Commun. Math. Phys. 148, 169–176 (1992)

Communications in
Mathematical
Physics


# Topological Particle Field Theory, General Coordinate Invariance and Generalized Chern–Simons Actions

**Noboru Kawamoto and Yoshiyuki Watabiki**
Department of Physics, Kyoto University, Kyoto 606, Japan

Received October 14, 1991; in revised form January 10, 1992

**Abstract.** We show that recently proposed generalized Chern–Simons action can be identified with the field theory action of a topological point particle. We find the crucial correspondence which makes it possible to derive the field theory actions from a special version of the generalized Chern–Simons actions. We provide arguments that the general coordinate invariance in the target space and the flat connection condition as a topological field theory can be accommodated in a very natural way. We propose series of new gauge invariant observables.

Topological field theories so far proposed can be mainly classified into three classes: the standard Chern–Simons type, the action having a form of total derivative, and the vanishing action with a specific gauge fixing like, flat connection, self-duality condition, etc. [1, 2]. Here we propose a new type of topological field theory. We start from a vanishing particle theory action and impose a particular gauge fixing and consider the corresponding field theory action, which turns out to coincide with the generalized Chern–Simons actions derived in the previous paper [3] which we refer to as paper I. The general coordinate invariance and the topological nature of the actions are natural consequences of the formulations. There have been several investigations to point out a particular connection between the three-dimensional Chern–Simons action and a particle mechanics [4] and to clear up the various issues of the topological particle [5].

We first consider a point particle theory which is invariant under the following transformation:

$$x^\mu(\tau) \to x^\mu(\tau) + a^\mu(\tau)\,, \tag{1}$$

where $a^\mu(\tau)$ is an arbitrary function of $\tau$ which parametrizes the world line of a point particle moving in the $N$-dimensional target space. An obvious candidate of the action will be given by

$$S^{[1]} = 0\,, \tag{2}$$

which is apparently invariant under the transformation (1) and thus has a vast corresponding symmetry. As a gauge fixing condition we take

$$\dot{x}^\mu(\tau) = 0\,, \tag{3}$$

where the dot denotes a derivative with respect to $\tau$. The global symmetry corresponding to $\tau$-independent mode of $x^\mu(\tau)$ which we denote $x^\mu$ is still left. Then the residual transformation corresponds to the standard general coordinate transformation

$$x^\mu \to x^\mu + a'^\mu(x) \,. \tag{4}$$

According to the gauge fixing condition (3) we can construct BRST invariant gauge fixed action

$$\begin{aligned} S_B^{[1]} &= -\int d\tau \delta_B \{\eta_\mu(\tau)\dot{x}^\mu(\tau)\} \\ &= \int d\tau \{p_\mu(\tau)\dot{x}^\mu(\tau) + \eta_\mu(\tau)\dot{\theta}^\mu(\tau)\} \,, \end{aligned} \tag{5}$$

where $\theta^\mu(\tau)$, $\eta_\mu(\tau)$ and $p_\mu(\tau)$ are ghost, anti-ghost and auxiliary fields respectively. The corresponding BRST transformation is given by

$$\begin{aligned} \delta_B x^\mu(\tau) &= \theta^\mu(\tau), \qquad & \delta_B p_\mu(\tau) &= 0 \,, \\ \delta_B \theta^\mu(\tau) &= 0, \qquad & \delta_B \eta_\mu(\tau) &= -p_\mu(\tau) \,, \end{aligned} \tag{6}$$

where $\delta_B$ is BRST operator. The equations of motion given by the action (5) require that only the $\tau$-independent modes $x^\mu$, $\theta^\mu$, $\eta_\mu$, $p_\mu$ for $x^\mu(\tau)$, $\theta^\mu(\tau)$, $\eta_\mu(\tau)$, $p_\mu(\tau)$ are dynamical parameters. Then the first quantization of the equal-time commutator leads to

$$\begin{aligned} [x^\mu, p_\nu] &= i\delta^\mu_\nu \,, \\ \{\theta^\mu, \eta_\nu\} &= i\delta^\mu_\nu \,, \end{aligned} \tag{7}$$

where $p_\mu$ can be identified with $-i\partial_\mu$. The BRST charge is then given by

$$Q_B = i\theta^\mu p_\mu = \theta^\mu \partial_\mu \,, \tag{8}$$

where the nilpotency of the BRST charge is obvious. We note that the gauge fixed action $S_B^{[1]}$ is reparametrization invariant with respect to $\tau$ even after the gauge fixing (3).

We now construct point particle field theory actions. In constructing field theory actions, there are some freedoms for the characteristics of actions. We consider a bosonic particle field $\Phi(x, \theta)$ and a fermionic particle field $\Psi(x, \theta)$, which are functionals of the coordinate $x^\mu$ and the ghost coordinate $\theta^\mu$. We introduce a certain gauge symmetry in such a way that the particle fields are gauge algebra valued functionals, $\Phi = T_a\Phi^a$, $\Psi = T_a\Psi^a$, where $T_a$ is a generator of the gauge algebra.

We then require that the field theory actions satisfy the gauge invariance and BRST invariance and carry a quadratic kinetic term. As a gauge symmetry we consider the following two types of gauge transformation:

$$\delta\Psi = Q_B v + [\Psi, v] \,, \tag{9}$$

and

$$\begin{aligned} \delta\Psi &= Q_B v + [\Psi, v] + \varepsilon_2\{\Phi, \rho\} \,, \\ \delta\Phi &= Q_B \rho + \{\Psi, \rho\} + [\Phi, v] \,, \end{aligned} \tag{10}$$

where $v(x, \theta)$ and $\rho(x, \theta)$ are bosonic and fermionic gauge parameters, respectively. The gauge transformation (9) is a standard one while the gauge transformation (10) is an unusual one in the sense that it includes the term $\{\Phi, \rho\}$, which makes it impossible to close the gauge algebra within the adjoint representation of Lie algebra. We have to consider a gauge algebra which is closed under a commutator and anticommutator. A specific example of the algebra is given by Clifford algebra [6].

We find the following two types of particle field theory actions which satisfy the above criteria:

$$\bar{S}^F = \int [dx][d\theta]\mathrm{Tr}\left(\frac{1}{2}\Psi Q_B \Psi + \frac{1}{3}\Psi^3\right),$$

$$S^B = \int [dx][d\theta]\mathrm{Tr}\left(\Phi Q_B \Psi + \Phi\Psi^2 - \frac{\varepsilon_2}{3}\Phi^3\right), \tag{11}$$

where $[dx] = dx^1 \ldots dx^N$ and $[d\theta] = d\theta^N \ldots d\theta^1$ and $\varepsilon_2 = \pm 1$. The trace is taken with respect to the gauge algebra. The actions $\bar{S}^F$ and $S^B$ are invariant under the gauge transformations (9) and (10), respectively. The BRST invariance of the actions in (11) is obvious because the BRST transformation (6) reproduces only the total derivative terms, i.e., $\delta_B \bar{S}^F = \delta_B S^B = 0$. The BRST invariance of the particle field theory actions assures the remaining symmetry of the gauge fixing, the general coordinate invariance.

We now decompose the gauge fields $\Psi$, $\Phi$ and gauge parameters $v$, $\rho$ into fermionic and bosonic counterparts

$$\begin{aligned}
\Psi(x,\theta) &= A(x,\theta) + \varepsilon_1\sqrt{\varepsilon_1\varepsilon_2}\,\hat{\psi}(x,\theta)\,,\\
\Phi(x,\theta) &= -\varepsilon_2\psi(x,\theta) + \varepsilon_1\sqrt{\varepsilon_1\varepsilon_2}\,\hat{A}(x,\theta),\\
v(x,\theta) &= \hat{a}(x,\theta) + \varepsilon_1\varepsilon_2\sqrt{\varepsilon_1\varepsilon_2}\,\alpha(x,\theta)\,,\\
\rho(x,\theta) &= \hat{\alpha}(x,\theta) - \varepsilon_1\sqrt{\varepsilon_1\varepsilon_2}\,a(x,\theta)\,,
\end{aligned} \tag{12}$$

where $\hat{\psi}(x,\theta)$, $\hat{A}(x,\theta)$, $\hat{a}(x,\theta)$ and $\hat{\alpha}(x,\theta)$ have even number of $\theta$ while $\psi(x,\theta)$, $A(x,\theta)$, $a(x,\theta)$ and $\alpha(x,\theta)$ have odd number of $\theta$. $\varepsilon_1$, $\varepsilon_2 = \pm 1$ is related to the "quaternion algebra" which will be defined later. To be explicit, the gauge fields are decomposed by

$$\begin{aligned}
\hat{A}(x,\theta) &= \frac{T_a}{2}\left(A^{(0)a}(x) + \frac{1}{2}A^{(2)a}_{\mu\nu}(x)\theta^\mu\theta^\nu + \ldots\right),\\
A(x,\theta) &= \frac{T_a}{2}\left(A^{(1)a}_{\mu}(x)\theta^\mu + \frac{1}{3!}A^{(3)a}_{\mu\nu\rho}(x)\theta^\mu\theta^\nu\theta^\rho + \ldots\right),\\
\hat{\psi}(x,\theta) &= \frac{T_a}{2}\left(\psi^{(0)a}(x) + \frac{1}{2}\psi^{(2)a}_{\mu\nu}(x)\theta^\mu\theta^\nu + \ldots\right),\\
\psi(x,\theta) &= \frac{T_a}{2}\left(\psi^{(1)a}_{\mu}(x)\theta^\mu + \frac{1}{3!}\psi^{(3)a}_{\mu\nu\rho}(x)\theta^\mu\theta^\nu\theta^\rho + \ldots\right),
\end{aligned} \tag{13}$$

where $A^{(p)a}_{\mu_1\cdots\mu_p}(x)$ denotes a bosonic $p$-rank tensor while $\psi^{(p)a}_{\mu_1\cdots\mu_p}(x)$ denotes a fermionic $p$-rank tensor. The gauge parameters, $\hat{a}(x,\theta)$, $\hat{\alpha}(x,\theta)$, $a(x,\theta)$ and $\alpha(x,\theta)$, can be decomposed similarly. Note that

$$\int [dx][d\theta]\theta^{\mu_1}\dots\theta^{\mu_N}\mathcal{O}(x) = \mathrm{sgn}(\mu_1,\dots,\mu_N)\int [dx]\mathcal{O}(x)$$
$$= \int dx^{\mu_1}\wedge\dots\wedge dx^{\mu_N}\mathcal{O}(x)\,. \tag{14}$$

Substituting the relations of (12) into the actions in (11) and integrating the ghost variables, we obtain the following actions, $\bar{S}_o^b$ and $\bar{S}_e^f$ from $\bar{S}^F$, and $S_e^b$ and $S_o^f$ from $S^B$:

$$\bar{S}_o^b = \int \mathrm{Tr}\left\{\frac{1}{2}A\,dA + \frac{1}{3}A^3 - \frac{\varepsilon_1\varepsilon_2}{2}\hat{\psi}(d\hat{\psi} + [A,\hat{\psi}])\right\},$$

$$\bar{S}_e^f = \int \mathrm{Tr}\left\{\hat{\psi}(dA + A^2) + \frac{\varepsilon_1\varepsilon_2}{3}\hat{\psi}^3\right\},$$

$$S_e^b = \int \mathrm{Tr}\left\{\hat{A}(dA + A^2 + \varepsilon_2\psi^2 + \varepsilon_1\varepsilon_2\hat{\psi}^2) - \frac{\varepsilon_1}{3}\hat{A}^3 - \varepsilon_2\psi(d\hat{\psi} + [A,\hat{\psi}])\right\},$$

$$S_o^f = \int \mathrm{Tr}\left\{\psi(dA + A^2 - \varepsilon_1\hat{A}^2 + \varepsilon_1\varepsilon_2\hat{\psi}^2) + \frac{\varepsilon_2}{3}\psi^3 + \varepsilon_1\hat{A}(d\hat{\psi} + [A,\hat{\psi}])\right\}, \tag{15}$$

where the ghost variable $\theta^\mu$ is replaced by $dx^\mu$; $A = A(x, dx)$, $\hat{A} = \hat{A}(x, dx)$, $\psi = \psi(x, dx)$, and $\hat{\psi} = \hat{\psi}(x, dx)$. Thus the scalar gauge fields have just turned into antisymmetric tensor gauge fields through the replacement.

The pure bosonic part of the action $\bar{S}_o^b$ with $N = 3$ is equivalent to the standard three-dimensional Chern–Simons action, where $A$ is one form. In general $\bar{S}_o^b$ is an arbitrary odd-dimensional action, includes gauge fermions, and carries overall bosonic statistics which was already discussed in ref. [7]. The standard three-dimensional Chern–Simons action is thus a special case of the more general action. $\bar{S}_e^f$ is the fermionic counterpart of the odd-dimensional action $\bar{S}_o^b$.

$S_e^b$ is an even-dimensional bosonic action while $S_o^f$ is an odd-dimensional fermionic action. These actions $S_e^b$ and $S_o^f$ in Eq. (15) are exactly the same actions and thus have the same forms of gauge invariance as those we obtained from the generalized Chern–Simons action in paper I where "quaternionic structure" played a fundamental role in the derivation. Here is a natural question how these two different formulations lead to the same actions. To answer this question we first summarize the previous derivation briefly and show that we can find a unified treatment to include the particle field theory derivation.

To construct a gauge invariant generalized Chern–Simons action, we need two types of gauge fields and parameters $\lambda_1 \in \Lambda_1$ and $\lambda_k \in \Lambda_k$ which carry suffices of a gauge algebra and satisfy the following three conditions:

i) $\Lambda_1\Lambda_1 \sim \Lambda_1$, $\Lambda_1\Lambda_k \sim \Lambda_k$, $\Lambda_k\Lambda_k \sim \Lambda_1$, where the first equation means the following: if $\lambda_1, \lambda_1' \in \Lambda_1$ then $\lambda_1 \lambda_1' \in \Lambda_1$ and similar for the other two equations.

ii) $\{\vec{Q}, \lambda_k\} = Q\lambda_k$, $[\vec{Q}, \lambda_1] = Q\lambda_1$, where $Q^2 = 0$ and $Q \in \Lambda_k$.

iii) $\mathrm{Tr}(\lambda_k \lambda_1) = \mathrm{Tr}(\lambda_1 \lambda_k)$.

We can then construct a generalized Chern–Simons action

$$S = \int \mathrm{Tr}\left(\frac{1}{2}\lambda_k Q \lambda_k + \frac{1}{3}\lambda_k^3\right), \tag{16}$$

which is invariant up to surface terms under the following gauge transformation:

$$\delta\lambda_k = Q\lambda_1 + [\lambda_k, \lambda_1] . \tag{17}$$

We use the quaternionic representation for a gauge field and a parameter and the derivative operator

$$\begin{aligned} \mathscr{A} &= \psi\mathbf{1} + \hat{\psi}\mathbf{i} + A\mathbf{j} + \hat{A}\mathbf{k} \in \Lambda_k, \qquad Q = \mathbf{j}d \in \Lambda_k , \\ \mathscr{V} &= \hat{a}\mathbf{1} + a\mathbf{i} + \hat{\alpha}\mathbf{j} + \alpha\mathbf{k} \in \Lambda_1 , \end{aligned} \tag{18}$$

where $d = dx^\mu \partial_\mu$. As a special case we also consider the following combination:

$$\begin{aligned} \bar{\mathscr{A}} &= \hat{\psi}\mathbf{i} + A\mathbf{j} \in \bar{\Lambda}_k, \quad Q = \mathbf{j}d \in \bar{\Lambda}_k , \\ \bar{\mathscr{V}} &= \hat{a}\mathbf{1} + \alpha\mathbf{k} \in \bar{\Lambda}_1 . \end{aligned} \tag{19}$$

Here the gauge fields and parameters are classified into four categories by the "quaternion algebra": $\hat{A}$, $\hat{a}$ (bosonic even form); $A$, $a$ (bosonic odd form); $\hat{\psi}$, $\hat{\alpha}$ (fermionic even form); $\psi$, $\alpha$ (fermionic odd form). The algebra is defined by

$$\begin{aligned} &\mathbf{1}^2 = \mathbf{1}, \quad \mathbf{i}^2 = \varepsilon_1 \mathbf{1}, \quad \mathbf{j}^2 = \varepsilon_2 \mathbf{1}, \quad \mathbf{k}^2 = -\varepsilon_1\varepsilon_2 \mathbf{1} , \\ &\mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = -\varepsilon_2 \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = -\varepsilon_1 \mathbf{j} , \end{aligned} \tag{20}$$

where $(\varepsilon_1, \varepsilon_2) = (-1, -1)$, $(-1, +1)$, $(+1, -1)$, $(+1, +1)$. In the case $(\varepsilon_1, \varepsilon_2) = (-1, -1)$, $\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}$ satisfy the algebra of quaternion while the rest of the three cases correspond to $gl(2, \mathbf{R})$ Lie algebra. We symbolically call these algebras "quaternion algebra."

Gauge fields and/or parameters, $\lambda_k$, $\lambda_1$, $\bar{\lambda}_k$, and $\bar{\lambda}_1$ have the same "quaternionic structure" as $\mathscr{A}$, $\mathscr{V}$, $\bar{\mathscr{A}}$, and $\bar{\mathscr{V}}$, respectively. As long as $\mathbf{1}$, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ satisfy the quaternion algebra, we can show that the corresponding conditions i) and ii) and the following relations are satisfied:

$$\begin{aligned} &\mathrm{Tr}_{\mathbf{1}}(\lambda_k\lambda_1) = \mathrm{Tr}_{\mathbf{1}}(\lambda_1\lambda_k), \qquad \mathrm{Tr}_{\mathbf{k}}(\lambda_k\lambda_1) = \mathrm{Tr}_{\mathbf{k}}(\lambda_1\lambda_k) , \\ &\mathrm{Tr}_{\mathbf{i}}(\bar{\lambda}_k\bar{\lambda}_1) = \mathrm{Tr}_{\mathbf{i}}(\bar{\lambda}_1\bar{\lambda}_k), \qquad \mathrm{Tr}_{\mathbf{j}}(\bar{\lambda}_k\bar{\lambda}_1) = \mathrm{Tr}_{\mathbf{j}}(\bar{\lambda}_1\bar{\lambda}_k) , \end{aligned} \tag{21}$$

where we pick up $\mathbf{1}^{\text{th}}$, $\mathbf{k}^{\text{th}}$, $\mathbf{i}^{\text{th}}$, and $\mathbf{j}^{\text{th}}$ components in the trace, $\mathrm{Tr}_{\mathbf{1}}$, $\mathrm{Tr}_{\mathbf{k}}$, $\mathrm{Tr}_{\mathbf{i}}$, and $\mathrm{Tr}_{\mathbf{j}}$, respectively.

It is now straightforward to reproduce the actions in (15) by using the general procedure to construct generalized Chern–Simons actions and the corresponding gauge transformation (16) and (17). Substituting $\lambda_k = \mathscr{A}$ and $\lambda_1 = \mathscr{V}$ into Eqs. (16)

and (17), we directly obtain the generalized Chern–Simons actions: $S_e^b$ ($\mathbf{k}^{\mathrm{th}}$ component of the trace), $S_o^f$ ($\mathbf{1}^{\mathrm{th}}$ component of the trace) and the corresponding gauge transformation, which are the same as given in paper I. Substituting $\bar{\lambda}_k = \bar{\mathscr{A}}$ and $\bar{\lambda}_1 = \bar{\mathscr{V}}$ into Eqs. (16) and (17), we obtain $\bar{S}_o^b$ ($\mathbf{j}^{\mathrm{th}}$ component of the trace), $\bar{S}_e^f$ ($\mathbf{i}^{\mathrm{th}}$ component of the trace) and the corresponding gauge transformation. We have thus obtained all combinations of dimensions and statistics.

We now consider how to derive the particle field theory action following to the general procedure using the quaternion representation. First we find the following crucial correspondence:

$$\begin{aligned}
&\mathbf{i}\hat{\psi}(x,\theta) \to -\sqrt{-1}\,\mathbf{i}\hat{\psi}\left(x, \frac{\mathbf{k}}{|\mathbf{k}|}dx\right),\\
&\mathbf{i}A(x,\theta) \to \mathbf{i}A\left(x, \frac{\mathbf{k}}{|\mathbf{k}|}dx\right),\\
&\mathbf{k}\hat{A}(x,\theta) \to -\sqrt{-1}\,\mathbf{k}\hat{A}\left(x, \frac{\mathbf{k}}{|\mathbf{k}|}dx\right),\\
&\mathbf{k}\psi(x,\theta) \to \mathbf{k}\psi\left(x, \frac{\mathbf{k}}{|\mathbf{k}|}dx\right),
\end{aligned} \tag{22}$$

and thus

$$\theta^\mu \to \frac{\mathbf{k}}{|\mathbf{k}|}dx^\mu , \tag{23}$$

where $|\mathbf{k}| = \sqrt{-\varepsilon_1\varepsilon_2}$. We then classify the $\theta^\mu$-dependent gauge fields and parameters as

$$\begin{aligned}
&\mathscr{A}^\theta = \mathbf{k}\Phi(x,\theta) + \mathbf{i}\Psi(x,\theta) \in \Lambda_k^\theta ,\\
&\mathscr{V}^\theta = \mathbf{1}v(x,\theta) + \mathbf{j}\rho(x,\theta) \in \Lambda_1^\theta ,\\
&\bar{\mathscr{A}}^\theta = \mathbf{i}\Psi(x,\theta) \in \bar{\Lambda}_k^\theta ,\\
&\bar{\mathscr{V}}^\theta = \mathbf{1}v(x,\theta) \in \bar{\Lambda}_1^\theta ,
\end{aligned} \tag{24}$$

where we have the correspondence, $\mathscr{A}^\theta(x,\theta) \to -\varepsilon_1\sqrt{-\varepsilon_1\varepsilon_2}\,\mathscr{A}(x,dx)$ for example. We obtain the corresponding relations i) for $\Lambda^\theta$ and $\bar{\Lambda}^\theta$ systems and

$$\mathrm{Tr}_{\mathbf{k}}(\lambda_k^\theta\lambda_1^\theta) = \mathrm{Tr}_{\mathbf{k}}(\lambda_1^\theta\lambda_k^\theta), \qquad \mathrm{Tr}_{\mathbf{i}}(\bar{\lambda}_k^\theta\bar{\lambda}_1^\theta) = \mathrm{Tr}_{\mathbf{i}}(\bar{\lambda}_1^\theta\bar{\lambda}_k^\theta) . \tag{25}$$

It is important to observe that the BRST charge falls into the class $\Lambda_k^\theta$ and $\bar{\Lambda}_k^\theta$, and corresponds to the differential operator

$$\mathbf{i}Q_B = \mathbf{i}\theta^\mu\partial_\mu \ (\in \Lambda_k^\theta, \bar{\Lambda}_k^\theta) \to Q = \mathbf{j}d . \tag{26}$$

A point particle field theory action $\bar{S}^F$ in (11) can be obtained by substituting $\lambda_k = \bar{\lambda}_k^\theta = \bar{\mathscr{A}}^\theta$ into Eq. (16) and taking the $\mathbf{i}^{\mathrm{th}}$ component of the trace. The action $S^B$ in (11) can be obtained by substituting $\lambda_k = \lambda_k^\theta = \mathscr{A}^\theta$ into Eq. (16) and taking the $\mathbf{k}^{\mathrm{th}}$ component of the trace. The gauge transformations (9) and (10) can be obtained by the corresponding substitutions into Eq. (17).

In this way, we have derived the topological particle field theory actions directly from the special versions of the generalized Chern–Simons action. Although the derivation is straightforward in this way, we need the arguments of the topological particle theory given in the beginning of this paper to understand the meanings of the $\theta$-representation.

The curvatures of the generalized gauge field $\mathscr{A}$ and $\mathscr{A}^\theta$ are, respectively, given by

$$\mathscr{F} = Q\mathscr{A} + \mathscr{A}^2, \qquad \mathscr{F}^\theta = Q\mathscr{A}^\theta + (\mathscr{A}^\theta)^2 . \tag{27}$$

The vanishing curvature conditions, $\mathscr{F} = 0$ and $\mathscr{F}^\theta = 0$ are nothing but the equations of motion of the generalized Chern–Simons actions.

We can now show the following relations which are familiar in the standard gauge theory:

$$\begin{aligned} &\mathrm{Tr}_{\mathbf{i}}(\mathscr{F}^n) = \mathrm{Tr}_{\mathbf{i}}(Q\Omega_{2n-1}), \quad \mathrm{Tr}_{\mathbf{j}}(\mathscr{F}^n) = \mathrm{Tr}_{\mathbf{j}}(Q\Omega_{2n-1}) , \\ &\mathrm{Tr}_{\mathbf{j}}\{(\mathscr{F}^\theta)^n\} = \mathrm{Tr}(Q_B\Omega^\theta_{2n-1}) . \end{aligned} \tag{28}$$

It should be noted that the terms in the trace belong to $\Lambda_1$ or $\Lambda_1^\theta$ and the gauge invariant sectors are the above mentioned sectors of the trace. We obtain the similar relation for the pure fermionic case. In case $n = 2$, $\Omega_3$ and $\Omega_3^\theta$, respectively, coincide with the generalized Chern–Simons action and the particle field theory action. We now propose the series of gauge invariant "physical observables"

$$\begin{aligned} &O_e^b = \int \mathrm{Tr}_{\mathbf{k}}\,\Omega_{2n-1}, \quad O_o^f = \int \mathrm{Tr}_{\mathbf{l}}\,\Omega_{2n-1} , \\ &O^\theta = \int [dx][d\theta]\,\mathrm{Tr}\,\Omega^\theta_{2n-1} , \end{aligned} \tag{29}$$

where $O_e^b$ and $O_o^f$ are bosonic even-dimensional and fermionic odd-dimensional observables, respectively, while $O^\theta$ is the field theoretical counterpart depending on the dimensions. We claim that these are new "observables" which can be found easily through the current formulation. We can construct similar recursive relations for the gauge invariant "observables" as in refs. [1] and [8].

It is interesting to note how the general coordinate invariance is accommodated in the current formulation. The general coordinate invariance of the generalized Chern–Simons actions is naturally built in because the actions accompany only the exterior derivative operators and thus possess no explicit metric dependence. On the other hand those generalized Chern–Simons actions have been derived from the topological particle field theory actions which are BRST invariant and thus possess the general coordinate invariance as a residual transformation of the gauge fixing for the vanishing action of a point particle. We may provide the following heuristic arguments to understand the general coordinate invariance and topological aspects of the formulations.

Let us first imagine a closed particle path in $N$-dimensional target space which is parametrized by the coordinate $x^\mu(\tau)$. The general coordinate invariance can be understood as identifying homotopically equivalent closed loops in the base manifold. Let us then consider a closed loop which can be deformed to a point by a general coordinate transformation. The holonomy along the loop vanishes if we move on the vanishing curvature surface, i.e., if the equations of motion are satisfied. In other words, the non-trivial topological nature of the base manifold can be reflected to the non-trivial holonomy over the flat connection of a certain gauge algebra. We thus claim that the topological nature is naturally built in because the flat connection condition is nothing but the equations of motion of the generalized Chern–Simons action and the particle field theory actions.

We claim that the topological particle field theory formulation together with the generalized Chern–Simons formulation provide a new type of topological field theory. Since the general coordinate invariance is very naturally accommodated, we can actually show that even-dimensional topological gravity can be elegantly treated by the current formulation [6].

## References

1. Witten, E.: Commun. Math. Phys. **117**, 353 (1988); ibid. **118**, 411 (1988); ibid. **121**, 351 (1989); Phys. Lett. **B206**, 601 (1988)
2. Brooks, R., Montano, D., Sonnenschein, J.: Phys. Lett. **B214**, 91 (1988); Labastida, J.M.F., Pernici, M., Witten, E.: Nucl. Phys. **B310**, 611 (1988); Horne, J.H.: Nucl. Phys. **B318**, 22 (1989); Horowitz, G.T.: Commun. Math. Phys. **125**, 417 (1989); Horowitz, G.T., Srednicki, M.: Commun. Math. Phys. **130**, 83 (1990); Blau, M., Thompson, G.: Ann. Phys. **205**, 130 (1991); Sonnenschein, J.: Phys. Rev. **D42**, 2080 (1990); Myers, R.C., Periwal, V.: Nucl. Phys. **B333**, 536 (1990); Nakamichi, A., Sugamoto, A., Oda, I.: In: Tokyo Superstrings Workshop, (1990) pp. 177
3. Kawamoto, N., Watabiki, Y.: Commun. Math. Phys. **144**, 641 (1992)
4. Howe, P.S., Townsend, P.K.: Class. Quant. Grav. **7**, 1655 (1990)
5. Birmingham, D., Rakowski, M., Thompson, G.: Nucl. Phys. **B315**, 577 (1989); Labastida, J.M.F.: Commun. Math. Phys. **123**, 641 (1989); Iwazaki, A.: Prog. Theor. Phys. **81**, 1104 (1989); Brooks, R.: Nucl. Phys. **B325**, 481 (1989); Birmingham, D., Blau, M., Rakowski, M., Thompson, G.: Phys. Rep. **209**, 129 (1991)
6. Kawamoto, N., Watabiki, Y.: Phys. Rev. **D45**, 605 (1992); Kyoto Univ. preprint, KUNS 1076, HE(TH) 91/08 (1991)
7. Myers, R.C., Periwal, V.: Phys. Lett. **B225**, 352 (1989)
8. Witten, E.: Nucl. Phys. **B340**, 281 (1990); Myers, R.C.: Nucl. Phys. **B343**, 705 (1990)

Communicated by H. Araki