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Classical Solutions of the Quantum Yang–Baxter Equation

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Abstract. The classical analogue is developed here for part of the construction in which knot and link invariants are produced from representations of quantum groups. Whereas previous work begins with a quantum group obtained by deforming the multiplication of functions on a Poisson Lie group, we work directly with a Poisson Lie group G and its associated symplectic groupoid. The classical analog of the quantum R-matrix is a lagrangian submanifold $\mathscr{R}$ in the cartesian square of the symplectic groupoid. For any symplectic leaf S in G, $\mathscr{R}$ induces a symplectic automorphism σ of $S \times S$ which satisfies the set-theoretic Yang–Baxter equation. When combined with the "flip" map exchanging components and suitably implanted in each cartesian power S^n, σ generates a symplectic action of the braid group B_n on S^n. Application of a symplectic trace formula to the fixed point set of the action of braids should lead to link invariants, but work on this last step is still in progress.

1. Introduction

A **quasitriangular Hopf algebra** [Dr1] is a Hopf algebra $\mathscr{A}$ together with an invertible element R of $\mathscr{A} \otimes \mathscr{A}$ (the **quantum R-matrix**) satisfying the "exchange" condition

$$\Delta'(a) = R\Delta(a)R^{-1}, \quad a \in A , \tag{1}$$

and the compatibility condition

$$(\Delta \otimes \mathrm{id})(R) = R_{13}R_{23}, \quad (\mathrm{id} \otimes \Delta)(R) = R_{13}R_{12} . \tag{2}$$

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These conditions imply that R satisfies the **quantum Yang–Baxter equation,**

$$R_{13}R_{23}R_{12} = R_{12}R_{23}R_{13} . \tag{3}$$

In the equations above, $\Delta\colon \mathscr{A} \to \mathscr{A} \otimes \mathscr{A}$ is the coproduct, Δ' is its composition with the automorphism of $\mathscr{A} \otimes \mathscr{A}$ which exchanges the factors, and R_{ij} is the image of $R \otimes \mathrm{id}$ under the automorphism of $\mathscr{A} \otimes \mathscr{A} \otimes \mathscr{A}$ induced by the permutation of $\{1, 2, 3\}$ which maps 1 to i and 2 to j.

The "classical analogue" of a quasitriangular Hopf algebra has heretofore been considered in the context of Lie algebras. A **Lie bialgebra** is a dual pair $(\mathfrak{g}, \mathfrak{g}^*)$ of Lie algebras for which the dual of multiplication on $\mathfrak{g}^*$, $c\colon \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$, is a cocycle with respect to the adjoint representation. The Lie bialgebra is called quasitriangular if it is equipped with an element r of $\mathfrak{g} \otimes \mathfrak{g}$ (the **classical r-matrix**) satisfying conditions which are "infinitesimal versions" of those for R, the most important of them being the **classical Yang–Baxter equation**

$$[r_{12}, r_{23}] + [r_{13}, r_{23}] + [r_{12}, r_{13}] = 0 . \tag{4}$$

The subscripts in (4) can be interpreted as in the quantum case once we consider $\mathfrak{g}$ as embedded in $U(\mathfrak{g})$.

Hopf algebras known as *quantum groups* appear as deformations with respect to a parameter $\hbar$ of algebras of functions on ordinary Lie groups. Passage to a Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ represents a first-order localization not only at $\hbar = 0$, but also at the identity element of the group. Globally, one should consider a Poisson structure on the corresponding Lie group G which represents the first derivative with respect to $\hbar$ of the Hopf algebra multiplication, and whose linearization at the identity of G is the Lie algebra structure on $\mathfrak{g}^*$. This structure makes G into a Poisson Lie group; i.e. the multiplication map $G \times G \to G$ is a Poisson map.

The aim of this paper is to answer an obvious question arising from the viewpoint expressed in the previous paragraphs. Since the classical r-matrix lives only on the Lie bialgebra, what is the corresponding object $\mathscr{R}$ at the level of the Poisson Lie group G? Our answer to this question is based on the following heuristic line of argument:

- $\mathscr{R}$ should be a geometric object which, when quantized, becomes the quantum R-matrix.
- The Hopf algebra $\mathscr{A}$ should be obtained by geometric quantization of the symplectic groupoid Γ_G of the Poisson Lie group G.
- When a symplectic manifold S is quantized to give a linear space, elements of the linear space are represented geometrically by lagrangian submanifolds of S, perhaps carrying some extra structure like a density.
- The quantization of the product $S \times T$ of two symplectic manifolds is the tensor product of their quantizations.

The conclusion we draw from the points above is that the object $\mathscr{R}$ should be a lagrangian submanifold of $\Gamma_G \times \Gamma_G$, the symplectic groupoid of $G \times G$. It should satisfy algebraic conditions closely resembling those of the *quantum* R-matrix. We will call our object the **(global) classical $\mathscr{R}$-matrix.**

Our program may be described further by reference to the following table.

Classical		Quantum
quasitriangular Lie bialgebras		
↓		
quasitriangular Poisson Lie groups	→	quasitriangular Hopf algebras
↓		↓
quasitensor categories of symplectic realizations	→	quasitensor categories of representations
↓		↓
symplectic actions of braid groups	→	representations of braid groups
↓		↓
link invariants		link invariants

The horizontal arrows represent processes generally known as "quantization." Previous work on the construction of link invariants has proceeded down the right-hand side of the table, using the methods of linear representation theory. Our aim is to delay the quantization process as far as possible, ultimately carrying out the entire process in the left-hand column of the table.

The present paper is devoted to describing the first three downward arrows in the left-hand column. Research is still in progress for the last step, in which we hope to use symplectic trace formulas to construct link invariants from symplectic actions of the braid group.

To conclude this introduction, we will give a brief description of the construction of quasitensor categories of realizations, since this approach gives another rationale for defining $\mathscr{R}$ the way we do.

A **symplectic realization**[1] $S = (S, J)$ of a Poisson manifold P is defined as a symplectic manifold S together with a Poisson map $J: S \to P$ which is complete in the sense that the pullback to S of every compactly supported function on P has a complete hamiltonian vector field. For instance, a symplectic realization of a dual Lie algebra $\mathfrak{g}^*$ (which with the operation of addition is dual as a Poisson group to G with the zero Poisson structure) is the momentum map for a hamiltonian action of (simply connected) G on a symplectic manifold. It is well known that such an action is the classical analogue of a representation of the group G or, what is nearly equivalent, of the universal enveloping algebra $U(\mathfrak{g})$. Since $U(\mathfrak{g})$ may be thought of as a quantum deformation of the Poisson algebra of (polynomial) functions on $\mathfrak{g}^*$, it is a natural generalization of this situation to consider realizations of a general Poisson manifold P as the classical analogues of representations of a quantum deformation of the functions on P. In this paper, we will not deal with the difficult problem of constructing representations from realizations; rather, we will simply carry out in the context of realizations the "categorical" constructions usually applied in representation theory.

An isomorphism between realizations is, of course, a symplectic diffeomorphism $\phi: S_\lambda \to S_\mu$ for which $J_\mu \circ \phi = J_\lambda$. With the isomorphisms as morphisms, the realizations form a category. It is also useful to consider more general morphisms, namely lagrangian submanifolds of $\bar{S}_\lambda \times S_\mu$ (the bar denotes reversing the sign of the symplectic structure) contained in the coisotropic submanifold $(J_\lambda \times J_\mu)^{-1}$ (diagonal in $\bar{P} \times P$), though these do not form a category because of the usual clean intersection requirements for good compositions [X1, X2].

[1] The term derives from Lie [Li]

Corresponding to the tensor product of representation spaces, we can take symplectic realizations $J_\lambda: S_\lambda \to P$ and $J_\mu: S_\mu \to P$ and form the product $S = S_\lambda \times S_\mu$ of symplectic manifolds. $(S, J_\lambda \times J_\mu)$ is then a realization of $P \times P$ but not of P. The most natural way to produce a realization of P itself is to use some Poisson map m from $P \times P$ to P. Given such a map, we obtain a "tensor product" operation on realizations, writing $S_\lambda \otimes S_\mu$ for $(S, m \circ (J_\lambda, J_\mu))$.

If m is the multiplication of a Poisson Lie group, then this tensor product is naturally associative, but it is not commutative. By analogy with the quantum case, we may seek to restore commutativity by choosing for each pair of realizations S_λ and S_μ a symplectic automorphism $\mathscr{R}_{\lambda\mu}$ of $S_\lambda \times S_\mu$ whose composition with the exchange diffeomorphism $S_\lambda \times S_\mu \to S_\mu \times S_\lambda$ is an isomorphism from $S_\lambda \otimes S_\mu$ to $S_\mu \otimes S_\lambda$. The operators $\mathscr{R}_{\lambda\mu}$ for various realizations should then satisfy compatibility conditions of "Yang-Baxter" type.

Still following the pattern of the quantum case, we seek to generate the $\mathscr{R}_{\lambda\mu}$ for *all* realizations by applying a certain universal object associated with P itself. The geometric nature of this universal object may be inferred from the fact that it should induce a symplectic diffeomorphism on the product of any two "realization spaces." In general, the objects which produce diffeomorphisms of the realization spaces of a Poisson manifold Q are the lagrangian submanifolds of the symplectic groupoid Γ_Q which are sections of both the source and target maps. These objects are called **lagrangian bisections**. Since $(S_\lambda \times S_\mu, J_\lambda \times J_\mu)$ is a realization of $P \times P$, we conclude that our universal object should be a lagrangian bisection of $\Gamma_P \times \Gamma_P$. This object, the global classical $\mathscr{R}$ matrix, should then satisfy conditions which are formally identical to those in the quantum case but which must be interpreted in the calculus of bisections on groupoids. (This solves the problem posed by Drinfel'd in [Dr2] of finding set theoretical solutions to the quantum Yang–Baxter equation.)

Given the global classical $\mathscr{R}$ matrix and a particular realization S of the Poisson Lie group P, one obtains just as in the quantum case a symplectic action of the k-stranded braid group on S^k.

The last step in constructing invariants of links is to extract invariant quantities from actions of the braid group. When the action is a linear representation, the usual procedure is to take traces (and then to normalize in certain ways [RT, Tu]). When the action is symplectic, one might try to quantize it and then take the trace of the resulting representation, but there is also a more geometric approach. It is well known that the traces of the quantizations of certain symplectic automorphisms can be calculated by "trace formulas" of Atiyah–Bott type which involve only the fixed points of the automorphisms. For our purposes, it may be unnecessary to know that these geometric calculations produce the traces of representations; rather it may be sufficient to know that the "symplectic traces" satisfy enough of the formal properties of linear traces. Our work on this last step is still in progress, and we cannot tell yet how successful it will be.

2. Quasitriangular Lie Bialgebras and Poisson Lie Groups

This section is devoted to a general study of quasitriangular Lie bialgebras and Poisson Lie groups. We refer the reader to [Dr1, LW2, STS] for background material.

The following conventions will be used throughout this paper. For any $X \in \mathfrak{g}$, $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$ denotes the usual adjoint representation, and $\mathrm{ad}^*_X : \mathfrak{g}^* \to \mathfrak{g}^*$ denotes the map defined by

$$\langle \mathrm{ad}^*_X \xi, Y \rangle = - \langle \xi, \mathrm{ad}_X Y \rangle$$

for any $X \in \mathfrak{g}$ and $\xi \in \mathfrak{g}^*$. Ad_g stands for the usual adjoint action of G on the Lie algebra $\mathfrak{g}$ (i.e., the derivative of $l_g \circ r_{g-1}$ at the identity), and $\mathrm{Ad}^*_g : \mathfrak{g}^* \to \mathfrak{g}^*$ for its dual map. Hence, we have the following identities: $\exp(\mathrm{ad} X) = \mathrm{Ad}_{\exp X}$ and $\exp(\mathrm{ad}^* X) = \mathrm{Ad}^*_{(\exp X)^{-1}}$.

Recall that a Lie group G is called a Poisson Lie group if it is also a Poisson manifold such that the multiplication m: $G \times G \to G$ is a Poisson map, where $G \times G$ is equipped with the product Poisson structure. The infinitesimal object associated to a Poisson Lie group is a so-called Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ [Dr1, LW2, STS]. An interesting class of Lie bialgebras, the quasitriangular Lie bialgebras, plays a fundamental rule in the theory of Poisson Lie groups. In what follows, we briefly recall some basic notions regarding Poisson Lie groups and quasitriangular Lie bialgebras (see [Dr1, Dr3, KoMa, RS] for more details).

An element $r \in \mathfrak{g} \otimes \mathfrak{g}$ is said to be a solution of the classical Yang–Baxter equation if

$$[r_{12}, r_{23}] + [r_{13}, r_{23}] + [r_{12}, r_{13}] = 0 .$$

Here, as usual, r_{12} is the image of $r \in \mathfrak{g} \otimes \mathfrak{g}$ under the mapping $\mathfrak{g} \otimes \mathfrak{g} \to U\mathfrak{g} \otimes U\mathfrak{g} \otimes U\mathfrak{g}$ which sends $a \otimes b$ into $a \otimes b \otimes 1$, and similarly for r_{13}, r_{23}.

Any $r \in \mathfrak{g} \otimes \mathfrak{g}$ can be identified with a bilinear form on $\mathfrak{g}^*$. It then determines a mapping $r_+ : \mathfrak{g}^* \to \mathfrak{g}$ by the rule $v(r_+(u)) = r(u, v)$ for u and v in $\mathfrak{g}^*$. The dual of r_+, which we denote by $-r_-$, is then $(Pr)_+$, where P is the "flip" operator which interchanges factors in $\mathfrak{g} \otimes \mathfrak{g}$.

Assume now that $r \in \mathfrak{g} \otimes \mathfrak{g}$ satisfies the classical Yang–Baxter equation and the symmetrized tensor $I = r + P(r)$ is ad-invariant in the sense that $\mathrm{ad}_X I = 0$ for all $X \in \mathfrak{g}$, where ad_X is the endomorphism of $\mathfrak{g} \otimes \mathfrak{g}$ induced naturally from the adjoint map on the Lie algebra $\mathfrak{g}$ by $\mathrm{ad}_X(a \otimes b) = (\mathrm{ad}_X a) \otimes b + a \otimes \mathrm{ad}_X b$. Then $(\mathfrak{g}, \mathfrak{g}^*)$ becomes a Lie bialgebra, with the Lie bracket on $\mathfrak{g}^*$ being defined by

$$[\xi, \eta] = \mathrm{ad}^*_{r_+(\xi)} \eta - \mathrm{ad}^*_{r_-(\eta)} \xi, \quad \forall \xi, \eta \in \mathfrak{g}^* .$$

Such a Lie bialgebra is called **quasitriangular** [Dr1]. It is called **triangular** if r is antisymmetric. (Some authors [KoMa] reserve the term quasitriangular for the case in which I is nondegenerate; we prefer to include triangularity as a special case of quasitriangularity.)

Lemma 2.1. *Suppose that* $\mathfrak{g}$ *is a quasitriangular Lie bialgebra. Then we have the following identities for any* $X \in \mathfrak{g}$ *and* $\xi \in \mathfrak{g}^*$:

$$[X, r_+(\xi)] = - \mathrm{ad}^*_\xi X + r_+(\mathrm{ad}^*_X \xi) ,$$

and

$$[X, r_-(\xi)] = - \mathrm{ad}^*_\xi X + r_-(\mathrm{ad}^*_X \xi) ,$$

where the bracket means the Lie bracket of $\mathfrak{g}$.

Proof. For any $\eta \in \mathfrak{g}^*$.

$$\begin{aligned}\langle -\mathrm{ad}^*_\xi X + r_+(\mathrm{ad}^*_X \xi), \eta\rangle &= \langle -\mathrm{ad}^*_\xi X, \eta\rangle + \langle r_+(\mathrm{ad}^*_X\xi), \eta\rangle \\ &= \langle X, [\xi, \eta]\rangle - \langle \xi, [X, r^*_+\eta]\rangle \\ &= \langle X, \mathrm{ad}^*_{r_+(\xi)}\eta + \mathrm{ad}^*_{r^*_+(\eta)}\xi\rangle - \langle \xi, [X, r^*_+\eta]\rangle \\ &= \langle [X, r_+(\xi)], \eta\rangle .\end{aligned}$$

This proves the first identity. The proof of the second one is similar. □

The following lemma is easily checked. It is stated for the case where I is nondegenerate in [RS].

Lemma 2.2. *For any quasitriangular Lie bialgebra, the linear maps r_+ and r_- defined above are both Lie algebra homomorphisms.*

An immediate consequence of Lemma 2.1 is the following result of Drinfel'd [Dr3], in which the Lie algebra $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$ is the double associated to a given Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ [Dr1, LW2, STS].

Proposition 2.3. *The Lie algebra homomorphisms $r_\pm$ naturally extend to Lie algebra homomorphisms $f_\pm$ from $\mathfrak{d}$ onto $\mathfrak{g}$ defined by: $f_\pm(X + \xi) = X + r_\pm \xi$.*

Lemma 2.4. *For any $\xi, \eta \in \mathfrak{g}^*$,*

$$r_+(\mathrm{ad}^*_{r_-\eta}\xi) = \mathrm{ad}^*_\eta(r_+\xi) ,$$

and

$$r_-(\mathrm{ad}^*_{r_+\eta}\xi) = \mathrm{ad}^*_\eta(r_-\xi) .$$

Proof. For any $\mu \in \mathfrak{g}^*$,

$$\begin{aligned}\langle r_+(\mathrm{ad}^*_{r_-\eta}\xi), \mu\rangle &= \langle \mathrm{ad}^*_{r_-\eta}\xi, r^*_+\mu\rangle \\ &= -\langle \xi, [r_-\eta, r^*_+\mu]\rangle \\ &= \langle \xi, r_-[\eta, \mu]\rangle \\ &= -\langle r_+\xi, \mathrm{ad}_\eta \mu\rangle \\ &= \langle \mathrm{ad}^*_\eta(r_+\xi), \mu\rangle .\end{aligned}$$

The other identity is proved similarly. □

The following proposition gives the Lie bracket relation between the images of r_+ and r_-.

Proposition 2.5. *For any $\xi, \eta \in \mathfrak{g}^*$,*

$$[r_+\xi, r_-\eta] = -\mathrm{ad}^*_\eta(r_+\xi) + \mathrm{ad}^*_\xi(r_-\eta) .$$

Proof. For any $\mu \in \mathfrak{g}^*$, we have

$$\begin{aligned}\langle \mathrm{ad}^*_\eta(r_+\xi) - \mathrm{ad}^*_\xi(r_-\eta), \mu\rangle &= \langle \mathrm{ad}^*_\eta(r_+\xi), \mu\rangle + \langle r_-\eta, [\xi, \mu]\rangle \\ &= \langle \mathrm{ad}^*_\eta(r_+\xi), \mu\rangle + \langle r_-\eta, \mathrm{ad}^*_{r_+(\xi)}\mu - \mathrm{ad}^*_{r_-(\mu)}\xi\rangle \\ &= \langle \mathrm{ad}^*_\eta(r_+\xi), \mu\rangle - \langle [r_+\xi, r_-\eta], \mu\rangle + \langle r_-[\mu, \eta], \xi\rangle \\ &= \langle \mathrm{ad}^*_\eta(r_+\xi), \mu\rangle - \langle [r_+\xi, r_-\eta], \mu\rangle - \langle [\mu, \eta], r_+\xi\rangle \\ &= -\langle [r_+\xi, r_-\eta], \mu\rangle .\end{aligned}$$

This completes the proof. □

Let $\kappa_\pm = f_\pm^{-1}(0) \subset \mathfrak{d}$ be the kernels of the Lie algebra homomorphisms $f_\pm$, which are clearly ideals in $\mathfrak{d}$. As an immediate consequence of the proposition above, we have the following important

Theorem 2.6.

$$[\kappa_+, \kappa_-] = 0 .$$

Proof. Suppose that l_+ and l_- are any elements in κ_+ and κ_-, respectively. We may assume that $l_+ = -r_+\xi + \xi$, and $l_- = -r_-\eta + \eta$ for certain $\xi, \eta \in \mathfrak{g}^*$. Then,

$$\begin{aligned}
[l_+, l_-] &= [-r_+\xi + \xi,\ -r_-\eta + \eta] \\
&= ([r_+\xi, r_-\eta] + \mathrm{ad}^*_\eta(r_+\xi) - \mathrm{ad}^*_\xi(r_-\eta)) \\
&\quad + ([\xi, \eta] - \mathrm{ad}^*_{r_+(\xi)}\eta + \mathrm{ad}^*_{r_-(\eta)}\xi) \\
&= 0 .
\end{aligned}$$

□

Remark 2.7. In the case where $(\mathfrak{g}, \mathfrak{g}^*)$ is a triangular Lie bialgebra, it is easy to see that $r_+ = r_-$ and $f_+ = f_-$; therefore $\kappa_+ = \kappa_-$. So, $\kappa_\pm$ is an abelian ideal. This is false in general, since $[\xi, \eta] \neq \mathrm{ad}^*_{r_+(\xi)}\eta - \mathrm{ad}^*_{r_+(\eta)}\xi$.

We now turn our attention to groups.

Definition 2.8. *A Poisson Lie group G is called quasitriangular if its corresponding Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ is quasitriangular and if the Lie algebra homomorphisms r_+ and r_- from $\mathfrak{g}^*$ to $\mathfrak{g}$ lift to Lie group homomorphisms R_+ and R_- from G^* to G.*

If G is quasitriangular, we define maps ϕ and ψ from G^* to G by $\phi(x) = R_+(x^{-1})$, $\psi(x) = R_-(x^{-1})$, for any $x \in G^*$.

Proposition 2.9. *Both ϕ and ψ are Poisson morphisms.*

In fact, we have the following general result.

Proposition 2.10. *Suppose that G and H are Poisson Lie groups, and ρ: $G \to H$ is a group homomorphism (or anti-homomorphism). Let $d\rho$: $\mathfrak{g} \to \mathfrak{h}$ be the corresponding Lie algebra homomorphism (or anti-homomorphism). ρ is a Poisson map if and only if $(d\rho)^*$: $\mathfrak{h}^* \to \mathfrak{g}^*$ is a Lie algebra homomorphism.*

One can prove this proposition directly by using coisotropic calculus, i.e., by showing that the graph of the map is a coisotropic submanifold in $G \times H^-$. The following proof, using a cocycle argument, was pointed out to us by J.-H. Lu.

Proof. Let π_G and π_H be the Poisson tensors on G and H respectively, and let $K_G(g) = r_{g^{-1}}\pi_G(g)$ and $K_H(h) = r_{h^{-1}}\pi_H(h)$ be the corresponding group 1-cocycles on G and H. Consider the two maps K_1 and K_2 from G to $\mathfrak{h} \wedge \mathfrak{h}$, given respectively by the compositions $G \xrightarrow{K_G} \mathfrak{g} \wedge \mathfrak{g} \xrightarrow{d\rho} \mathfrak{h} \wedge \mathfrak{h}$ and $G \xrightarrow{\rho} H \xrightarrow{K_H} \mathfrak{h} \wedge \mathfrak{h}$.

It is easy to see that both K_1 and K_2 are group 1-cocycles on G with values in $\mathfrak{h} \wedge \mathfrak{h}$. The corresponding Lie algebra 1-cocycles are given, respectively, by the compositions of the following maps: $\mathfrak{g} \xrightarrow{\delta_\mathfrak{g}} \mathfrak{g} \wedge \mathfrak{g} \xrightarrow{d\rho} \mathfrak{h} \wedge \mathfrak{h}$ and $\mathfrak{g} \xrightarrow{d\rho} \mathfrak{h} \xrightarrow{\delta_\mathfrak{h}} \mathfrak{h} \wedge \mathfrak{h}$, where $\delta_\mathfrak{g}$ and $\delta_\mathfrak{h}$ are, respectively, the Lie algebra 1-cocycles defining the cocommutator on $\mathfrak{g}$ and $\mathfrak{h}$. It is obvious that $d\rho \circ \delta_\mathfrak{g} = \delta_\mathfrak{h} \circ d\rho$, since $(d\rho)^*$: $\mathfrak{h}^* \to \mathfrak{g}^*$ is a Lie algebra homomorphism. Hence $K_1(g) = K_2(g)$. That is, $(T\rho)\pi_G = \pi_H$. □

Applying the exponential map and using the definitions of ϕ and ψ, we have the following immediate corollary.

Corollary 2.11. *For any* $\xi, \eta \in \mathfrak{g}^*$,

$$r_+(\mathrm{Ad}^*_{\psi(u)}\xi) = \mathrm{Ad}^*_{u^{-1}}(r_+\xi) ,$$

and

$$r_-(\mathrm{Ad}^*_{\phi(u)}\xi) = \mathrm{Ad}^*_{u^{-1}}(r_-\xi) .$$

Suppose that G is a complete quasitriangular Poisson Lie group with the Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$. Let G^* be its dual Poisson Lie group, and $D = G \times G^*$ the simply connected double group corresponding to the Poisson group pair (G, G^*) [LW2, STS]. By $F_\pm$, we denote the Lie group homomorphisms from D to G corresponding to the Lie algebra homomorphisms $f_\pm$, and $K_\pm$ their kernels. For any $d = gu = \bar{u}\bar{g} \in D$ with $g, \bar{g} \in G$ and $u, \bar{u} \in G^*$, $F_\pm$ are given, respectively, by

$$F_+(d) = g\phi(u^{-1}) = \phi(\bar{u}^{-1})\bar{g} ,$$

and

$$F_-(d) = g\psi(u^{-1}) = \psi(\bar{u}^{-1})\bar{g} .$$

An immediate consequence of Theorem 2.6 is the following:

Theorem 2.12. *K_+ and K_- commute with each other with respect to the group structure of the double group D. In particular, for any $u, v \in G^*$,*

$$u\phi(u) \cdot \psi(v)v = \psi(v)v \cdot u\phi(u) ,$$

and

$$u\psi(u) \cdot \phi(v)v = \phi(v)v \cdot u\psi(u) ,$$

where the dot means the multiplication in the double group D.

3. Symplectic Groupoids of Quasitriangular Poisson Lie Groups

In this section, we will study some basic properties of the symplectic groupoids of quasitriangular Poisson Lie groups. Besides their important role in the discussions of the next section, these properties may be quite interesting themselves. First of all, we need to recall some basic notions.

Suppose that G is a complete quasitriangular Poisson Lie group and D its corresponding double group. There is a natural symplectic structure π_+ on D, which makes D into a symplectic double groupoid. That is, D with the symplectic structure π_+ becomes a symplectic groupoid over G, and D with $-\pi_+$ is a symplectic groupoid over G^*. These two groupoid structures are compatible with each other to make D into a double groupoid. We refer the reader to [LW1] for more details. Note that in the rest of this paper, we always use α_1 and β_1 to denote the source and target maps of the first groupoid structure on D, that is, the one over G, and use α_2 and β_2 to denote the source and target maps of the second groupoid structure of D. We also use the notations $(D \rightrightarrows G, \alpha_1, \beta_1)$ and $(D^- \rightrightarrows G^*, \alpha_2, \beta_2)$ to denote these two groupoid structures on D. In order to distinguish the multiplications of these two groupoid structures on D, we shall use $x * y$ to denote the product of x and $y \in D$ in the first groupoid structure, and simply xy in the second. Also, we shall use the following conventions for dressing transformations: the left and right dressing transformations of G on G^* are given, respectively, by $\lambda_g u = \alpha_2(gu)$ and $\rho_g(u) = \beta_2(ug)$, for any $g \in G$ and $u \in G^*$; similarly, by left and right dressing transformations of G^* on G, we mean the actions of G^* on G given respectively by $\lambda_u g = \alpha_1(ug)$ and $\rho_u g = \beta_1(gu)$.

The following lemma from [LW1] gives explicit formulas for π_+, which are essential to our computations in the sequel.

Lemma 3.1. *If $d \in D$ can be factored as $d = gu$ for some $g \in G$ and $u \in G^*$, then π_+ is given explicitly by*

$$(l_{g^{-1}} r_{u^{-1}} \pi_+(d))(\xi_1 + X_1, \xi_2 + X_2) = \langle X_1, \xi_2 \rangle - \langle X_2, \xi_1 \rangle + (l_{g^{-1}} \pi_G(g))(\xi_1, \xi_2) + (r_{u^{-1}} \pi_{G^*}(u))(X_1, X_2), \quad \forall X_1, X_2 \in \mathfrak{g} \text{ and } \xi_1, \xi_2 \in \mathfrak{g}^*,$$

where π_G and π_{G^*} are the Poisson structures on G and G^*, respectively.

Similarly, if $d = ug$ for some $u \in G^*$ and $g \in G$, then

$$(l_{u^{-1}} r_{g^{-1}} \pi_+(d))(X_1 + \xi_1, X_2 + \xi_2) = \langle X_1, \xi_2 \rangle - \langle X_2, \xi_1 \rangle + (r_{g^{-1}} \pi_G(g))(\xi_1, \xi_2) + (l_{u^{-1}} \pi_{G^*}(u))(X_1, X_2), \quad \forall X_1, X_2 \in \mathfrak{g} \text{ and } \xi_1, \xi_2 \in \mathfrak{g}^* .$$

It is standard in Poisson geometry that, associated to any 1-form θ on a Poisson manifold P, there is a vector field X_θ on P, the contraction of θ with the Poisson tensor of P.

The following proposition describes such vector fields on (D, π_+) associated to certain special 1-forms.

Proposition 3.2. (1) *If $d = gu \in D = G \times G^*$, then for any $\theta \in \Omega^1(G)$,*

$$X_{\alpha_1^* \theta}(d) = (X_\theta(g), \ - r_u(l_g^* \theta)) \in T_d D \cong T_g G \oplus T_u G^* .$$

Here, $l_g^* \theta$ is considered as an element in $\mathfrak{g}^*$, the Lie algebra of G^*, and $r_u(l_g^* \theta)$ is the right invariant vector field on G^* corresponding to $l_g^* \theta \in \mathfrak{g}^*$.

(2) *Similarly, if $d = ug \in D = G^* \times G$, then for any $\eta \in \Omega^1(G^*)$,*

$$X_{\alpha_2^* \eta}(d) = (- X_\eta(u), r_g(l_u^* \eta)) \in T_d D \cong T_u G^* \oplus T_g G .$$

Proof. Let us assume that $(l_{g^{-1}} r_{u^{-1}}) X_{\alpha_1^* \theta} = X + \eta$, for some $X \in \mathfrak{g}$ and $\eta \in \mathfrak{g}^*$. Then, for any $\xi_2 \in \mathfrak{g}^*$ and $X_2 \in \mathfrak{g}$, we have

$$(l_{g^{-1}} r_{u^{-1}} \pi_+(d))(l_g^* \theta, \xi_2 + X_2) = \langle X + \eta, \xi_2 + X_2 \rangle .$$

Lemma 3.1 implies that

$$- \langle X_2, l_g^* \theta \rangle + (l_{g^{-1}} \pi_G(g))(l_g^* \theta, \xi_2) = \langle X + \eta, \xi_2 + X_2 \rangle .$$

By letting $\xi_2 = 0$, one gets immediately that $\eta = - l_g^* \theta$. On the other hand, since $(T\alpha_1) X_{\alpha_1^* \theta} = X_\theta$, we have $X = l_{g^{-1}} X_\theta(g)$.

The second part can be proved similarly. □

We can express the vector field $X_{\alpha_2^*\eta}$ explicitly under the factorization $D = G \times G^*$ as well.

Proposition 3.3. *If $d = gu \in D = G \times G^*$, then for any $\eta \in \Omega^1(G^*)$,*

$$X_{\alpha_2^*\eta}(d) = (r_g(r_{\bar{u}}^*(\eta)), 0) \in T_g G \oplus T_u G^*, \quad \forall \eta \in \Omega^1(G^*) , \tag{5}$$

where $\bar{u} = \alpha_2(d)$, and $r_{\bar{u}}^*(\eta)$ is considered as an element in the Lie algebra $\mathfrak{g}$, as usual.

Proof. It suffices to prove this proposition for any right invariant one-form $\eta = Z^r$ on G^*, where $Z \in \mathfrak{g}$. According to Corollary 3.6 in [Lu2], the morphism $\mathfrak{g} \to \mathscr{X}(D)$ given by $Z \to X_{\alpha_2^* Z^r}$, for any $Z \in \mathfrak{g}$, defines an infinitesimal left $\mathfrak{g}$-action on D (note that α_2 is an anti-Poisson map from (D, π_+) to G^*). On the other hand, let ϕ_t denote the family of automorphisms of D defined by $\phi_t(d) = [(\exp tZ)g]u$, for any $d = gu \in D$. Obviously, ϕ_t is a flow on D. In order to prove Eq. (5) for $\eta = Z^r$, it is sufficient to show that ϕ_t is the flow generated by $X_{\alpha_2^* Z^r}$. For this purpose, below we will compute explicitly the derivative of ϕ_t at $t = 0$ under the decomposition $D = G^*G$, since we already know the expression of $X_{\alpha_2^* Z^r}$ under such a decomposition of D by Proposition 3.2. Assume that $d = gu = \bar{u}\bar{g}$. Then, under the decomposition $D = G^*G$, we can write $\phi_t(d) = (\exp tZ)\bar{u}\bar{g} = [\lambda_{\exp tZ}\bar{u}][r_{\bar{g}}\rho_{\bar{u}}(\exp tZ)]$. Now clearly, $\left.\frac{d}{dt}\right|_{t=0} (\lambda_{\exp tZ}\bar{u}) = -X_{Z^r}(\bar{u})$, and $\left.\frac{d}{dt}\right|_{t=0} [r_{\bar{g}}\rho_{\bar{u}}(\exp tZ)] = r_{\bar{g}}\mathrm{Ad}_{\bar{u}}^* Z$. So $\left.\frac{d}{dt}\right|_{t=0} \phi_t(d) = (-X_{Z^r}(\bar{u}), r_{\bar{g}}\mathrm{Ad}_{\bar{u}}^* Z) \in T_d D \cong T_{\bar{u}} G^* \oplus T_{\bar{g}} G$, which exactly equals $X_{\alpha_2^* Z^r}$ according to Proposition 3.2 (2). This completes the proof. □

The following theorem is an immediate consequence of the proposition above.

Theorem 3.4. *For any $\xi \in \mathfrak{g}^*$, let $\theta = l_{g^{-1}}^* \xi \in \Omega^1(G)$ be the corresponding left invariant 1-form on G. If $d = gu \in G \times G^*$, the vector fields $X_{\alpha_2^*\phi^*\theta}$ and $X_{\alpha_2^*\psi^*\theta}$ on D can be explicitly expressed by:*

$$X_{\alpha_2^*\phi^*\theta}(d) = (r_g(r_-\xi), 0) \in T_d D \cong T_g G \oplus T_u G^* ,$$

and

$$X_{\alpha_2^*\psi^*\theta}(d) = (r_g(r_+\xi), 0) \in T_d D \cong T_g G \oplus T_u G^* .$$

We end this section with an interesting by-product of Proposition 3.3, which reveals the relation between two expressions for certain 1-forms on D using the two decompositions $D = G^*G$ and $D = GG^*$.

Corollary 3.5. *Assume that $d = \bar{u}\bar{g} = gu \in D$ for some $\bar{g}, g \in G$ and $\bar{u}, u \in G^*$. For any given cotangent vector $\eta \in T_{\bar{u}}^* G^*$, we assume, under the decomposition $D = G \times G^*$ that, $l_g^* r_u^* [(\alpha_2^*\eta)(d)] = \xi + X$, for some $X \in \mathfrak{g}$ and $\xi \in \mathfrak{g}^*$. Then,*

$$\mathrm{Ad}_g(X + (l_{g^{-1}}\pi_G(g))^{\#}\xi) = r_{\bar{u}}^*(\eta) \in \mathfrak{g} ,$$

or, equivalently,

$$\mathrm{Ad}_g(X - r_\pm(\xi)) + r_\pm(\mathrm{Ad}_{g^{-1}}^*\xi) = r_{\bar{u}}^*(\eta) ,$$

where $(l_{g^{-1}}\pi_G(g))^{\#}\xi$ denotes the element in $\mathfrak{g}$ obtained by contracting $l_{g^{-1}}\pi_G(g) \in \mathfrak{g} \otimes \mathfrak{g}$ with $\xi \in \mathfrak{g}^*$.

Proof. For any $\xi_2 \in \mathfrak{g}^*$ and $X_2 \in \mathfrak{g}$,

$$\begin{aligned}\langle l_{g^{-1}} r_{u^{-1}} X_{\alpha_2^* \eta}(d), \xi_2 + X_2 \rangle &= \langle X_{\alpha_2^* \eta}(d), l_{g^{-1}}^* r_{u^{-1}}^* (\xi_2 + X_2) \rangle \\ &= \pi_+ (\alpha_2^* \eta, l_{g^{-1}}^* r_{u^{-1}}^* (\xi_2 + X_2)\rangle \\ &= (l_{g^{-1}} r_{u^{-1}} \pi_+ (d))(\xi + X, \xi_2 + X_2) \\ &\quad \text{(by Lemma 3.1)} \\ &= \langle X, \xi_2 \rangle - \langle X_2, \xi \rangle + (l_{g^{-1}} \pi_G(g))(\xi, \xi_2) \\ &\quad + (r_{u^{-1}} \pi_{G^*}(u))(X, X_2) .\end{aligned}$$

By letting $X_2 = 0$ and using Proposition 3.3, we have

$$\langle \mathrm{Ad}_{g^{-1}}(r_{\bar{u}}^*(\eta)), \xi_2 \rangle = \langle X, \xi_2 \rangle + (l_{g^{-1}} \pi_G(g))(\xi, \xi_2) .$$

Hence,

$$X + (l_{g^{-1}} \pi_G(g))^{\#} \xi = \mathrm{Ad}_{g^{-1}} r_{\bar{u}}^* (\eta) ,$$

which implies the desired identities. □

4. The Global $\mathscr{R}$-Matrix

In this section, which is the main part of this paper, we will demonstrate how to lift the Poisson morphism ϕ (or ψ) induced from a classical quasitriangular r-matrix to the groupoid level to obtain a lagrangian bisection called the global $\mathscr{R}$-matrix which retains, in a formal way, many important properties of the quantum R-matrix.

In order to explain the meaning of this global $\mathscr{R}$-matrix, we start with the simplest case where both ϕ and ψ are assumed to be diffeomorphisms. Also, as in the last two sections, both Poisson groups G and G^* are assumed to be simply connected and complete. It is known [CDW] that the α-simply connected symplectic groupoid for a given Poisson manifold is unique if it exists. Since, by assumption, ϕ: $G^* \to G$ is a Poisson diffeomorphism, it is lifted to a symplectic groupoid isomorphism $\mathscr{F}$ from the groupoid $(D^- \rightrightarrows G^*, \alpha_2, \beta_2)$ to $(D \rightrightarrows G, \alpha_1, \beta_1)$:

$$\begin{tikzcd} D^- \arrow[r, "\mathscr{F}"] \arrow[d, shift right, "\alpha_2"'] \arrow[d, shift left, "\beta_2"] & D \arrow[d, shift right, "\alpha_1"'] \arrow[d, shift left, "\beta_1"] \\ G^* \arrow[r, "\phi"] & G \end{tikzcd}$$

Furthermore, we know that D is a double groupoid, with the second groupoid structure being induced from the group structure of the base space [LW1]. Since ϕ is a group anti-homomorphism, $\mathscr{F}$ is a symplectic groupoid anti-homomorphism with respect to the second groupoid structures. The restriction of $\mathscr{F}$ to the identity space of the second groupoid structure is an anti-Poisson diffeomorphism from G to G^*, which is given by $\psi^{-1} \circ i$. Here i: $G \to G$ is the inversion of G. In other words, we have the following diagram of symplectic groupoid anti-homomorphisms:

$$\begin{tikzcd}[row sep=large, column sep=huge]
D \arrow[r, "\mathscr{F}"] \arrow[d, shift right, "\alpha_1"'] \arrow[d, shift left, "\beta_1"] & D^- \arrow[d, shift right, "\alpha_2"'] \arrow[d, shift left, "\beta_2"] \\
G \arrow[r, "\psi^{-1}\circ i"] & G^*
\end{tikzcd}$$

We denote by $\mathscr{R}$ the graph of $\mathscr{F}$, i.e., $\mathscr{R} = \{(d, \mathscr{F}(d))|d \in D\}$. For any $d \in D$, we assume that $\beta_2(d) = u$ and $\beta_2(\mathscr{F}(d)) = v$. It follows from the above discussion that

$$\alpha_1(\mathscr{F}(d)) = \phi(\alpha_2(d)) , \tag{6}$$

and

$$\beta_2(\mathscr{F}(d)) = (\psi^{-1} \circ i)(\alpha_1(d)) = \psi^{-1}[(\alpha_1(d))^{-1}] . \tag{7}$$

Equation (7) implies that $\alpha_1(d) = \psi(v^{-1})$. Hence $d = \psi(v^{-1})u$ and $\mathscr{F}(d) = \phi(\tilde{u})v$, where $\tilde{u} = \alpha_2(d) = \lambda_{\psi(v^{-1})}u$. Hence, we have the following.

Proposition 4.1. *$\mathscr{R}$ consists of all elements of the form $(\psi(v^{-1})u, \phi(\tilde{u})v) \in D$, with u and v being any elements of G^*, and $\tilde{u}$ equal to $\lambda_{\psi(v^{-1})}u$.*

A Poisson morphism which is not a diffeomorphism may not be liftable to a symplectic groupoid homomorphism. However, by the method of characteristics a general Poisson morphism may be lifted to a canonical relation between symplectic groupoids. First, we let $D = G \times G^*$; then $D \times D = G \times G^* \times G \times G^*$. We denote by p_i the projection from D onto its i^{th} factor, and we denote by p the natural projection of D onto $G^* \times G^*$. Let E be the graph of ϕ, sitting in $D \times D$, i.e., $E = \{(e, u, \phi(u), e)|u \in G^*\}$, and let

$$K_\alpha = \{(d_1, d_2)|d_1, d_2 \in D \text{ satisfy } \phi(\alpha_2(d_1)) = \alpha_1(d_2)\} ,$$

$$K_\beta = \{(d_1, d_2)|d_1, d_2 \in D \text{ satisfy } \phi(\beta_2(d_1)) = \beta_1(d_2)\} .$$

According to Theorem 1.3 of Chapter 3 in [CDW], there exists a unique maximal lagrangian immersion $\mathscr{L}$, containing E and contained in K_α.

Theorem 4.2. *Suppose that G is a simply connected complete quasitriangular Poisson Lie group, G^* its simply connected dual. Then $\mathscr{L}$ is a lagrangian submanifold. In fact, it is a graph over $G^* \times G^*$ with respect to the projection p, and has the following explicit form $\mathscr{L} = \{(\psi(v^{-1}), u, \phi(\tilde{u}), v) \in D \times D | \forall u, v \in G^*\}$.*

Proof. It is known that $K_\alpha \subset (D^-)^- \times D = D \times D$ is a coisotropic submanifold. Its characteristic flow is given by $(X^-_{\alpha_2^*\phi^*\theta}, X_{\alpha_1^*\theta})$ for all 1-forms $\theta \in \Omega^1(G)$, where $X^-_{\alpha_2^*\phi^*\theta}$ is the vector field on D obtained by contracting the 1-form $\alpha_2^*\phi^*\theta$ with the opposite symplectic structure $-\pi_+$ on D, i.e., $X^-_{\alpha_2^*\phi^*\theta} = -X_{\alpha_2^*\phi^*\theta}$.

By $\tilde{\lambda}_t(d)$ and $\lambda_t(d)$, we denote the flow of $X_{\alpha_2^*\phi^*\theta}$ and $X_{\alpha_1^*\theta}$ respectively. For any $u \in G^*$, consider the map p_4: $p_2^{-1}(u) \to G^*$. Since p_2: $\mathscr{L} \to G^*$ is a projection, $\dim p_2^{-1}(u) = \dim G^*$ by dimension counting. We shall prove that p_4 has the path lifting property. To show this, it is sufficient to consider a path in G^* of the form $\zeta_t(e)$, which starts from the identity $e \in G^*$ and is a product of flows generated by right invariant vector fields $\xi_i \in \mathfrak{g}^*$. For this purpose, we let $\theta_i = -l^*_{g^{-1}}\xi_i \in \Omega^1(G)$ be the corresponding left invariant 1-forms on G, and let $\tilde{\lambda}_t(u)$ and $\lambda_t(\phi(u))$ be the corresponding products of flows on D generated respectively by $X_{\alpha_2^*\phi^*\theta_i}$ and $X_{\alpha_1^*\theta_i}$. It is clear that $(\tilde{\lambda}_t(u), \lambda_t(\phi(u)) \in p_2^{-1}(u)$.

Since for $d = gv$, $X_{\alpha_1^*\theta_i}(d) = (X_{\theta_i}(g), r_v\xi_i)$ according to Proposition 3.2, it is clear that $(\tilde{\lambda}_t(u), \lambda_t(\phi(u))$ is a lift of $\zeta_t(e)$.

It follows immediately from the path lifting property that $p_4\colon p_2^{-1}(u) \to G^*$ is a submersion. Furthermore, since $\dim p_2^{-1}(u) = \dim G^*$, p_4 is a local diffeomorphism; therefore a covering map. However G^* is simply connected, so p_4 is a diffeomorphism. Hence $\mathscr{L}$ is a graph over $G^* \times G^*$.

In order to express $\mathscr{L}$ explicitly, we first note that according to Theorem 3.4,

$$X_{\alpha_2^*\phi^*\theta_i}(d) = (-r_g(r_-\xi_i), 0) \in T_gG \oplus T_u^*G^*$$

for any $d = gu$; and therefore its flow at $d = gu$ is given by:

$$\tilde{\lambda}_t^i(gu) = (R_-(\exp t\xi_i)g, u) = (\psi(\exp - t\xi_i)g, u) \in G \times G^* = D\ .$$

Thus, it is not hard to see that $p_1 \circ p_4^{-1}\colon G^* \to G$ is a group homomorphism, which is in fact $\psi \circ i$, the composition of ψ with the inversion map i of the group G^*. Thus, for any $v \in G^*$, if $(d_1, d_2) \in p_2^{-1}(u)$ such that $p_4(d_1, d_2) = v$, then $d_1 = \psi(v^{-1})u$. Furthermore, according to the construction of K_α, $\alpha_1(d_2) = \phi(\alpha_2(d)) = \phi(\tilde{u})$, with $\tilde{u}$ being equal to $\lambda_{\psi(v^{-1})}u$. Hence, $(d_1, d_2) = (\psi(v^{-1}), u, \phi(\tilde{u}), v)$. □

If the Poisson groups G and G^* are complete but are not assumed to be simply connected, the submanifold $\mathscr{R}$ of $D \times D$ consisting of all elements of the form $(\psi(v^{-1}), u, \phi(\tilde{u}), v)$ is still contained in $\mathscr{L}$, hence it is still a lagrangian submanifold of $D \times D$ ($\mathscr{R}$ may not be equal to $\mathscr{L}$). It is such a lagrangian submanifold $\mathscr{R}$ that will become the major object of our study in this paper.

Definition 4.3. *For a complete quasitriangular Poisson group G, the lagrangian submanifold $\mathscr{R} = \{(\psi(v^{-1}), u, \phi(\tilde{u}), v) \mid \tilde{u} = \lambda_{\psi(v^{-1})}u, \forall u, v \in G^*\}$ of $D \times D$ is called its global $\mathscr{R}$-matrix.*

Similarly, the submanifold $\{(\phi(v^{-1}), u, \psi(\tilde{u}), v) \mid \tilde{u} = \lambda_{\phi(v^{-1})}u, \forall u, v \in G^*\}$ is also a lagrangian submanifold of $D \times D$, which we will denote by $\mathscr{R}_-$.

We end this section with the following very useful formulas about dressing transformations, as a consequence of the discussion above.

Corollary 4.4. *For any $u, v \in G^*$,*

$$\phi(\lambda_{\psi(v)}u) = \rho_v\phi(u)$$

and

$$\psi(\lambda_{\phi(v)}u) = \rho_v\psi(u)\ .$$

Similarly,

$$\phi(\rho_{\psi(v)}u) = \lambda_v\phi(u)\ ,$$

and

$$\psi(\rho_{\phi(v)}u) = \lambda_v\psi(u)\ .$$

Proof. According to Theorem 1.3 of Chap. 3 in [CDW], $\mathscr{L}$ is automatically contained in K_β, so is $\mathscr{R} \subset \mathscr{L}$. Hence, for any $(d_1, d_2) \in \mathscr{R}$,

$$\beta_1(d_2) = \phi(\beta_2(d_1))\ . \tag{8}$$

Therefore,

$$\beta_1(\phi(\tilde{u})v) = \phi(\beta_2(\psi(v^{-1})u)) = \phi(u)\ .$$

That is, $\phi(\tilde{u})v = u_1\phi(u)$ for some $u_1 \in G^*$. Hence, $u_1^{-1}\phi(\tilde{u}) = \phi(u)v^{-1}$. So, $\phi(\tilde{u}) = \rho_{v^{-1}}\phi(u)$. This proves the first identity. The second one is proved similarly.

As for the third identity, we assume that $w_1 = \rho_{\psi(v)}u$ and $u\psi(v) = h_1 w_1$ for certain $h_1 \in G$. Hence, $\psi(v)w_1^{-1} = u^{-1}h_1$. In other words, $u^{-1} = \lambda_{\psi(v)}w_1^{-1}$. It follows from the first identity that $\phi(u^{-1}) = \rho_v\phi(w_1^{-1})$. That is, there is some $h_2 \in G$ such that $\phi(w_1^{-1})v = h_2\phi(u^{-1})$. Therefore, $v\phi(u) = \phi(w_1)h_2$. The third identity thus follows immediately. The last one can be proved in the same way. □

5. Properties of the global $\mathscr{R}$-matrix

As a continuation of the last section, we shall study here some important properties of the global $\mathscr{R}$-matrix. Readers familiar with quantum group theory will find many similarities between the properties listed here and those of a quasitriangular quantum R-matrix.

The following theorem shows that the action of $\mathscr{R}$ on $G^* \times G^*$ intertwines the group multiplication and its opposite.

Theorem 5.1. *For any* $(d_1, d_2) \in \mathscr{R}$, *we have*

$$\alpha_2(d_1)\alpha_2(d_2) = \beta_2(d_2)\beta_2(d_1)\,.$$

Or equivalently, we have the following identity for any $u, v \in G^*$,

$$(\lambda_{\psi(v^{-1})}u)(\lambda_{\phi(\tilde{u})}v) = vu\,,$$

where, again, $\tilde{u}$ *denotes* $\lambda_{\psi(v^{-1})}u$.

Proof. Assume that $d_1 = \psi(v^{-1})u$ and $d_2 = \phi(\tilde{u})v$ for some $u, v \in G^*$. Hence,

$$\psi(v^{-1})u = \tilde{u}g_1 \quad \text{for some} \quad g_1 \in G\,. \tag{9}$$

Assume further that $\phi(\tilde{u})v = u_2 g_2$, for some $u_2 \in G^*$ and $g_2 \in G$. It follows from Corollary 4.4 that $g_2 = \rho_v\phi(\tilde{u}) = \rho_v(\rho_{v^{-1}}\phi(u)) = \phi(u)$. Hence

$$u_2 = \phi(\tilde{u})vg_2^{-1} \tag{10}$$

$$= \phi(\tilde{u})v\phi(u^{-1})\,. \tag{11}$$

From Eq. (9), it follows, by applying the morphism F_+, that $\psi(v^{-1})\phi(u^{-1}) = \phi(\tilde{u}^{-1})g_1$. Hence,

$$\phi(\tilde{u}) = g_1\phi(u)\psi(v)\,. \tag{12}$$

Therefore,

$$\begin{aligned}
\tilde{u}u_2 &= \tilde{u}\phi(\tilde{u})v\phi(u^{-1}) \quad \text{(by Eq. (12))}\\
&= \tilde{u}g_1\phi(u)\psi(v)v\phi(u^{-1}) \quad \text{(by Eq. (9))}\\
&= \psi(v^{-1})u\phi(u)\psi(v)v\phi(u^{-1}) \quad \text{(by using Theorem 2.12)}\\
&= \psi(v^{-1})[(\psi(v)v)(u\phi(u))]\phi(u^{-1})\\
&= vu\,.
\end{aligned}$$

□

Before we go into more detailed properties of the $\mathscr{R}$-matrix, we need to recall some conventions related to symplectic groupoids.

Let $(\Gamma \rightrightarrows P, \alpha, \beta)$ be a symplectic groupoid. A lagrangian submanifold of Γ for which the restrictions of α and β are diffeomorphisms onto P is called a lagrangian bisection. The lagrangian bisections form a Lie group $U(\Gamma)$ under the multiplication of subsets induced from the product on Γ. In fact, the Lie algebra of $U(\Gamma)$ can be identified in a natural way with the space $Z^1(P)$ of closed 1-forms on P, with the bracket induced from the Poisson bracket on $C^\infty(P)$. Any lagrangian bisection $\mathscr{L} \in U(\Gamma)$ induces an automorphism $\mathrm{Ad}_{\mathscr{L}}$ of the symplectic groupoid Γ, by sending each $\gamma \in \Gamma$ to $\mathscr{L}\gamma\mathscr{L}^{-1}$. Its restriction to P is a Poisson diffeomorphism which sends each $p \in P$ to $\hat{\alpha}(\hat{\beta}^{-1}(p))$, where $\hat{\alpha}$ and $\hat{\beta}$ are the restrictions of α and β to $\mathscr{L}$.

When $\Gamma = D^-$, regarded as the symplectic groupoid of the Poisson Lie group G^*, the global $\mathscr{R}$-matrix $\mathscr{R}$ belongs to $U(D^- \times D^-) = U(D \times D)$, where $D^- \times D^-$ is considered as the symplectic groupoid product. Now, let $\Delta \subset D \times D \times D^-$ be the graph of multiplication of the groupoid $(D \rightrightarrows G, \alpha_1, \beta_1)$, and let $\Delta' = P_{12}(\Delta) \subset D \times D \times D^-$, where P_{12} is the permutation which exchanges the first and second coordinates. It is clear that Δ' is still a lagrangian submanifold in $D \times D \times D^-$.

The following theorem is the global classical analogue of the exchange condition (1).

Theorem 5.2.

$$\Delta' = \mathrm{Ad}_{\mathscr{R}_{12}^{-1}} \Delta \ ,$$

where $\mathscr{R}_{12} = \mathscr{R} \times G^* \subset D \times D \times D$ is considered as an element of $U(D \times D \times D)$.

Proof. For any given $x, y \in D$ such that $\beta_1(y) = \alpha_1(x)$, we let $(x', y') = \mathrm{Ad}_{\mathscr{R}}(x, y) \in D \times D$. It suffices to show that $x' * y' = y * x$. In this proof, by γ^{-1} for any $\gamma \in D$, we mean the inverse of γ with respect to the groupoid $(D^- \rightrightarrows G^*, \alpha_2, \beta_2)$. Assume that $\beta_1(y) = \alpha_1(x) = g$,

$$x = gu = u_1 g_1 \tag{13}$$

and

$$y = vg = h_1 v_1 \ , \tag{14}$$

for some $u, v, u_1, v_1 \in G^*$ and $g_1, h_1 \in G$. By definition,

$$(x', y') = \mathrm{Ad}_{\mathscr{R}}(x, y) = (d_1, d_2)(x, y)(\tilde{d}_1^{-1}, \tilde{d}_2^{-1}) \ ,$$

for some $(d_1, d_2) \in \mathscr{R}$ and $(\tilde{d}_1, \tilde{d}_2) \in \mathscr{R}$, which are composable with (x, y). Then, we have $\beta_2(d_2) = \alpha_2(y) = v$ and $\beta_2(d_1) = \alpha_2(x) = u_1$. Therefore, $(d_1, d_2) = (\psi(v^{-1})u_1, \phi(\tilde{u}_1)v)$, where $\tilde{u}_1 = \lambda_{\psi(v^{-1})} u_1$. Hence there is $h_2 \in G$ such that

$$\psi(v^{-1})u_1 = \tilde{u}_1 h_2 \ . \tag{15}$$

On the other hand, we know that

$$\beta_2(\tilde{d}_1) = \alpha_2(\tilde{d}_1^{-1}) = \beta_2(x) = u \tag{16}$$

and

$$\beta_2(\tilde{d}_2) = \alpha_2(\tilde{d}_2^{-1}) = \beta_2(y) = v_1 \ . \tag{17}$$

Therefore, $(\tilde{d}_1, \tilde{d}_2) = (\psi(v_1^{-1})u, \phi(\tilde{u})v_1)$ with $\tilde{u} = \lambda_{\psi(v_1^{-1})} u$, i.e.,

$$\psi(v_1^{-1})u = \tilde{u}h, \quad \text{for some } h \in G \ . \tag{18}$$

We shall divide the rest of the proof into two steps.

Step 1. First of all, we need to show that $\beta_1(x') = \alpha_1(y')$.

It is noted that $\beta_1(x') = \beta_1(d_1 x \tilde{d}_1^{-1}) = \beta_1(d_1)\beta_1(x)\beta_1(\tilde{d}_1)^{-1} = h_2 g_1 h^{-1}$, and $\alpha_1(y') = \alpha_1(d_2 y \tilde{d}_2^{-1}) = \alpha_1(d_2)\alpha_1(y)\alpha_1(\tilde{d}_2)^{-1} = \phi(\tilde{u}_1)h_1\phi(\tilde{u}^{-1})$. It follows from Eq. (15) that

$$\tilde{u}_1 h_2 g_1 = \psi(v^{-1})u_1 g_1 = \psi(v^{-1})x = \psi(v^{-1})gu\,. \tag{19}$$

From Eq. (14), it follows, by applying the map F_-, that

$$\psi(v^{-1})g = h_1\psi(v_1^{-1})\,. \tag{20}$$

Hence,

$$\begin{aligned}\tilde{u}_1 h_2 g_1 &= (\psi(v^{-1})g)u \quad \text{(by Eq. (20))}\\ &= h_1\psi(v_1^{-1})u \quad \text{(by Eq. (18))}\\ &= h_1\tilde{u}h\,.\end{aligned}$$

Therefore, $h_2 g_1 h^{-1} = \tilde{u}_1^{-1} h_1 \tilde{u}$, from which by applying F_+, we have $h_2 g_1 h^{-1} = \phi(\tilde{u}_1)h_1\phi(\tilde{u}^{-1})$. Hence, $\beta_1(x') = \alpha_1(y')$.

Step 2. A direct computation yields that

$$\begin{aligned}\alpha_1(x' * y') &= \alpha_1(x')\\ &= \alpha_1(d_1)\alpha_1(x)\alpha_1(\tilde{d}_1)^{-1}\\ &= \psi(v^{-1})g\psi(v_1^{-1})^{-1} \quad \text{(by Eq. (20))}\\ &= h_1\psi(v_1^{-1})\psi(v_1^{-1})^{-1}\\ &= h_1\\ &= \alpha_1(y * x)\,.\end{aligned}$$

Moreover, since

$$\begin{aligned}\beta_2(x' * y') &= \beta_2(x')\beta_2(y')\\ &= \beta_2(d_1 x \tilde{d}_1^{-1})\beta_2(d_2 y \tilde{d}_2^{-1})\\ &= \alpha_2(\tilde{d}_1)\alpha_2(\tilde{d}_2)\,,\end{aligned}$$

and

$$\begin{aligned}\beta_2(y * x) &= \beta_2(y)\beta_2(x) \quad \text{(by Eqs. (16) and (17))}\\ &= \beta_2(\tilde{d}_2)\beta_2(\tilde{d}_1)\,,\end{aligned}$$

we have $\beta_2(x' * y') = \beta_2(y * x)$ according to Theorem 5.1. Therefore, $x' * y' = y * x$. This completes our proof. $\square$

For any lagrangian bisection $\mathscr{L} \in U(D \times D)$, we define $(\Delta \otimes \mathrm{id})\mathscr{L} \subset D \times D \times D$ to be the composition $\mathscr{L} \circ \Delta'$ of $\mathscr{L} \subset D \times D$ with $\Delta' \subset D \times D \times D^-$. Here, Δ' is considered as a canonical relation from $D \times D$ to D, $\mathscr{L}$ as a canonical relation from D to D^-. Therefore, the resulting composition is a canonical relation (under the assumption of clean intersection) from $D \times D$ to D^-. That is, $(\Delta \otimes \mathrm{id})\mathscr{L}$ is the subset of $D \times D \times D$ consisting of all elements (d_1, d_2, d_3) such that $(d_2 * d_1, d_3) \in \mathscr{L}$.

Similarly, by $(\mathrm{id} \otimes \Delta)\mathscr{L}$, we denote the composition $\Delta'' \circ \mathscr{L}$ of $\Delta'' = P_{13}(\Delta) \subset D^- \times D \times D$ with $\mathscr{L} \subset D \times D$, where P_{13} is the permutation of the first and third coordinates. Namely, $(\mathrm{id} \otimes \Delta)\mathscr{L}$ consists of all elements $(d_1, d_2, d_3) \in D \times D \times D$ such that $(d_1, d_3 * d_2) \in \mathscr{L}$.

Proposition 5.3. *Both* $(\Delta \otimes \mathrm{id})\mathscr{L}$ *and* $(\mathrm{id} \otimes \Delta)\mathscr{L}$ *for any* $\mathscr{L} \in U(D \times D)$ *are well-defined lagrangian bisections in* $U(D \times D \times D)$.

Proof. In order to show that they are well-defined bisections, we assume that

$$\mathscr{L} = \{(\phi_1(u, v), u, \phi_2(u, v), v) | \forall u, v \in G^*\} ,$$

with ϕ_1 and ϕ_2 being some smooth maps from $G^* \times G^*$ to $G \times G$. Suppose that (d_1, d_2, d_3) is any element in $(\Delta \otimes \mathrm{id})\mathscr{L}$. Assume that $d_1 = g_1 u, d_2 = g_2 v$ and $d_3 = g_3 w$ for $g_1, g_2, g_3 \in G$ and $u_1, u_2, u_3 \in G^*$. Then, $\alpha_1(d_2 * d_1) = \alpha_1(d_2) = g_2$ and $\beta_2(d_2 * d_1) = \beta_2(d_2)\beta_2(d_1) = vu$. From the definition of $\mathscr{L}$, it follows that $g_2 = \phi_1(vu, w)$ and $g_3 = \phi_2(vu, w)$. Finally, $g_1 = \alpha_1(d_1) = \beta_1(d_2) = \rho_v(\phi_1(vu, w))$.

Conversely, for any $u, v, w \in G^*$, the element $(d_1, d_2, d_3) \in D \times D \times D$ with g_1, g_2, g_3 being given by the above formulas belong to $(\Delta \otimes \mathrm{id})\mathscr{L}$. Hence, $(\Delta \otimes \mathrm{id})\mathscr{L}$ is a well-defined bisection, so is $(\mathrm{id} \otimes \Delta)\mathscr{L}$. Finally, by lagrangian calculus, it is simple to see that both of them are lagrangian submanifolds. □

For the global $\mathscr{R}$-matrix $\mathscr{R} \in U(D \times D)$, we have the following analogue of the compatibility condition (2).

Theorem 5.4. *Suppose that G is a complete quasitriangular Poisson Lie group, and* $\mathscr{R} \in U(D \times D)$ *is its global* $\mathscr{R}$*-matrix. Then,*

$$(\Delta \otimes \mathrm{id})\mathscr{R} = \mathscr{R}_{13}\mathscr{R}_{23} \tag{21}$$

and

$$(\mathrm{id} \otimes \Delta)\mathscr{R} = \mathscr{R}_{13}\mathscr{R}_{12} , \tag{22}$$

where, as usual, $\mathscr{R}_{12} = \mathscr{R} \times G^* \subset D \times D \times D$, *with* G^* *being identified with the base space of the symplectic groupoid* $(D^- \rightrightarrows G^*, \alpha_2, \beta_2)$, $\mathscr{R}_{23} = G^* \times \mathscr{R} \subset D \times D \times D$, *and* $\mathscr{R}_{13} = P_{23}(\mathscr{R}_{12})$.

Before we prove this theorem, we need the following

Lemma 5.5. *If* $(d_1, d_2) \in \mathscr{R}$, *then*

$$\alpha_1(d_1) = \psi(\beta_2(d_2))^{-1} ,$$

$$\beta_1(d_1) = \psi(\alpha_2(d_2))^{-1} ;$$

and

$$\alpha_1(d_2) = \phi(\alpha_2(d_1)) ,$$

$$\beta_1(d_2) = \phi(\beta_2(d_1)) .$$

Proof. The last two identities follow directly from the construction of $\mathscr{R}$, and the first one is quite obvious.

For the second identity, we assume that $(d_1, d_2) = (\psi(v^{-1})u, \phi(\tilde{u})v)$ for some $u, v \in G^*$, and

$$\psi(v^{-1})u = \tilde{u}h_1 , \tag{23}$$

$$\phi(\tilde{u})v = u_2 h_2 , \tag{24}$$

for certain $h_1, h_2 \in G$.

Equation (23) implies that $\psi(v^{-1})\psi(u^{-1}) = \psi(\tilde{u}^{-1})h_1$. Hence,

$$\begin{aligned}\beta_1(d_1) = h_1 &= \psi(\tilde{u})\psi(v^{-1})\psi(u^{-1})\\ &= \psi((vu)^{-1}\tilde{u}) \quad \text{(by Theorem 5.1)}\\ &= \psi((\tilde{u}u_2)^{-1}\tilde{u})\\ &= \psi(u_2^{-1})\\ &= \psi(\alpha_2(d_2))^{-1}\ .\end{aligned}$$

□

Proof of Theorem 5.4. Since both $(\Delta \otimes \mathrm{id})\mathscr{R}$ and $\mathscr{R}_{13}\mathscr{R}_{23}$ are bisections of $D \times D \times D$, it suffices to show that $\mathscr{R}_{13}\mathscr{R}_{23} \subset (\Delta \otimes \mathrm{id})\mathscr{R}$. Assume that (x, y, z) is any element in $\mathscr{R}_{13}\mathscr{R}_{23}$, then by definition, $x = d'_1$, $y = d''_1$, and $z = d'_2 d''_2$ for some (d'_1, d'_2) and $(d''_1, d''_2) \in \mathscr{R}$. So it is necessary that $\beta_2(d'_2) = \alpha_2(d''_2)$. Therefore, according to Lemma 5.5,

$$\beta_1(y) = \beta_1(d''_1) = \psi(\alpha_2(d''_2))^{-1} = \psi(\beta_2(d'_2))^{-1} = \alpha_1(d'_1) = \alpha_1(x)\ . \tag{25}$$

Below, we shall show that $(y * x, z) \in \mathscr{R}$.

Assume that $(d'_1, d'_2) = (\psi(s^{-1})u, \phi(\tilde{u})s)$ and $(d''_1, d''_2) = (\psi(w^{-1})v, \ \phi(\tilde{v})w)$. Then,

$$\beta_2(y * x) = \beta_2(y)\beta_2(x) = \beta_2(d''_1)\beta_2(d'_1) = vu\ ,$$

and

$$\alpha_1(y * x) = \alpha_1(y) = \alpha_1(d''_1) = \psi(w^{-1})\ .$$

On the other hand, we know that

$$\beta_2(z) = \beta_2(d'_2 d''_2) = \beta_2(d''_2) = w\ ,$$

and

$$\alpha_1(z) = \alpha_1(d'_2 d''_2) = \alpha_1(d'_2)\alpha_1(d''_2) = \phi(\tilde{u})\phi(\tilde{v}) = \phi(\tilde{v}\tilde{u})\ .$$

It remains to show that $\tilde{v}\tilde{u} = \lambda_{\psi(w^{-1})}(vu)$.

Now by definition, there are h_1 and $h_2 \in G$, such that $\psi(s^{-1})u = \tilde{u}h_1$ and $\psi(w^{-1})v = \tilde{v}h_2$. In fact, here $h_2 = \beta_1(d''_1) = \alpha_1(d'_1) = \psi(s^{-1})$, according to Eq. (25). Therefore, $\psi(w^{-1})vu = \tilde{v}h_2 u = \tilde{v}\psi(s^{-1})u = \tilde{v}\tilde{u}h_1$, i.e., $\tilde{v}\tilde{u} = \lambda_{\psi(w^{-1})}vu$. This completes our proof of the first identity.

The second identity can be proved along the same lines. □

Remark 5.6. In the case where ϕ is an isomorphism, and hence $\mathscr{R}$ is the graph of the symplectic groupoid automorphism $\mathscr{F}$, it is easy to see that $\mathscr{R}_{13}\mathscr{R}_{23}$ consists of all elements of the form $(x, y, \mathscr{F}(x)\mathscr{F}(y))$ for any $x, y \in D$ such that $\beta_1(y) = \alpha_1(x)$, and

$$\mathscr{R}_{13}\mathscr{R}_{12} = \{(xy, \mathscr{F}(y), \mathscr{F}(x)) | \forall x, y \ s.t.\ \beta_2(x) = \alpha_2(y)\}\ .$$

Therefore, $(\Delta \otimes \mathrm{id})\mathscr{R} = \mathscr{R}_{13}\mathscr{R}_{23}$ is exactly equivalent to saying that $\mathscr{F}$ is a groupoid antihomomorphism between $(D \rightrightarrows G, \alpha_1, \beta_1)$ and $(D^- \rightrightarrows G^*, \alpha_2, \beta_2)$, and $(\mathrm{id} \otimes \Delta)\mathscr{R} = \mathscr{R}_{13}\mathscr{R}_{12}$ amounts to say that $\mathscr{F}$ is a groupoid homomorphism between $(D^- \rightrightarrows G^*, \alpha_2, \beta_2)$ and $(D \rightrightarrows G, \alpha_1, \beta_1)$.

We conclude this section with the following analogue of (3).

Theorem 5.7. *$\mathscr{R}$ satisfies the "quantum Yang–Baxter equation", i.e.,*

$$\mathscr{R}_{13}\mathscr{R}_{23}\mathscr{R}_{12} = \mathscr{R}_{12}\mathscr{R}_{23}\mathscr{R}_{13} ,$$

where the multiplication is understood as the multiplication in the group $U(D \times D \times D)$.

Proof. It is not hard to see, as in the above, that $\mathscr{R}_{23}\mathscr{R}_{13} = \Delta \circ \mathscr{R}$, the composition of the lagrangian $\Delta \subset D \times D \times D^-$ with $\mathscr{R} \subset D \times D$. According to Theorem 5.2,

$$\begin{aligned} \mathscr{R}_{12}^{-1}(\mathscr{R}_{23}\mathscr{R}_{13})\mathscr{R}_{12} &= \mathrm{Ad}_{\mathscr{R}_{12}^{-1}}(\mathscr{R}_{23}\mathscr{R}_{13}) \\ &= \mathrm{Ad}_{\mathscr{R}_{12}^{-1}}(\Delta \circ \mathscr{R}) \\ &= (\mathrm{Ad}_{\mathscr{R}_{12}^{-1}}\Delta) \circ \mathscr{R} \\ &= \Delta' \circ \mathscr{R} \\ &= (\Delta \otimes \mathrm{id})\mathscr{R} \\ &= \mathscr{R}_{13}\mathscr{R}_{23} , \end{aligned}$$

which implies the Yang–Baxter equation. □

6. Quasitensor "Categories" of Symplectic Realizations

Symplectic realizations play the role for Poisson manifolds which representations do for noncommutative algebras. A **symplectic realization** of a Poisson manifold P is defined to be a symplectic manifold S together with a Poisson map $J: S \to P$. The realization is said to be **complete** if J is a complete Poisson morphism in the sense that the hamiltonian vector field of the pullback to S of every compactly supported function on P is complete. For a given Poisson manifold P, complete symplectic realizations of P become a "category" $\zeta(P)$, in which the objects are complete symplectic realizations of P, the morphisms from a symplectic realization $S_\lambda \xrightarrow{J_\lambda} P$ to a symplectic realization $S_\mu \xrightarrow{J_\mu} P$ are lagrangian submanifolds contained in the inverse image of the diagonal of $P \times \bar{P}$ under the map $J_\lambda \times J_\mu$ (the bar denotes reversing the sign of the symplectic structure), and the composition of morphisms is set-theoretic composition of relations [X1, X2]. As is pointed out in [X1], $\zeta(P)$ is not a true category, since a transversality assumption is required for the composition of morphisms to be a morphism.

Suppose that P is an integrable Poisson manifold in the sense that P admits a symplectic groupoid $(\Gamma \rightrightarrows P, \alpha, \beta)$. By a symplectic left Γ-module we mean a symplectic realization $J: S \to P$ with a symplectic left Γ-action. A theorem in [X1] asserts that if Γ is α-simply connected, then any complete symplectic realization of P becomes a left Γ-module in a natural way. In other words, the "category" of complete symplectic realizations of P is exactly isomorphic to the "category" of left Γ-modules.

Recall that $U(\Gamma)$ denotes the set of all lagrangian bisections of Γ. $U(\Gamma)$ becomes an infinite dimensional Lie group with the multiplication being naturally induced from that on Γ.[2] If S is a symplectic left Γ-module, then the group $U(\Gamma)$ acts symplectically on S. We therefore have the following:

[2] The "Lie algebra" of $U(\Gamma)$ is isomorphic to the closed 1-forms on P. To form a group whose Lie algebra is $C^\infty(P)$, we must extend Γ by prequantization [WX]

Proposition 6.1. *Let Γ be an α-simply connected symplectic groupoid over the Poisson manifold P. Given a complete symplectic realization $J\colon S \to P$ there is a canonical group homomorphism, denoted by ρ_S, from $U(\Gamma)$ to $\mathscr{D}(S)$, the group of all symplectic automorphisms of S.*

Remark 6.2. One can give the following quantum interpretation of the preceding proposition. If the Poisson manifold P is a geometric model for some noncommutative algebra, then the group $U(\Gamma)$ models the unitary elements of that algebra.

When $P = G^*$, the dual of a simply connected complete Poisson Lie group G, its symplectic groupoid Γ_{G^*} is diffeomorphic to the double group $D = G \bowtie G^*$. In this case, symplectic left Γ_{G^*}- modules (hence complete symplectic realizations of G^*) can be described in terms of some familiar objects: the so-called Poisson G-spaces. Recall that a Poisson manifold S with a left G-action is called a Poisson G-space if the action map $G \times S \to S$ is a Poisson map. A Poisson morphism $J\colon S \to G^*$ is said to be a momentum mapping for the Poisson G-space, if

$$X \in \mathfrak{g} \mapsto -\pi_S^\#(J^*(X^r)) \in \mathscr{X}(S) \tag{26}$$

is the infinitesimal generator of the G-action, where X^r denotes the right-invariant one form on G^* with value $X \in \mathfrak{g}$ at the identity.[3] The following proposition reveals the relation between symplectic left Γ_{G^*}-modules and Poisson G-spaces.

Proposition 6.3. *If $J\colon S \to G^*$ is a symplectic left Γ_{G^*}-module, then S is a Poisson G-space with the action being defined by*

$$gx = (g, J(x)) \cdot x\ , \tag{27}$$

for any $g \in G$ and $x \in S$, where $(g, J(x))$ is considered as an element in $\Gamma_{G^*} \cong G \bowtie G^*$ and the dot in the right-hand side refers to the groupoid Γ_{G^*}-action on S. Moreover, J is the momentum mapping of the induced Poisson G-action, in the sense described above. Conversely, if a symplectic manifold S is a Poisson G-space with a momentum mapping $J\colon S \to G^*$, Eq. (27) defines a symplectic left Γ_{G^*}-action on G.

Proof. According to Theorem 3.1 in [X1], the Poisson morphism $J\colon S \to G^*$, considered as a symplectic realization, induces a unique symplectic left Γ_{G^*}-action on S. On the other hand, J induces a Poisson group G-action on S, with the infinitesimal generators being given by Eq. (26) according to Corollary 3.6 in [Lu2]. All we need to show is that these two actions are related by Eq. (27). For this purpose, in the following, let us recall the explicit expression for the symplectic left Γ_{G^*}-action on S. For any $X \in \mathfrak{g}$ and any $x \in S$, let $\phi_t^{\alpha_2}(u)$, with $u = J(x)$, be the flow on Γ_{G^*} generated by the Poisson vector field corresponding to the one-form $\alpha_2^*(X^r)$, and let $\phi_t^J(x)$ be the flow on S generated by the Poisson vector field corresponding to the one-form $J^*(X^r)$. According to Theorem 3.1 in [X1], $\phi_t^{\alpha_2}(u) \cdot x = \phi_t^J(x)$, where the dot on the left-hand side denotes the symplectic groupoid Γ_{G^*}-action on S. Clearly, the right-hand side equals $\exp(-tX) \cdot x$, by the definition of momentum mappings. Therefore, it suffices to show that $\phi_t^{\alpha_2}(u) = (\exp(-tX), u) \in D = G \times G^*$, which is indeed true according to Proposition 3.3 (note that the Poisson structure on Γ_{G^*} is $-\pi_+$). □

[3] Our sign conventions for Poisson G-spaces and momentum mappings are different from the usual ones in the literature [STS, Lu2]. We use them because they match those for groupoids.

Remark 6.4. Returning to the quantum analogies in Remark 6.2, we see that when S is a Poisson G-space, there is an induced action of $U(\Gamma_{G^*})$ on S. A partially quantized version of this statement goes as follows. When G has the zero Poisson structure, then $U(\Gamma_{G^*})$ is a model for the group $\mathscr{G}$ of unitary elements in the universal enveloping algebra of $\mathfrak{g}$ (or in a completion of this enveloping algebra which is a convolution algebra of distributional densities on G). A linear representation of G induces in a natural way a linear representation of the group $\mathscr{G}$. In fact, G can be identified with the subgroup of $\mathscr{G}$ consisting of delta distributions at the elements of G. In symplectic terms, Γ_{G^*} in this case is just T^*G, and G is embedded in $U(\Gamma_{G^*})$ as the set of fibres of the cotangent bundle.

Now if G has a nonzero Poisson structure, the only fibres of the projection from $G \times G^*$ to G which are lagrangian are those which lie over the subgroup G_0 of G where the Poisson structure vanishes. As a result, there is no embedding of G in $U(\Gamma_{G^*})$, and hence no action of G on S by symplectic transformations. The only "points" in G are the elements of G_0, and so the natural group of automorphisms of S is not G but rather the "universal enveloping group" $U(\Gamma_{G^*})$.

Analogously, when the quantization of a group G acts on a vector space, one has a representation, not of the underlying classical group G, but rather of an algebra which includes only part of G among its unitary elements.

Corresponding to the tensor product of representations of a Hopf algebra, we can define a "tensor product" of symplectic realizations of a Poisson group. Suppose that $J_\lambda\colon S_\lambda \to G^*$ and $J_u\colon S_\mu \to G^*$ are symplectic realizations of G^*, then $J_\lambda \times J_\mu\colon S_\lambda \times S_u \to G^* \times G^*$ is a symplectic realization of $G^* \times G^*$. Since the group multiplication $m\colon G^* \times G^* \to G^*$ is a Poisson map, $S_\lambda \times S_\mu \xrightarrow{m\circ(J_\lambda \times J_\mu)} G^*$ is thus a symplectic realization of G^*. We write this realization as $S_\lambda \otimes S_\mu$, and the realization map $m \circ (J_\lambda \times J_\mu)$ as $J_\lambda \otimes J_\mu$. Hence, we obtain a "tensor product" operation on the set of all realizations of G^*. The following proposition shows that this "tensor product" operation is closed among complete realizations when the Poisson group G itself is complete.

Proposition 6.5. *Let G be a complete Poisson group. If both symplectic realizations $J_\lambda\colon S_\lambda \to G^*$ and $J_\mu\colon S_\mu \to G^*$ are complete, then their tensor product $S_\lambda \otimes S_\mu$ is also complete.*

This proposition is a direct consequence of the following:

Lemma 6.6. *If G is a complete Poisson group, then the multiplication $m\colon G^* \times G^* \to G^*$ is a complete Poisson map.*

Proof. It suffices to show that the Poisson map $m\colon G^* \times G^* \to G^*$ induces a well-defined Poisson G-action on $G^* \times G^*$. In fact, we claim that m is the momentum mapping of the following G-action:

$$g \cdot (u, v) = (\lambda_g u, \lambda_{\rho_u g} v)\ .$$

It is simple to check directly that the above formula indeed defines a group action on $G^* \times G^*$. Its infinitesimal generator is given by:

$$X \in \mathfrak{g} \mapsto (\pi^\#(X^r)(u), \pi^\#((T\rho_u X)^r)(v)) \in T_u G^* \oplus T_v G^*\ .$$

Below, we shall compute the infinitesimal $\mathfrak{g}$-action on $G^* \times G^*$ induced from the multiplication map $m: G^* \times G^* \to G^*$ as a Poisson morphism. It is easy to see that for any cotangent vector $\theta \in T^*_{uv}G^*$, $m^*\theta = r^*_v\theta + l^*_u\theta \in T^*_u G^* \oplus T^*_v G^*$. In particular, if θ is equal to X^r, the right invariant one form with value $X \in \mathfrak{g}$ at the identity, $m^*\theta|_{(u,v)} = r^*_{u^{-1}}X + r^*_{v^{-1}}(\mathrm{Ad}^*_u X) \in T^*_u G^* \oplus T^*_v G^*$. It thus remains to show that $T\rho_u X = \mathrm{Ad}^*_u X$.

In order to show this identity, we assume that in the double group D, $(\exp tX)u = u_t h_t$, for some $u_t \in G^*$ and $h_t \in G$. It thus follows that $u^{-1}(\exp tX)u = (u^{-1}u_t)h_t$. Hence $T\rho_u X = \frac{d}{dt}\Big|_{t=0} h_t = T\mathrm{Pr}_2(\mathrm{Ad}_{u^{-1}}X)$, where Pr_2 is the natural projection from D to G under the decomposition $D = G^* \bowtie G$. Hence, the conclusion follows immediately as a direct consequence of Proposition 2.34 (3) in [Lu1]. $\square$

In the rest of the paper, we will confine ourselves to complete symplectic realizations of G^*.

In general, the tensor product $\otimes : S_\lambda \times S_\mu \mapsto S_\lambda \otimes S_\mu$ is not commutative except when G^* is commutative, i.e. when the Poisson structure on G is zero and G^* is the usual Lie–Poisson space $\mathfrak{g}^*$ with multiplication being the usual addition as a vector space. However, as is the case for quantum groups, if G is quasitriangular, an isomorphism of realizations between $S_\lambda \otimes S_\mu$ and $S_\mu \otimes S_\lambda$ can be restored via the global classical $\mathscr{R}$-matrix.

Theorem 6.7. *Suppose that* $S_\lambda \xrightarrow{J_\lambda} G^*$ *and* $S_\mu \xrightarrow{J_\mu} G^*$ *are complete symplectic realizations; then*

$$\begin{tikzcd}
S_\lambda \otimes S_\mu \arrow[r, "\mathscr{R}_{\lambda\mu}"] \arrow[d, "J_\lambda \otimes J_\mu"'] & S_\mu \otimes S_\lambda \arrow[d, "J_\mu \otimes J_\lambda"] \\
G^* \arrow[r, "\mathrm{id}"] & G^*
\end{tikzcd}$$

is an isomorphism of symplectic realizations. Here $R_{\lambda\mu} = \rho_{S_\mu \times S_\lambda}(R) \circ \sigma$, *where* σ: $S_\lambda \times S_\mu \to S_\mu \times S_\lambda$ *is the exchange of components, and* $\rho_{S_\mu \times S_\lambda}$: $U(\Gamma_{G^*} \times \Gamma_{G^*}) \mapsto \mathscr{D}(S_\mu \times S_\lambda)$ *is the group homomorphism of Proposition* 6.1.

Proof. $\mathscr{R}_{\lambda\mu}$ is obviously a symplectic diffeomorphism from $S_\lambda \times S_\mu$ to $S_\mu \times S_\lambda$. It suffices to check the commutativity of the above diagram. For any $(x, y) \in S_\lambda \otimes S_\mu$, $\mathscr{R}_{\lambda\mu}(x, y) = (d_1, d_2)\cdot(y, x)$ for some compatible $(d_1, d_2) \in \mathscr{R}$, where the dot stands for the action of the groupoid $\Gamma_{G^*} \times \Gamma_{G^*}$ on $S_\mu \times S_\lambda$. Hence $(J_\mu \otimes J_\lambda)\mathscr{R}_{\lambda\mu}(x, y) = \alpha_2(d_1)\alpha_2(d_2)$. On the other hand, $(J_\lambda \otimes J_\mu)(x, y) = J_\lambda(x)J_\mu(y) = \beta_2(d_2)\beta_2(d_1)$, where the second equality follows from the compatibility between (d_1, d_2) and (y, x). Thus, the conclusion follows immediately from Theorem 5.1. $\square$

Remark 6.8. It is simple to see that $\mathscr{R}_{\lambda\mu} \circ \mathscr{R}_{\mu\lambda} = \rho_{S_\mu \times S_\lambda}(\mathscr{R}_{12}\mathscr{R}_{21})$, which will be seen in the next section not to be the identity in general.

It is also worth noting that the quantum Yang–Baxter equation for our classical global $\mathscr{R}$-matrix implies that the "category" of symplectic realizations of G^*, becomes a quasitensor "category," exactly as in the case of a quasitriangular quantum group [Dr1]. We will end this section with some further remarks.

Remark 6.9. When G is a zero Poisson Lie group, its dual is just the Lie Poisson manifold $\mathfrak{g}^*$. The tensor product of two realizations $J_\lambda\colon S_\lambda \to \mathfrak{g}^*$ and $J_\mu\colon S_\mu \to \mathfrak{g}^*$ is formed by $(J_\lambda \otimes J_\mu)(x, y) = J_\lambda(x) + J_\mu(y)$. If both realizations are complete and thus induce G-actions on S_μ and S_λ respectively, then the G-action on $S_\lambda \times S_\mu$ induced from the tensor product of realizations $S_\lambda \otimes S_\mu$ is exactly the diagonal action. For a general Poisson group G, such an action is given by a twisted diagonal action of the following form:

$$g \cdot (x, y) = (g \cdot x, (\rho_{J(x)} g) \cdot y) ,$$

where $\rho_{J(x)} g$ is the right dressing action of $J(x)$ on g. (It would be interesting to know the quantum analogue of this twisted diagonal action.)

In the context of Poisson G-spaces, i.e., when both $S_\lambda \otimes S_\mu$ and $S_\mu \otimes S_\lambda$ are considered as Poisson G-spaces induced from the realization morphisms, Theorem 6.7 can be interpreted as saying that $\mathscr{R}_{\lambda\mu}$ defines an isomorphism of Poisson G-spaces $S_\lambda \otimes S_\mu$ and $S_\mu \otimes S_\lambda$.

Remark 6.10. When G is $SU(n)$ with the standard Bruhat–Poisson structure [LW2], its dual G^* is $SB(n, \mathbb{C})$, which has recently been shown to be Poisson diffeomorphic to $\mathfrak{su}^*(n)$ [GW]. Hence, the "category" of symplectic realizations of $SB(n, \mathbb{C})$ is exactly isomorphic to the "category" of realizations of $\mathfrak{su}^*(n)$. It is well-known that the latter consists of all hamiltonian $SU(n)$-spaces. In other words, in the context of G-spaces, any Poisson $SU(n)$-space having a momentum mapping arises from an ordinary hamiltonian $SU(n)$-space with the action being twisted in a certain way. However, one should note that the tensor products of realizations are quite different in these two situations.

7. Set-Theoretic Quantum Yang–Baxter Equation and Symplectic Braid Group Actions

Finding solutions to the set-theoretic quantum Yang–Baxter equation is an open problem in quantum group theory recently posed by Drinfel'd [Dr2]. By such a solution, we mean a map $R\colon S \times S \to S \times S$, where S is any set, satisfying the quantum Yang–Baxter equation:

$$R_{13} R_{23} R_{12} = R_{12} R_{23} R_{13} ,$$

where $R_{ij}\colon S \times S \times S \to S \times S \times S$ is the "implantation" of R on the i^{th} and j^{th} factors of the cartesian product. Clearly, one can adapt this equation to various contexts. In particular, when S is a symplectic manifold, $R\colon S \times S \to S \times S$ should be a symplectic diffeomorphism. The following theorem indicates that the relation between the global $\mathscr{R}$-matrix and solutions to the set-theoretic Yang–Baxter equation is similar to that between a universal R-matrix of a quantum group and solutions to the quantum Yang–Baxter equation induced from particular representations.

Theorem 7.1. *Let G be a complete quasitriangular Poisson Lie group and $J\colon S \to G^*$ any complete symplectic realization of G^*. Then $\rho_{S \times S}(\mathscr{R})\colon S \times S \to S \times S$ is a solution to the set-theoretic quantum Yang–Baxter equation.*

Proof. According to Theorem 5.7, $\mathscr{R}$ satisfies the quantum Yang–Baxter equation in the group of bisections of groupoids. The desired result then follows immediately from the fact (Proposition 6.1) that the ρ's are group homomorphisms. □

An easy consequence is the following:

Corollary 7.2. *If G is a complete quasitriangular group, then the map $\mathscr{R}_{G^*G^*}$: $G^* \times G^* \to G^* \times G^*$, $(u, v) \to (\lambda_{\psi(v^{-1})}u,\ \rho_{\phi(u^{-1})}v)$ is a solution to the set-theoretic Yang–Baxter equation. Moreover, $\mathscr{R}_{G^*G^*}$ preserves the Poisson structure and leaves $\mathcal{O} \times \mathcal{O}$ invariant for any symplectic leaf $\mathcal{O} \subset G^*$.*

Proof. Let $\tilde{u} = \lambda_{\psi(v^{-1})}u$. Then Corollary 4.4 implies that $\phi(\tilde{u}) = \phi(\lambda_{\psi(v^{-1})}u) = \rho_{v^{-1}}\phi(u)$. Therefore, there is some $w \in G^*$ such that $\phi(u)v^{-1} = w\phi(\tilde{u})$, or $\phi(\tilde{u})v = w^{-1}\phi(u)$. This implies that $\lambda_{\phi(\tilde{u})}v = w^{-1} = (\lambda_{\phi(u)}v^{-1})^{-1} = \rho_{\phi(u^{-1})}v$. □

As in the case of quantum groups, solutions to the set-theoretic Yang–Baxter equation are closely related to braid group actions. More precisely, suppose that R: $S \times S \to S \times S$ is a solution to the set-theoretic Yang–Baxter equation. Let $\hat{R} = R \circ \sigma$ with σ: $S \times S \to S \times S$ being the exchange of components, and let $\hat{R}_1$ and $\hat{R}^2$ be the maps from $S \times S \times S$ to itself defined by $\hat{R}_1 = \hat{R} \times \mathrm{id}$ and $\hat{R}_2 = \mathrm{id} \times \hat{R}$, respectively.

Proposition 7.3. *If R: $S \times S \to S \times S$ satisfies the set-theoretic quantum Yang–Baxter equation, then $\hat{R}$ satisfies the braid relation:*

$$\hat{R}_1\hat{R}_2\hat{R}_1 = \hat{R}_2\hat{R}_1\hat{R}_2\ .$$

Proof. It can be checked directly that

$$\hat{R}_1 = \sigma^{123}_{231} \circ R_{23} \circ \sigma^{123}_{321}\ ,$$

and

$$\hat{R}_2 = \sigma^{123}_{213} \circ R_{13} \circ \sigma^{123}_{312}\ ,$$

where $\sigma^{i_1i_2i_3}_{j_1j_2j_3}$ denotes the transformation of $S \times S \times S$ which maps $(x_{i_1}, x_{i_2}, x_{i_3})$ to $(x_{j_1}, x_{j_2}, x_{j_3})$. Hence,

$$\begin{aligned}\hat{R}_1\hat{R}_2\hat{R}_1 &= \hat{R}_1\sigma^{123}_{213}R_{13}R_{23}\sigma^{123}_{321}\\ &= R_{12}R_{13}R_{23}\sigma^{123}_{321}\ .\end{aligned}$$

Similarly, we have $\hat{R}_2\hat{R}_1\hat{R}_2 = R_{23}R_{13}R_{12}\sigma^{123}_{321}$. Thus the conclusion follows immediately from the Yang–Baxter equation. □

Let $\hat{R}_i(n)$ be the endomorphism of the cartesian power S^n defined by:

$$\hat{R}_i(n)(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-1}, \hat{R}(x_i, x_{i+1}), x_{i+2}, \ldots, x_n)\ .$$

Proposition 7.3 implies that the assignment of $\hat{R}_i(n)$ to the i^{th} generator b_i of the braid group B_n defines an action of B_n on S^n for each n. In particular, we have the following

Theorem 7.4. *If G is a complete quasitriangular Poisson Lie group and $J: S \to G^*$ any complete symplectic realization of G^*, then the action of the classical $\mathscr{R}$ matrix on $S \times S$ induces a symplectic action of the braid group B_n on S^n.*

In particular, if S is a symplectic leaf $\mathcal{O}$ of G^* (i.e., an orbit of the dressing action), one has the following:

Theorem 7.5. *If G is a complete quasitriangular Poisson Lie group, then $(G^*)^n$ admits an action of B_n which preserves the Poisson structure and which leaves $\mathcal{O}^n$ invariant for any symplectic leaf $\mathcal{O} \subset G^*$.*

Remark 7.6. Given a quasitriangular Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ with classical r-matrix r, one can deform r in an obvious way to obtain a one parameter family of quasitriangular r-matrices: $r_t = tr$. The corresponding one parameter family of Lie bialgebras is $(\mathfrak{g}, \mathfrak{g}_t^*)$, where $\mathfrak{g}_t^*$ is the ordinary dual Lie algebra $\mathfrak{g}^*$ with a deformed Lie bracket: $[\xi, \eta]_t = t[\xi, \eta]$. Let $\mathfrak{g}_t$ denote the deformed Lie algebra of $\mathfrak{g}$ with the bracket given by $[X, Y]_t = t[X, Y]$, and $j_t\colon \mathfrak{g} \to \mathfrak{g}$ the map given by $j(X) = \frac{1}{t}X$ for all $X \in \mathfrak{g}$. Then, j_t is a Lie algebra isomorphism from $\mathfrak{g}$ to $\mathfrak{g}_t$. The dual of j_t, given by $j_t^*(\xi) = \frac{1}{t}\xi$, is easily seen to be a Lie algebra isomorphism from $\mathfrak{g}^*$ to $\mathfrak{g}_t^*$, so $(\mathfrak{g}, \mathfrak{g}_t^*)$ is isomorphic to $(\mathfrak{g}_t, \mathfrak{g}^*)$ as a Lie bialgebra. Globally, G_t^* is isomorphic as a Poisson group to G^* with the deformed Poisson structure $t\pi$. In the cases, such as those in [GW], where G^* is linearizable, all the G_t^* for $t \neq 0$ are Poisson diffeomorphic to (G^*, π); therefore they still have Γ_{G^*} as symplectic groupoid. The general constructions in the previous sections thus give rise to a family of lagrangian bisections $\mathscr{R}_t \in U(\Gamma_{G^*} \times \Gamma_{G^*})$, which satisfy all the properties of $\mathscr{R}$ in Sect. 5 and approach the identity bisection as t goes to zero. Hence, all the constructions in the section above can be carried out for this one-parameter family of $\mathscr{R}$-matrices. In particular, as in Theorem 7.4 (or Theorem 7.5), one obtains a one-parameter family of braid group actions on S^n (or $(G^*)^n$) and the actions reduce to the trivial symmetric group action as t approaches zero. Perhaps some useful information can be extracted from the derivative of this action with respect to t at $t = 0$.

8. Properties of the Braid Group Actions

This section is devoted to a further study of properties of the global classical $\mathscr{R}$-matrix and its induced braid group actions as described in the last section. One may find the prototypes from quantum group theory for these properties in various references [Dr3, RT]. Since our braid group actions are closely related to the classical $\mathscr{R}$-matrix, many properties of such actions are deduced from the corresponding properties of the classical $\mathscr{R}$-matrix. The first question we are going to investigate is how far away such a braid group action induced from a $\mathscr{R}$-matrix is from the trivial action of the symmetric group. In order to answer this question, it suffices to compute the square of $\mathscr{R}_{\mu\mu}$, which is clearly equal to $\rho_{S_\mu \times S_\mu}(\mathscr{R}_{12}\mathscr{R}_{21})$.

Proposition 8.1. *The lagrangian bisection $\mathscr{R}_{12}\mathscr{R}_{21} \subset \Gamma_{G^*} \times \Gamma_{G^*}$ consists of all elements of the form $(\psi(\tilde{u}^{-1})\phi(\tilde{u})v, \phi(\tilde{u}_1)\psi(v^{-1})u)$ for all $u, v \in G^*$, where $\tilde{u}$ and $\tilde{u}_1$ are given by $\tilde{u} = \lambda_{\psi(v^{-1})}u$ and $\tilde{u}_1 = \lambda_{\psi(\tilde{u}^{-1})\phi(\tilde{u})}v$.*

Proof. Let (r_1, r_2) be the element in $\mathscr{R}_{12}\mathscr{R}_{21}$ such that $\beta_2(r_1) = v$ and $\beta_2(r_2) = u$. We assume that

$$(r_1, r_2) = (\psi(v_1^{-1})u_1, \phi(\tilde{u}_1)v_1) * (\phi(\tilde{u})v, \psi(v^{-1})u)\,,$$

for some compatible $u_1, v_1 \in G^*$. Here, as usual, $\tilde{u} = \lambda_{\psi(v^{-1})}u$ and $\tilde{u}_1 = \lambda_{\psi(v_1^{-1})}u_1$. Hence

$$u_1 = \alpha_2(\phi(\tilde{u})v) = \lambda_{\phi(\tilde{u})}v\,, \tag{28}$$

and

$$v_1 = \alpha_2(\psi(v^{-1})u) = \tilde{u} = \lambda_{\psi(v^{-1})}u \; . \tag{29}$$

So $\alpha_1(r_1) = \psi(v_1^{-1})\phi(\tilde{u}) = \psi(\tilde{u}^{-1})\phi(\tilde{u})$, and $\alpha_1(r_2) = \phi(\tilde{u}_1)\psi(v^{-1})$. It thus follows that

$$\tilde{u}_1 = \lambda_{\psi(v_1^{-1})}u_1 = \lambda_{\psi(\tilde{u}^{-1})}\lambda_{\phi(\tilde{u})}v = \lambda_{\psi(\tilde{u}^{-1})\phi(\tilde{u})}v \; ,$$

which yields the conclusion. □

The following corollary gives an explicit description of the action of a generator of the braid group on $G^* \times G^*$, and hence on the cartesian square of any of its symplectic leaves.

Corollary 8.2. *Let τ denote the generating transformation of the B_2 action on $G^* \times G^*$ induced from the $\mathscr{R}$-matrix of a complete quasitriangular Poisson group G. Then τ^2 is given explicitly by*:

$$\tau^2(v, u) = (\tilde{u}_1, \tilde{u}_1^{-1}vu) \; ,$$

where $\tilde{u}_1 = \lambda_{\psi(\tilde{u}^{-1})\phi(\tilde{u})}v$, and $\tilde{u} = \lambda_{\psi(v^{-1})}u$.

Proof. Proposition 8.1 implies that $\tau^2(v, u) = \rho(\mathscr{R}_{12}\mathscr{R}_{21})(v, u) = (\lambda_{\psi(\tilde{u})\phi(\tilde{u})}v, \lambda_{\phi(\tilde{u}_1)\psi(v^{-1})}u)$. So it remains to check that $\lambda_{\phi(\tilde{u}_1)\psi(v^{-1})}u = \tilde{u}_1^{-1}vu$.

To prove this, we let $w = \lambda_{\phi(\tilde{u}_1)\psi(v^{-1})}u = \lambda_{\phi(\tilde{u}_1)}v_1$. Since $(\psi(v_1^{-1})u_1, \phi(\tilde{u}_1)v_1)$ belongs to $\mathscr{R}$, we have $\tilde{u}_1 w = v_1 u_1$ according to Theorem 5.1. Similarly, $(\psi(v^{-1})u, \phi(\tilde{u})v) \in \mathscr{R}$ implies that $\tilde{u}\lambda_{\phi(\tilde{u})}v = vu$, or $v_1 u_1 = vu$ by Eqs. (29) and (28). It thus follows immediately that $w = \tilde{u}_1^{-1}vu$. □

Remark 8.3. If G is a triangular Poisson Lie group, we know that $\phi = \psi$. Proposition 8.1 thus implies that $\mathscr{R}_{12}\mathscr{R}_{21}$ is the identity lagrangian bisection, or equivalently, $\mathscr{R}_{12} = \mathscr{R}_{21}^{-1}$. This is an analogue of the ordinary unitary condition on a quantum R-matrix. Moreover, in this case, $\mathscr{R}_{\mu\mu}^2$ is the identity map for any realization S_μ, so the braid group action on S_μ^n introduced in the last section reduces to the trivial action of symmetric group.

The following proposition describes a construction of a "central" lagrangian section out of the classical $\mathscr{R}$-matrix.

Let $\Delta \subset \Gamma_{G^*} \times \Gamma_{G^*}^- \times \Gamma_{G^*}^-$ be the graph of the groupoid multiplication of $(\Gamma_{G^*} \rightrightarrows G^*, \alpha_2, \beta_2)$. Consider Δ formally as a morphism from Γ_{G^*} to $\Gamma_{G^*} \times \Gamma_{G^*}$ and $\mathscr{R}$ as a morphism from $\Gamma_{G^*} \times \Gamma_{G^*}$ to a point. We thus can form their composition $\mathscr{L} = \mathscr{R} \circ \Delta$, a morphism from Γ_{G^*} to a point, i.e. a lagrangian submanifold of Γ_{G^*}. Below, we shall show that $\mathscr{L}$ is a "central" bisection (see [Dr3] [RT] for a similar construction for quasitriangular quantum groups).

Proposition 8.4. (1) *$\mathscr{L}$ is a well-defined lagrangian bisection of Γ_{G^*} contained in the isotropy groupoid of Γ_{G^*}. Moreover, $\mathscr{L}$ can be written explicitly as $\mathscr{L} = \{\psi(v^{-1})\phi(v)v \,|\, v \in G^*\}$.* (2) *$\mathscr{L}$ commutes with any bisection of Γ_{G^*}.*

The following lemma is crucial to the proof of this proposition.

Lemma 8.5. *For any $u \in G^*$, we have $\lambda_{\psi(u^{-1})\phi(u)}u = u$.*

Proof. Assume that

$$\psi(u^{-1})\phi(u)u = v_1 h_1 \; , \tag{30}$$

for some $v_1 \in G^*$ and $h_1 \in G$. It suffices to show that $v_1 = u$.

It follows from Eq. (30) by applying the morphism $\mathscr{F}_+$ that $\psi(u^{-1}) = \phi(v_1^{-1})h_1$, that is, $h_1 = \phi(v_1)\psi(u^{-1})$. Thus Eq. (30) becomes

$$\psi(u^{-1})\phi(u)u = v_1\phi(v_1)\psi(u^{-1}) .$$

Multiplying both sides of the above equation by u^{-1} from the left and using Theorem 2.12, we have $\phi(u)\psi(u^{-1}) = u^{-1}v_1\phi(v_1)\psi(u^{-1})$, which implies that $u\phi(u) = v_1\phi(v_1)$. It thus follows that $u = v_1$. □

Proof of Proposition 8.4.
(1) Let $d \in \mathscr{L}$ be the element in $\mathscr{L}$ satisfying $\beta_2(d) = v$. Assume that $d = \psi(v^{-1})u * \phi(\tilde{u})v$ for some $u \in G^*$, where $\tilde{u} = \lambda_{\psi(v^{-1})}u$ as usual. It suffices to show that such u exists uniquely for any given $v \in G^*$, and that the resulting d is equal to $\psi(v^{-1})\phi(v)v$. According to Corollary 4.4, we have

$$\rho_v \phi(\tilde{u}) = \phi(\lambda_{\psi(v)}\tilde{u}) = \phi(\lambda_{\psi(v)}\lambda_{\psi(v^{-1})}u) = \phi(u) .$$

On the other hand, it follows from the compatibility between $\psi(v^{-1})u$ and $\phi(\tilde{u})v$ that

$$u = \alpha_2(\phi(\tilde{u})v) = \lambda_{\phi(\tilde{u})}v . \tag{31}$$

Thus, we have $\phi(\tilde{u})v = u\phi(u)$, which implies that $\phi(\tilde{u}) = \phi(v)$, by applying the morphism $\mathscr{F}_+$. Hence,

$$\tilde{u} = \lambda_{\psi(v^{-1})}u = \lambda_{\psi(v^{-1})}\lambda_{\phi(\tilde{u})}v = \lambda_{\psi(v^{-1})\phi(v)}v = v ,$$

where the last step follows from Lemma 8.5. So there exists some $h \in G$ such that $\psi(v^{-1})u = vh$, which simply implies that $h = \psi(u^{-1})$. Therefore, we have $\psi(v)v = u\psi(u)$, i.e., $u = \lambda_{\psi(v)}v$. Conversely, it is easy to check that the element u defined by such an equation will satisfy our requirement. Moreover, $\alpha_1(d) = \psi(v^{-1})\phi(\tilde{u}) = \psi(v^{-1})\phi(v)$. This completes the proof of Part 1.

(2) Suppose that $\mathscr{K} \subset \Gamma_{G^*}$ is an arbitrary bisection. Assume that $\mathscr{K}$ has the form $\{k(v)v \mid \forall v \in G^*\}$, where $k(v)$ is some smooth map from G^* to G. In order to prove that $\mathscr{L} \cdot \mathscr{K} = \mathscr{K} \cdot \mathscr{L}$, it suffices to show that $l(v_1)k(v) = k(v_2)l(v)$ in G, for any $v \in G^*$, where

$$v_1 = \alpha_2(k(v)v) = \lambda_{k(v)}v , \tag{32}$$

and

$$v_2 = \alpha_2(l(v)v) = \lambda_{l(v)}v . \tag{33}$$

According to Lemma 8.5, we have $v_2 = v$. We write $g = k(v)$. It follows from Eq. (32) that $gv = v_1h$ for some $h \in G$. Applying the morphisms $\mathscr{F}_+$ and $\mathscr{F}_-$ to both sides of this equation, one gets

$$g\phi(v^{-1}) = \phi(v_1^{-1})h$$

and

$$g\psi(v^{-1}) = \psi(v_1^{-1})h .$$

By eliminating h from the two equations above, we obtain the following identity:

$$\psi(v_1^{-1})\phi(v_1)g = g\psi(v^{-1})\phi(v) ,$$

that is, $l(v_1)g = gl(v)$, which is exactly the identity we desired. □

The proposition above will be applied to the study of fixed point sets in Sect. 10.

The last property we will prove in this section is the commutativity of the classical $\mathcal{R}$-matrix with certain diagonal constant lagrangian bisections. The corresponding commutativity of their induced actions on a particular symplectic realization is obtained as a consequence.

Let $a \in G$ be any element of G, and let $L_a \subset \Gamma_{G^*} \cong G \times G^*$ be the constant bisection corresponding to the element a, i.e., $L_a = \{(a, u) | \forall u \in G^*\}$. A simple computation leads to the following:

Proposition 8.6. *L_a is a lagrangian bisection if and only if $\mathrm{Ad}_a \Lambda = \Lambda$, where, as usual, Λ denotes the anti-symmetric part of the classical r-matrix $\frac{1}{2}(P(r) - r) \in \mathfrak{g} \wedge \mathfrak{g}$.*

We will denote by $\mathscr{C}$ the set of all elements in G satisfying the condition: $\mathrm{Ad}_a \Lambda = \Lambda$. It is quite clear that $\mathscr{C}$ is a subgroup of G with its Lie algebra characterized by the following condition: X is in the Lie algebra of $\mathscr{C}$ if and only if $\mathrm{ad}_X \Lambda = 0$. An equivalent condition characterizing such a Lie subalgebra is that the image of X under the cocommutator $\mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ is zero. The dressing action of any element in $\mathscr{C}$ has a very simple form. In fact, it coincides with the conjugation in the double group D.

Proposition 8.7. *If $a \in \mathscr{C}$, then $\lambda_a u = aua^{-1}$, for all $u \in G^*$, where the multiplication in the right side should be taken inside the double group D.*

Proof. Assume that $u_1 = \lambda_a u$. Thus, $au = u_1 h_1$ for some $h_1 \in G$. So h_1 is in fact equal to $\rho_u a$. Since $\mathrm{Ad}_a \Lambda = \Lambda$, it follows that $\pi_G(a) = 0$, which simply implies that the dressing vector field vanishes at the point a. Thus, $h_1 = a$. So $\lambda_a u = u_1 = aua^{-1}$. □

Theorem 8.8. *Suppose that G is a complete quasitriangular Poisson Lie group, and let $\mathcal{R} \in U(\Gamma_{G^*} \times \Gamma_{G^*})$ be its global classical $\mathcal{R}$-matrix. Then $\mathcal{R}$ commutes with $L_a \times L_a$ in $U(\Gamma_{G^*} \times \Gamma_{G^*})$ for any $a \in \mathscr{C}$.*

Proof. It is trivial to see that the bisection $(L_a \times L_a) \cdot \mathcal{R}$ has the form $\{(a\psi(v^{-1})u, a\phi(\tilde{u})v) | \forall u, v \in G^*, \tilde{u} = \lambda_{\psi(v^{-1})} u\}$. Let (r_1, r_2) be the element in $\mathcal{R} \cdot (L_a \times L_a)$ such that $\beta_2(r_1) = u$ and $\beta_2(r_2) = v$. Then, $r_1 = (\psi(v_1^{-1})u_1) * (au)$ and $r_2 = (\phi(\tilde{u}_1)v_1) * (av)$, where $u_1 = \alpha_2(au) = aua^{-1}$, and $v_1 = \alpha_2(av) = ava^{-1}$. It thus follows that $\psi(v_1) = a\psi(v)a^{-1}$. Hence, $\psi(v_1^{-1})u_1 = a\psi(v^{-1})ua^{-1}$. Therefore, $\tilde{u}_1 = \lambda_{\psi(v_1^{-1})} u_1 = \lambda_a \lambda_{\psi(v^{-1})} u = a\tilde{u}a^{-1}$. Finally, using the identities above, we have $\alpha_1(r_1) = \psi(v_1^{-1})a = a\psi(v^{-1})$, and $\alpha_1(r_2) = \phi(\tilde{u}_1)a = a\phi(\tilde{u})$. This completes the proof. □

An immediate consequence of this theorem is the following:

Corollary 8.9. *Under the same assumption as in Theorem 7.4, the induced braid group action on S_μ^n commutes with the diagonal action of $\underbrace{L_a \times \cdots \times L_a}_{n}$ for any $a \in \mathscr{C}$.*

Remark 8.10. According to Drinfel'd [Dr3], the classical r-matrix of a quasitriangular Lie bialgebra naturally gives rise to a one-parameter subgroup of $\mathscr{C}$ as follows. If $r = \sum a_i \otimes b_i \in \mathfrak{g} \otimes \mathfrak{g}$ is a quasitriangular r-matrix, then $H = \sum [a_i, b_i] \in \mathfrak{g}$ belongs to the Lie algebra of $\mathscr{C}$. H thus defines a one parameter subgroup of $\mathscr{C}$. In fact, this one parameter subgroup induces an interesting hamiltonian action on the symplectic double groupoid. Let $\phi_t(r) = \mathrm{Ad}_{L_{\exp tH}} r$. Then ϕ_t defines a one parameter family of symplectic groupoid automorphisms of Γ_{G^*}, with its action on the identity space being the dressing action $\lambda_{\exp th}$. However, it is not hard to check that ϕ_t is also a groupoid automorphism with respect to the second groupoid structures Γ_G, and the action on the identity space G of the second groupoid is exactly $\mathrm{Ad}_{\exp tH}$. In fact, the one-parameter family of dressing transformations $\lambda_{\exp tH}$ is a hamiltonian action with momentum mapping μ: $G^* \to R$ being the character of G^* corresponding to the Lie algebra homomorphism $\mathfrak{g}^* \to R$ defined by the linear function l_H. Thus ϕ_t is also a hamiltonian action, whose momentum mapping is $\mu(\alpha_2(x)) - \mu(\beta_2(x))$.

9. Factorizable Poisson Groups and Doubles

The doubles of Poisson Lie groups are examples of quasitriangular Poisson Lie groups. In this section, we discuss doubles as examples of the smaller class of factorizable Lie algebras and groups [Dr1, RS].

Let $r \in \mathfrak{g} \otimes \mathfrak{g}$ be the r-matrix of a quasitriangular Lie bialgebra $\mathfrak{g}$. If the symmetrization $I = r + P(r)$ is nondegenerate, the bialgebra is called **factorizable** [RS]. In this case, the corresponding linear mapping $j \overset{\text{def}}{=} r_+ - r_-$: $\mathfrak{g}^* \to \mathfrak{g}$ is bijective.

Now let G be a Poisson Lie group corresponding to the bialgebra above, and let G^* be its simply connected dual. The Lie algebra homomorphisms $r_\pm : \mathfrak{g}^* \to \mathfrak{g}$ lift to group homomorphisms $R_\pm : G^* \to G$, and we may therefore define the map $J: G^* \to G$ by $J(u) = R_+(u)R_-(u)^{-1}$, whose derivative at the identity element of G^* is j. (Neither j nor J is a homomorphism.) When J is a global diffeomorphism, we say that the group G is *factorizable*, since we have for each element x in G the factorization $x = x_+ x_-^{-1}$, where $x_\pm = R_\pm(J^{-1}(x))$.

Proposition 9.1. *Any simply connected factorizable Poisson Lie group is complete.*

Proof. Suppose that G is a simply connected factorizable Poisson Lie group. According to Proposition 1.5 [RS], the simply connected double group D of G is isomorphic to $G \times G$, where the embeddings $\phi_1 : G^* \to G \times G$ and $\phi_2 : G \to G \times G$ of G^* and G into D are given by $w \mapsto (R_+ w, R_- w)$ and $x \mapsto (x, x)$, respectively. According to Proposition 2.43 [Lu1], in order to prove that G is complete, it suffices to show that the map $\phi_1 \times \phi_2 : G^* \times G \to G \times G$ is a global diffeomorphism, where $\phi_1 \times \phi_2$ is given explicitly by $(\phi_1 \times \phi_2)(w, x) = \phi_1(w)\phi_2(x) = ((R_+ w)x, (R_- w)x)$. This is indeed true because G is factorizable. In fact, the map from $G \times G$ to $G^* \times G$ sending each $(y, z) \in G \times G$ to $(J^{-1}(yz^{-1}), (yz^{-1})_+^{-1} y) \in G^* \times G$ will be such an inverse of $\phi_1 \times \phi_2$. This completes our proof. □

Factorizability enables us to describe the classical $\mathscr{R}$-matrix quite explicitly.

Theorem 9.2. (1) *If G is a factorizable Poisson Lie group, under the identification of G^* with G via J, the classical $\mathscr{R}$-matrix $\mathscr{R} \subset G \bowtie G \times G \bowtie G$ takes the form*

$$\mathscr{R} = \{(y_-, x, (y_- x y_-^{-1})_+^{-1}, y) | \forall x, y \in G\} .$$

(2) *The map $G \times G \to G \times G$ given by:*

$$(x, y) \mapsto (y_- x y_-^{-1}, (y_- x y_-)_+^{-1} y (y_- x y_-^{-1})_+) \tag{34}$$

is a Poisson diffeomorphism when G is equipped with the Poisson structure of G^*, and it satisfies the set-theoretic quantum Yang–Baxter equation.

Now let $(\mathfrak{h}, \mathfrak{h}^*)$ be an arbitrary Lie bialgebra, and let $\mathfrak{d} = \mathfrak{h} \oplus \mathfrak{h}^*$ be its double. It is well-known that both $\mathfrak{h}$ and $\mathfrak{h}^*$ are Lie subalgebras of $\mathfrak{d}$ and the natural pairing $\langle (X, \xi), (Y, \eta) \rangle = \langle \xi, Y \rangle + \langle \eta, X \rangle$ is $\mathfrak{d}$-invariant. Let D be the simply connected Lie group with Lie algebra $\mathfrak{d}$. Then D is in fact a quasitriangular Poisson Lie group, where the r-matrix $r \in \mathfrak{d} \otimes \mathfrak{d}$ is described as follows (see [RS]). Suppose that P is the canonical projection from $\mathfrak{d}$ onto $\mathfrak{h}$ in the decomposition $\mathfrak{d} = \mathfrak{h} \oplus \mathfrak{h}^*$; then P can be considered as an element in $\mathfrak{d} \otimes \mathfrak{d}$ which we will denote by r. It is simple to check directly that r satisfies the classical Yang–Baxter equation and indeed gives rise to the canonical Poisson structure on the double group D. Note that $\mathfrak{d}^*$ is isomorphic to $\mathfrak{h} \oplus (\mathfrak{h}^*)^o$ as a Lie algebra, where $(\mathfrak{h}^*)^o$ means the Lie algebra with the opposite Lie bracket.

It is trivial to see that the Lie algebra homomorphisms $r_\pm$ from $\mathfrak{d}^*(\cong \mathfrak{h} \oplus (\mathfrak{h}^*)^o)$ to $\mathfrak{d}(\cong \mathfrak{h} \oplus \mathfrak{h}^*)$ are given by $r_+(X, \xi) = (X, 0)$ and $r_-(X, \xi) = (0, -\xi)$, respectively for any $\xi \in \mathfrak{h}^*$ and $X \in \mathfrak{h}$. So $J = r_+ - r_-$ is a linear isomorphism, which yields that $\mathfrak{d}$ is a factorizable Lie algebra.

Let H and H^* be the simply connected Lie groups corresponding to $\mathfrak{h}$ and $\mathfrak{h}^*$. Suppose that H is complete. Then H^* is also complete according to [Lu1, Ma]. Then $D = H \bowtie H^*$. According to the observation above, D^*, the simply connected dual of D, is isomorphic to the direct product $H \times (H^*)^o$ as a Lie group (the Poisson structures are in general different).

Clearly, the lifts $R_+, R_-\colon D^* \to D$ of r_+ and r_- on the group level are given respectively by $R_+(g, u) = g$ and $R_-(g, u) = u^{-1}$. The map $J\colon D^* \to D$ is given by $J(x) = R_+(x)R_-(x)^{-1}$. Then J is given by $J(g, u) = gu$, under the decomposition $D = H \bowtie H^*$. So J is a global diffeomorphism. In other words, D is factorizable.

An immediate consequence of this proposition is the following:

Corollary 9.3. *The double group of a complete Poisson Lie group is still complete.*

It is well-known that the left dressing action of G on G^*, for a factorizable Poisson Lie group G, coincides with the conjugation action Ad_x under the identification of G^* with G by J [RS, STS]. A routine computation using this observation leads to the following:

Lemma 9.4. *Suppose that H is any complete Poisson Lie group and $D = H \bowtie H^*$ its double. For any $(h, v) \in D = H \bowtie H^*$ and $(g, u) \in D^* = H \times (H^*)^o$, assume that $(\tilde{g}, \tilde{u}) = \lambda_{(h,v)}(g, u)$. Then $\tilde{g} = h\lambda_v(g \cdot \lambda_{uv^{-1}} h^{-1})$.*

Theorem 9.5. (1) *Suppose that H is a complete Poisson Lie group. Let $D = H \bowtie H^*$ be its double group and $D^* = H \times (H^*)^o$. Then the classical $\mathscr{R}$-matrix $\mathscr{R} \subset \Gamma_{D^*} \times \Gamma_{D^*} = D \bowtie D^* \times D \bowtie D^*$ has the form $\{(1_H, v^{-1}), (g, u); ((\lambda_{v^{-1}} g)^{-1}, 1_{H^*}), (h, v) | g, h \in H$ and $u, v \in H^*\}$.*

(2) *If D^* is identified with D via J and Γ_{D^*} is identified with the direct product $D \times D$ as described before, $\mathscr{R} \subset \Gamma_{D^*} \times \Gamma_{D^*} \cong D \times D \times D \times D$ is thus given by*

$$\mathscr{R} = \{(v^{-1}g, v^{-1}u^{-1}, \tilde{g}^{-1}h, \tilde{g}^{-1}v^{-1}) | \forall u, v \in H^*, g, h \in H\},$$

where $\tilde{g} = \lambda_{v^{-1}} g$.

Example 9.6. Suppose that H is $SU(2)$ with the standard Bruhat–Poisson structure [LW2]. Its dual is the "book group" $SB(2, \mathbb{C})$ consisting of all upper triangular matrices with positive diagonal and determinant 1. So its double group D is factorizable and isomorphic to $SL(2, \mathbb{C})$. Moreover $D^* = SU(2) \times SB(2, \mathbb{C})^o$. The factorization $x = x_+ x_-^{-1}$ for $x \in SL(2, \mathbb{C})$ is exactly the Gram–Schmidt process with $x_+ \in SU(2)$ and $x_- \in SB(2, \mathbb{C})$. Theorem 9.2 implies that the Poisson diffeomorphism τ: $SL(2, \mathbb{C}) \times SL(2, \mathbb{C}) \to SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$, $\tau(x, y) = (y_- x y_-^{-1}, (y_- x y_-)_+^{-1} y (y_- x y_-^{-1})_+)$ satisfies the quantum Yang–Baxter equation, and hence a braid group action on $SL(2, \mathbb{C})^n$ can be built up with the generator b_1 acting on $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ by the map $\tau \circ \sigma$.

Example 9.7. Let $G = SL(2, \mathbb{C})$, and let H, X_+, X_- be the standard generators of $\mathfrak{sl}(2, \mathbb{C})$. Then $r = X^+ \otimes X^- + \frac{1}{4} H \otimes H \in \mathfrak{sl}(2, \mathbb{C}) \otimes \mathfrak{sl}(2, \mathbb{C})$ is a quasitriangular r-matrix, which defines a complex Poisson Lie group structure on $SL(2, \mathbb{C})$. The dual of G is $B_+ * B_-$ which consists of all the elements of the form

$$\left\{ \begin{pmatrix} \lambda & f \\ 0 & \lambda^{-1} \end{pmatrix}, \begin{pmatrix} \lambda^{-1} & 0 \\ -g & \lambda \end{pmatrix} \middle| \forall f, g \in \mathbb{C}, \lambda \in \mathbb{C}^* \right\}$$

and is considered as a subgroup of $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$. $R_\pm : B_+ * B_- \to SL(2, \mathbb{C})$ are, respectively, the projection to the first and second factors. Let $K \in SL(2, \mathbb{C})$ be the image of the map $J: B_+ * B_- \to SL(2, \mathbb{C})$, $J(x) = R_+(x) R_-(x)^{-1}$. Then K is an open dense subset of $SL(2, \mathbb{C})$. Note that the decomposition for $x \in K$ into $x_+ x_-^{-1}$ is not unique. In fact there are two ways of decomposing x and the resulting x_+ and x_- differ by a sign. So the map defined by Eq. (34) is still a well-defined map, whenever its right-hand side is defined. However, we do not know the appropriate domain for this map nor, more important, whether it still satisfies the quantum Yang–Baxter equation.

10. Fixed Point Sets

In this section, as a first step toward obtaining link invariants from the braid group actions induced from $\mathscr{R}$-matrices, we will study fixed point sets and some other related aspects of these actions.

It is well-known [B] that a braid can be closed in a standard way to form an oriented link. Two braids give rise to equivalent links if and only if they are equivalent under Markov moves. There are two types of Markov moves: one is by conjugation $A \to BAB^{-1}$; the other is by increasing the number of strands in a braid by a simple twist: $A \to Ab_n^{\pm}$, for $A \in B_n$, where b_n is the n^{th} generator of B_{n+1}.

Theorem 10.1. *Suppose that G is a complete quasitriangular Poisson Lie group, and $\mathcal{O}_\mu \subset G^*$ is any symplectic leaf of G^*. Let B_n act on $\mathcal{O}_\mu^n$ as defined in Theorem* 7.4. *If $A, B \in \coprod_n B_n$ define equivalent links, then the fixed point sets of A and B are diffeomorphic.*

Proof. It suffices to check this theorem for the second type of Markov move. Suppose that $(x_1, \ldots, x_{n+1}) \in \mathcal{O}_\mu^{n+1}$ is a fixed point of Ab_n. Then, we have

$$(A \times \mathrm{id})(x_1, \ldots, x_{n-1}, \hat{\mathscr{R}}_{\mu\mu}(x_n, x_{n+1})) = (x_1, \ldots, x_{n+1}),$$

that is,

$$(A \times \mathrm{id})(x_1, \ldots, x_{n-1}, d_1 \cdot x_{n+1}, d_2 \cdot x_n) = (x_1, \ldots, x_{n+1}),$$

for some compatible $(d_1, d_2) \in \mathscr{R}$. In other words, we have

$$A(x_1, \ldots, x_{n-1}, d_1 \cdot x_{n+1}) = (x_1, \ldots, x_{n-1}, x_n), \tag{35}$$

and

$$d_2 \cdot x_n = x_{n+1} \,. \tag{36}$$

From Eqs. (35) and (36), it follows that $A(x_1, x_2, \ldots, (d_1 * d_2) \cdot x_n) = (x_1, x_2, \ldots, x_n)$. Since $d_1 * d_2$ lies in $\mathscr{L}$, which acts trivially on G^* according to Proposition 8.4, $(x_1, \ldots, x_n)$ is a fixed point of A and moreover $x_{n+1} = d_2 \cdot x_n = \lambda_{\phi(x_n)} x_n$. Conversely, it is simple to see that if $(x_1, \ldots, x_n)$ is a fixed point of A, then $(x_1, \ldots, x_n, \lambda_{\phi(x_n)} x_n)$ will be a fixed point of Ab_n. □

Recall that a differentiable map $\theta\colon X \to X$ is said to have clean fixed points if the fixed point set X_θ of θ is a submanifold of X whose tangent bundle is the fixed point set of the bundle map $T\theta\colon TX \to TX$. [GU2]. If a symplectic map has clean fixed points, there is a canonical density on the fixed point submanifold, which is a key ingredient in defining the symplectic trace [GU1, GU2].

As usual, we use $\mathcal{O}$ to denote a symplectic leaf of G^*. Then, according to Theorem 7.5, $\mathcal{O}^n$ admits a symplectic B_n-action.

Theorem 10.2. *If the action of $A \in B_n$ on $\mathcal{O}^n$ has clean fixed points, so does the action of Ab_n on $\mathcal{O}^{n+1}$.*

Proof. Assume that $(x_1^0, \ldots, x_n^0)$ is any clean fixed point of A. Then $(x_1^0, \ldots, x_n^0, x_{n+1}^0)$, with $x_{n+1}^0 = \lambda_{\phi(x_n^0)} x_n^0$, is a fixed point of Ab_n according to Theorem 10.1. In what follows, we will show that it is clean. By definition $(Ab_n)(x_1, \ldots, x_{n+1}) = (A(x_1, \ldots, d_1 x_{n+1}), d_2 x_n)$, where (d_1, d_2) is the point in $\mathscr{R}$ compatible with (x_{n+1}, x_n) in the sense that $\beta_2(d_1) = x_{n+1}$ and $\beta_2(d_2) = x_n$. Therefore, $(\delta_{x_1}, \ldots, \delta_{x_{n+1}})$ is a fixed point of $T(Ab_n)$ iff

$$(TA)(\delta_{x_1}, \ldots, \delta_{x_{n-1}}, \delta_{d_1 x_{n+1}}) = (\delta_{x_1}, \ldots, \delta_{x_{n-1}}, \delta_{x_n}) \tag{37}$$

and

$$\delta_{x_{n+1}} = \delta_{d_2 x_n} \,. \tag{38}$$

According to Corollary 7.2, $d_2 x_n = \rho_{\phi(x_{n+1}^{-1})} x_n$. Equation (38) leads to

$$\begin{aligned} \delta_{x_n} &= \rho_{\phi(x_{n+1})} \cdot \delta_{\rho_{\phi(x_{n+1}^{-1})} x_n} + \delta_{\rho_{\phi(x_{n+1})} x_{n+1}^0} \\ &= \rho_{\phi(x_{n+1})} \cdot \delta_{x_{n+1}} + \delta_{\rho_{\phi(x_{n+1})} x_{n+1}^0} \\ &= \delta_{\rho_{\phi(x_{n+1})} x_{n+1}} \,. \end{aligned}$$

That is, $(\delta_{x_n}, \delta_{x_{n+1}})$ is tangent to the fixed point set of the action of b_1 on $\mathcal{O} \times \mathcal{O}$. Therefore, $\delta_{d_1 x_{n+1}} = \delta_{x_n}$. It thus follows that $(\delta_{x_1}, \ldots, \delta_{x_n})$ is a fixed point of TA. The conclusion thus follows immediately, since the fixed points of A are all clean. □

In general, it is very hard to describe the fixed point set for the action of a particular braid. However, for the action of the n^{th} power of the generator of B_2, one can obtain quite a neat answer.

Theorem 10.3. *Suppose that G is a factorizable Poisson Lie group so that G^* is identified with G via J. Let b_1 be the braid group generator acting on $G \times G$ as in Theorem 9.2 (2). For any $u, v \in G$, assume that $b_1^n(v, u) = (x, y)$. Then,*

$$x = \begin{cases} (w_2 w_1)^{-k} w_2 (w_2 w_1)^k & \text{if } n = 2k \\ (w_2 w_1)^{-k} w_1 (w_2 w_1)^k & \text{if } n = 2k + 1 \end{cases}$$

and $J^{-1}(x) J^{-1}(y) = J^{-1}(v) J^{-1}(u)$, where $w_1 = v_- u v_-^{-1}$ and $w_2 = v$.

Proof. We shall prove this by induction. Obviously, this theorem holds for $n = 1$ (Theorem 9.2). Suppose that it is true for $n = 2k$. Let $b_1(v, u) = (v', u')$, $w'_1 = v'_- u'(v'_-)^{-1}$ and $w'_2 = v'$. We note that w'_1 is in fact the first component of $b_1(v', u') = b_1^2(v, u)$. It thus follows from Corollary 8.2 that $w'_1 = w_1^{-1} w_2 w_1$. Obviously, $w'_2 = v' = w_1$. Let $(x, y) = b_1^{2k+1}(v, u) = b_1^{2k}(v', u')$. Thus, according to the inductive assumption, $x = (w'_2 w'_1)^{-k} w'_2 (w'_2 w'_1)^k = (w_2 w_1)^{-k} w_1 (w_2 w_1)^k$. On the other hand, the relation $J^{-1}(x) J^{-1}(y) = J^{-1}(v) J^{-1}(u)$ follows directly from Theorem 5.1. The other case (when n is odd) can be proved similarly. □

Corollary 10.4. *$(v, u) \in G \times G$ is a fixed point of b_1^n iff it satisfies the following "Artin relation" [Br-S]:*

$$w_{[n]} \cdots w_1 w_2 w_1 = w_{[n-1]} \cdots w_2 w_1 w_2 ,$$

where both sides of the equation have n factors of alternating w_1 and w_2, and w_1, w_2 are defined as in Theorem 10.3. *Also, the symbol $[n]$ denotes* 1 *if n is odd, and* 2 *if n is even.*

Although the corollary above gives a quite neat description for the fixed points of b_1^n, it would be still hard (almost impossible) to find the fixed points explicitly except in certain special examples (see Remark 10.7).

When a symplectic map θ has clean fixed points, one can look at its induced normal map, which is closely related to the symplectic trace. As the first step, we shall look at the normal map of the single braid action.

Assume that $\mathcal{O}$ is a symplectic leaf of G^*. Let $f\colon \mathcal{O} \to \mathcal{O}$ be the map defined by $f(v) = \lambda_{\phi(v)} v$. According to Theorem 10.1, the fixed point set X_{b_1} of b_1 coincides with the graph of f. We proceed, in the following, to calculate the normal map of b_1. For any given point (v_0, u_0) in the graph of f, the normal bundle $N_{(v_0, u_0)}$ of X_{b_1} at (v_0, u_0) is canonically identified with $T_{u_0}\mathcal{O}$ via the correspondence $\Phi\colon N_{(v_0, u_0)} \to T_{u_0}\mathcal{O}$ which sends each class $[\delta_{v_0}, \delta_{u_0}]$ in $N_{(v_0, u_0)}$ to $\delta_{u_0} - (Tf)\delta_{v_0} \in T_{u_0}\mathcal{O}$. Let $b_1^{\perp}\colon T_{u_0}\mathcal{O} \to T_{u_0}\mathcal{O}$ denote the normal map of b_1 with respect to this correspondence Φ.

Theorem 10.5. *$b_1^{\perp}\colon T_{u_0}\mathcal{O} \to T_{u_0}\mathcal{O}$ is equal to $-T\lambda_{\psi(u_0^{-1})\phi(u_0)}$.*

Proof. To calculate $b_1^{\perp}$, we fix v_0, and let u be in a neighborhood of u_0. Assume that

$$b_1(v_0, u) = (s, t) . \tag{39}$$

Then, $s = \lambda_{\psi(v_0^{-1})} u$ and $t = \lambda_{\phi(s)} v_0$. Now taking the derivative of the equation $f(s) = \lambda_{\phi(s)} s$ at $u = u_0$ (then $s = v_0$), one gets $(Tf)\delta_s = T\lambda_{\phi(v_0)}\delta_s + \delta_{\lambda_{\phi(s)} v_0}$. Therefore, $\delta_t - (Tf)\delta_s = -T\lambda_{\phi(v_0)}\delta_s = -T\lambda_{\phi(v_0)\psi(v_0^{-1})}\delta_u$. Taking the derivative of Eq. (39), we have $(Tb_1)(0, \delta_u) = (\delta_s, \delta_t)$. Since $[0, \delta_u]$ and $[\delta_s, \delta_t]$ go to δ_u and $\delta_t - (Tf)\delta_s$, respectively, under the correspondence Φ, $b_1^{\perp}$ sends each δ_u to $-T\lambda_{\phi(v_0)\psi(v_0^{-1})}\delta_u$. Finally, it is easy to see that $\phi(v_0)\psi(v_0^{-1}) = \psi(u_0^{-1})\phi(u_0)$. This completes the proof. □

When G is factorizable, $\lambda_{\psi(u_0^{-1})\phi(u_0)}$ coincides with the adjoint action of $\psi(u_0^{-1})\phi(u_0)$. Thus we have,

Corollary 10.6. *Suppose that G is a factorizable Poisson Lie group, $\mathcal{O}$ a conjugacy class in G. The normal map of b_1 at the fixed point $(h, h_-^{-1} h_+)$ is equivalent to $-T(\mathrm{Ad}_{g^{-1}})\colon T_g\mathcal{O} \to T_g\mathcal{O}$, the negative of the tangent map of $\mathrm{Ad}_{g^{-1}}$, where $g = h_-^{-1} h_+$.*

Remark 10.7. Let $\mathcal{O}$ be the conjugacy class of a generic element in $SL(2, \mathbb{C})$ of the form $K = \mathrm{diag}(k, 1/k)$ as in Example 9.6. For any $g \in \mathcal{O}$, a group element P which conjugates g to K determines an isomorphism from $T_g\mathcal{O}$ to $T_K\mathcal{O}$, and the morphism $-T(\mathrm{Ad}_{g^{-1}})$: $T_g\mathcal{O} \to T_g\mathcal{O}$ becomes $-T(\mathrm{Ad}_{K^{-1}})$: $T_K\mathcal{O} \to T_K\mathcal{O}$ under any such isomorphism. The conjugacy class of K may be identified with the manifold of pairs of independent lines in $\mathbb{C}^2$, which is an open set in $\mathbb{C}P^1 \times \mathbb{C}P^1$. The two coordinate directions in $\mathbb{C}P^1 \times \mathbb{C}P^1$ define transverse foliations on $\mathcal{O}$, and these are the eigenbundles of the normal map $b_1^\perp$, with eigenvalues $-k^2$ and $-1/k^2$. In other words, when $k \neq \pm i$, $b_1^\perp - I$ is an invertible map. This fact implies that the fixed point set of b_1, i.e., the graph of the map $f(g) = g_-^{-1} g_+$, is a symplectic submanifold in $\mathcal{O} \times \mathcal{O}$.

As for b_1^2, the fixed point of b_1^2: $\mathcal{O} \times \mathcal{O} \to \mathcal{O} \times \mathcal{O}$ are defined by the equation: $w_1 w_2 = w_2 w_1$, which has exactly two components of solutions: $w_1 = w_2$ and $w_1 = w_2^{-1}$ (for $SL(n, \mathbb{C})$, the number of solutions is the order of its Weyl group). The first component is just the fixed point set of b_1, a symplectic submanifold when $k \neq \pm i$; the normal map is just $(b_1^\perp)^2$. It is not hard to see that the other component is indeed a lagrangian submanifold of $\mathcal{O} \times \mathcal{O}$. Therefore, the normal map is exactly equal to the identity map. So coincidentally, these two components correspond to the two extreme cases of fixed point sets of a symplectic diffeomorphism. It also can be shown that these fixed points are clean as long as $b_1^\perp - I$ is invertible (i.e., $k \neq \pm i$).

An essential next step in our program is to study the symplectic traces of the braid group actions and in particular to determine how they change under Markov moves. Work on this problem is now in progress.

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References

[B] Birman, J.: Braids, links and mapping class groups. Ann. Math. Studies **82**, Princeton, NJ: Princeton Univ. Press 1975

[Br-S] Brieskorn, E., Saito, K.: Artin-Gruppen und Coxeter-Gruppen. Invent. Math. **17**, 245–271 (1972)

[CDW] Coste, A., Dazord, P., Weinstein, A.: Groupoïdes symplectiques. Publications du Départment de Mathématiques, Université Claude Bernard Lyon I (1987)

[Dr1] Drinfel'd, V.G.: Quantum groups. Proc. ICM, Berkeley, **1**, 789–820 (1986)

[Dr2] Drinfel'd, V.G.: On some unsolved problems in quantum group theory. Lecture Notes in Math. Berlin, Heidelberg, New York: Springer (to appear)

[Dr3] Drinfel'd, V.G.: On almost cocommutative Hopf algebras. Leningrad Math. J. **1**, 321–342 (1990)

[GW] Ginzburg, V., Weinstein, A.: Lie–Poisson structure on some Poisson Lie groups. J. A.M.S. **5**, 445–453 (1992)

[GU1] Guillemin, V, Uribe, A.: Monodromy in the quantum spherical pendulum. Commun. Math. Phys. **122**, 563–574 (1989)

[GU2] Guillemin, V., Uribe, A.: Reduction and the trace formula. J. Diff. Geom. **32**, 315–347 (1990)

[KoMa] Kosmann, Schwarzbach, Y., Magri, F.: Poisson-Lie groups and complete integrability, part 1, Drinfel'd bialgebras, dual extensions and their canonical representations. Annal. Inst. Henri Poincaré, Série A (Physique Théorique), **49**(4), 433–460 (1988)

[Li] Lie, S.: Theorie der Transformationsgruppen. (Zweiter Abschnitt, unter Mitwirkung von Prof. Dr. Friedrich Engel). Leipzig: Teubner 1890
[Lu1] Lu, J.-H.: Multiplicative and affine Poisson structures on Lie groups. Ph.D. thesis, University of California, Berkeley (1990)
[Lu2] Lu, J.-H.: Momentum mappings and reduction of Poisson actions, Séminaire Sud-Rhodanien à Berkeley, Symplectic geometry, groupoids, and integrable systems. MSRI publications, pp. 209–226 Berlin, Heidelberg, New York: Springer 1991
[LW1] Lu, J.H., Weinstein, A.: Groupoïdes symplectiques doubles des groupes de Lie–Poisson. C. R. Acad. Sc. Paris **309**, 951–954 (1989)
[LW2] Lu, J.H., Weinstein, A.: Poisson Lie groups, dressing transformations, and Bruhat decompositions. J. Diff. Geom. **31**, 501–526 (1990)
[Ma] Majid, S.: Matched pairs of Lie groups associated to solutions of the Yang–Baxter equations. Pacific J. Math. **141**(2), 311–332 (1990)
[RS] Reshetikhin, N., Semenov-Tian-Shansky, M.A.: Quantum R-matrices and factorization problems. J. Geom. Phys. **5**, 533–550 (1988)
[RT] Reshetikhin, N., Turaev, V.G.: Ribbon graphs and their invariants derived from quantum groups. Commun. Math. Phys. **127**, 1–26 (1990)
[STS] Semenov-Tian-Shansky, M.A.: Dressing transformations and Poisson Lie group actions. Publ. RIMS, Kyoto University **21**, 1237–1260 (1985)
[Tu] Turaev, V.G.: The Yang–Baxter equation and invariants of links. Invent. Math. **92**, 527–553 (1988)
[WX] Weinstein, A., Xu, P.: Extensions of symplectic groupoids and quantization. J. Reine Angew. Math. **417**, 159–189 (1991)
[X1] Xu, P: Morita equivalence of Poisson manifolds. Commun. Math. Phys. **142**, 493–509 (1991)
[X2] Xu, P.: Symplectic realizations of Poisson manifolds. Ann. Scient. Ec. Norm. Sup. (to appear)

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