly cancellation and its physical consequences for two-dimensional condensed matter physics will appear in [19]. Among the results of our analysis are the following ones: The coefficients $\sigma^{(1)}$ and $\sigma^{(2)}$ of the terms (3.53″) in $S^*_\Omega(\tilde{a}, \tilde{w})$ are related to the Hall (or transverse) conductivity, $\sigma^{(1)} + \sigma^{(2)} = \sigma = \frac{h}{e^2}\,\sigma_H$. It is a rational number belonging to a certain discrete set that depends in an explicit way on the number of independent chiral $U(1)$-currents on ∂M_3. If $(\sigma^{(1)} + \sigma^{(2)}) \notin \mathbb{Z}$ the system has excitations of fractional electric charge and fractional (intermediate) statistics. If the coefficient, k, of the $SU(2)$-Chern-Simons term $\frac{1}{4\pi} \int_{M_3} \mathrm{tr}\left(w \wedge dw + \frac{2}{3} w \wedge w \wedge w \right)$ – which will turn out to be the Hall (or transverse) conductivity for the spin currents $\vec{s}$ – does not vanish then there are, in general, neutral excitations carrying $SU(2)$-spin with fractional statistics, so-called *spinons*. It turns out that spinons can be *fermions* $\left(\text{presumably realized in the } \sigma_H = \frac{5}{2}\frac{e^2}{h} \text{ quantum Hall fluid}\right)$ or *semions* ("half-fermions"; realized in Halperin spin-singlet quantum Hall fluids with $\sigma_H =$

$\frac{2}{4l+1}\frac{e^2}{h}, l = 1, 2 \ldots$) – besides more exotic possibilities which do not appear to be realized in the electron gas, but are encountered in two-dimensional systems of particles with higher spin. All this is discussed in detail in [19].

Let us briefly sketch one way towards understanding the quantization of the plateau-values of the Hall conductivity $\sigma_H = \sigma \frac{e^2}{h}$, i.e., of the coefficient of the term $\frac{1}{4\pi} \int_{M_3} \tilde{a} \wedge d\tilde{a}$ in the effective action developed in [2]. For simplicity we neglect spin effects, setting $\mu_e = 0$; but see [19] for the general case. The effective action in the scaling limit is then given by (3.24), (3.24′), i.e.,

$$\frac{i}{\hbar} S^*_\Omega(\tilde{a}) = -\left(i \int_{M_3} *j_c \wedge \tilde{a} + i \frac{\sigma}{4\pi} \int_{M_3} \tilde{a} \wedge d\tilde{a} + \text{b.t.} \right). \tag{3.67}$$

Let us suppose the system has only one conserved electromagnetic current, j. By Eqs. (2.38) and (3.15″) there then exists a quantized vector potential, $\mathscr{A}$, such that

$$j = *d\mathscr{A}. \tag{3.68}$$

We should ask which gauge theory for the gauge potential $\mathscr{A}$ reproduces the form (3.67) of the effective action in the scaling limit? The unique answer, found in [2], is that the gauge theory for $\mathscr{A}$ is given by the path integral

$$\mathscr{Z}(\tilde{a})^{-1} \int \exp\left(\frac{-i}{4\pi\sigma} \int \mathscr{A} \wedge d\mathscr{A} + \frac{i}{2\pi} \int \mathscr{A} \wedge d\tilde{a} + (\ldots) \right) \mathscr{D}\mathscr{A}, \tag{3.69}$$

where $(\ldots)$ refers to irrelevant terms and boundary terms. This gauge theory has excitations (static and point-like in the scaling limit) of charge

$$q = \int_{t=\text{const}} (\partial_1 \mathscr{A}_2 - \partial_2 \mathscr{A}_1) d^2\underline{x} = \pm 1,$$

i.e., of the charge of a hole or an electron (in units where $e = 1$). For these excitations to be fermions – as they must be if spin is neglected – it is necessary and sufficient that

$$\sigma_H \frac{h}{e^2} = \sigma = \frac{1}{2l+1}, \quad \text{for some } l \in \mathbb{Z}. \tag{3.70}$$

Besides electrons and holes the theory then describes excitations of fractional charge,

$$q = \pm \frac{n}{2l+1}, \qquad n = 1, \ldots, 2l,$$

which have fractional statistics. For details see [2], and for a general analysis (involving several independently conserved electromagnetic currents) see [13, 3, 19]. The general analysis reproduces *all* known plateau-values of σ_H!

In the next section we discuss the "transport equations" and sum rules for the current Green functions that follow from the form (3.53) and (3.53′) of the effective action $S^*_\Omega(\tilde{a}, \tilde{w})$.

4. Linear Response Theory and Current Sum Rules for Incompressible Electron Fluids

In this section, we determine the dependence on the external electromagnetic field $(\vec{E}, \vec{B})$ of expectation values of the electromagnetic and spin currents in essentially stationary states of a two-dimensional, incompressible electron fluid at very low temperatures. Using the form of the effective action in the scaling limit, found in Sect. 3, we calculate the current expectation values to leading order in the scale parameter λ.

From the Ward identities, Eqs. (3.4), (3.5), and the behaviour of currents, gauge potentials and the effective action under scale transformations determined in Eqs. (3.6)–(3.9) and (3.11), we derive the basic equations of "response theory":

$$\langle j^\mu(x)\rangle_{a,w} = \frac{-1}{\hbar\lambda^2}\left(\frac{\delta S_\Omega^*}{\delta\tilde{a}_\mu(\lambda^{-1}x)}\right)(\lambda a, \lambda w) + \mathcal{O}(\lambda^{-3}), \tag{4.1}$$

and

$$\langle s_A^\mu(x)\rangle_{a,w} = \frac{-1}{\hbar\lambda^2}\left(\frac{\delta S_\Omega^*}{\delta\tilde{w}_{\mu A}(\lambda^{-1}x)}\right)(\lambda a, \lambda w) + \mathcal{O}(\lambda^{-3}), \tag{4.2}$$

where λ is the scale parameter, and

$$a(x) = a_c(x) + \tilde{a}(x), \qquad w(x) = w_c(x) + \tilde{w}(x) \tag{4.3}$$

are the total electromagnetic vector potential and $SU(2)$-gauge potential, respectively, in "mathematical units." The basic hypothesis is that the system is incompressible when $a = a_c$, $w = w_c$. We are interested in predicting the response of the system to turning on additional external fields $\tilde{a}$, $\tilde{w}$ of order λ^{-1}, see Eq. (3.8). In our final equations we shall display only those terms contributing to $\langle j^\mu(x)\rangle_{a,w}$ and $\langle s_A^\mu(x)\rangle_{a,w}$ that are *linear* in $\tilde{a}$, $\tilde{w}$. (They are the leading terms in λ.)

As in Sect. 3, Eqs. (3.6)–(3.8), we propose to work in rescaled variables,

$$\left.\begin{aligned} x = \lambda\xi, \qquad \xi \in \Omega, \qquad \Omega \text{ fixed}, \\ a_\mu(x) \to a_\mu^{(\lambda)}(x) = \lambda^{-1}[a_{c,\mu}(\xi;\lambda) + \tilde{a}_\mu(\xi)], \\ w_\mu(x) \to w_\mu^{(\lambda)}(x) = \lambda^{-1}[w_{c,\mu}(\xi;\lambda) + \tilde{w}_\mu(\xi)], \\ \lambda^2 j^\mu(x) \to j^\mu(\xi), \qquad \lambda^2 s_A^\mu(x) \to s_A^\mu(\xi). \end{aligned}\right\}$$

Then Eqs. (4.1) and (4.2) read

$$\langle j^\mu(\xi)\rangle_{a,w} = -\hbar^{-1}(\delta/\delta\tilde{a}_\mu(\xi))S_\Omega^*(a,w), \tag{4.1'}$$

and

$$\langle s_A^\mu(\xi)\rangle_{a,w} = -\hbar^{-1}(\delta/\delta\tilde{w}_{\mu A}(\xi))S_\Omega^*(a,w), \tag{4.2'}$$

up to corrections of order λ^{-1} which we shall usually not display explicitly.

Before evaluating (4.1′) and (4.2′), we recall from (3.53) and (3.53′) the form of the effective action in the scaling limit. We write it in a form well suited for the following discussion. In an $SU(2)$-gauge where $w_{c,\mu A} = -\delta_{\mu 0}\delta_{A3}\frac{\mu_e}{2c}B_{c,3}$ [see

(3.26)] the effective action is given by

$$
\begin{aligned}
&-\frac{1}{\hbar}\,S^*_\Omega(\tilde a,\tilde w;a_c,w_c)\\
&=\int\limits_{M_3} j^\mu_c\tilde a_\mu d^3\xi+\int\limits_{M_3} m^\mu_3\tilde w_{\mu 3}d^3\xi\\
&+\frac{\sigma}{4\pi}\int\limits_{M_3}\varepsilon^{\mu\nu\varrho}\tilde a_\mu\partial_\nu\tilde a_\varrho d^3\xi+\frac{\chi}{2\pi}\int\limits_{M_3}\varepsilon^{\mu\nu\varrho}\tilde a_\mu\partial_\nu\tilde w_{\varrho 3}d^3\xi\\
&+\frac{\sigma_s}{4\pi}\int\limits_{M_3}\varepsilon^{\mu\nu\varrho}\tilde w_{\mu 3}\partial_\nu\tilde w_{\varrho 3}d^3\xi\\
&-\frac{k}{2\pi}\int\limits_{M_3}\varepsilon^{\mu\nu\varrho}\left\{w_{\mu A}\partial_\nu w_{\varrho A}-\frac{2}{3}\,\varepsilon_{ABC}w_{\mu A}w_{\nu B}w_{\varrho C}\right\}d^3\xi\\
&+\sum_{A=1}^{2}\int\limits_{M_3}\tau^{\mu\nu}_1\tilde w_{\mu A}\tilde w_{\nu A}d^3\xi+\sum_{A,B=1}^{2}\int\limits_{M_3}\tau^{\mu\nu}_2\varepsilon_{AB}\tilde w_{\mu A}\tilde w_{\nu B}d^3\xi\\
&+\sum_{A,B,C=1}^{3}{}'\int\limits_{M_3}\eta^{\mu\nu\varrho}_{ABC}\tilde w_{\mu A}\tilde w_{\nu B}\tilde w_{\varrho C}d^3\xi+\text{b.t.}
\end{aligned}
\tag{4.4}
$$

Here and throughout this section, summation convention is understood, for $\mu,\nu,\varrho=0,1,2$ and $A,B,C=1,2,3$, if "$\sum$" is not displayed explicitly. Furthermore, for $A,B=1,2$, ε_{AB} is the sign of the transposition (AB) of (12). [In the case where rotation invariance holds in the scaling limit the τ-terms can be reduced further, as shown in (3.46′).]

The formula for $\langle j^\mu(\xi)\rangle_{a,w}$ is somewhat simpler to evaluate than the one for $\langle s^\mu_A(\xi)\rangle_{a,w}$; so we start with the former. In expression (4.4) for $S^*_\Omega(a,w)$ only the first, third and fourth term depend on $\tilde a$. Combining (4.1′) with (4.4) we find that

$$\langle j^\mu(\xi)\rangle_{a,w}=j^\mu_c(\xi)+\frac{\sigma}{2\pi}\,\varepsilon^{\mu\nu\varrho}\partial_\nu(a-a_c)_\varrho+\frac{\chi}{2\pi}\,\varepsilon^{\mu\nu\varrho}\partial_\nu(w-w_c)_{\varrho 3}\,. \tag{4.5}$$

In order to elucidate the physical content of Eq. (4.5), we now pass from "mathematical units" to physical units. The connection between a_μ and w_μ and the corresponding physical gauge potentials A_μ and $\vec W_\mu$, respectively, is given in Eqs. (2.8)–(2.11):

$$a_\mu=\frac{e}{\hbar c}\,A_\mu\,,\qquad \vec w_\mu=\frac{\mu_e}{2c}\,\vec W_\mu\,, \tag{4.6}$$

with $\mu_e\approx\dfrac{-e}{m_0c}$ the magnetic moment of the electron (up to a factor $\hbar/2$). If the electromagnetic field $(\vec E,\vec B)$ is the only physical, external gauge field acting on the electron fluid then

$$W_{0A}(\xi)=-\,B_A(\xi)\,,\quad\text{and}\quad W_{lA}(\xi)=-\,\frac{1}{2}\,\varepsilon_{lAB}E_B(\xi)\,, \tag{4.7}$$

$A=1,2,3$, $l=1,2$. Note that, since $\vec B$ and $\vec E$ are electromagnetic field strengths, they have scaling- (or mass-) dimension 2. However, μ_e has scaling dimension -1, so that $\vec{w}'_\mu$ has again scaling dimension 1, as required of a gauge potential; see the discussion following (3.8).

We define the electromagnetic current, $\mathscr{J}^\mu$, in standard physical units by

$$\mathscr{J}^0 \equiv \varrho = ej^0 \,, \qquad \underline{\mathscr{J}} = ec\underline{j} \,. \tag{4.8}$$

Inserting (4.6) and (4.8) into (4.5), we find the equations

$$\langle \varrho(\xi) \rangle_{\vec{E},\vec{B}} = \varrho_c(\xi) + \frac{\sigma_H}{c}\, \tilde{B}_3(\xi) + \frac{\chi}{4\pi}\, \frac{e\mu_e}{c}\, \mathrm{curl}\, \underline{\tilde{W}}_3(\xi) \,, \tag{4.9}$$

where $\sigma_H = \sigma \frac{e^2}{h}$ is the Hall conductivity; (σ is dimensionless and can thus only depend on dimensionless parameters of the electron fluid, in particular on the filling factor ν). Expressing $\vec{W}_\mu$ in terms of $\vec{E}$ and $\vec{B}$, as in Eq. (4.7), (4.9) becomes

$$\langle \varrho(\xi) \rangle_{\vec{E},\vec{B}} = \varrho_c(\xi) + \frac{\sigma_H}{c}\, \tilde{B}_3(\xi) - \frac{\chi}{8\pi}\, \frac{e\mu_e}{c}\, \underline{\nabla} \cdot \underline{\tilde{E}}(\xi) \,, \tag{4.10}$$

where

$$\tilde{B}_A = B_A - B_{c,A} \,, \qquad \tilde{E}_A = E_A - E_{c,A} \,, \quad \text{and} \quad \underline{\nabla} \cdot \underline{\tilde{E}} = \frac{\partial}{\partial \xi_1}\, \tilde{E}_1 + \frac{\partial}{\partial \xi_2}\, \tilde{E}_2 \,.$$

We note that the Maxwell term

$$\frac{1}{2}\left(\frac{e}{\hbar c}\right)^2 \left[g^{(0)} \int_{M_3} \underline{\tilde{E}}^2(\xi)\, d^2\xi + g^{(1)} \int_{M_3} \tilde{B}_3^2(\xi)\, d^3\xi \right] , \tag{4.11}$$

where

$$\frac{e}{\hbar c}\, \tilde{E}_j = \partial_j \tilde{a}_0 - \partial_0 \tilde{a}_j \,, \quad \text{and} \quad \frac{e}{\hbar c}\, \tilde{B}_3 = \partial_1 \tilde{a}_2 - \partial_2 \tilde{a}_1 \,,$$

[see expression (3.23)], would yield another contribution

$$-g^{(0)} \frac{e^2}{\hbar c}\, \underline{\nabla} \cdot \underline{\tilde{E}}(\xi) \tag{4.12}$$

to the right-hand side of (4.10). The coupling constants $g^{(0)}$ and $g^{(1)}$ have scaling dimension -1, i.e., are lengths, and are characteristic of the widht of the system in the z-direction transverse to the plane of the system. One would expect that, in general, $g^{(0)}$, and $g^{(1)}$ are much larger than $\frac{\hbar c}{e^2}\frac{e|\mu_e|}{c} \approx \lambda_{\text{Compton}}$, where $\lambda_{\text{Compton}} = \frac{\hbar}{m_0 c}$ is the Compton wavelength of the electron. Combining (4.10) and (4.12) we have that

$$\langle \varrho(\xi) \rangle_{\vec{E},\vec{B}} = \varrho_c(\xi) + \frac{\sigma_H}{c}\, \tilde{B}_3(\xi) - l_0 \underline{\nabla} \cdot \underline{\tilde{E}}(\xi) \,, \tag{4.13}$$

with

$$l_0 = g^{(0)} \frac{e^2}{\hbar c} + \frac{\chi}{8\pi}\, \frac{e\mu_e}{c} \,. \tag{4.14}$$

For $l_0 = 0$, Eq. (4.13) reduces to an equation exploited in [2].

Setting $\mu \equiv i = 1, 2$ in (4.5) and using Eqs. (4.6)–(4.8) and (4.11), we arrive at the equation

$$\langle \mathscr{J}^i(\xi) \rangle_{\vec{E},\vec{B}} = \mathscr{J}_c^i(\xi) + \sigma_H \varepsilon^{ij} \tilde{E}_j(\xi) + l_0 \delta^{ij} \frac{\partial}{\partial \tau}\, \tilde{E}_j(\xi) + c l_1 \varepsilon^{ij} \partial_j \tilde{B}_3(\xi) \,, \tag{4.15}$$

where

$$l_1 = g^{(1)} \frac{e^2}{\hbar c} - \frac{\chi}{4\pi} \frac{e\mu_e}{c}, \tag{4.16}$$

ε^{ij} is the sign of the transposition (ij) of (12), and $\tau = \lambda^{-1} t$ is the rescaled time variable.

It should be noted that in Eqs. (4.13) and (4.15) *only* the component $\tilde{B}_3$ of the magnetic field *perpendicular* to the plane of the system and the components $\tilde{E}_1$ and $\tilde{E}_2$ of the electric field *parallel* to the plane of the system appear. Furthermore, these equations are manifestly consistent with the *continuity equation* for the electromagnetic current.

Next, we calculate the expectation value of the spin current in an external electromagnetic field. The general formula follows from the form (4.4) of the effective action. We present it in mathematical units:

$$\begin{aligned} \langle s_A^\mu(\xi)\rangle_{a,w} = {} & \delta_{A3} m_3^\mu(\xi) + \delta_{A3} \frac{\chi}{2\pi} \varepsilon^{\mu\nu\varrho} \partial_\nu \tilde{a}_\varrho(\xi) + \delta_{A3} \frac{\sigma_s}{2\pi} \varepsilon^{\mu\nu\varrho} \partial_\nu \tilde{w}_{\varrho 3}(\xi) \\ & - \frac{k}{\pi} \varepsilon^{\mu\nu\varrho} \{(\partial_\nu w_{\varrho A}(\xi) - \varepsilon_{ABC} w_{\nu B}(\xi) w_{\varrho C}(\xi)\} \\ & + 2(1-\delta_{A3}) \left\{ \tau_1^{\mu\nu}(\xi) \tilde{w}_{\nu A}(\xi) + \sum_{B=1}^{2} \varepsilon_{AB} \tau_2^{\mu\nu}(\xi) \tilde{w}_{\nu B}(\xi) \right\} \\ & + 3 \sum_{B,C=1}^{3}{}' \eta_{ABC}^{\mu\nu\varrho}(\xi) \tilde{w}_{\nu B}(\xi) \tilde{w}_{\varrho C}(\xi) . \end{aligned} \tag{4.17}$$

Terms quadratic in $\tilde{w}$ can be discarded within linear response theory. Thus the term $\frac{k}{\pi} \varepsilon^{\mu\nu\varrho} \varepsilon_{ABC} w_{\nu B}(\xi) w_{\varrho C}(\xi)$ can be replaced by

$$\frac{2k}{\pi} (1-\delta^{\mu 0})(1-\delta_{A3}) \sum_{i,B=1}^{2} w_{c,03}(\xi) \varepsilon^{\mu i} \varepsilon_{AB} \tilde{w}_{iB}(\xi), \tag{4.18}$$

for our choice, $w_{c,\mu A} = \delta_{A3}\delta_{\mu 0} w_{c,03}$, of the background gauge potential. For $\mu = 1, 2$ and $i = 1, 2$, $\varepsilon^{\mu i}$ is the sign of the transposition (μi) of (12). Moreover, the term proportional to $\eta_{ABC}^{\mu\nu\varrho}$ is quadratic in the fluctuation potential $\tilde{w}$ and hence can be dropped.

We should emphasize that Eq. (4.17) is *not* $SU(2)$-gauge invariant, but transforms under the adjoint representation of the $SU(2)$-gauge group. In particular, we recall that the spin current $m_3^\mu(\xi)$ is really the 3-component of an $su(2)$-vector $\vec{m}^\mu(\xi) = (0, 0, m_3^\mu(\xi))$, whose 1- and 2-components only vanish because w_c has only a non-vanishing 3-component; see (3.29) and (3.30). Similarly, the coefficient χ of the second term on the right-hand side of (4.17) is the 3-component of the $su(2)$-vector $\vec{\chi} = (0, 0, \chi)$, which is constant and whose 1- and 2-components again vanish only because of our special choice of w_c; see (3.54). Furthermore, the τ-terms should be understood as multiplying the orthogonal projection of $\vec{w}_\mu(\xi)$ onto the two-dimensional plane perpendicular to $\vec{m}^0(\xi)$.

With these remarks, we may rewrite Eq. (4.17) in an $SU(2)$-covariant form:

$$\begin{aligned}\langle \vec{s}^{\mu}(\xi)\rangle_{a,w} = \vec{m}^{\mu}(\xi) &+ \frac{\vec{\chi}}{2\pi}\,\varepsilon^{\mu\nu\varrho}\partial_{\nu}\tilde{a}_{\varrho}(\xi) + \frac{\sigma_s}{2\pi}\,\varepsilon^{\mu\nu\varrho}\partial_{\nu}\vec{\tilde{w}}_{\varrho}(\xi)\\ &- \frac{k}{\pi}\,\varepsilon^{\mu\nu\varrho}\{\partial_{\nu}\vec{w}_{\varrho}(\xi) - \vec{w}_{\nu}(\xi)\wedge\vec{w}_{\varrho}(\xi)\}\\ &+ 2\tau_1^{\mu\nu}(\xi)\,\{\vec{\tilde{w}}_{\nu}(\xi) - (\vec{\tilde{w}}_{\nu}(\xi)\cdot\hat{m}^0(\xi))\hat{m}^0(\xi)\}\\ &+ 2\tau_2^{\mu\nu}(\xi)\vec{\tilde{w}}_{\nu}(\xi)\wedge\hat{m}^0(\xi) + \dots\,,\end{aligned} \tag{4.17'}$$

where $\hat{m}^0(\xi) = \dfrac{\vec{m}^0(\xi)}{|\vec{m}^0(\xi)|}$.

In order to understand the physical content of formula (4.17′), we now specialize it to different components of $\vec{s}^{\mu}$ and rewrite it in physical units. We define the spin density by

$$\mathscr{S}^0_A(\xi) = \frac{\hbar}{2}\,s^0_A(\xi) \tag{4.19}$$

and the spin current density by

$$\mathscr{S}^i_A(\xi) = \frac{\hbar c}{2}\,s^i_A(\xi)\,, \qquad i = 1,2\,. \tag{4.20}$$

Using Eqs. (4.6) and (4.7) and omitting terms quadratic in $\vec{W}$, we find the following equation for $\mu = 0$ and $A = 3$:

$$\begin{aligned}\langle \mathscr{S}^0_3(\xi)\rangle_{\vec{E},\vec{B}} = M^0(\xi) &+ (\sigma^{\text{spin}}_{H1} - \sigma^{\text{spin}}_{H2})\,\frac{1}{2c}\,\underline{\nabla}\cdot\underline{\tilde{E}}(\xi)\\ &+ \frac{\chi}{4\pi}\,\frac{e}{c}\,\tilde{B}_3(\xi) + \dots\,,\end{aligned} \tag{4.21}$$

where the dots refer to terms of order $O\left(\hbar\left(\frac{\mu_e}{c}\right)^2(\vec{E}^2 + \vec{B}^2)\right)$, and

$$\sigma^{\text{spin}}_{H1} = \frac{k}{4\pi}\,\mu_e\hbar\,, \qquad \sigma^{\text{spin}}_{H2} = \frac{\sigma_s}{8\pi}\,\mu_e\hbar\,. \tag{4.22}$$

We recall that $|\mu_e|\,\frac{\hbar}{2} \equiv \mu_{\text{Bohr}} = 0.579\cdot 10^{-8}\,\text{eV Gauss}^{-1}$.

Let us briefly interpret the different terms in (4.21) physically.

(1) Up to a factor of μ_e, $M^0(\xi) = \frac{\hbar}{2}\,m^0_3(\xi)$ is the *magnetization* of the system in an external field $\vec{E} = \vec{E}_c = 0$, $\vec{B} = \vec{B}_c (= (0,0,B_c))$.

(2) The term $(\sigma^{\text{spin}}_{H1} - \sigma^{\text{spin}}_{H2})\,\frac{1}{2c}\,\underline{\nabla}\cdot\underline{\tilde{E}}(\xi)$ results from spin-orbit interactions and parity breaking and describes one aspect of the *"quantized Hall effect for spin currents."* As shown in Sect. 3, (3.63), the coefficient k is always an *integer*; so in the case of vanishing σ_s, this is an integer Hall effect. In general, repeating the discussion sketched at the end of Sect. 3 for the second Chern-Simons term in (3.53′), we may infer the rationality of σ_s; see (3.70). This then gives the *quantization* of the total coefficient $(\sigma^{\text{spin}}_{H1} - \sigma^{\text{spin}}_{H2})$. (Under normal circumstances, this term will be unobservably small.)

(3) The term $\frac{\chi}{4\pi}\frac{e}{c}\tilde{B}_3(\xi) \equiv \chi_\perp \mu_e^{-1}\tilde{B}_3(\xi)$ describes the response of the spin density $\langle \mathscr{S}_3^0(\xi)\rangle_{\vec{E},\vec{B}}$ in the direction perpendicular to the plane of the system to a change, $\tilde{B}_3(\xi)$, in the external magnetic field. Thus the coefficient

$$\chi_\perp = \frac{\chi}{4\pi}\frac{\mu_e}{c} \tag{4.23}$$

is the *magnetic susceptibility* in the direction transverse to the plane of the system. At the end of this section we shall see that the coefficient χ is quanized, so the *susceptibility* $\chi_\perp$ is *quantized*, too.

Next, we determine the spin density $\langle \mathscr{S}_A^0(\xi)\rangle_{\vec{E},\vec{B}}$ in a direction, $A = 1,2$, parallel to the plane of the system. From (4.17′), (4.6), (4.7), and (4.19) we obtain that

$$\begin{aligned}\langle \mathscr{S}_A^0(\xi)\rangle_{\vec{E},\vec{B}} = &-\sigma_{H1}^{\text{spin}}\frac{1}{2c}\,\partial_A\tilde{E}_3(\xi) + \chi_\parallel(x)\mu_e^{-1}\tilde{B}_A(\xi)\\ &+\frac{\mu_e\hbar}{4c}\left\{\sum_{B=1}^{2}\varepsilon_{AB}\tau_1^{0B}(\xi) - \tau_2^{0A}(\xi)\right\}\tilde{E}_3(\xi) + \dots\,,\end{aligned} \tag{4.24}$$

where the parallel magnetic susceptibility is given by

$$\chi_\parallel = -\frac{\mu_e^2\hbar}{2c}\,\tau_1^{00}(\xi)\,. \tag{4.25}$$

The first term on the right-hand side of (4.24) is another manifestation of the (integer) quantized Hall effect for the spin current.

Below, we shall derive restrictions on the coefficients τ_1 and τ_2 which follow from full $SU(2)$-gauge invariance of the theory.

We proceed to calculate the expectation values of the different components of the spin current densities. Let us start with $\langle S_3^i(\xi)\rangle_{\vec{E},\vec{B}}$, $i = 1,2$. From (4.17′), (4.6), (4.7), and (4.20) we find the equation

$$\begin{aligned}\langle \mathscr{S}_3^i(\xi)\rangle_{\vec{E},\vec{B}} = &\, M^i(\xi) + \sigma_{H1}^{\text{spin}}\varepsilon^{ij}\partial_j(B_{c,3} + \tilde{B}_3)(\xi)\\ &-\sigma_{H2}^{\text{spin}}\varepsilon^{ij}\partial_j\tilde{B}_3(\xi) - (\sigma_{H1}^{\text{spin}} - \sigma_{H2}^{\text{spin}})\delta^{ij}\,\frac{1}{2c}\frac{\partial}{\partial\tau}\,\tilde{E}_j(\xi)\\ &+\chi_\perp c\mu_e^{-1}\varepsilon^{ij}\tilde{E}_j(\xi) + \dots\,, \quad \text{for}\quad i = 1,2\,.\end{aligned} \tag{4.26}$$

The dots stand for terms of order $\mathscr{O}\left(\frac{\mu_e^2\hbar}{c}(\vec{\tilde{E}}^2 + \vec{\tilde{B}}^2)\right)$; $M^i \equiv \frac{\hbar c}{2}m_3^i$ represents a possible persistent spin current circulating in the system. the second, third and fourth term describe *the quantized Hall effect for spin currents.* We note that the second term describes again an integer quantized Hall effect, because $k \in \mathbb{Z}$, as follows from $SU(2)$-gauge invariance. The last term on the right-hand side of (4.26) is a cousin of the ordinary electromagnetic Hall effect. Terms like the second, third and the last one on the right-hand side of (4.26) are already predicted by classical physics. The surprising feature of quantum mechanics is that the coefficients $\sigma_{H1}^{\text{spin}}$ and $\sigma_{H2}^{\text{spin}}$ are "quantized," (i.e., belong to discrete sets of real numbers), for incompressible systems. Suppose we study a spin-singlet quantum Hall fluid or a two-dimensional, rotating incompressible spin fluid of neutral particles, such as a rotating film of superfluid ^3He, in an external electromagnetic field. Then the last term in (4.26) is absent ($\chi = 0$), while, in general, the first four terms are still present $\Big($replacing $\mu_e\,\frac{\hbar}{2}$ by the magnetic moment of the constituents of the corresponding quantum fluid$\Big)$. Such systems are studied in [15, 21].

Finally, we consider the expectation value of the spin current density $\mathscr{S}_A$, for $A = 1, 2$. Equation (4.17′) combined with (4.6), (4.7), and (4.20), readily implies that

$$\begin{aligned}\langle \mathscr{S}^i_A(\xi)\rangle_{\vec{E},\vec{B}} &= \sigma^{\text{spin}}_{H1}\delta^i_A \frac{1}{2c}\,\partial_\tau \tilde{E}_3(\xi) + \sigma^{\text{spin}}_{H1}\varepsilon^{ij}\partial_j \tilde{B}_A(\xi)\\ &\quad - \frac{\mu_e\hbar}{2}\sum_{B=1}^{2}\{\delta_{AB}\tau_1^{0i}(\xi) - \varepsilon_{AB}\tau_2^{0i}(\xi)\}\,\tilde{B}_B(\xi)\\ &\quad + \kappa^i_A(\xi)\tilde{E}_3(\xi) + \dots\,, \end{aligned} \tag{4.27}$$

for $i, A = 1, 2$, where

$$\kappa^i_A(\xi) = \frac{\mu_e\hbar}{4}\left\{\frac{k}{2\pi}\,\frac{\mu_e}{c}\,\varepsilon^i_A B_{c,3}(\xi) + \sum_{B=1}^{2}\varepsilon_{AB}\tau_1^{iB}(\xi) - \varepsilon^i_A\tau_2^{12}(\xi)\right\}. \tag{4.28}$$

The first two terms on the right-hand side of (4.27) describe again an *integer* quantized Hall effect for spin currents. The third term will be studied more closely in our subsequent discussion. It is absent if the system is rotation invariant in the scaling limit: see (3.43). The strange last term comes from the τ-terms and the term $\frac{k}{6\pi}\int \operatorname{tr}(w\wedge w\wedge w)$ in the effective action. It describes some kind of "zitterbewegung" which appears to be unobservably small.

Next, we derive further constraints on the coefficients of the different terms in the effective action given in Eq. (4.4) which depend on the physical situation under consideration. We first recall Eq. (3.58), the "covariant conservation" law of the spin current, which we showed to be a consequence of $SU(2)$-gauge invariance in Sect. 3. In rescaled variables, Eq. (3.58) takes the following form; [see (3.6)–(3.8), we return to work in "mathematical units"]:

$$\begin{aligned}\partial_\mu\langle \vec{s}^{\,\mu}(\xi)\rangle_{a,w} &\equiv \frac{\partial}{\partial\xi^0}\langle \vec{s}^{\,0}(\xi)\rangle_{a,w} + \underline{\nabla}\cdot\langle \underline{\vec{s}}(\xi)\rangle_{a,w}\\ &= 2\vec{\tilde{w}}_\mu(\xi;\lambda)\wedge\langle \vec{s}^{\,\mu}(\xi)\rangle_{a,w}\,, \end{aligned} \tag{4.29}$$

where $\underline{\vec{s}} = \langle \vec{s}^{\,1}, \vec{s}^{\,2}\rangle$, $\underline{\nabla} = \left(\frac{\partial}{\partial\xi^1}, \frac{\partial}{\partial\xi^2}\right)$, $\vec{\tilde{w}}_\mu = \frac{1}{2i}\operatorname{tr}(w_\mu\vec{\sigma})$, and $\wedge$ denotes the usual vector product. In components, Eq. (4.29) reads

$$\begin{aligned}&\frac{\partial}{\partial\xi^0}\langle s^0_A(\xi)\rangle_{a,w} + \underline{\nabla}\cdot\langle \underline{s}_A(\xi)\rangle_{a,w}\\ &\quad= 2\varepsilon_{ABC}\{w_{c,\mu B}(\xi;\lambda) + \tilde{w}_{\mu B}(\xi)\}\,\langle s^\mu_C(\xi)\rangle_{a,w}\,. \end{aligned} \tag{4.29′}$$

We now determine the behaviour of both sides of Eq. (4.29′) when the scale parameter λ becomes large, using Eqs. (3.6)–(3.9) and (4,17). For the left-hand side (l.h.s.) we find

$$\begin{aligned}\text{l.h.s.} &= \delta_{A3}\partial_\mu m^\mu_3(\xi)\\ &\quad + 2(1-\delta_{A3})\sum_{B=1}^{2}\varepsilon_{AB}\left\{\frac{k}{\pi}\,\varepsilon^{ij}\partial_i(w_{c,03}\tilde{w}_{jB})(\xi) + \partial_\mu(\tau_2^{\mu\nu}\tilde{w}_{\nu_B})(\xi)\right\}\\ &\quad + 2(1-\delta_{A3})\partial_\mu(\tau_1^{\nu\mu}\tilde{w}_{\nu A})(\xi) + \dots\,, \end{aligned} \tag{4.30}$$

where the values of i and j range over 1 and 2, and the dots stand for terms of lower order in λ; (see below). Next, we study the right-hand-side (r.h.s.) of (4.29′). We recall that $\tau_1^{\mu\nu}$ is symmetric, while $\tau_2^{\mu\nu}$ is anti-symmetric in μ and ν. We then find

$$\begin{aligned}
\text{r.h.s.} = &-2(1-\delta_{A3})\sum_{B=1}^{2}\varepsilon_{AB}w_{c,03}(\xi)\,\langle s_B^{\mu}(\xi)\rangle_{a,w} + 2\varepsilon_{ABC}\tilde{w}_{\mu B}(\xi)\,\langle s_C^{\mu}(\xi)\rangle_{a,w} \\
= &\ 2(1-\delta_{A3})\sum_{B=1}^{2}\varepsilon_{AB}m_3^{\mu}(\xi)\tilde{w}_{\mu B}(\xi) \\
&+\frac{2k}{\pi}(1-\delta_{A3})\sum_{B=1}^{2}\varepsilon_{AB}\varepsilon^{ij}\partial_i(w_{c,03}\tilde{w}_{jB})(\xi) \\
&-4(1-\delta_{A3})\left\{\sum_{B=1}^{2}\varepsilon_{AB}\tau_1^{0\nu}(\xi)\tilde{w}_{\nu B}(\xi)-\tau_2^{0j}(\xi)\tilde{w}_{jA}(\xi)\right\}w_{c,03}(\xi) \\
&-4(1-\delta_{A3})\left\{\sum_{B=1}^{2}\varepsilon_{AB}\tau_1^{\mu\nu}(\xi)\tilde{w}_{\nu B}(\xi)-\tau_2^{\mu\nu}(\xi)\tilde{w}_{\nu A}(\xi)\right\}\tilde{w}_{\mu 3}(\xi) \\
&-6(1-\delta_{A3})\sum_{B=1}^{2}\varepsilon_{AB}\sum_{C,D=1}^{3}{}' \eta_{BCD}^{0\nu\varrho}(\xi)\tilde{w}_{\nu C}(\xi)\tilde{w}_{\varrho D}(\xi)w_{c,03}(\xi)+\dots \qquad (4.31)
\end{aligned}$$

Equations (4.30) and (4.31) are valid in an $SU(2)$-gauge where w_c satisfies $w_{c,\mu A}(\xi) = -\delta_{\mu 0}\delta_{A3}\,\frac{\mu_e}{2c}\,B_{c,3}(\xi)$, and we recall that

$$w_{c,\mu A}(\xi) \equiv \text{“}\lim_{\lambda\to\infty}\text{”}\,\lambda w_{c,\mu A}(\lambda\xi) = \text{“}\lim_{\lambda\to\infty}\text{”}\,w_{c,\mu A}(\xi;\lambda); \qquad (4.32)$$

see (3.8) and (3.36). Combining (4.32) and the discussion in Sect. 3 on the scaling properties of the current correlators [see (3.13)], we may order the terms in (4.30) and (4.31) according to their scaling dimension D, (behaving like λ^{-D}, as $\lambda \nearrow \infty$). For the terms in Eq. (4.30), one finds the values $D = -2$, -1, and -1, respectively. the *dots* stand for *marginal* and *irrelevant* terms ($D \geq 0$). Likewise, the terms in (4.31) have the values $D = -2, -1, -2, -1$, and -1. We note that subleading terms in the spin current, (behaving like λ^{-1} as $\lambda \nearrow \infty$, and not considered in this paper) could give rise to marginal ($D = 0$) contributions to the right-hand side of (4.29′) when combined with $w_{c,\mu}(\xi;\lambda)$. This is the reason for displaying only the $D = -2$ and -1 terms in Eqs. (4.30) and (4.31). Equating the terms of equal dimension D on the left-hand side, (4.30), and the right-hand side, (4.31), we find the following constraints [in the $SU(2)$-gauge considered above]:
(a) Setting $A = 3$, the $(D = -2)$-terms give

$$\partial_\mu m_3^{\mu}(\xi) = 0\,. \qquad (4.33)$$

This constraint has already been found in (3.34), as a consequence of $U(1)_{\text{spin}}$-gauge invariance.

(b) For $A = 1,2$, the $(D = -2)$-terms imply

$$0 = \sum_{B=1}^{2} \varepsilon_{AB}\{m_3^{\mu}(\xi) - 2\tau_1^{0\mu}(\xi)\, w_{c,03}(\xi)\}\, \tilde{w}_{\mu B}(\xi)$$

$$+ 2w_{c,03}(\xi) \sum_{j=1}^{2} \tau_2^{0j}(\xi)\, \tilde{w}_{jA}(\xi)\,. \tag{4.34}$$

(c) Finally, for $A = 1,2$ the $(D = -1)$-terms lead to

$$\partial_\mu \left\{ \tau_1^{\mu\nu} \tilde{w}_{\nu A} + \sum_{B=1}^{2} \varepsilon_{AB} \tau_2^{\mu\nu} \tilde{w}_{\nu B} \right\} (\xi)$$

$$= -2 \left\{ \sum_{B=1}^{2} \varepsilon_{AB} \tau_1^{\mu\nu}(\xi)\, \tilde{w}_{\nu B}(\xi) - \tau_2^{\mu\nu}(\xi)\, \tilde{w}_{\nu A}(\xi) \right\} \tilde{w}_{\mu 3}(\xi)$$

$$- 3 \sum_{B=1}^{2} \varepsilon_{AB} \sum_{C,D=1}^{3}{}' \eta_{BCD}^{0\nu\varrho}(\xi)\, w_{c,03}(\xi)\, \tilde{w}_{\nu C}(\xi)\, \tilde{w}_{\varrho D}(\xi)\,. \tag{4.35}$$

We propose to discuss the implications of the constraints (b) and (c) in several physically distinct situations. Unless stated otherwise, we always assume the two-dimensional system to be incompressible for certain *non-zero* values of the background potential w_c. Unless specified otherwise, the indices can take values as follows: $\mu, \nu = 0,1,2$, $k, l = 1,2$, and $A = 1,2,3$.

Case (1). For *arbitrary* fluctuation potentials $\tilde{w}_{\mu A}(\xi)$ (in some Schwartz space, $S(M_3)$, over M_3), (c) implies

$$\tau_1^{\mu\nu}(\xi) = 0 = \tau_2^{\mu\nu}(\xi) \quad \text{and} \quad \eta_{kl3}^{0\mu\nu}(\xi) = 0\,, \tag{4.36}$$

and from (b) we find that

$$\tau_1^{0\mu}(\xi) = \frac{m_3^{\mu}(\xi)}{2w_{c,03}(\xi)}\,. \tag{4.37}$$

Together with (4.36), this implies that

$$M^{\mu}(\xi) \equiv \frac{\hbar}{2}\, m_3^{\mu}(\xi) = 0\,, \quad \text{for all } \mu\,, \tag{4.38}$$

i.e., the magnetization M^0 and the persistent spin current $\underline{M}$ of the system in the background field w_c vanish. This means that the groundstate of the system is essentially a *spin-singlet state*. We would expect, however, that, generically, systems subject to a strong external magnetic field ($w_c \neq 0$) exhibit a *non-zero magnetization*, $M_3^0 \neq 0$. Our conclusion (4.38) might thus appear to be puzzling! We have to analyze where the solution to this puzzle lies. To this end, we must draw attention to a somewhat subtle aspect of our analysis that we have not elucidated, so far, namely the *differentiability properties* of the effective actions $S_{\lambda\Omega}^{\text{eff}}(a, w)$, $1 \leq \lambda < \infty$, in the fluctuation fields $\tilde{a}$, $\tilde{w}$, for a given background field a_c, w_c.

Presently, we are only interested in the differentiability properties with respect to the $SU(2)$-gauge potential $w = w_c + \tilde{w}$, and thus we suppress the $U(1)$-gauge potential $a = a_c + \tilde{a}$ in the following. To be more precise, we have assumed, so far, that $S_{\lambda\Omega}^{\text{eff}}(w)$ be *four times* continuously (Fréchet) *differentiable* in w on a Schwartz space neigbourhood, $\mathscr{S}_1(\lambda M_3)$, of $w_c(x)$. The space $\mathscr{S}_1(\lambda M_3)$ consists of fluctuation potentials of the form $\tilde{w}_{\mu A}(x) \equiv \tilde{w}^{(\lambda)}_{\mu A}(x) = \lambda^{-1}\tilde{w}_{\mu A}(\xi)$ [see (3.8)], where $\lambda\xi = x$, and $\tilde{w}_{\mu A}(\xi) \in \mathscr{S}(M_3)$, the space of smooth fluctuation potentials of "rapid decrease" on M_3. We have assumed, for example, the existence of a continuous linear map $D_{\mu A}S^{\text{eff}}_{\lambda\Omega}(w_c) \equiv \dfrac{\delta S^{\text{eff}}_{\lambda\Omega}}{\delta w_{\mu A}}$ on $\mathscr{S}_1(\lambda M_3)$ such that

$$\frac{|S^{\text{eff}}_{\lambda\Omega}(w_c+\tilde{w}) - S^{\text{eff}}_{\lambda\Omega}(w_c) - D_{\mu A}S^{\text{eff}}_{\lambda\Omega}(w_c)\tilde{w}_{\mu A}|}{\|\tilde{w}_{\mu A}\|_{\mathscr{S}_1}} \to 0\,, \tag{4.39}$$

as $\|\tilde{w}_{\mu A}\|_{\mathscr{S}_1} \to 0$, for all $\tilde{w}_{\mu A} \in \mathscr{S}_1(\lambda M_3)$.

Just as well as full (Fréchet) differentiability of $S^{\text{eff}}_{\lambda\Omega}$ on $\mathscr{S}_1(\lambda M_3)$ at w_c, we could only have assumed the existence of *directional* (Gateaux) *derivatives* of $S^{\text{eff}}_{\lambda\Omega}$ at w_c in *particular* directions $\tilde{w}_{\mu A} \in \mathscr{S}_i(\lambda M_3)$, with $\mathscr{S}_i(\lambda M_3)$, $i = 2, 3, \ldots$, certain *subspaces* of $\mathscr{S}_1(\lambda M_3)$. As a matter of fact, the puzzle connected with Eq. (4.38) suggests that a selfconsistent analysis of the physics of the system will show *which space* of fluctuation potentials is to be considered, i.e., what kind of differentiability properties of $S^{\text{eff}}_{\lambda\Omega}(w)$ to expect. We emphasize that, in the analysis of Sect. 3, it was *not* necessary to precisely specify the space of fluctuation fields, $\tilde{w}$, for which Eq. (3.53) for the effective action S^*_Ω holds, since in Eq. (3.53) we have found the *most general* form of S^*_Ω compatible with general principles. It is only in dicussing the *full implications* of $SU(2)$-gauge invariance, Eq. (4.29), that specifying more precise differentiability properties of $S^{\text{eff}}_{\lambda\Omega}$ in w, for large values of λ, becomes essential. A selfconsistent analysis shows that these differentiability properties of the effective action are closely related to specific physical properties of the quantum fluid in a given background field w_c. We propose to consider some typical examples.

Case (2). Since we are primarily interested in studying essentially stationary states of incompressible quantum fluids, it is natural to investigate the consequences of the assumption that the effective action, for a specific choice of gauge, is four times continuously differentialbe in w at w_c *only* on a space, $\mathscr{S}_2(\lambda M_3)$, of fluctuation fields $\tilde{w}$ which are time-independent or, at least, are so slowly varying in time that time derivatives of $\tilde{w}$ can be neglected in constraint (c), Eq. (4.35). More precisely, we assume $S^{\text{eff}}_{\lambda\Omega}$ $(1 \le \lambda < \infty)$ to be four times (Fréchet) differentiable in a neighbourhood of $\tilde{w} = 0$ of the spaces

$$\begin{aligned}\mathscr{S}_2(\lambda M_3) = \{&\tilde{w}_{\mu A} \in \mathscr{S}_1(\lambda M_3) : \tilde{w}_{\mu A}(x) = \lambda^{-1}\tilde{w}_{\mu A}(\xi;\lambda),\\ &\text{with } \tilde{w}_{\mu A}(\xi;\lambda) \in \mathscr{S}(M_3)\,, \partial_0\tilde{w}_{\mu A}(\xi;\lambda) = O(\lambda^{-1}\}\,.\end{aligned} \tag{4.40}$$

The terms proportional to $\dfrac{\partial}{\partial x^0}\,\tilde{w}_{\mu A}(x)$ therefore scale with an additional factor of λ^{-1} and drop out of constraint (c), as $\lambda \to \infty$. The implications of (4.29) in this case are as in (4.36), *except* that the coefficients $\tau^{00}_1(\xi)$ and $\eta^{000}_{kk3}(\xi)$ need *not* vanish, but must *only* satisfy the equations

$$\partial_0\tau^{00}_1(\xi) = 0\,, \quad \text{and} \quad \eta^{000}_{kk3}(\xi) = -\,\frac{\tau^{00}_1(\xi)}{3w_{c,03}(\xi)}\,. \tag{4.41}$$

Furthermore, Eq. (4.37) follows as in Case (1).

We conclude that

$$M^{\mu}(\xi) = \left(\frac{\hbar}{2}\, m_3^0(\xi), \underline{0}\right), \quad \text{with} \quad m_0^3 \neq 0\,, \quad \text{in general,} \tag{4.42}$$

i.e., the system may exhibit a *non-zero magnetization* but *no* persistent spin (super-) current in the background field w_c. Note that, from (4.41), (4.37), and (4.42) and constraint (a), it follows that, for consistency,

$$\partial_0 w_{c,03}(\xi) = 0\,.$$

It may be useful to discuss these findings a little further: For $S^{\text{eff}}_{\lambda\Omega}(w_c + \tilde{w})$ to be several times continuously differentiable in $\tilde{w}$ at $\tilde{w} = 0$ on some space $\mathscr{S}_i(\lambda M_3)$, a weak form of *incompresibility* must hold for all potentials $w = w_c + \tilde{w}$, with $\tilde{w} \in \mathscr{S}_i(\lambda M_3)$. Equation (4.38) tells us, therefore, that incompressibility of a system with *non-zero* magnetization in a background potential w_c is *unstable* against perturbations $w_c \to w_c + \tilde{w}$, for arbitrarily *small* but *strongly time-dependent* potentials $\tilde{w}$. In contrast, the result in Case (2) says that a form if incompressibility for a system in a suitable background potential w_c may be *stable* against tiny perturbations $w_c \to w_c + \tilde{w}$, provided $\tilde{w}$ is only *very weakly time-dependent, and in an $SU(2)$-gauge where w_c is time-independent.*

Case (3). In addition to the restrictions on $\tilde{w}_{\mu A}$ in Case (2) we might also choose the *spatial* variations of $\tilde{w}_{\mu A}$ to be very small, i.e., to assume differentiability of $S^{\text{eff}}_{\lambda\Omega}$ on

$$\begin{aligned}\mathscr{S}_3(\lambda M_3) = \{\tilde{w}_{\mu A} \in \mathscr{S}_1(\lambda M_3) \colon & \tilde{w}_{\mu A}(x) = \lambda^{-1}\tilde{w}_{\mu A}(\xi;\lambda), \\ & \text{with } \tilde{w}_{\mu A}(\xi;\lambda) \in \mathscr{S}(M_3) \text{ and} \\ & \partial_\nu \tilde{w}_{\mu A}(\xi;\lambda) = O(\lambda^{-1}), \text{ for all } \mu,\nu\}\,. \end{aligned} \tag{4.43}$$

From constraint (b) we then derive again Eq. (4.37), together with

$$\tau_2^{0k}(\xi) = 0\,. \tag{4.44}$$

All components of $\tau_1^{\mu\nu}$ can, in principle, be non-zero, (taking into account, of course, (3.42)), but they must satisfy

$$\partial_\mu \tau_1^{\mu\nu}(\xi) = 0\,, \quad \text{and} \quad \partial_l \tau_2^{lk}(\xi) = 0\,. \tag{4.45}$$

Furthermore, there are relations between η- and τ-components of a similar type as in (4.41) which we do not wish to display explicitly. In conclusion, systems to which Case (3) applies can exhibit a *non-zero* magnetization $M^0(\xi)$ *and*, possibly, support a *persistent spin* (super-) *current*. Combining Eqs. (4.45), (4.37) and constraint (a), it follows that, for consistency, the (rescaled) background field $w_{c,03}(\xi)$ [see (4.32)] must be *constant* on M_3. [This determines, in part, our choice of an $SU(2)$-gauge!]

Case (4). Since in the $SU(2)$-gauge in which we are working, the identifications (4.7), hold, i.e.,

$$\tilde{w}_{0A}(\xi) = -\frac{\mu_e}{2c}\,\tilde{B}_A(\xi)\,, \quad \text{and} \quad \tilde{w}_{lA}(\xi) = -\frac{\mu_e}{4c}\,\varepsilon_{lAB}\tilde{E}_B(\xi)\,, \tag{4.46}$$

one may wish to repeat our analysis in Cases (1)–(3), assuming differentiability of the effective actions *only* on subspaces of fluctuation potentials as described here and, in addition, requiring the fluctuation potentials to be of the form (4.46). The results are similar to those in Cases (1)–(3), but slightly less restrictive. As an example we consider a situation similar to that corresponding to Case (3). We define

$$\mathscr{S}_4(\lambda M_3) = \{\tilde{w}_{\mu A} \in \mathscr{S}_3(\lambda M_3) : \tilde{w}_{0A}(\xi;\lambda) = b_A(\xi;\lambda), \tilde{w}_{lA}(\xi;\lambda) = \varepsilon_{lAB} e_B(\xi;\lambda), \\ \text{with } \partial_\mu b_A(\xi;\lambda), \partial_\mu e_A(\xi;\lambda) = O(\lambda^{-1}), \text{ for all } \mu \text{ and } A\}.$$

In this case, *no* component of $\tau_1^{\mu\nu}$ and $\tau_2^{\mu\nu}$ has to vanish [other than $\tau_2^{\mu\nu}$ which vanish by antisymmetry; see (3.42)]. The only restrictions are

$$\partial_\mu \tau_1^{0\mu}(\xi) = 0, \qquad \partial_l \tau_2^{0l}(\xi) = 0, \tag{4.47}$$

and

$$\partial_\mu \left\{ \tau_1^{\mu k} + \sum_{l=1}^{2} \varepsilon^{kl} \varrho_2^{\mu l} \right\}(\xi) = 0. \tag{4.48}$$

All components of $M^\mu(\xi)$ can be non-zero, and we have the following relations:

$$m_3^0(\xi) = 2w_{c,03}(\xi)\tau_1^{00}(\xi), \tag{4.37'}$$

and

$$m_3^k(\xi) = 2w_{c,03}(\xi) \left\{ \tau_1^{0k} + \sum_{l=1}^{2} \varepsilon^{kl} \tau_2^{0l} \right\}(\xi). \tag{4.49}$$

Again, one can derive relations between η- and τ-components which, however, are of little interest in linear response theory and are therefore not presented here.

Case (5). Let us finally consider a two-dimensional quantum fluid which is incompressible in a *vanishing* background potential, i.e., for $w_c \equiv 0$, and let us assume that the effective action is four times differentiable on some space $\mathscr{S}_i(\lambda M_3)$. Then we infer from constraint (b) [Eq. (4.34)] that $M^\mu(\xi)$ *must vanish identically.* Our conclusion is *independent* of the particular choice of the space $\mathscr{S}_i(\lambda M_3)$, for $i = 1, 2, 3, 4$. This result is a variant of the *Goldstone theorem* [28]: If any component of $M^\mu(\xi)$, in particular the magnetization $\mu_e M^0(\xi)$, does *not* vanish in the limit where w_c tends to 0 then the system *cannot* be incompressible in a vanishing background magnetic field. In other words, the system must exhibit gapless excitations, the Goldstone bosons, coupled to the groundstate by the spin current.

It is necessary to discuss the main formulas of linear response theory, see Eqs. (4.13), (4.15), (4.21), (4.24), (4.26), and (4.27), in some more detail and to ask whether there are relations between the four fundamental parameters, σ, χ, σ_s, and k of the theory.

First, we note once more that Eqs. (4.13) and (4.15) confirm that

$$(\langle \varrho(\xi) \rangle_{\vec{E},\vec{B}}, \langle \underline{\mathscr{J}}(\xi) \rangle_{\vec{E},\vec{B}})$$

satisfies the continuity equation, (i.e., is a conserved, classical current), on account of Faraday's induction law (in $2+1$ dimensions)

$$\frac{1}{c}\frac{\partial}{\partial \tau} \tilde{B}_3(\xi) + \partial_1 \tilde{E}_2(\xi) - \partial_2 \tilde{E}_1(\xi) = 0. \tag{4.50}$$

Second, if the background field $\vec{E}_c$, $\vec{B}_c$ is chosen to be of the form $\vec{E}_c = 0$, $\vec{B}_c = (0, 0, B_{c,3}(\xi))$, then the current

$$(\langle S_3^0(\xi) \rangle_{\vec{E},\vec{B}}, \langle \underline{S}(\xi) \rangle_{\vec{E},\vec{B}})$$

satisfies the continuity equation to first order in $\vec{\tilde{E}}$ and $\vec{\tilde{B}}$, as expected. This is seen from Eqs. (4.21) and (4.26), by using (4.33) and (4.50).

Finally, formulas (4.24) and (4.27) show that, for $A = 1$, or 2, and in the situation of, for example, Case (2),

$$\frac{\partial}{\partial\tau}\langle S_A^0(\xi)\rangle_{\vec{E},\vec{B}} + \underline{\nabla}\cdot\langle \underline{S}_A(\xi)\rangle_{\vec{E},\vec{B}} = \mu_e^{-1}\chi_{\|}(\xi)\frac{\partial}{\partial\tau}\tilde{B}_A(\xi) + \varepsilon_A^i\partial_i(\kappa\tilde{E}_3)(\xi), \quad (4.51)$$

where we have used that $\partial_\tau\chi_{\|} = 0$, as can be seen from definition (4.25) and Eq. (4.41), and κ is defined by

$$\kappa(\xi) = k\,\frac{\mu_e^2\hbar}{8\pi c}\,B_{c,3}(\xi). \quad (4.51')$$

Let us suppose that $\vec{B}_c(\xi)$ is constant, so that $\kappa(\xi)$ is constant as well. If $\vec{\tilde{E}}(\xi) = \vec{\tilde{E}}(\xi^0, \xi^1, \xi^2)$ is independent of ξ^3, for $\xi^3 \approx 0$, (i.e., in the vicinity of the plane of the system), we have that

$$\varepsilon_A^i\partial_i(\kappa\tilde{E}_3)(\xi) = -\,\kappa(\mathrm{curl}\,\vec{\tilde{E}})_A(\xi).$$

Then if

$$\mu_e^{-1}c\chi_{\|} = -\,\kappa = \text{const.} \quad (4.52)$$

it follows from Faraday's law in $3+1$ dimensions, i.e.,

$$\frac{1}{c}\frac{\partial}{\partial\tau}\tilde{B}_A(\xi) + (\mathrm{curl}\,\vec{\tilde{E}})_A(\xi) = 0,$$

that the right-hand side of Eq. (4.51) vanishes, i.e., that $(\langle \mathscr{S}_A^0(\xi)\rangle_{\vec{E},\vec{B}}\,\langle \underline{\mathscr{S}}_A(\xi)\rangle_{\vec{E},\vec{B}})$ is conserved to first order in $\vec{\tilde{B}}$ and $\vec{\tilde{E}}$.

It follows from (4.25), (4.51′), and (4.37) that Eq. (4.52) would hold, provided

$$k = -\frac{8\pi c^2}{\mu_e^2\hbar}\frac{M^0}{B_{c,3}^2}. \quad (4.53)$$

If the system does not exhibit spontaneous magnetization, as $\vec{B}_c \to 0$, M^0 is proportional to $B_{c,3}$, for $B_{c,3}$ small, and (4.53) would imply that

$$k = \frac{\text{const.}}{B_{c,3}} = \text{const.'}\nu, \quad (4.54)$$

for small $B_{c,3}$, where ν is the filling factor. However, for an incompressible quantum Hall fluid, k must be an integer [see (3.62), (3.63)], and relations (4.53), (4.54) will therefore be *at best* approximately valid. Thus the currents $(\langle \mathscr{S}_A^0(\xi)\rangle_{\vec{E},\vec{B}}, \langle \underline{\mathscr{S}}_A(\xi)\rangle_{\vec{E},\vec{B}})$ are, in general, *not* conserved, even in first order in $\vec{\tilde{E}}$, $\vec{\tilde{B}}$, as one might expect; (see Sect. 2). Approximate conservation of these currents would imply approximate validity of Eqs. (4.53) and (4.54), i.e., $k \propto \nu$, or $ke^2/\sigma_H h \approx$ const., for large ν, because $\sigma_H \approx \dfrac{e^2}{k}\nu$, for large values of ν. This would mean that, for large filling factors ν, the number of spin-singlet bands would be large. There are no obvious reasons why this should be the case, but these remarks pose, at least, an interesting problem – relations between k and ν.

These considerations bring us to the next topic, that of *relations between the fundamental parameters*, σ, χ, σ_s and k, characterizing a two-dimensional, incompressible electron fluid; [see formula (4.4) for the effective action S^*_Ω, or Eqs. (3.53), (3.53′) and (3.54)]. Here it is convenient to work with dimensionless quantities. The conductivities σ_H, $\sigma_{Hi}^{\text{spin}}$, $i = 1, 2$, etc., can easily be computed from σ, χ, σ_s and k. The problem of relations between σ, χ, σ_s and k requires a more careful study of the quantum dynamics of the system than we wish to present in the present paper. We therefore just summarize some elementary considerations and defer a detailed analysis to another publication, [19].

The integer k counts the number of spin-singlet (edge current) bands of the quantum Hall fluid. If the fluid has a single (edge current) band which is a spin singlet then

$$k = 1\,, \qquad \sigma = \frac{2}{4l+1}\,, \qquad l = 0, 1, 2, \ldots\,, \qquad \sigma_s = 0\,, \qquad \chi = 0\,; \tag{4.55}$$

this follows from results in [27]; see [2]. For $k \geq 2$, there can be mixing between different spin-singlet bands, and the formula for σ becomes rather complicated.

If the quantum Hall fluid has only one fully polarized (edge current) band then

$$k = 0\,, \qquad \sigma = \sigma_s = \chi = \frac{1}{2l+1}\,, \qquad l = 0, 1, 2, \ldots\,. \tag{4.56}$$

If there are two oppositely polarized (edge current) bands then $k = 0$, $\chi = 0$, $\sigma = \sigma_s$, but the formula for σ becomes more complicated.

Quite generally, σ, σ_s and χ are found to be *rational numbers*, and there are relations between them generalizing those in (4.55), (4.56). These results follow from a detailed study of the representation theory of chircal edge current algebras and of anomaly cancellation [26]; see [13, 19].

Finally, we propose to discuss the most important *sum rules* for current Green functions that can be derived from the form (4.4) of the effective action S^*_Ω.

From the structure of the terms in (4.4) we derive, using identity (3.4) and definition (3.5), the following sum rules for current Green functions; (we are working tin the thermodynamic limit, $\Omega \nearrow \mathbb{R}^2$):

(a)

$$\int \langle T[\varrho(x)\,\varrho(y)]\rangle^c_{a_c, w_c}\, d^3y = 0\,, \tag{4.57}$$

and

$$\int \langle T[\underline{j}(x) \cdot \underline{j}(y)]\rangle^c_{a_c, w_c}\, d^3y = 0\,. \tag{4.58}$$

Taking into account the next to leading (Maxwell) term in the effective action, we also have that

$$\int \langle \varrho(\underline{x}, t)\,\varrho(\underline{y}, t)\rangle^c_{a_c, w_c}\, d^2\underline{y} = 0 \tag{4.59}$$

which is the Stillinger-Lovett sum rule expressing a weak form of *screening*.

(b) The Hall conductivity of the electron fluid can be found from the sum rule $\left(\sigma_H = \sigma\,\frac{e^2}{h}\right)$

$$\int \varepsilon_{\mu\nu\varrho}(x-y)^\mu \,\langle T[j^\mu(x) j^\varrho(y)]\rangle^c_{a_c, w_c}\, d^3y = \frac{3i}{\pi}\,\sigma\,. \tag{4.60}$$

(c) From the absence of a term cubic in $\tilde{a}$ in S^*_Ω we conclude that

$$\int \langle T[j^\mu(x) j^\nu(y) j^\varrho(z)]\rangle^c_{a_c,w_c} d^3y d^3z = 0\,, \tag{4.61}$$

for all μ, ν and ϱ.

Next, we derive some sum rules for the spin currents. For example:

(d) For $A = 3$ and $\mu = 0, 1, 2$, and for $A = 1, 2$, $\mu \neq 0$ in Case (3)

$$\int \langle T[s^\mu_A(x)\, s^\mu_A(y)]\rangle^c_{a_c,w_c} d^3y = 0\,. \tag{4.62}$$

For $A = 3$ and $\mu = 0$, we can also derive the following improved sum rule (next-to-leading-order terms in the effective action):

$$\int \langle s^0_3(\underline{x}, t)\, s^0_3(\underline{y}, t)\rangle^c_{a_c,w_c} d^2\underline{y} = 0\,. \tag{4.63}$$

(e) The Hall conductivity for the spin current is found from

$$\int \varepsilon_{\mu\nu\varrho}(x-y)^\mu \langle T[s^\nu_A(x)\, s^\varrho_A(y)]\rangle^c_{a_c,w_c} d^3y = 6i\left(\frac{\sigma_s}{2\pi}\,\delta_{A3} - \frac{k}{\pi}\right). \tag{4.64}$$

Moreover, for $A \neq B \in \{1, 2\}$,

$$\int \langle T[s^1_A(x)\, s^2_B(y)]\rangle^c_{a_c,w_c} d^3y = 2i\varepsilon_{AB}\left(\frac{k}{\pi}\, w_{c,03}(x) + \tau^{12}_2(x)\right). \tag{4.65}$$

(f) We also obtain mixed $(j - s)$ sum rules:

$$\int \langle T[j^\mu(x)\, s^\mu_A(y)]\rangle^c_{a_c,w_c} d^3y = 0\,, \tag{4.66}$$

and

$$\int \varepsilon_{\mu\nu\varrho}(x-y)^\mu \langle T[j^\mu(x)\, s^\varrho_3(y)]\rangle^c_{a_c,w_c} d^3y = \frac{3i}{\pi}\,\chi\,. \tag{4.67}$$

(g) Let us finally note that there is another kind of sum rules which are consequences of $SU(2)$-*gauge invariance*: For an arbitrary polynomial, $F(\vec{s})$, in the spin currents $\vec{s}$, one has that

$$\langle F(\vec{s})\rangle_{a_c,w_c} = \langle F(R^{-1}(g)\vec{s})\rangle_{a_c,w_c}\,, \tag{4.68}$$

where g is an $SU(2)$-gauge transformation, and R is the adjoint representation of $SU(2)$. Equation (4.68) is an $SU(2)$-*Ward identity*. Since the left-hand side of (4.68) is independent of g, arbitrary derivatives of the right-hand side of (4.68) in g must vanish. Expanding the right-hand side of (4.68) in $g - 1 \approx X \in su(2)$, setting $\tilde{w} := {}^g w_c - w_c$, we find the infinitesimal versions of the Ward identities which have the form of "sum rules." They are rather striking consequences of the non-abelian gauge invariance of the system. An example of this kind of "sum rule" is the covariant conservation of the spin current disucssed in Sect. 3, see (3.58).

A detailed discussion of the quantization of the constants σ, σ_s, k and χ and extensions of our methods to other incompressible systems, including three-dimensional ones, is deferred to forthcoming papers.

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