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On the Integrability of the Super-KdV Equation

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Abstract. Supersymmetric analogues of the Gelfand–Dikii polynomials are derived. An alternative proof of the super-KdV equation is given using identities obeyed by the polynomials. They also allow the recursive generation of the infinite number of conserved quantities for the super-KdV flow.

1. Introduction

The Korteweg–de Vries (KdV) equation is a nonlinear differential equation describing the "time" evolution of a function $u(x)$ of one "spatial" variable,

$$\partial_t u = \frac{1}{2}(\partial^3 u - 6u\,\partial u)\,, \tag{1.1}$$

where $\partial = \partial_x$. This equation is integrable, in that there exist infinitely many conserved quantities, and it is intimately related to the differential operator $L = -\partial^2 + u$. Recently, the KdV hierarchy has been found to be related to matrix models [1] and to two-dimensional topological gravity [2]. Central to this interpretation are the Gelfand–Dikii polynomials [3] and the recursion relations they obey. The string equations of matrix models and the correlation functions of topological gravity are conveniently expressed in terms of these polynomials.

In the supersymmetric version of the KdV equation (the sKdV equation), the variable x acquires a Grassmann partner θ, so $X \equiv (x, \theta)$ are coordinates in a one dimensional superspace. The sKdV equation is a nonlinear differential equation describing the "time" evolution of a Grassmann-valued superfield $\hat{U}(X) = \hat{u}(x) + \theta u(x)$, where $u(x)$ is an ordinary function and $\hat{u}(x)$ is a Grassmann-valued function (hats denote Grassmann-valued quantities). Defining the operator $D = \partial_\theta + \theta\partial$, the sKdV equation is [4, 5, 6]:

$$\partial_t \hat{U} = \frac{1}{2}(\partial^3 \hat{U} - 3(\partial \hat{U})D\hat{U} - 3\hat{U}\partial D\hat{U})\,. \tag{1.2}$$

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By decomposing with respect to θ, one obtains the pair of coupled differential equations

$$\partial_t u = \frac{1}{2}(\partial^3 u - 6u\partial u + 3\hat{u}\partial^2\hat{u}) ,$$

$$\partial_t \hat{u} = \frac{1}{2}(\partial^3 \hat{u} - 3u\partial\hat{u} - 3\hat{u}\partial u) .$$

These equations are invariant under the infinitesimal supersymmetry transformation $\delta\hat{u} = \varepsilon u$, $\delta u = \varepsilon\partial\hat{u}$ (where ε is a constant Grassmann parameter), which is equivalent to the superspace translation $\delta x = \theta\varepsilon$, $\delta\theta = \varepsilon$.

It is known [6, 7] that the sKdV equation is related to the differential operator $\hat{L} = -\partial D + \hat{U}$, and the existence of infinitely many conserved quantities for the sKdV flow has been proved using pseudodifferential operator techniques [4, 7]. In this paper, we reexamine the integrability of the sKdV equation by developing the analogues of the Gelfand–Dikii polynomials, which can be related to the coefficients in the asymptotic expansion of a heat kernel associated with the superspace operator $\hat{L}$. There are in fact two series of polynomials in the superfield $\hat{U}$ and its derivatives which are linked by various identities. These allow the conserved quantities for the sKdV equation to be generated recursively. A more complicated recursive algorithm has been given by Bilal and Gervais in [6] by using the super-Riccati equation, and it would be of interest to relate the two approaches.

On the physical side, one might hope that the analogues of the Gelfand–Dikii polynomials and the recursion relations they obey bear some relation to the correlation functions of two-dimensional topological supergravity.

2. Review of the Ordinary KdV Equation

The existence of infinitely many conserved quantities for the KdV flow described in (1.1) can be proved using the language of pseudodifferential operators [8] or in terms of a set of relations obeyed by the Gelfand–Dikii polynomials [3]. In both cases, one exploits the relation of the KdV equation to the differential operator $L = -\partial^2 + u$. Here we review the second approach before going on to develop its supersymmetric analogue.

The resolvent $R(x, x'; \lambda) = \dfrac{1}{L+\lambda}\delta(x, x')$ of the operator L admits an asymptotic expansion $R(x, x'; \lambda) = \sum_{n=0}^{\infty} R_n(x, x')\lambda^{-n-1/2}$ in the limit $\lambda \to \infty$. The Gelfand–Dikii polynomials $R_n(x)$, which are polynomials in the function $u(x)$ and its derivatives, are the coincidence limits of the coefficients which appear in this asymptotic expansion, $R_n(x) = \lim_{x\to x'} R_n(x, x')$. The resolvent is related by a Laplace transformation to the heat kernel $K(x, x'; \xi) = e^{-\xi L}\delta(x, x')$,

$$R(x, x'; \lambda) = \int_0^\infty d\xi\, e^{-\xi\lambda} K(x, x'; \xi) .$$

From this it follows that $R_n(x) = \dfrac{(2n-1)!!}{2^{n+1}} a_n(x)$, where $(2n-1)!! = (2n-1)\cdot(2n-3)\ldots 3\cdot 1$ for $n \geqq 1$ and 1 for $n = 0$, and the $a_n(x)$ are the coefficients in the asymptotic expansion of the coincidence limit of the heat kernel in the limit of small ξ,

$$\lim_{x\to x'} K(x, x'; \xi) = \frac{1}{\sqrt{4\pi\xi}} \sum_{n=0}^{\infty} a_n(x)\, \xi^n .$$

We choose to work with the a_n rather than the R_n.

The coefficients $a_n(x)$ can be generated iteratively via the recursion relation

$$\left(n + \frac{1}{2}\right) \partial a_{n+1} = \frac{1}{4} (\partial^3 - 4u\partial - 2(\partial u))\, a_n \tag{2.1}$$

with the initial condition $a_0 = 1$. This follows from the identity

$$\partial_\xi \partial K(x, x; \xi) = \frac{1}{4} (\partial^3 - 4u\partial - 2(\partial u))\, K(x, x; \xi) \tag{2.2}$$

obeyed by the coincidence limit of the heat kernel. The identity is most conveniently proved by introducing the representation

$$K(x, x'; \xi) = \sum_n e^{-\xi\lambda_n^2}\, \phi_n(x)\, \phi_n(x') , \tag{2.3}$$

where the ϕ_n are a complete set of eigenfunctions of the operator L, $L\phi_n = \lambda_n^2\, \phi_n$, with the completeness relation taking the form $\delta(x, x') = \sum_n \phi_n(x)\phi_n(x')$. The proof makes use of the identity $0 = \phi_n \partial^3 \phi_n - (\partial\phi_n)\, \partial^2 \phi_n - (\partial u)\, \phi_n^2$ which follows from $(\partial\phi_n)\, L\phi_n - \phi_n \partial L\phi_n = 0$. The coefficients a_n also satisfy the identity

$$\frac{\delta a_n}{\delta u} = -\, a_{n-1} , \tag{2.4}$$

where $\frac{\delta a_n}{\delta u}$ is defined by $\delta \int dx\, a_n(x) = \int dx\, \delta u(x) \frac{\delta a_n(x)}{\delta u(x)}$. Equation (2.4) is a consequence of the equation

$$\delta \int dx K(x, x; \xi) = -\, \xi \int dx\, dx'\, \delta(x', x)\, \delta u(x)\, K(x, x'; \xi) .$$

To make contact with the KdV equation again, one introduces the Poisson bracket

$$[u(x), u(x')]_{\mathrm{PB}} = \frac{1}{2} (\partial^3 - 4u\partial - 2(\partial u))\, \delta(x, x')$$

and the quantities $H_n = (2n-1)!! \int dx\, a_{n+1}(x)$. Using $[u(x), H_n]_{\mathrm{PB}} = -\,(2n+1)!!\, \partial a_{n+1}$, the KdV equation (1.1) can then be written in the Hamiltonian form

$$\partial_t u(x) = [u(x), H_1]_{\mathrm{PB}}$$

with respect to the Hamiltonian $H_1 = \frac{1}{2} \int dx\, u^2$. With the help of the recursion relation (2.1) and repeated integration by parts, one finds $[H_m, H_n]_{\mathrm{PB}} = [H_{m-1}, H_{n+1}]_{\mathrm{PB}}$. Iterating this relation, $[H_m, H_n]_{\mathrm{PB}} = [H_0, H_{m+n}]_{\mathrm{PB}} = 0$, so the quantities H_n are in involution with respect to the Poisson bracket. Also,

$\partial_t H_n = [H_n, H_1]_{\mathrm{PB}} = 0$, so that the quantities H_n are conserved by the KdV flow (1.1). This is the usual form of the statement of the integrability of the KdV equation.

The KdV hierarchy is obtained by introducing additional "time" parameters $t_2, t_3 \ldots$ (with $t_0 = x, t_1 = t$) with the dependence of u on these variables defined by

$$\partial_{t_n} u(x) = [u(x), H_n]_{\mathrm{PB}} = -(2n+1)!!\,\partial a_{n+1} .$$

The consistency condition $\partial_{t_n} \partial_{t_m} u = \partial_{t_m} \partial_{t_n} u$ follows from the Jacobi identity obeyed by the Poisson bracket.

In the next sections we carry out a similar programme for the sKdV equation.

3. The Supersymmetric Heat Kernel

As mentioned in the introduction, the sKdV equation is related to the superspace differential operator $\hat{L} = -\partial D + \hat{U}$. As this operator maps Grassmann-valued superfields into commuting superfields and vice versa, it is not suitable for the construction of a heat kernel. Instead we consider the operator

$$\Delta = \hat{L} D = -\partial^2 + \hat{U} D \tag{3.1}$$

(we could equally as well have considered the operator $\tilde{\Delta} = D\hat{L}$, see below). The heat kernel $\hat{K}(X, X'; \xi)$ (with ξ a real parameter) associated with Δ satisfies the heat equation

$$(\partial_\xi + \Delta(X))\hat{K}(X, X'; \xi) = 0$$

with the boundary condition $\hat{K}(X, X'; 0) = \hat{\delta}(X, X')$, where $\Delta(X)$ acts on the argument X only. Here, $\hat{\delta}(X, X')$ is the supersymmetric delta function, defined by $\int dX\, \Phi(X)\hat{\delta}(X, X') = \Phi(X')$, where $\int dX = \int dx\, d\theta$ is the supersymmetric integration measure, $\int d\theta\theta = 1, \int d\theta 1 = 0$. Invariance under supersymmetry dictates $\hat{\delta}(X, X') = \delta(x - x' - \theta\theta')(\theta - \theta')$.

The solution to the heat equation with the given boundary conditions is formally

$$\hat{K}(X, X'; \xi) = e^{-\xi\Delta(X)} \cdot \hat{\delta}(X, X') , \tag{3.2}$$

where the operators act on the delta function rather than through it. By sandwiching $\Delta(X) \cdot \hat{\delta}(X, X')$ between two superfields, integrating over X and X' and integrating by parts repeatedly, one can show that

$$\Delta(X) \cdot \hat{\delta}(X, X') = \tilde{\Delta}(X') \cdot \hat{\delta}(X, X') , \tag{3.3}$$

where $\tilde{\Delta} = D\hat{L} = -\partial^2 - \hat{U}D + (D\hat{U})$ (the superfield $\hat{U}(X)$ and its derivatives are assumed to vanish at $\pm\infty$ so that integration by parts can be carried out without generating surface terms). Thus the heat kernel can equivalently be represented as

$$\hat{K}(X, X'; \xi) = e^{-\xi\tilde{\Delta}(X')} \cdot \hat{\delta}(X, X') . \tag{3.4}$$

Note that this heat kernel is to be distinguished from

$$\hat{\tilde{K}}(X, X'; \xi) = e^{-\xi\tilde{\Delta}(X)} \cdot \hat{\delta}(X, X') . \tag{3.5}$$

However, using $\hat{\delta}(X, X') = -\hat{\delta}(X', X)$, we have

$$\lim_{X \to X'} \hat{K}(X, X'; \xi) = -\lim_{X \to X'} \hat{\tilde{K}}(X, X'; \xi),$$

so the coefficients in the asymptotic expansions of these two kernels are the same up to sign.

In the limit of small ξ, the coincidence limit of the heat kernel (3.2) possesses the asymptotic expansion

$$\lim_{X \to X'} \hat{K}(X, X'; \xi) = \frac{1}{\sqrt{4\pi\xi}} \sum_{n=0}^{\infty} \hat{A}_n(X)\xi^n. \tag{3.6}$$

Representing the delta function $\hat{\delta}(X, X')$ in the form $\int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(x-x'-\theta\theta')}(\theta - \theta')$, the heat kernel takes the form

$$\hat{K}(X, X'; \xi) = \frac{1}{\sqrt{4\pi^2\xi}} \int_{-\infty}^{\infty} dk\, e^{-k^2} e^{ik\xi^{-1/2}(x-x'-\theta\theta')} \times e^{\xi(\partial^2 - \hat{U}(X)D) + ik\xi^{1/2}(2\partial - \hat{U}(X)\cdot(\theta-\theta'))} \cdot (\theta - \theta'), \tag{3.7}$$

where a rescaling $k \to \xi^{-1/2}k$ has been performed. It is now straightforward if tedious to evaluate the coefficients $\hat{A}_n$, the first few of which are

$$\begin{aligned}
\hat{A}_0 &= 0, \\
\hat{A}_1 &= -\hat{U}, \\
\hat{A}_2 &= -\frac{1}{6}[\partial^2\hat{U} - 3\hat{U}(D\hat{U})], \\
\hat{A}_3 &= -\frac{1}{60}[\partial^4\hat{U} - 5\hat{U}(\partial^2 D\hat{U}) - 5(D\hat{U})(\partial^2\hat{U}) \\
&\quad - 5(\partial\hat{U})(\partial D\hat{U}) + 10\hat{U}(D\hat{U})^2].
\end{aligned} \tag{3.8}$$

Also of importance here will be the coefficients $A_n(X)$ defined by the asymptotic expansion

$$\lim_{X \to X'} D\hat{K}(X, X'; \xi) = \frac{1}{\sqrt{4\pi\xi}} \sum_{n=0}^{\infty} A_n(X)\xi^n. \tag{3.9}$$

Again, direct calculation yields

$$\begin{aligned}
A_0 &= 1, \\
A_1 &= -D\hat{U}, \\
A_2 &= -\frac{1}{6}[\partial^2 D\hat{U} - 3(D\hat{U})^2 + 2\hat{U}\partial\hat{U}], \\
A_3 &= -\frac{1}{60}[\partial^4 D\hat{U} - 4\hat{U}\partial^3\hat{U} - 10(D\hat{U})(\partial^2 D\hat{U}) + (\partial\hat{U})(\partial^2\hat{U}) \\
&\quad - 5(\partial D\hat{U})^2 - 15\hat{U}(D\hat{U})(\partial\hat{U}) + 10(D\hat{U})^3]
\end{aligned}$$

for the first few coefficients.

The heat kernel satisfies a number of identities, of which the principal one is

(I) $$\lim_{X\to X'} \partial\hat{K}(X, X'; \xi) = \frac{1}{2} DK(X),$$

where we adopt the notation $K(X) = \lim_{X\to X'} D\hat{K}(X, X'; \xi)$ and $\hat{K}(X) = \lim_{X\to X'} \hat{K}(X, X'; \xi)$ (the dependence on ξ is suppressed for notational convenience). This identity can be checked explicitly to low orders in the expansion in ξ using the representation (3.7) for the heat kernel, where care must be taken to include the terms where ∂ acts on the factor $e^{ik\xi^{-1/2}(x-x'-\theta\theta')}$. The general proof is given in Appendix A. Secondary identities are

(II) $$\partial K(X) = -\hat{L}\hat{K}(X),$$

(III) $$\partial\partial_\xi \hat{K}(X) = \hat{O}K(X)$$

with

$$\hat{O} = \frac{1}{4}(\partial^2 D - 3\hat{U}\partial - (D\hat{U})D - 2(\partial\hat{U}))\,. \tag{3.10}$$

We will also need to make use of the result

(IV) $$\frac{\delta\hat{K}(X)}{\delta\hat{U}(X)} = -\xi K(X),$$

where the functional derivative $\delta/\delta\hat{U}$ is defined by

$$\delta \int dX\, \hat{K}(X) = \int dX\, \delta\hat{U}(X) \frac{\delta\hat{K}(X)}{\delta\hat{U}(X)}$$

for an arbitrary variation $\delta\hat{U}(X)$. (This means that if $\hat{P}(X)$ is a polynomial in $\hat{U}(X)$ and its derivatives $D\hat{U}(X)$, $\partial\hat{U}(X)$, . . . , then

$$\frac{\delta\hat{P}(X)}{\delta\hat{U}(X)} = \frac{\tilde{\partial}\hat{P}(X)}{\tilde{\partial}\hat{U}(X)} + D\frac{\tilde{\partial}\hat{P}(X)}{\tilde{\partial}(D\hat{U}(X))} - \partial\frac{\tilde{\partial}\hat{P}(X)}{\tilde{\partial}(\partial\hat{U}(X))} - \dots,$$

where $\tilde{\partial}$ denotes derivatives of $\hat{P}$ with respect to its arguments (to distinguish it from $\partial \equiv \partial_x$)). These identities are proven in Appendices B–D.

Using the asymptotic expansions (3.6) and (3.9) for $\hat{K}(X)$ and $K(X)$, (II) and (III) imply the following relations:

(II′) $$\partial A_n = -\hat{L}\hat{A}_n,$$

(III′) $$\left(n + \frac{1}{2}\right)\partial\hat{A}_{n+1} = \hat{O}A_n\,.$$

With the initial conditions $\hat{A}_0 = 0$ and $A_0 = 1$, these can be used to generate the sequences of coefficients $\hat{A}_n$ and A_n recursively. This is one of the key results in the paper, as it will allow the recursive generation of conserved quantities for the sKdV equation. Similarly, identity (IV) yields

(IV′) $$\frac{\delta\hat{A}_{n+1}}{\delta\hat{U}} = -A_n\,.$$

4. Conserved Quantities for the sKdV Flow

In this section, we use the results of the previous section to prove that the quantities

$$H_n = (2n-1)!! \int dX \, \hat{A}_{n+1}(X) \quad n = 0, 1, 2, \ldots \tag{4.1}$$

(which are functionals of $\hat{U}$ and its derivatives) provide an infinite set of conserved quantities for the sKdV flow (1.2). The proof hinges on the result (proved in Appendix E) that the H_n are in involution with respect to the super-Poisson bracket

$$\{\hat{U}(X), \hat{U}(X')\}_{\mathrm{PB}} = -2\hat{O}\hat{\delta}(X, X') \tag{4.2}$$

with $\hat{O}$ as defined in (3.10). This is the "classical" version of the super-Virasoro algebra [5, 6], and as such the super-Poisson bracket is symmetric and fulfills the super-Jacobi identity (we use the notation $\{\ \}_{\mathrm{PB}}$ because of the symmetry of the super-Poisson bracket).

Using the result $\{\hat{U}(X), H_n\}_{\mathrm{PB}} = -(2n+1)!!\partial\hat{A}_{n+1}(X)$ (see Appendix E) and the expression in (3.8) for $\hat{A}_2$, the sKdV equation can be expressed in the Hamiltonian form

$$\partial_t \hat{U}(X) = \{\hat{U}(X), H_1\}_{\mathrm{PB}} \,.$$

The time dependence of the H_n is given by $\partial_t H_n = \{H_n, H_1\}_{\mathrm{PB}}$, which vanishes since the H_n are in involution with respect to the super-Poisson bracket. Thus the H_n provide an infinite set of quantities in involution which are preserved by the sKdV flow, which is the usual statement of the integrability of the sKdV equation.

It should be noted that the above proof can also be carried through using the heat kernel $\hat{\tilde{K}}(X, X'; \xi)$ in (3.5) associated with the operator $\tilde{\Delta} = D\hat{L}$. Defining $\hat{\tilde{K}}(X) = \lim_{X \to X'} \hat{\tilde{K}}(X, X'; \xi)$ and $\tilde{K}(X) = \lim_{X \to X'} D\hat{\tilde{K}}(X, X'; \xi)$, one finds

$$\hat{\tilde{K}}(X) = -\hat{K}(X)\,,$$
$$\tilde{K}(X) = K(X) - D\hat{K}(X)\,.$$

The first relation has already been discussed in Sect. 3, and the second results from writing $D\hat{\tilde{K}}(X, X'; \xi)$ in the form $[D, e^{-\xi D\hat{L}}]\,\hat{\delta}(X, X') + De^{-\xi \hat{L} D}\,\hat{\delta}(X, X')$. The identities (II), (III) and (IV) are accordingly modified and follow from

$$\lim_{X \to X'} \partial\hat{\tilde{K}}(X, X'; \xi) = \frac{1}{2}\partial\hat{\tilde{K}}(X) + \frac{1}{2}D\tilde{K}(X)\,,$$

which can be proved in a manner similar to (I).

Conclusion

In this paper, we have derived the supersymmetric analogue of the Gelfand–Dikii polynomials and have given an alternative proof of the integrability of the super-KdV equation using the various identities obeyed by the polynomials. These identities also allow the infinite set of quantities conserved by the sKdV flow to be generated recursively. The situation is more complicated than in the case of the ordinary KdV equation, as there are two series of polynomials, the $\hat{A}_n$ and the A_n, and the recursion relation links the two series.

In the case of the ordinary KdV equation, there exists a bi-Hamiltonian structure, in that the KdV equation can be expressed in Hamiltonian form with respect to two different Poisson structures [3, 8]. This can be viewed as arising as a consequence of the recursion relation (2.1) and the fact that the coefficients a_n are the densities from which the conserved quantities for the KdV flow are constructed. It is known that the sKdV equation does not possess a local[1] bi-Hamiltonian structure [7], and in the present context, this is related to the fact that the recursion relation (III′) relates $\hat{A}_{n+1}$ and A_n, while the conserved quantities are constructed purely in terms of the $\hat{A}_n$.

Appendix A: Proof of (I)

We begin by noting that in the case of the ordinary KdV equation treated in Sect. 2, the corresponding identity reads

$$\lim_{x\to x'} \partial K(x, x'; \xi) = \frac{1}{2}\partial \lim_{x\to x'} K(x, x'; \xi) .$$

This is easily proved using the eigenfunction representation (2.3) of the heat kernel associated with L. As no obvious analogue of this representation exists for the supersymmetric heat kernel, we are forced to resort to functional methods for the proof of (I).

As seen in Sect. 3, the heat kernel associated with $\Delta = \hat{L}D$ is formally given by

$$\hat{K}(X, X'; \xi) = e^{-\xi\hat{L}D}\,\hat{\delta}(X, X') .$$

By sandwiching $\partial e^{-\xi(\hat{L}D)}\cdot\hat{\delta}(X, X')$ between two superfields, integrating over X and X' and integrating by parts repeatedly, one finds

$$\partial e^{-\xi(\hat{L}D)}\cdot\hat{\delta}(X, X') = -\,e^{-\xi(D'\hat{L}')}\,\partial'\cdot\hat{\delta}(X, X') ,$$

where the primes on operators indicate that they act on the argument X'. This allows us to write $\partial\hat{K}(X, X'; \xi)$ in the form

$$\partial\hat{K}(X, X'; \xi) = \frac{1}{2}(\partial e^{-\xi(\hat{L}D)} - e^{-\xi(D'\hat{L}')}\,\partial')\,\hat{\delta}(X, X') .$$

Now, in the limit $X\to X'$,

$$e^{-\xi(D'\hat{L}')}\partial'\cdot\hat{\delta}(X, X') \to e^{-\xi(D\hat{L})}\partial\cdot\hat{\delta}(X', X) = -\,e^{-\xi(D\hat{L})}\partial\cdot\hat{\delta}(X, X') .$$

Thus

$$\lim_{X\to X'} \partial\hat{K}(X, X'; \xi) = \lim_{X\to X'} \frac{1}{2}(\partial e^{-\xi(\hat{L}D)} + e^{-\xi(D\hat{L})}\partial)\,\hat{\delta}(X, X')$$

$$= \lim_{X\to X'} \frac{1}{2}\{D, De^{-\xi(\hat{L}D)}\}\,\hat{\delta}(X, X') .$$

[1] However, equations (II′) and (III′) can be rewritten formally as $(n+\frac{1}{2})\partial\hat{A}_{(n+1)} = -\,\hat{O}\partial^{-1}\hat{L}\hat{A}_n$, which exhibits a *nonlocal* bi-Hamiltonian structure for the super-KdV equation. This nonlocal bi-Hamiltonian structure has been written down by W. Oevel and Z. Popowicz [9]. I thank the referee for pointing this out

This is equivalent to $\frac{1}{2} D \lim_{X \to X'} D\hat{K}(X, X'; \xi) = \frac{1}{2} DK(X)$, as the anticommutator means that the operator D on the left does not act through onto the delta function but acts only on the coefficients of the operators in the power series expansion of $De^{-\xi(\hat{L}D)}$.

There is actually a slight subtlety here, related to the terms $\hat{U}(X) \cdot (\theta - \theta')$ in the plane wave representation (3.7) of the heat kernel. In this representation, $\lim_{X \to X'} \{D, De^{-\xi(\hat{L}D)}\} \hat{\delta}(X, X')$ simplifies to

$$\lim_{\theta \to \theta'} \frac{1}{\sqrt{4\pi^2 \xi}} \int_{-\infty}^{\infty} dk\, e^{-k^2} \{D, (D + ik\xi^{-1/2}(\theta - \theta'))$$
$$\times e^{\xi(\partial^2 - \hat{U}(X)D) + ik\xi^{1/2}(2\partial - \hat{U}(X)(\theta - \theta'))}\} \cdot (\theta - \theta') .$$

In principal, to a given order in ξ in the expansion of the exponential, there are terms where the left-most operator D annihilates the $(\theta - \theta')$ factor in $\hat{U}(X) \cdot (\theta - \theta')$ which would not be present if we first took the limit $\theta \to \theta'$ and then acted with the D. However, such terms are precisely cancelled by terms in which the left-most D acts on the factor $ik\xi^{-1/2}(\theta - \theta')$, which causes the factor $(2\partial - \hat{U}(X) \cdot (\theta - \theta'))$ to be replaced by a factor $(\partial^2 - \hat{U}(X)D)$ in the expansion of the exponential to the given order in ξ. In other words, a term of the form $\lim_{\theta \to \theta'} D \cdot O_1 \cdot (-\hat{U}(X) \cdot (\theta - \theta')) \cdot O_2 \cdot 1$ (with O_1 and O_2 operators) is cancelled by a corresponding term $\lim_{\theta \to \theta'} O_1 \cdot (\partial^2 - \hat{U}(X)D) \cdot O_2 \cdot (\theta - \theta')$ in a given order in the expansion in ξ.

Appendix B: Proof of (II)

First we need the preliminary results

$$\lim_{X \to X'} (D + D')\, \hat{K}(X, X'; \xi) = D\hat{K}(X) ,$$

$$\lim_{X \to X'} (D - D')\hat{K}(X, X'; \xi) = 2K(X) - D\hat{K}(X) ,$$

which follow from consideration of the quantity $\int dX \int dX' \Phi(X)((D \pm D') \times \hat{K}(X, X'; \xi))\, \hat{\delta}(X, X')$ (with $\Phi(X)$ an arbitrary superfield) by integration by parts and use of the identity $D \cdot \hat{\delta}(X, X') = -D' \cdot \hat{\delta}(X, X')$. In a similar manner, one finds

$$\lim_{X \to X'} (\partial + \partial')\hat{K}(X, X'; \xi) = \partial\hat{K}(X) ,$$

and, using (I),

$$\lim_{X \to X'} (\partial - \partial')\hat{K}(X, X'; \xi) = DK(X) - \partial\hat{K}(X) ,$$

$$\lim_{X \to X'} (\partial + \partial')(\partial - \partial')\hat{K}(X, X'; \xi) = \partial DK(X) - \partial^2\hat{K}(X) .$$

The proof of (II) follows from the equation

$$0 = (\Delta(X) - \tilde{\Delta}(X'))\hat{K}(X, X'; \xi)\,, \tag{B.1}$$

which is proved by integrating against superfields and integrating by parts. Substituting for Δ and $\tilde{\Delta}$,

$$0 = \Big[-(\partial + \partial')(\partial - \partial') + \frac{1}{2}(\hat{U}(X) + \hat{U}(X'))(D + D')$$
$$+ \frac{1}{2}(\hat{U}(X) - \hat{U}(X'))(D - D') - (D'\hat{U}(X'))\Big]\hat{K}(X, X'; \xi)\,.$$

Taking the limit $X \to X'$ and using the preliminary results, one obtains

$$0 = D[-\partial K + \partial D\hat{K} - \hat{U}\hat{K}]$$

from which (II) follows.

Appendix C: Proof of (III)

Identity (III) is the supersymmetric version of the result (2.2) for the ordinary KdV equation. Again, the absence of a suitable representation for the supersymmetric heat kernel in terms of eigenfunctions means that the simple proof in the nonsupersymmetric case has no analogue and functional methods have to be employed.

Using Eq. (B.1) from Appendix B,

$$(\partial + \partial')\partial_\xi \hat{K}(X, X'; \xi) = -\frac{1}{2}(\partial + \partial')(\Delta(X) + \tilde{\Delta}(X'))\hat{K}(X, X'; \xi)\;.$$

Adding the identity

$$0 = \frac{1}{4}(\partial - \partial')(\Delta(X) - \tilde{\Delta}(X'))\hat{K}(X, X'; \xi)$$

which also follows from (B.1), one obtains

$$(\partial + \partial')\partial_\xi \hat{K}(X, X'; \xi) = \frac{1}{4}(O_1 + O_2 + O_3 + O_4 + O_5)\hat{K}(X, X'; \xi)\,,$$

with

$$\begin{aligned}
O_1 &= (\partial + \partial')^3\,,\\
O_2 &= -\hat{U}(X)\partial D - 3\hat{U}(X)\partial' D + 3\hat{U}(X')\partial D' + \hat{U}(X')\partial' D'\,,\\
O_3 &= -2(D'\hat{U}(X'))(\partial + \partial') - (D'\hat{U}(X'))(\partial - \partial')\,,\\
O_4 &= -(\partial\hat{U}(X))D + (\partial'\hat{U}(X'))D'\,,\\
O_5 &= -(\partial' D'\hat{U}(X'))\,.
\end{aligned}$$

Rewriting

$$\begin{aligned}
O_2 = &-(\hat{U}(X) - \hat{U}(X'))(\partial + \partial')(D + D')\\
&-(\hat{U}(X) + \hat{U}(X'))(\partial + \partial')(D - D')\\
&+\frac{1}{2}(\hat{U}(X) + \hat{U}(X'))(\partial - \partial')(D + D')\\
&+\frac{1}{2}(\hat{U}(X) - \hat{U}(X'))(\partial - \partial')(D - D')
\end{aligned}$$

and

$$O_4 = -\frac{1}{2}((\partial\hat{U}(X)) - (\partial'\hat{U}(X')))(D + D')$$

$$-\frac{1}{2}((\partial\hat{U}(X)) + (\partial'\hat{U}(X')))(D - D'),$$

and using the results of Appendix B, one obtains in the limit $X \to X'$,

$$\partial\partial_\xi\hat{K}(X) = \frac{1}{4}[\partial^3 + \hat{U}\partial D - (D\hat{U})\partial + (\partial\hat{U})D - (\partial D\hat{U})]\hat{K}(X)$$

$$+\frac{1}{4}[-3\hat{U}\partial - (D\hat{U})D - 2(\partial\hat{U})]K(X),$$

which yields the identity (III) upon use of (II).

Appendix D: Proof of (IV)

With $\int dX\,\hat{K}(X) = \int dX \int dX'\,\hat{\delta}(X', X)\hat{K}(X, X';\,\xi)$, one obtains from (3.2) after integration by parts

$$\delta \int dX\,\hat{K}(X) = -\xi \int dX \int dX'\hat{\delta}(X', X)\delta\hat{U}(X)D\hat{K}(X, X';\,\xi)$$

$$= -\xi \int dX\,\delta\hat{U}(X)K(X),$$

as required.

Appendix E: Proving $\{H_m, H_n\}_{\mathrm{PB}} = 0$

One has

$$\{\hat{U}(X), H_m\}_{\mathrm{PB}} = \{\hat{U}(X), \int dX'\,\hat{U}(X')\}_{\mathrm{PB}}\frac{\delta H_m}{\delta\hat{U}(X')} = -2(2m-1)!!\,\hat{O}A_m(X),$$

where use has been made of $\delta H_m/\delta\hat{U}(X) = -(2n-1)!!A_m(X)$, which follows from (IV'). (Using (III'), this is equivalent to $\{\hat{U}(X), H_m\}_{\mathrm{PB}} = -(2m+1)!!\,\partial\hat{A}_{m+1}(X))$.

Then

$$\{H_m, H_n\}_{\mathrm{PB}} = \{H_m, \int dX\,\hat{U}(X)\}_{\mathrm{PB}}\frac{\delta H_n}{\delta\hat{U}(X)}$$

$$= 2(2m-1)!!\,(2n-1)!! \int dX\,A_m(X)\hat{O}A_n(X).$$

Using (III'), integrating by parts and then using (II'), one obtains

$$(2m-1)!!\,(2n+1)!! \int dX(\hat{L}\hat{A}_m(X))\hat{A}_{n+1}(X).$$

Integrating by parts, using (II′) and integrating by parts again yields

$$-(2m-1)!!\,(2n+1)!!\int dX(\partial\hat{A}_m(X))\,A_{n+1}\,.$$

Applying (III′) and integrating by parts gives

$$2(2m-3)!!\,(2n+1)!!\int dX\,A_{m-1}\,\hat{O}A_{n+1}\,,$$

which is $\{H_{m-1}, H_{n+1}\}_{\mathrm{PB}}$. By repeatedly applying this procedure, $\{H_m, H_n\}_{\mathrm{PB}} = \{H_0, H_{m+n}\}_{\mathrm{PB}}$. If we attempt to carry out the above procedure again, we obtain $\partial\hat{A}_0$ in the integrand, which vanishes since $\hat{A}_0 = 0$, proving the result.

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Note added in proof. The integrability of the sKdV equation has recently been treated in [10] using a reduction of the super-KP hierarchy. I thank C.M. Yung for pointing this out.