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# The Polyakov Path Integral Over Bordered Surfaces

## III. The BRST Extended Closed String Off-Shell Amplitudes

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**Abstract.** The geometrical approach to the functional integral over Faddeev-Popov ghost fields is developed and applied to construct the BRST extension of the off-shell closed string amplitudes in the constant curvature gauge. In this gauge the overlap path integral for off-shell amplitudes is evaluated. It leads to the nonlocal sewing procedure generating all off-shell amplitudes from the cubic interaction vertex. The general scheme of the reconstruction of a covariant closed string field theory from the off-shell amplitudes is discussed within the path integral framework.

## 1. Introduction

In the present paper we complete our study of the Polyakov path integral over bordered surfaces initiated in [1, 2]. The interest in this object can be traced back to Alvarez's pioneering paper [3] where the string ansatz for the Wilson loop was considered. The main development in calculating this functional integral was achieved in the context of the closed string off-shell amplitudes [4–11]. This approach was aimed to derive a covariant closed string field theory (CCSFT) from off-shell amplitudes defined in terms of a functional integral over surfaces connecting closed contours in the target space [11]. In spite of a very suggestive physical and geometrical picture and of important progress in the calculating techniques involved [11, 12] this program did not succeed. It seems that it does not mean a principal invalidity of the basic idea but rather reflects the fact that the functional integral techniques are much less developed than for instance the operator ones [13]. In fact the major recent achievement in constructing CCSFT – the nonpolynomial theory [14–18] – is based on the operator formulation of conformal field theories on punctured Riemann surfaces. There is yet another, well developed approach to CCSFT – the improved [19] covariantized light cone theory [20] in which the relation between an off-shell string diagram and a path integral over bordered surfaces is even less transparent.

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In our previous papers [1, 2] the path integral representation of the off-shell bosonic string amplitudes was constructed in the case of zero ghosts boundary conditions. The main aim of the present work is to extend the geometrical methods of Refs 1, 2 in order to derive the full dependence of the off-shell amplitudes on ghosts variables. This method is then applied to construct the BRST extended off-shell amplitudes in the constant curvature gauge. In particular the overlap path integral for these amplitudes is calculated.

Although the operator approach is believed to be more general than the path integral one [13] and provides a powerful tool (for instance, in solving 2-dim gravity [21]) it seems that the functional formalism developed in [1, 2] and accomplished in the present paper gives new insight in the structure of the closed string Feynman diagrams. Let us also note that the geometrical approach we use to handle the path integral over bordered surfaces is more general and gives some new tools in the covariant functional quantization of gauge systems in the Schrödinger representation.

We began, in Sect. 2, by describing a geometrical framework for the path integral over Faddeev-Popov ghost fields. As it is known from the Yang-Mills theory [22] and the Polyakov path integral over closed surfaces [23] there exists behind the Faddeev-Popov procedure a well defined geometrical construction. This construction involving the infinite dimensional Riemannian geometry of the space of fields is motivated by some finite dimensional integral formula [24]. Following this line of thinking we present a finite dimensional counterpart of the "exponentiation" of the Faddeev-Popov determinant by a Gaussian integral over ghosts. This is done within the Berezin-Leites-Kostant approach to supermanifolds [25] and leads to the geometrical interpretation of the Faddeev-Popov ghosts different from the standard one [26] developed in the context of the Yang-Mills theory. Recall that in the covariant Y–M theory one prefers to work with the space of all connections promoting nonphysical field components to propagating variables by means of a gauge fixing term in the Lagrangian. Then in order to "cancel" the effect of this procedure one has to add a ghost system. The basic principle ensuring a consistency of this formulation is the BRST invariance of the resulting system [27]. This is in contrast with the BDHP bosonic string where we are content with the integration over a gauge slice in the space of world sheet metrics (just because at least in $d = 26$ it leads to a finite dimensional integral). Therefore the only need for ghost fields is to handle a nontrivial insertion of the Faddeev-Popov determinant in a resulting measure. This difference reflects itself for instance in a different structure of a kinetic term of ghost fields.

In Sect. 3 the geometrical scheme developed in Sect. 2 is applied in the case of the closed string partition function. This leads to an equivalent system involving ghost fields. A detailed discussion of the infinite dimensional supermanifold of boundary conditions of this system is presented in Sect. 4. In particular the action of the group of residual gauge transformations is described. The transformation properties of the ghost variables justify our interpretation of "boundary reparametrizations" given in [1, 2]. We close this section by deriving the expression for the BRST extended off-shell amplitudes in the constant curvature gauge.

In Sect. 5 we consider an overlap path integral of two off-shell amplitudes in the constant curvature gauge. Using the results of [12] we are able to calculate it explicitly. It was shown that the integrand of the final expression reproduces exactly the integrand of the off-shell amplitude over the surface obtained by gluing the initial surfaces along a single common boundary. The range of integration is however essentially larger than a corresponding restricted fundamental domain. Although because of the infinite overcounting mentioned above the naive overlap integral does not solve the sewing problem, the sewing of integrands is a nontrivial result. Let us stress one interesting point of this calculation. To form the overlap integral we use the inner product in the space of functionals over the supermanifold of boundary conditions which is given by an extension of the functional integral considered in [1, 2]. The nontrivial measure for integration over the length parameter in this inner product comes from the "1-dimensional" Faddeev-Popov procedure through the $\zeta$-function regularization of the Faddeev-Popov determinant [1]. This measure was derived within the geometrical approach to the path integral without referring to the off-shell amplitudes in the constant curvature gauge nor to the sewing procedure. It is remarkable that this measure is precisely of the form required to achieve the sewing of the Weil-Petersson volume forms.

In the last section the problem of deriving interaction vertices from the set of all off-shell amplitudes is discussed. We present a brief comparison of two solutions of this problem given by the constant curvature gauge and the minimal area one supporting the nonpolynomial approach. We conclude this section by speculations about a possible connection between the constant curvature gauge and the covariantized light cone approach.

## 2. Graded Manifolds and the Geometry of the Faddeev-Popov Ghosts

Let us consider a trivial principal fibre bundle $P(B, \pi, G)$ over a compact base manifold $B$ and with a compact Lie group $G$ as a structure group. Suppose that there is a smooth $G$-invariant Riemannian structure $g$ on the total space $P$ of the bundle and a family $\{h^p\}_{p\in P}$ of right invariant Riemannian metrics on $G$ such that for every $p \in P$, $a \in G$, $\delta a$, $\delta a' \in T_eG$

$$h_e^{pa}(\delta a, \delta a') = h_e^p(\mathrm{Ad}(a)\delta a, \mathrm{Ad}(a)\delta a') \tag{2.1}$$

[Ad(.) denotes the adjoint action of $G$ on its Lie algebra $G' = T_eG$-]

For any global section $\sigma : \beta \to \Sigma \equiv \sigma(B) \subset P$ we construct the family of linear maps $\{\Delta_s\}_{s\in\Sigma}$:

$$\Delta_s = \Pi_s^{W^\perp} \circ \tau_s : T_eG \to W_s^\perp \subset T_sP \,,$$

where

$$\Pi_s^{W^\perp} : T_sP \to W_s^\perp$$

is the orthogonal projection onto the orthogonal (with respect to $g$) complement $W_s^\perp$ of the space $T_s\Sigma$ tangent to the gauge slice $\Sigma$ and $\tau_s : T_eG \to T_sP$ denotes a linear map defined by

$$\tau_s = \beta^*_{s|e} \,,$$

where

$$\beta_s : G \ni a \to s \cdot a \in \pi^{-1}(\pi(s)) \,.$$

The family $\{\Delta_s^+\}_{s\in\Sigma}$ of adjoint maps is defined by the relations:

$$h_e^s(\Delta_s^+ \delta w, \delta a) = g_s(\delta w, \Delta_s \delta a); \qquad \delta a \in T_eG \,, \;\; \delta w \in W_s^\perp \,.$$

For any $G$-invariant function $f$ on $P$ we have the following version of the Fubini theorem [24]:

$$\int\limits_P d\omega^g \left( \int\limits_G d\omega^{h^p} \right)^{-1} f(p) = \int\limits_\Sigma d\omega^\Sigma (\det \Delta_s^+ \Delta_s)^{1/2} f(s) \,, \tag{2.2}$$

where $d\omega^g$, $d\omega^{h^p}$, $d\omega^\Sigma$ denote the volume forms related to the Riemannian metrics $g$, $h^p$ and to the induced metric on $\Sigma$ respectively.

Let $\Sigma$ be some smooth gauge slice in $P(B,\pi,G)$. Consider a vector bundle $E(\Sigma)$ over $\Sigma$ defined as the direct sum:

$$E(\Sigma) = N\Sigma \oplus \Sigma \times G',$$

where $N\Sigma$ denotes the bundle normal to $\Sigma$ and $\Sigma \times G'$ is a trivial bundle with the standard fibre $G'$. Note that in our case $N\Sigma$ is also trivial.

The diffeomorphism $\phi_a \equiv R_{a|\Sigma} : \Sigma \to \Sigma \cdot a$ induced by the right action $R$ of $G$ on $P$ extends to a vector bundle morphism $\Phi_a : E(\Sigma) \to E(\Sigma \cdot a)$ defined on each fibre of $E(\Sigma)$ by:

$$W_s^\perp \oplus G' \ni (\delta w, \delta a) \xrightarrow{\Phi_a} (R_a^* \delta w, \mathrm{Ad}(a^{-1})\delta a) \in W_{sa}^\perp \oplus G'.$$

Let us introduce an Euclidean structure $\mu^\Sigma$ on $E(\Sigma)$ given by:

$$\mu_s^\Sigma((\delta w, \delta a),(\delta w', \delta a')) = g_s(\delta w, \delta w') + h_e^s(\delta a, \delta a').$$

It follows from (2.1) that $\Phi_a$ is an Euclidean vector bundle isomorphism.

Now we consider a Berezin-Leites-Kostant graded manifold $\mathrm{Gr}(E(\Sigma))$ defined by the sheaf of sections of the bundle $\wedge E(\Sigma)^*$. For every $a \in G$ the vector bundle morphism $\Phi_a$ generates a BLK-morphism of graded manifolds. Let $\{\tilde\eta^j\}_{j=1}^n$, $\{\tilde\xi^j\}_{j=1}^n$ denote systems of global sections of $(N\Sigma \cdot a)^*$ and $(\Sigma \cdot a \times G')^*$ respectively, forming a basis at each fibre of $E(\Sigma)^*$. These systems form a set of odd generators of the graded commutative algebra $\Gamma(\wedge E(\Sigma \cdot a)^*)$ and serve a system of odd coordinates for the graded manifold $\mathrm{Gr}(E(\Sigma \cdot a))$. In these coordinates the BLK morphism

$$\Phi_a : \mathrm{Gr}(E(\Sigma)) \to \mathrm{Gr}(E(\Sigma \cdot a))$$

takes the following form:

$$\begin{aligned} (\Phi_a^* \tilde f_0)(s) &= \tilde f_0(s \cdot a), \\ (\Phi_a^* \tilde\eta^i)(s) &= R_a^* \tilde\eta^i(s), \\ (\Phi_a^* \tilde\xi^j)(s) &= \mathrm{Ad}^*(a^{-1})\tilde\xi^j(s \cdot a), \end{aligned} \tag{2.3}$$

where $\tilde f_0 \in C^\infty(\Sigma \cdot a) \subset \Gamma(\wedge E(\Sigma \cdot a)^*)$.

Some remarks concerning notation are in order. The BLK morphism $\Phi_a$ between graded manifolds can be defined [25] as a homomorphism $\Phi_a^* : \Gamma(\wedge E(\Sigma \cdot a)^*) \to \Gamma(\wedge E(\Sigma)^*)$ of graded commutative algebras of global functions. The formula (2.3) describes $\Phi_a^*$ by its values on generators of the algebra $\Gamma(\wedge E(\Sigma \cdot a)^*)$ (for simplicity we omit an explicit description of even generators given by local charts of $\Sigma \cdot a$). Note that since $E(\Sigma \cdot a)$ is a trivial bundle $\tilde\eta$, $\tilde\xi$ are global odd coordinates on $\mathrm{Gr}(E(\Sigma \cdot a))$. There is however another way of notation commonly used in the physical literature and based on the idea of "points with anticommuting coordinates." Within the graded manifold approach one can give the following interpretation of this notation. For every $\tilde f \in \Gamma(\wedge E(\Sigma \cdot a)^*)$ we have the expansion:

$$\tilde f = \sum_{k,l=1}^{n} \tilde f_{i_1 \ldots i_k j_1 \ldots j_l} \tilde\eta^{i_1} \wedge \cdots \wedge \tilde\eta^{i_k} \wedge \tilde\xi^{j_1} \wedge \cdots \wedge \tilde\xi^{j_l}, \tag{2.4}$$

where, as before, $\{\tilde{\eta}^i\}_{i=1}^n$, $\{\tilde{\xi}^j\}_{j=1}^n$ denote bases of sections in the bundles $(N\Sigma \cdot a)^*$ and $(\Sigma \cdot a \times G')^*$. Regarding

$$\tilde{\eta} \equiv (\tilde{\eta}^1, \ldots, \tilde{\eta}^n), \qquad \tilde{\xi} \equiv (\tilde{\xi}^1, \ldots, \tilde{\xi}^n)$$

as vectors with anticommuting components one can interpret the expansion (2.4) as a formal Taylor expansion of the function $f(\tilde{s}, \tilde{\eta}, \tilde{\xi})$ of the even ($\tilde{s}$) and the odd ($\tilde{\eta}, \tilde{\xi}$) variables, at the point $(\tilde{s}, 0, 0)$.

Let $\{\eta^i\}_{i=1}^n$, $\{\xi^j\}_{j=1}^n$ denote bases of global sections of the bundles $(N\Sigma)^*$ and $(\Sigma \times G')^*$ providing a global system of odd coordinates in $\mathrm{Gr}(E(\Sigma))$. In these coordinates $\Phi_a^* \tilde{\eta}^i$, $\Phi_a^* \tilde{\xi}^j$ can be expressed as follows:

$$\begin{aligned} \Phi_a^* \tilde{\eta}^i(s) &= \langle R_a^* \tilde{\eta}^i(s) \mid \delta w_k(s) \rangle \eta^k(s) \\ &= \langle \tilde{\eta}^i(s \cdot a) \mid R_a^* \delta w_k(s \cdot a) \rangle \eta^k(s) , \\ \Phi_a^* \tilde{\xi}^j(s) &= \langle \mathrm{Ad}^*(a^{-1}) \tilde{\xi}^j(s \cdot a) \mid \delta v_l(s) \rangle \xi^l(s) \\ &= \langle \tilde{\xi}^i(s \cdot a) \mid \mathrm{Ad}(a^{-1}) \delta v_l(s) \rangle \xi^l(s) , \end{aligned} \tag{2.5}$$

where $\{\delta w_k\}_{k=1}^n$, $\{\delta v_l\}_{l=1}^n$ denote the dual bases:

$$\langle \eta^i(s), \delta w_k(s) \rangle = \delta_k^i , \qquad \langle \xi^j(s), \delta v_l(s) \rangle = \delta_l^j . \tag{2.6}$$

Note that Eqs. (2.5) can be formally regarded as the transformation rules for vectors:

$$\eta = (\eta^1, \ldots, \eta^n), \qquad \xi = (\xi^1, \ldots, \xi^n)$$

with respect to the map $R_a^*$ and $\mathrm{Ad}(a^{-1}) R_{a^{-1}}^*$ respectively. With this interpretation the algebra homomorphism $\Phi_a^*$ can be written in the following more familiar form:

$$(\Phi_a^* \tilde{f})(s, \eta, \xi) = f(s \cdot a, R_a^* \eta, \mathrm{Ad}(a^{-1}) R_{a^{-1}}^* \xi) .$$

This in order suggests even more compact commonly used notation for the morphism $\Phi_a^*$:

$$\tilde{s} = s \cdot a , \qquad \tilde{\eta} = R_a^* \eta , \qquad \tilde{\xi} = \mathrm{Ad}(a^{-1}) R_{a^{-1}}^* \xi . \tag{2.7}$$

Although in the present framework the transformation rule (2.7) makes sense only through the interpretation given above, correctly used it provides a very useful shorthand notation of the morphism defined by (2.3).

The central object of our consideration is the following Berezin integral over $\mathrm{Gr}(E(\Sigma))$:

$$\begin{aligned} &\int\limits_{\mathrm{Gr}(E(\Sigma))} d\bar{\omega}^{\Sigma} \exp(g_s(\eta, \Delta_s \xi)) f(s) \\ &= \int\limits_{\Sigma} d\omega^{\Sigma} \int\limits_{\mathrm{Gr}(E_s(\Sigma))} d\omega_\eta^s d\omega_\xi^s \exp(g_s(\eta, \Delta_s \xi)) f(s) , \end{aligned} \tag{2.8}$$

where $\mathrm{Gr}(E_s(\Sigma))$ denotes $(0, 2n)$-dimensional graded manifold and the Berezin "volume forms" are defined by:

$$\begin{aligned} d\omega_\eta^s &\equiv (\det g_s(\delta w_i, \delta w_j))^{-1/2} d\eta^1 \cdot \ldots \cdot d\eta^n , \\ d\omega_\xi^s &\equiv (\det h_s(\delta a_i, \delta a_j))^{-1/2} d\xi^1 \cdot \ldots \cdot d\xi^n , \end{aligned} \tag{2.9}$$

where $\{\delta w_j\}_{j=1}^n$, $\{\delta a_j\}_{j=1}^n$ are given by (2.6). The symbol $g_s(\eta, \Delta_s\xi)$ in (2.8) is interpreted as an element of the algebra $\Gamma(\wedge E(\Sigma)^*)$ according to the formula:

$$g_s(\eta, \Delta_s\xi) \equiv \sum_{i,j} g_s(\delta w_i, \Delta_s \delta a_j)\eta^i \wedge \xi^j \,. \tag{2.10}$$

One can easy verify that for any function $\tilde{f} \in \Gamma(\wedge E(\Sigma \cdot a)^*)$ and for any $a \in G$ the following relation holds:

$$\int_{\mathrm{Gr}(E(\Sigma))} d\bar{\omega}^{\Sigma} \exp(g_s(\eta, \Delta_s\xi))\Phi_a^* \tilde{f} = \int_{\mathrm{Gr}(E(\Sigma\cdot a))} d\bar{\omega}^{\Sigma\cdot a} \exp(g_{\tilde{s}}(\tilde{\eta}, \Delta_{\tilde{s}}\tilde{\xi}))\tilde{f} \,.$$

Moreover for every $G$-invariant function $f \in C^\infty(P)$:

$$\int_P d\omega^g \left( \int_G d\omega^{h^p} \right)^{-1} f(p) = \int_{\mathrm{Gr}(E(\Sigma))} d\bar{\omega}^{\Sigma} \exp(g_s(\eta, \Delta_s\xi)) f(s) \,. \tag{2.11}$$

The formal generalization of the formula above provides the geometrical setting of the Faddeev-Popov procedure with the Faddeev-Popov determinant "exponentiated" by the path integral over ghost variables. This interpretation of the ghost variables as coordinates in some infinite dimensional graded manifold is closely related to the treatment of anticommuting fields in supersymmetric theories proposed in [28].

## 3. The Closed String Partition Function

As an illustration of the geometrical description sketched in the previous section let us consider the $h$-loop closed string partition function:

$$Z_h = \int_{\mathscr{M}_h} \mathscr{D}g \int_{\mathscr{X}_h} \mathscr{D}x \left( \int_{\mathscr{D}_h} \mathscr{D}f \times \int_{\mathscr{W}_h} \mathscr{D}\phi \right)^{-1} \exp(-S[g,x]) \,, \tag{3.1}$$

where $\mathscr{M}_h$ is the space of all Riemannian metrics on some fixed oriented 2-dim manifold $M_h$ of genus $h$; $\mathscr{X}_h$ is the space of all mappings $x: M_h \to \mathbb{R}^{26}$; $\mathscr{D}_h$ denotes the group of orientation preserving diffeomorphisms of $M_h$ and $\mathscr{W}_h$ is the group of conformal rescalings of metrics (the additive group of real valued functions on $M_h$). The functional measures in (3.1) are regarded as infinite dimensional volume forms related to the ultralocal Riemannian structures $M(.,.)$, $X^g(.,.)$, $H^g(.,.)$, $W^g(.,.)$ defined on $\mathscr{M}_h$, $\mathscr{X}_h$, $\mathscr{D}_h$ and $\mathscr{W}_h$ respectively:

$$M_g(\delta g, \delta g') \equiv \int_{M_h} \sqrt{g} d^2 z g^{ac} g^{bd} \delta g_{ab} \delta g'_{cd} \,, \qquad \delta g, \delta g' \in \mathscr{T}_g \mathscr{M}_h;$$

$$X_x^g(\delta x, \delta x') \equiv \int_{M_h} \sqrt{g} d^2 z \delta x \delta x' \,, \qquad \delta x, \delta x' \in \mathscr{T}_x \mathscr{X}_h \cong \mathscr{X}_h;$$

$$H_{id}^g(\delta f, \delta f') \equiv \int_{M_h} \sqrt{g} d^2 z g_{ab} \delta f^a \delta f'^b \,, \qquad \delta f, \delta f' \in \mathscr{T}_{id} \mathscr{D}_h;$$

$$W_\phi^g(\delta\phi, \delta\phi') \equiv \int_{M_h} \sqrt{g} d^2 z \delta\phi \delta\phi' \,, \qquad \delta\phi, \delta\phi' \in \mathscr{T}_\phi \mathscr{W}_h \cong \mathscr{W}_h \,.$$

We consider the class of conformal gauge slices given by global sections

$$\sigma : \mathcal{T}_h \ni t \to g_t \in \mathcal{M}_h$$

of the principal fibre bundle:

$$\begin{array}{ccc} \mathcal{D}_h^0 \odot \mathcal{W}_h & \longrightarrow & \mathcal{M}_h \\ & & \downarrow \\ & & \mathcal{T}_h \end{array} . \tag{3.2}$$

Let us denote by $\mathcal{S}_\sigma = \sigma(\mathcal{T}_h)$ the gauge slice in the bundle (3.2), then the submanifold $\mathcal{S}_\sigma^c \equiv \mathcal{S}_\sigma \cdot \mathcal{W}_h \subset \mathcal{M}_h$ is a conformal gauge clise in the bundle:

$$\begin{array}{ccc} \mathcal{D}_h^0 & \longrightarrow & \mathcal{M}_h \\ & & \downarrow \\ & & \mathcal{M}_h / \mathcal{D}_h^0 \cong \mathcal{T}_h \times \mathcal{W}_h . \end{array}$$

At every point $g \in \mathcal{S}_\sigma \subset \mathcal{S}_\sigma^c$ there is an orthogonal decomposition of the space $\mathcal{T}_g \mathcal{S}_\sigma^c$ tangent to $\mathcal{S}_\sigma^c$ at $g$:

$$\mathcal{T}_g \mathcal{S}_\sigma^c = \mathcal{K}_g \oplus \mathcal{K}_g^\perp ,$$

where

$$\mathcal{K}_g = \{\delta g \in \mathcal{T}_g \mathcal{M}_h : \delta g = \delta\phi \cdot g, \delta\phi \in \mathcal{W}_h\}$$

and $\mathcal{K}_g^\perp$ is a finite dimensional space ($\dim \mathcal{K}_g^\perp = 6h - 6$).

Using the Faddeev-Popov method in a conformal gauge $\mathcal{S}_\sigma^c$ one can derive the following expression for $Z_h$ [24]:

$$Z_h = \int\limits_{S_\sigma} d\omega^{\sigma,\psi} \left( \frac{\det P_g^+ P_g}{\det H(P_g^+)} \right)^{1/2} L^{26} \left( \frac{\det' \mathcal{L}_g}{\int \sqrt{g} d^2 z} \right)^{-13} , \tag{3.3}$$

where $\mathcal{L}_g$ denotes the Laplace-Beltrami operator acting on scalar fields on $M_h$; $P_g$ is the conformal Lie derivative operator and $P_g^+$ is adjoint to $P_g$. Other symbols in (3.3) are defined by:

$$d\omega^{\sigma,\psi} \equiv d\omega^\sigma \cdot \det M_g(\delta\tilde{\chi}_i, \delta\psi_j) ,$$
$$H(P_g^+) = M_g(\delta\psi_i, \delta\psi_j) ,$$

where $d\omega^\sigma$ is the volume form on $\mathcal{S}_\sigma$ related to the induced Riemannian structure on $\mathcal{S}_\sigma$; $\{\delta\tilde{\chi}_i\}_{i=1}^{6h-6}$ is an orthonormal basis in $\mathcal{K}_g^\perp$ and $\{\delta\psi_j\}_{j=1}^{6h-6}$ is an arbitrary basis in $\ker P_g^+$.

For every point $(g, x) \in \mathcal{S}_\sigma \times \mathcal{X}_h$ we define a vector space:

$$\mathcal{V}_{g,x} = \mathcal{H}_g \oplus \mathcal{T}_{id} \mathcal{D}_h ,$$

where

$$\mathcal{H}_g = \{\delta g \in \mathcal{T}_g \mathcal{M}_h : g^{ab} \delta g_{ab} = 0\} .$$

The disjoint union:

$$\mathcal{E}^\sigma = \bigcup_{(g,x) \in \mathcal{S}_\sigma \times \mathcal{X}_h} \mathcal{V}_{g,x}$$

can be given the structure of a trivial vector bundle over $\mathcal{S}_\sigma \times \mathcal{X}_h$. On $\mathcal{E}^\sigma$ we introduce the Euclidean structure $E(.,.)$ defined by:

$$E_{(g,x)}((\delta h, \delta f), (\delta h', \delta f')) \equiv M_g(\delta h, \delta h') + H^g_{id}(\delta f, \delta f').$$

According to the finite dimensional scheme described in the previous section one can rewrite the expression (3.3) in the following form:

$$Z_h = \int_{\mathrm{Gr}(\mathcal{E}^\sigma)} \mathcal{D}\Omega^\sigma M_g(\delta\psi_1, \eta) \cdot \ldots \cdot M_g(\delta\psi_n, \eta) \times \exp(S[g,x] + M_g(\eta, P_g, \xi)), \tag{3.4}$$

where the functional $\mathcal{D}\Omega^\sigma$ over the graded infinite dimensional manifold $\mathrm{Gr}(\mathcal{E}^\sigma)$ is defined by:

$$\int_{\mathrm{Gr}(\mathcal{E}^\sigma)} \mathcal{D}\Omega^\sigma \equiv \int_{\mathcal{S}_\sigma} d\omega^{\sigma,\psi} \int_{\mathcal{X}_h} \mathcal{D}^g x \int_{\mathrm{Gr}(\mathcal{T}_g)} \mathcal{D}^g\eta \mathcal{D}^g\xi .$$

The Berezin infinite dimensional volume form $\mathcal{D}^g\eta\mathcal{D}^g\xi$ is related to the Euclidean structure $E(.,.)$ on $\mathcal{E}^\sigma$. Note that the integrand in (3.4) is interpreted as a function on the infinite dimensional graded manifold $\mathrm{Gr}(\mathcal{E}^\sigma)$, i.e. as an element of the graded commutative algebra $\Gamma(\wedge\mathcal{E}^{\sigma *})$, where $\mathcal{E}^{\sigma *}$ denotes the bundle dual to $\mathcal{E}^\sigma$.

The action of the diffeomorphism group $\mathcal{D}_h$ on the base space $\mathcal{S}_\sigma \times \mathcal{X}_h$ of $\mathcal{E}^\sigma$:

$$(g,x) \xrightarrow{f\in\mathcal{D}_h} (f^*g, f^*x)$$

extends to the action on $\mathcal{E}^\sigma$ by the Euclidean vector bundle isomorphisms

$$\Phi_f : (\mathcal{E}^\sigma, E^\sigma(.,.)) \to (\mathcal{E}^{f^*\sigma}, E^{f^*\sigma}(.,.))$$

given at each fibre $\mathcal{E}^\sigma_{g,x}$ of $\mathcal{E}^\sigma$ by:

$$(\delta h, \delta f) \xrightarrow{\Phi_f} (f^*\delta h \circ f, f_*\delta f \circ f^{-1}).$$

This in order induces a BLK morphisms of graded manifolds:

$$\Phi_f F[g,x,\eta,\xi] = F[f^*g, f^*x, f^*\eta \circ f, f_*\xi \circ f^{-1}]. \tag{3.5}$$

The expression (3.4) is invariant with respect to the transformations (3.5). Moreover, since the conformal anomaly vanishes it is also invariant under the BLK morphisms $\Phi_\varphi$, $\varphi \in \mathcal{W}_h$ of the following form:

$$\begin{aligned} &\Phi_\varphi : \mathrm{Gr}(\mathcal{E}^\sigma) \to \mathrm{Gr}(\mathcal{E}^{\exp\varphi\cdot\sigma}) \\ &\Phi_\varphi F[g,x,\eta,\xi] = F[\exp\varphi \cdot g, x, \eta, \xi]. \end{aligned} \tag{3.6}$$

Another form of (3.3) can be obtained by choosing a special subclass of gauge slices $\hat{\mathcal{S}}$ of the bundle (3.2) which are determined by sections:

$$\hat\sigma : \mathcal{T}_h \ni t \to \hat g_t \in \mathcal{M}_h^{-1} \tag{3.7}$$

with values in the space $\mathcal{M}_h^{-1}$ of metrics with the constant scalar curvature equal to $-1$. In this case the partition function takes the following form [24]:

$$Z_h = \int_{\hat{\mathcal{S}}} \hat\sigma^* d\omega^{\mathrm{WP}} \left(\frac{\det P_g^+ P_g}{\det H(P_g^+)}\right)^{1/2} L^{26} \left(\frac{\det' \mathcal{L}_g}{\int \sqrt{g} d^2 z}\right)^{-13}, \tag{3.8}$$

where $d\omega^{\mathrm{WP}}$ denotes the Weil-Petersson volume form on the Teichmüller space $\mathcal{T}_h$.

For the expression (3.8) one can construct slightly different path integral representation with a simplified treatment of ghosts zero modes. Using the orthogonal decomposition:

$$\mathcal{H}_g = \mathrm{im}\, P_g \oplus \ker P_g^+$$

we construct for any Sect. (3.7) the vector bundle:

$$\begin{aligned} \hat{\mathcal{E}}_\sigma &= \bigcup_{(g,x)\in \hat{\mathcal{S}}\times \mathcal{X}_h} \hat{\mathcal{V}}_{g,x}\,, \\ \hat{\mathcal{V}}_{g,x} &= \mathrm{im}\, P_g \oplus \mathcal{T}_{\mathrm{id}}\mathcal{D}_h \end{aligned} \tag{3.9}$$

with the Euclidean structure $\hat{E}(.,.)$ defined by obvious restriction of $E(.,.)$.

We have the following Gaussian path integral representation of $Z_h$:

$$Z_h = \int\limits_{\hat{\mathcal{S}}} \hat{\sigma}^* d\omega^{\mathrm{WP}} \int\limits_{\mathcal{X}_h} \mathcal{D}^g x \int\limits_{\mathrm{Gr}(\hat{\mathcal{V}}_{(g,x)})} \mathcal{D}^g \hat{\eta} \mathcal{D}^g \xi \exp(S[g,x] + M_g(\hat{\eta}, P_g\xi))\,. \tag{3.10}$$

## 4. The BRST Extension of the Off-Shell Closed String Amplitudes

In this section we will construct the BRST extension of the closed string amplitudes. The basic idea is to consider instead of the constrained system described by the action functional:

$$S[g,x] \equiv \frac{1}{2}\int\limits_M \sqrt{g} d^2 z g^{ab}\partial_a x^\mu \partial_b x^\mu$$

defined on the manifold of fields $\mathcal{M}_h \times \mathcal{X}_h$ an equivalent "Gaussian" system determined by the action functional:

$$\begin{aligned} S[g,x,\eta,\xi] &\equiv S[g,x] + S_{gh}[g,\eta,\xi] + S'_{gh}[g,\eta,\xi]\,, \\ S_{gh}[g,\eta,\xi] &\equiv M_g[\eta, P_g\xi] = \int\limits_M \sqrt{g} d^2 z g^{ab} g^{cd} \eta_{ac}(P_g\xi)_{bd}\,, \\ S'_{gh}[g,\eta,\xi] &\equiv -2\int\limits_{\partial M} e d\sigma n^a t^b \eta_{ab} t_c \xi^c \end{aligned} \tag{4.1}$$

defined on the graded manifold of fields $\mathrm{Gr}(\mathcal{E}^\sigma)$ or $\mathrm{Gr}(\hat{\mathcal{E}}^\sigma)$. These two possibilities are related to two different expressions (3.4) and (3.10) for the off-shell amplitudes. In the present paper we will conider only the constant curvature gauge for which the second possibility is relevant (1.2).

Note that the first two terms in (4.1) follow from the path integral representation (3.9) of the partition function. The boundary term $S'_{gh}[g,\eta,\xi]$ is added to ensure the existence of the extrema of the action functional (4.1) with nonhomogeneous boundary conditions for ghost variables [9].

We will start with the discussion of the geometry of the (graded) manifold of the boundary conditions for trajectories of the system (4.1). It proceeds along the standard lines described in [1.2]. Let $M_{h,b}$ denote the 2-dimensional oriented world sheet manifold with $h$-handles and $b$-boundary components. For a single boundary component $\Sigma$ of $M_{h,b}$ let us fix some orientation preserving diffeomorphisms

$$\varrho : S \to \Sigma ,$$

where $S$ is an oriented 1-dim circle and $\Sigma$ is endowed with the induced orientation. The boundary values along $\Sigma$ of a given "trajectory" $(g, x, \eta, \xi)$ over $M_{h,b}$ are defined by:

$$\begin{aligned} e^2 = \varrho^* g\,, \qquad \tilde{x} \equiv \varrho^* x\,, \\ \tilde{\eta} \equiv \varrho^*(n^a t^b \eta_{ab})\,, \qquad \tilde{\xi} \equiv \varrho^*(n_a \xi^a)\,, \end{aligned} \tag{4.2}$$

where $n$ and $t$ denote the unit vectors normal and tanget to $\Sigma$ respectively.

The space of all boundary conditions for "trajectories" of the system (4.1) is then the graded manifold $\mathrm{Gr}(\widetilde{\mathscr{E}})$ generated by the vector bundle:

$$\widetilde{\mathscr{E}} = (\mathscr{M}_S \times S \times \mathscr{X}_S) \times (C^\infty(S) \times C^\infty(S))\,.$$

(As it was discussed in [2] for some technical reasons the "bosonic" part of the space determined by (4.2) has to be extended by the factor $S$.)

The residual gauge transformations can be easily derived from the relations (3.5), (3.6), and (4.2). They are described by the group $\mathscr{G} \equiv (\mathscr{W}_S \odot U(1)) \odot \mathscr{D}_S$, where $\mathscr{W}_S$ denotes the additive group of real valued functions on $S$, $U(1) \equiv [\mathbb{R}, + \bmod 2\pi]$ and $\mathscr{D}_S$ is the group of orientation preserving diffeomorphisms of $S$. The $\mathscr{G}$-action on $\mathrm{Gr}(\widetilde{\mathscr{E}})$ by BLK morphisms is generated by the $\mathscr{G}$-action $R$ on $\widetilde{\mathscr{E}}$ by the vector bundle morphisms:

$$\begin{aligned} &R : \widetilde{\mathscr{E}} \times \mathscr{G} \to \widetilde{\mathscr{E}} \\ &R((e, s, \tilde{x}, a, b), (\varphi, \theta, \gamma)) \\ &\quad = \left( \exp(\varphi) \cdot \gamma^* e, \gamma^{-1}(s) + \frac{\not\theta}{2\pi}, \tilde{x} \circ \gamma, \exp(-2\varphi) a \circ \gamma, \exp(\varphi) b \circ \gamma \right), \end{aligned} \tag{4.3}$$

where we use the shorthand notation $s \to s + \frac{\not\theta}{2\pi}$ for the isometry $I(e, \theta)$ of $e$ determined by the distance $\frac{\not\theta}{2\pi}$ between $s$ and $I(e, \theta)(s)$.

The correspondence between a trajectory and its boundary value along $\Sigma \subset \partial M_{h,b}$ given by (4.2) depends on the choice of a diffeomorphism $\varrho : S \to \Sigma$. One can overcome this difficulty choosing instead of $\mathrm{Gr}(\widetilde{\mathscr{E}})$ the quotient space $\mathrm{Gr}(\widetilde{\mathscr{E}})/\mathscr{D}_S$. As it follows from the definition (4.3) of the $\mathscr{G}$-action on $\mathrm{Gr}(\widetilde{\mathscr{E}})$ the following relation holds:

$$\mathrm{Gr}(\widetilde{\mathscr{E}})/\mathscr{D}_S = \mathrm{Gr}(\mathscr{K})\,,$$

where $\mathscr{K}$ is a vector bundle defined as the base space of the principal fibre bundle:

$$\begin{array}{ccc} \mathscr{D}_S & \longrightarrow & \widetilde{\mathscr{E}} \\ & & \big\downarrow \Pi_{\mathscr{K}} \\ & & \mathscr{K} = \widetilde{\mathscr{E}}/\mathscr{D}_S\,. \end{array} \tag{4.4}$$

In order to parametrize $\mathscr{K}$ we will use a special class of global gauge slices (1-dim conformal gauges) in the bundle (4.4):

$$\mathscr{S}[\hat{e}, \hat{s}] \equiv \{(e, s, \tilde{x}, a, b) \in \widetilde{\mathscr{E}} : e = \mathrm{const} \cdot \hat{e}, s = \hat{s}\}\,. \tag{4.5}$$

Consider the trivial vector bundle:

$$\mathscr{P} = (\mathbb{R}_+ \times \mathscr{X}_S) \times (C^\infty(S) \times C^\infty(S)).$$

For every $(\hat{e}, \hat{s}) \in \mathscr{M}_S \times S$ we have the vector bundle isomorphism:

$$t[\hat{e}, \hat{s}] : \mathscr{P} \ni (\ell, \tilde{x}, a, b) \to (\ell \hat{\ell}^{-1} \hat{e}, \hat{s}, \tilde{x}, a, b) \in \mathscr{S}[\hat{e}, \hat{s}]$$

which yields the following parametrization of $\mathscr{K}$:

$$p[\hat{e}, \hat{s}] \equiv \Pi_{\mathscr{K}} \circ t[\hat{e}, \hat{s}] : \mathscr{P} \to \mathscr{K} . \tag{4.6}$$

The residual gauge transformations in the space $\mathrm{Gr}(\mathscr{K})$ are described in the one dimensional conformal gauge (4.5) by the subgroup $\mathbb{R}_+ \times \widetilde{\mathscr{D}}_S[\hat{e}, \hat{s}] \subset \mathscr{G}_S$ consisting of all transformations preserving the gauge slice $\mathscr{S}[\hat{e}, \hat{s}]$. $\mathbb{R}_+$ denotes the 1-dim group of constant rescalings of einbeins while $\widetilde{\mathscr{D}}_S[\hat{e}, \hat{s}]$ is given by:

$$\widetilde{\mathscr{D}}_S[\hat{e}, \hat{s}] \equiv \left\{ (\varphi, \theta, \gamma) \in \mathscr{G}_S : \exp(\varphi)\gamma^* \hat{e} = \hat{e}, \gamma^{-1}(\hat{s}) + \frac{\ell \theta}{2\pi} = \hat{s} \right\}.$$

For a fixed $\gamma \in \mathscr{D}_S$ the conditions for $\varphi$ and $\theta$ in the formula above have unique solutions and the map:

$$d[\hat{e}, \hat{s}] : \widetilde{\mathscr{D}}_S[\hat{e}, \hat{s}] \ni (\varphi, \theta, \gamma) \to \gamma \in \mathscr{D}_S$$

is a group isomorphism. Moreover for every $(\hat{e}, \hat{s}) \in \mathscr{M}_S \times S$ the following diagram is commutative:

$$\begin{array}{ccc} \mathscr{S}[\hat{e}, \hat{s}] \times \widetilde{\mathscr{D}}_S[\hat{e}, \hat{s}] & \xrightarrow{\;R'\;} & \mathscr{S}[\hat{e}, \hat{s}] \\ \big\downarrow {\scriptstyle t[\hat{e},\hat{s}] \times d[\hat{e},\hat{s}]} & & \big\downarrow {\scriptstyle t[\hat{e},\hat{s}]} \\ \mathscr{P} \times \mathscr{D}_S & \xrightarrow{\;R^P\;} & \mathscr{P} \end{array}$$

where $R'$ denotes the restriction of the action $R$ (4.3) and $R^P$ is defined by:

$$R^P((\ell, \tilde{x}, a, b,), \gamma) \equiv (\ell, \tilde{x} \circ \gamma, (\dot{\gamma})^2 a \circ \gamma, (\dot{\gamma})^{-1} b \circ \gamma) . \tag{4.7}$$

It follows that the residual gauge transformations in $\mathrm{Gr}(\mathscr{K})$ form the group $\widetilde{\mathscr{G}} \approx \mathbb{R}_+ \times \mathscr{D}_S$, where the action of $\mathbb{R}_+$ on $\mathscr{P}$ is given by:

$$(\ell, \tilde{x}, \tilde{\eta}, \tilde{\xi}) \xrightarrow{\;\Phi_\lambda\;} (\lambda \ell, \tilde{x}, \lambda^{-2} \tilde{\eta}, \lambda \tilde{\xi}) .$$

Note that the formula (4.7) gives the well known rule of transformations for $x$- and $b, c$-ghost variables with respect to the conformal transformations. In particular it yields the correct conformal weights: $0, 2, -1$ of the fields involved. The transformations (4.7) are frequently called the boundary reparametrizations. It is somehow misleading since, according to the considerations given above, these transformations are a consequence of the invariance of the original system (4.1) with respect to conformal rescalings of metrics and their form is a result of our choice of the 1-dim confcormal gauge (4.5). It should be stressed that the symmetry (4.7) has nothing to do with the $\mathscr{D}$-invariance which is completely "solved" by taking the quotient $\widetilde{\mathscr{E}}/\mathscr{D}_S$.

Now we proceed to the construction of the BRST extended off-shell closed string amplitudes. Let $M_{h,b}$ denote an oriented compact 2-dim manifold with $h$-handles and

$b$-boundary components diffeomorphic to a circle. On the relative Teichmüller space $T^R_{h,b}$ of $M_{h,b}$ we introduce Frenchel-Nielsen coordinates:

$$T^R_{h,b} \ni t \to (L_1, \ldots, L_b, \theta_1, \ell_1, \ldots, \theta_{3h+b-3}, \ell_{3h+b-3}) \in \mathbb{R}^b_+(\mathbb{R} \times \mathbb{R}_+)^{3h+b-3}.$$

For a given pattern for gluing $2h-2+b$ pants to obtain the surface $M_{h,b}$ the coordinates $L_1, \ldots, L_b$ are lengths (with respect to the hyperbolic geometry on $M_{h,b}$) of the boundary components $\Sigma_1, \ldots, \Sigma_b$ while $(\ell_i, \theta_i)$, $i = 1, \ldots, 3h+b-3$, are parameters of gluing [30]. In these coordinates the Weil-Petersson volume form has especially simple form:

$$d\omega^{\mathrm{WP}} = \prod_{j=1}^{b} dL_j \prod_{i=1}^{3h+b-3} \frac{d\theta_i}{2\pi} \ell_i d\ell_i .$$

(We use the convention where the Dehn twists corresponds to $\theta_i = 2k\pi$.) We introduce the restricted Weil-Petersson volume form:

$$d\hat{\omega}^{\mathrm{WP}} = \prod_{i=1}^{3h+b-3} \frac{d\theta}{2\pi} \ell_i d\ell_i$$

and the restricted fundamental domain:

$$\widetilde{T}^R_{h,b}[\ell_1, \ldots, \ell_b] \equiv \{t \in [T^R_{h,b}] : L_j(t) = \ell_j, j = 1, \ldots, b\},$$

where $[T^R_{h,b}]$ is a fundamental domain of the modular group. Let us consider the following principal fibre bundle:

$$\begin{array}{ccc} \mathscr{L}^i_{h,b} \odot \mathscr{W}^i_{h,b} & \longrightarrow & \mathscr{M}^i_{h,b} \\ & & \big\downarrow \Pi_{h,b} \\ & & T^R_{h,b}\,. \end{array} \tag{4.8}$$

The constant curvature gauge consists in the choice of a section

$$\Xi : T^R_{h,b} \ni t \to g^t \in \mathscr{M}^{-1}_{h,b} \subset \mathscr{M}^i_{h,b}$$

with values in the space $\mathscr{M}^{-1}_{h,b}$ of metrics with the scalar curvature equals $-1$ and with the property that every boundary component of $M_{h,b}$ is a geodesic line. In this gauge and for a given set $\{\dot{c}_1, \ldots, \dot{c}_b\}$ ($\dot{c}_j \in \mathscr{M}_S \times S \times \mathscr{X}_S/\mathscr{D}_S$) of boundary values for "bosonic part of a trajectory" we have the following expression for the off-shell closed string amplitude [2]:

$$A_h[\dot{c}_1, \ldots, \dot{c}_b] = \int\limits_{\widetilde{T}^R_{h,b}[\ell_1 \ldots \ell_b]} d\hat{\omega}^{\mathrm{WP}} \times \int\limits_{\Sigma_1} e^t_1 d\sigma_1 \times \cdots \times \int\limits_{\Sigma_b} e^t_b d\sigma_b$$
$$\times (\det{}_A P^+_{g^t} P_{g^t})^{1/2} (\det{}_D \mathscr{L}_{g^t})^{-13} \exp(-W[g^t, \sigma \mid \dot{c}_1 \ldots \dot{c}_b]). \tag{4.9}$$

For a given $\sigma = (\sigma_1, \ldots, \sigma_b) \in \Sigma_1 \times \cdots \times \Sigma_b$ the functional $W[g^t, \sigma \mid \dot{c}_1, \ldots, \dot{c}_b]$ is defined by:

$$W[g^t, \sigma \mid \dot{c}_1, \ldots, \dot{c}_b] = S[g^t, x_{cl}], \tag{4.10}$$

where $x_{cl}$ is the solution of the boundary value problem:

$$\begin{aligned} \mathcal{L}_{g^t} x_{cl} &= 0\,, \\ x_{cl|\Sigma_j} &= \tilde{x}_j \circ \gamma_j[g^t, \sigma \mid \hat{e}, \hat{s}]\,. \end{aligned} \tag{4.11}$$

In the formulae above $(\ell_j, \tilde{x}_j)$ is the parametrization of $\dot{c}_j$ in the 1-dim conformal gauge determined by $(\hat{e}, \hat{s})$ with $\int \hat{e} ds = 1$. The diffeomorphisms

$$\gamma_j[g^t, \sigma \mid \hat{e}, \hat{s}] : \Sigma_j \to S$$

are uniquely determined by the equations:

$$\begin{aligned} (\gamma_j[g^t, \sigma \mid \hat{e}, \hat{s}])^* \hat{e}^2 &= i_j^* g^t\,, \\ \gamma_j[g^t, \sigma \mid \hat{e}, \hat{s}]\,(\sigma_j) &= \hat{s}\,. \end{aligned}$$

where $i_j : \Sigma_j \to M_{h,b}$ denotes the inclusion of the $k_{\text{th}}$ boundary component.

As it was mentioned above the BRST extended off-shell amplitudes could be constructed by means of the path integral over some space of trajectories of the system (4.1) with prescribed boundary values for "bosonic" and ghosts variables. Because of the anticommuting nature of ghost variables the description of the relevant supermanifold of trajectories is slightly more complicated than in the bosonic case.

For every $k \equiv (k_1, \ldots, k_b)$; $k_j \in \mathcal{K}$ let us consider the following fibration:

$$\mathcal{E}[k] = \bigcup_{(g^t, \sigma, x) \in \mathcal{T}[k]} \mathcal{V}[g^t, \sigma, x \mid k]\,,$$

where

$$\begin{aligned} \mathcal{T}[k] = \{(g^t, \sigma, x) &\in \Xi(T^R_{h,b}[\ell]) \times \Sigma_1 \times \cdots \times \Sigma_b \times \mathcal{E}_{h,b} : \\ &x_{cl|\Sigma_j} = \tilde{x}_j \circ \gamma_j[g^t, \sigma \mid \hat{e}, \hat{s}]\}\,, \\ \mathcal{V}[g^t, \sigma, x \mid k] \equiv \{(\delta h, \delta f) &\in \mathcal{H}_{g^t} \times \mathcal{T}_{\text{id}} \mathcal{D}_{h,b} : M_{g^t}(\delta h, \ker P^+_{g^t}) = 0; \\ &n^a t^b \delta h_{ab|\Sigma_j} = a_j \circ \gamma_j[g^t, \sigma \mid \hat{e}, \hat{s}]; \\ &n_c \delta f^c_{|\Sigma_j} = b_j \circ \gamma_j[g^t, \sigma \mid \hat{e}, \hat{s}]\}\,, \end{aligned}$$

and

$$k_j = p[\hat{e}, \hat{s}]\,(\ell_j, \tilde{x}_j, a_j, b_j)\,, \qquad j = 1, \ldots, b\,.$$

Note that the definition above is independent of the choice of a 1-dim conformal gauge $(\hat{e}, \hat{s})$.

The bundle $\mathcal{E}[k]$ is a vector bundle if and only if for every $j = 1, \ldots, b$.

$$k_j = p[\hat{e}, \hat{s}]\,(\ell_j, \tilde{x}_j, 0, 0)\,.$$

In this case one can identify $k_j$ with $\dot{c}_j$ and the expression (4.9) can be rewritten in terms of the graded manifold $\mathrm{Gr}(\mathcal{E}[\dot{c}_1, \ldots, \dot{c}_b])$ of the trajectories.

In the case of nonhomogeneous boundary conditions for the ghost variables $\mathcal{E}[k]$ is an affine bundle and one cannot construct a corresponding graded manifold. However in the present case of the Gaussian integral one can make a shift by the classical solution. At every fibre $\mathscr{V}[g^t, \sigma, x \mid k]$ let us fix the "reference" point $(\delta h^{cl}, \delta f_{cl})$ uniquely determined as the solution of the following boundary value problem:

$$\begin{aligned} P^+_{g^t}\delta h^{cl} &= 0\,, \\ M_{g^t}(\delta h, \ker P^+_{g^t}) &= 0\,, \\ n^a t^b \delta h_{ab|\Sigma_j} &= a_j \circ \gamma_j[g^t, \sigma \mid \hat{e}, \hat{s}]; \\ P^+_{g^t} P_{g^t}\delta f_{cl} &= 0\,, \\ n_c \delta f^c_{|\Sigma_j} &= b_j \circ \gamma_j[g^t, \sigma \mid \hat{e}, \hat{s}]\,. \end{aligned} \tag{4.12}$$

It yields the isomorphism of the affine bundle $\mathscr{E}[k]$ onto the vector bundle $\mathscr{E}[k^0]$, where $k^0$ denotes the "bosonic" part of $k$,

$$k^0_j = p[\hat{e}, \hat{s}]\,(\mathscr{l}_j, \tilde{x}_j, 0, 0)\,, \qquad j = 1, \ldots, b\,.$$

The action $S_{gh} + S'_{gh}$ originally defined as a functional on $\mathscr{E}[k]$ transforms under this isomorphism to the functional

$$(S_{gh} + S'_{gh})\,[g^t, \sigma, \delta h, \delta f] + W_{gh}[g^t, \sigma \mid a, b]\,,$$

defined on the cartesian product $\mathscr{E}[k^0] \times \left(\overset{b}{\times}\mathscr{K}\right)$.

The functional $W_{gh}[g^t, \sigma \mid a, b]$ in the formula above is defined by

$$W_{gh}[g^t, \sigma \mid a, b] \equiv -2 \sum_{j=1}^{b} \int\limits_{\Sigma_j} e^t_j a_j t_c \delta f^c_{cl} d\sigma_j\,.$$

It is bilinear with respect to the variables $a = (a_1, \ldots, a_b)$, $b = (b_1, \ldots, b_b)$ and, according to the formula (2.10) can be regarded as a function on the graded manifold $\overset{b}{\underset{i=1}{\times}} \mathrm{Gr}(\mathscr{K})$. Let us observe that a more careful treatment requires a similar construction for the $x$-variables as well. It follows that the supermanifold of trajectories of the system (4.1) with prescribed boundary values should be regarded as a member of the family of graded manifolds parametrized by the graded manifold $\overset{b}{\underset{i=1}{\times}} \mathrm{Gr}(\mathscr{K})$.

Choosing some basis $\tilde{\eta} \oplus \tilde{\xi} = \{\tilde{\eta}_i\}_{i=1}^{\infty} \oplus \{\tilde{\xi}_j\}_{j=1}^{\infty}$ in the dual bundle $\mathscr{K}^*$ one can write the final expression for the BRST extended off-shell closed string amplitude in the constant curvature gauge:

$$\begin{aligned} &A^h[(k^0_1, \tilde{\eta}_1, \tilde{\xi}_1), \ldots, (k^0_b, \tilde{\eta}_b, \tilde{\xi}_b)] \\ &= \int\limits_{\widetilde{T}^R_{h,b}[\mathscr{l}_1 \ldots \mathscr{l}_b]} d\hat{\omega}^{\mathrm{WP}} \times \int\limits_{\Sigma_1} e^t_1 d\sigma_1 \times \cdots \times \int\limits_{\Sigma_b} e^t_b d\sigma_b (\det{}_A P^+_{g^t} P_{g^t})^{1/2} \times (\det{}_D \mathscr{L}_{g^t})^{-13} \\ &\quad \times \exp(-W[g^t, \sigma \mid k^0_1 \ldots k^0_b] - W_{gh}[g^t, \sigma \mid (\tilde{\eta}_1, \tilde{\xi}_1) \ldots (\tilde{\eta}_b, \tilde{\xi}_b)])\,. \end{aligned} \tag{4.13}$$

The expression above is independent of the choice of coordinates in the base manifold $\mathscr{C}$ of the bundle $\mathscr{K}$ nor of the choice of a basis of global sections of $\mathscr{K}$. The off-shell amplitude (4.13) is therefore a well defined functional on the $\infty$-dim supermanifold $\overset{b}{\underset{i=1}{\times}} \mathrm{Gr}(\mathscr{K})$. As an off-shell object it is not invariant with respect to the residual gauge transformations. In fact using the methods of [1, 2] one can show that the off-shell amplitudes defined in different gauges are related by these transformations.

## 5. Sewing Amplitudes

In order to discuss the sewing procedure for the amplitudes defined in the previous section it is convenient to use the parametrization of $\mathrm{Gr}(\mathscr{H})$ given by a 1-dimensional conformal gauge (4.5). Let us fix $(\hat{e}, \hat{s})$ on $S$ and a set of points $\sigma = (\sigma_1, \ldots, \sigma_b)$ such that for any $i$ $\sigma_i$ belongs to the $i^{\text{th}}$ boundary component $\Sigma_i$ of $M_{h,b}$. Changing variables

$$[-\pi, \pi] \ni \theta \rightarrow \sigma_i + (2\pi)^{-1} \hat{\ell}_i \theta_i$$

and using the $U(1)$ transformation properties of $W$ and $W_{gh}$, we have the following expression:

$$\begin{aligned} A^h[(\alpha_i, x_i, \tilde{\eta}_i, \tilde{\xi}_i)_{i=1}^b] &= \int\limits_{\widetilde{T}^R_{h,b}[\alpha_1 \hat{\ell} \ldots \alpha_b \hat{\ell}]} d\hat{\omega}^{\mathrm{WP}} \times \prod_{i=1}^b (2\pi)^{-1} \alpha_i \hat{\ell} \int\limits_{-\pi}^{\pi} d\theta_i \times (\det{}_A P^+_{g^t} P_{g^t})^{1/2} \times (\det{}_D \mathscr{L}_{g^t})^{-13} \\ &\quad \times \exp(-W[g^t, \sigma \mid \tilde{x}_i(. + \theta_i)_{i=1}^b] \\ &\quad - W_{gh}[g^t, \sigma \mid (\tilde{\eta}_i(. + \theta_i), \tilde{\xi}_i(. + \theta_i))_{i=1}^b]) , \end{aligned} \tag{5.1}$$

where $f_i(. + \theta_i) = f_i \circ I(\alpha_i \hat{\ell}, \theta)$; $f_i = \tilde{x}_i, \tilde{\eta}_i, \tilde{\xi}_i$.

The second ingredient we need for sewing is an inner product in the space of functionals on $\mathrm{Gr}(\mathscr{H})$. The "bosonic" part of such a product was discussed in [2]. Its BRST extension has in the 1-dim conformal gauge the following form:

$$\begin{aligned} (\Phi, \Psi) = \int\limits_0^\infty (\alpha \hat{\ell})^{-1} d\alpha \hat{\ell} \int \mathscr{D}^{\alpha \hat{e}} \tilde{x} \int \mathscr{D}^{\alpha \hat{e}} \tilde{\eta} \int \mathscr{D}^{\alpha \hat{e}} \tilde{\xi} \\ \times \Phi[\alpha, \tilde{x}(-.), \tilde{\eta}(-.), \tilde{\xi}(-.)] \Psi[\alpha, \tilde{x}, \tilde{\eta}, \tilde{\xi}] , \end{aligned} \tag{5.2}$$

where $f(-.) = f \circ r[\hat{e}, \hat{s}]$; $f = \tilde{x}_i, \tilde{\eta}_i, \tilde{\xi}_i$, and $r[\hat{e}, \hat{s}]$ denotes the orientation reversing isometry of $\hat{e}$ uniquely defined by the condition $r[\hat{e}, \hat{s}](\hat{s}) = \hat{s}$. The $\infty$-dim "Berezin volume forms" $\mathscr{D}^{\alpha \hat{e}} \tilde{\eta}$, $\mathscr{D}^{\alpha \hat{e}} \tilde{\xi}$ are related to the Euclidean structure:

$$E_{(\alpha, x)}((a, b), (a', b')) = \int \alpha \hat{e} a a' + \int \alpha \hat{e} b b' .$$

Let $A_1$, $A_2$ be the off-shell amplitudes over the disjoint model surfaces $M'_{h',b'}$, $M''_{h''b''}$, respectively. For simplicity we consider a procedure for sewing amplitudes $A_1$ and $A_2$ along a single boundary component of $M'$ and $M''$. In this case the path integral framework suggests the following overlap integral:

$$\begin{aligned} I_{12} = \int\limits_0^\infty (\alpha \hat{\ell})^{-1} d\alpha \hat{\ell} \int \mathscr{D}^{\alpha \hat{e}} \tilde{x} \int \mathscr{D}^{\alpha \hat{e}} \tilde{\eta} \int \mathscr{D}^{\alpha \hat{e}} \tilde{\xi} \\ \times A_1[(\alpha_i, \tilde{x}_i, \eta_i, \xi_i)_{i=1}^{b'-1}, (\alpha, \tilde{x}(-.), \tilde{\eta}(-.), \tilde{\xi}(-.))] \\ \times A_2[(\alpha, \tilde{x}, \tilde{\eta}, \tilde{\xi}), (\alpha_i, \tilde{x}_i, \tilde{\eta}_i, \tilde{\xi}_i)_{i=2}^{b''}] . \end{aligned} \tag{5.3}$$

The question is whether the formula above gives the off-shell amplitude over a model surface $M_{h,b}$ of genus $h = h' + h''$ and with the number of boundary components $b = b' + b'' - 2$.

After substitution of (5.1) into (5.3) one gets in particular the double integration over the twists $\theta'$, $\theta''$ on the boundary components $\Sigma_{b'} \subset \partial M_{h'b'}$ and $\Sigma_1 \subset \partial M_{h''b''}$ respectively. Due to the $U(1)$ invariance of the functional measure in (5.2) the integrand depends only on the sum $\theta' + \theta''$, and the integration over one twist decouples yielding the factor $\alpha\hat{\ell}$ which in order cancels the factor $(\alpha\hat{\ell})^{-1}$ in the integral over $\alpha$ in (5.3). Thus we have the following expression

$$
\begin{aligned}
I_{12} = & \int_0^\infty \alpha\hat{\ell}\, d\alpha\hat{\ell} \int_{-\pi}^{\pi} d\theta \int_{\widetilde{T}_1^R} d\hat{\omega}_1^{\mathrm{WP}} \int_{\widetilde{T}_2^R} d\hat{\omega}_2^{\mathrm{WP}} \prod_{i=1}^{b'+b''-2} (2\pi)^{-1} \alpha_i \hat{\ell} \int_{-\pi}^{\pi} d\theta_i \\
& \times J_{12}[g^{t'}, g^{t''}, \sigma', \sigma'', \theta \mid \tilde{x}_i(. + \theta_i)_{i=1}^{b'+b''-2}] \\
& \times K_{12}[g^{t'}, g^{t''}, \sigma', \sigma'', \theta \mid (\tilde{\eta}_i(. + \theta_i), \tilde{\xi}_i(. + \theta_i))_{i=1}^{b'+b''-2}] .
\end{aligned}
\tag{5.4}
$$

where

$$
\begin{aligned}
\widetilde{T}_1^R &= \widetilde{T}_1^R[\ell_1, \ldots, \ell_{b'-1}, \alpha\hat{\ell}] , \\
\widetilde{T}_2^R &= \widetilde{T}_2^R[\alpha\hat{\ell}, \ell_{b'}, \ldots, \ell_{b'+b''-2}]
\end{aligned}
$$

are restricted fundamental domains in the Teichmüller spaces of $M'$ and $M''$, respectively. The functionals $J_{12}$, $K_{12}$ are defined by the following overlap path integrals:

$$
\begin{aligned}
& J_{12}[g^{t'}, g^{t''}, \sigma', \sigma'', \theta \mid \tilde{x}_i(. + \theta_i)_{i=1}^{b'+b''-2}] \\
& \quad = \int \mathscr{D}^{\alpha\hat{e}} \tilde{x} (\det{}_D \mathscr{S}_{g^{t'}})^{-13} \exp(-W[g^{t'}, \sigma' \mid (\tilde{x}_i)_{i=1}^{b'-1}, \tilde{x}(-.)] \\
& \qquad \times (\det{}_D \mathscr{S}_{g^{t''}})^{-13} \exp(-W[g^{t'}, \sigma'' \mid \tilde{x}(. + \theta), (x_i)_{i=b'}^{b'+b''-2}] ,
\end{aligned}
\tag{5.5}
$$

$$
\begin{aligned}
& K_{12}[g^{t'}, g^{t''}, \sigma', \sigma'', \theta \mid (\tilde{\eta}_i(. + \theta_i), \tilde{\xi}_i(. + \theta_i))_{i=1}^{b'+b''-2}] \\
& \quad = \int \mathscr{D}^{\alpha\hat{e}} \tilde{\eta} \int \mathscr{D}^{\alpha\hat{e}\tilde{\xi}} (\det{}_A P^+_{g^{t'}} P_{g^{t'}})^{1/2} \\
& \qquad \times \exp(-W_{gh}[g^{t'}, \sigma' \mid (\tilde{\eta}_i, \tilde{\xi}_i))_{i=1}^{b'-1}, (\tilde{\eta}(-.), \tilde{\xi}(-.))]) (\det{}_A P^+_{g^{t''}} P_{g^{t''}})^{1/2} \\
& \qquad \times \exp(-W_{gh}[g^{t''}, \sigma'' \mid (\tilde{\eta}(. + \theta), \tilde{\xi}(. + \theta)), (\tilde{\eta}_i, \tilde{x}_i))_{i=b'}^{b'+b''-2}])
\end{aligned}
\tag{5.6}
$$

The form $\alpha\hat{\ell}\, d(\alpha\hat{\ell}) \wedge (2\pi)^{-1} d\theta \wedge d\hat{\omega}_1^{\mathrm{WP}} \wedge d\hat{\omega}_2^{\mathrm{WP}}$ originally defined on the cartesian product

$$
\mathbb{R}_+ \times \mathbb{R} \times T_1^R[\ell_1, \ldots, \ell_{b'-1}, 1] \times T_2^R[1, \ell_{b'}, \ldots, \ell_{b'+b''-2}]
$$

can be regarded as the restricted Weil-Petersson volume form $d\hat{\omega}^{\mathrm{WP}}$ on

$$
T_{hb}^R[\ell_1, \ldots, \ell_b] \, (h = h' + h'', b = b' + b'' - 2) .
$$

In fact identifying the boundary components $\Sigma'_b$ and $\Sigma''_1$ by means of an orientation reversing diffeomorphism $\gamma: \Sigma'_{b'} \to \Sigma''_1$ ($\gamma(\sigma'_{b'}) = \sigma''_1$) one obtains an oriented surface $M' \cup M''$ diffeomorphic to $M_{h,b}$. Using an orientation preserving diffeomorphism $f: M_{h,b} \to M' \cup M''$ one can construct a partition of $M_{h,b}$ into $2h - 2 + b$ pairs of pants from partitions of $M'$ and $M''$. With this choice of Frenchel-Nielsen coordinates in $T^R_{hb}$ the diffeomorphism $f$, induces an isomorphism

$$T^R_{hb}[\ell_1, \dots, \ell_b] = \mathbb{R}_+ \times \mathbb{R} \times T^R_1[\ell_1, \dots, \ell_{b'-1}, 1] \times T^R_2[1, \ell_{b'}, \dots, \ell_{b'+b''-2}]$$

and provides the identification

$$d\hat{\omega}^{\mathrm{WP}} = \alpha\hat{\ell}\, d(\alpha\hat{\ell}) \wedge (2\pi)^{-1} d\theta \wedge d\hat{\omega}^{\mathrm{WP}}_1 \wedge d\hat{\omega}^{\mathrm{WP}}_2 .$$

Note that the range of integration of $d\hat{\omega}^{\mathrm{WP}}$ in (5.4),

$$T_{12} = \mathbb{R}_+ \times [-\pi, \pi] \times \widetilde{T}^R_1 \times \widetilde{T}^R_2 ,$$

is not a restricted fundamental domain in $T^R_{hb}[\ell_1, \dots, \ell_b]$. In fact it contains infinitely many restricted fundamental domains which leads to an infinite overcounting in the formula (5.3).

Postponing a more detailed discussion of this point to the next section let us now turn to the overlap integrals (5.5) and (5.6). Our aim is to show that for any

$$t = (\alpha\hat{\ell}, \theta, t', t'') \in T_{12} \subset T^R_{hb}[\ell_1, \dots, \ell_b] ,$$

and for any collections $\sigma' = (\sigma'_1, \dots, \sigma'_{b'})$, $\sigma'' = (\sigma''_1, \dots, \sigma''_{b''})$ of points on $\partial M'$ and $\partial M''$ there exists on $M_{h,b}$ a smooth metric $g^t$ over $t(\Pi_{h,b}(g^t) = t)$ with the constant scalar curvature equal to $-1$, and a collection $\sigma = (\sigma_1, \dots, \sigma_b)$ of points on $\partial M_{h,b}$ such that the following formulae hold:

$$\begin{aligned} &J_{12}[g^{t'}, g^{t''}, \sigma', \sigma'', \theta \mid (\tilde{x}_i)^b_{i=1}] \\ &\quad = (\det{}_D \Delta_{g^t})^{-13} \exp(-W[g^t, \sigma \mid (\tilde{x}_i)^b_{i=1}]) , \end{aligned} \tag{5.7}$$

$$\begin{aligned} &K_{12}[g^{t'}, g^{t''}, \sigma', \sigma'', \theta \mid (\tilde{\eta}_i, \tilde{\xi}_i)^b_{i=1}] \\ &\quad = (\det{}_A P^+_{g^t} P_{g^t})^{1/2} \exp(-W_{gh}[g^t, \sigma \mid (\tilde{\eta}_i, \tilde{\xi}_i)^b_{i=1}]) . \end{aligned} \tag{5.8}$$

We will show that the sewing relations above are equivalent to the sewing relations for conformal field theories on $M_{h,b}$ (at a fixed conformal structure) recently proved in [12]. The reasoning is the same for both relations and we will present it only for (5.7). First, let us observe that the classical action functional $W[g, \sigma \mid (\tilde{x}_i)^b_{i=1}]$ regarded as a functional $W[M_{h,b}, g, \sigma \mid (\tilde{x}_i)^b_{i=1}]$ is invariant with respect to the action of arbitrary orientation preserving diffeomorphisms changing the model surface $M_{h,b}$:

$$W[M_{h,b}, g, \sigma \mid (\tilde{x}_i)^b_{i=1}] = W[f^{-1}(M_{h,b}), f^* g, f^{-1}(\sigma) \mid (\tilde{x}_i)^b_{i=1}] . \tag{5.9}$$

This is a simple consequence of the transformation properties of the PDHP string action and of the boundary conditions (4.11) under general diffeomorphisms. Using the diffeomorphism $f: M_{h,b} \to M' \cup M''$ considered above one can replace the functionals appearing in (5.5) by the following ones:

$$\begin{aligned} &W[f^{-1}(M'), f^* g^{t'}, f^{-1}(\sigma') \mid (\tilde{x}_i)^{b'-1}_{i=1}, \tilde{x}(-.)] , \\ &W[f^{-1}(M''), f^* g^{t''}, f^{-1}(\sigma'') \mid \tilde{x}(. + \theta), (\tilde{x}_i)^b_{i=b'}] . \end{aligned} \tag{5.10}$$

The final problem is whether the metrics $f^* g^{t'}$ on $M^1_{h,b} = f^{-1}(M')$ and $f^* g^{t''}$ on $M^2_{h,b} \equiv f^{-1}(M'')$ can be regarded as restrictions of some smooth metric $g^t$ on the whole surface $M_{h,b}$. Note that the metrics $f^* g^{t'}$, $f^* g^{t''}$ have the constant scalar curvature equal to $-1$. Moreover the common boundary is a geodesic of the same length with respect to both metrics. Thus by an appropriate choice of diffeomorphisms $f_1 : M^1_{h,b} \to M^1_{h,b}$, $f_2 : M^2_{h,b} \to M^2_{h,b}$ one can achieve the coincidence of induced einbeins, normal directions and other metric components along $\Sigma = M^1_{h,b} \cap M^2_{h,b}$ in such a way that $f_1^* f^* g^{t'}$ and $f_2^* f^* g^{t''}$ form a smooth metric $g'$ on $M_{h,b}$. Within the Frenchel-Nielsen coordinates on $T^R_{hb}$ induced by $f$, $g'$ is a constant curvature metric over $(\alpha\hat{\ell}, 0, t', t'')$. Using again the relation (5.9) one gets the following equivalent form of the functionals (5.10):

$$W[M^1_{h,b}, f_1^* f^* g^{t'}, \sigma_1 \mid (\tilde{x}_i)_{i=1}^{b'-1}, \tilde{x}(-.)] \,,$$
$$W[M^2_{h,b}, f_2^* f^* g^{t''}, \sigma_2 \mid \tilde{x}(. + \theta), (\tilde{x}_i)_{i=b'}^{b}] \,,$$

where $\sigma_{1i} = f_1^{-1} f^{-1}(\sigma'_i)$, $\sigma_{2i} = f_2^{-1} f^{-1}(\sigma''_i)$.

Finally by the twist $\theta$ along $\Sigma$ one can proceed from the metric $g'$ to the metric $g^t$ over $t = (\alpha\hat{\ell}, \theta, t', t'')$. In terms of this metric the formulae (5.7), (5.6) can be rewritten in the following form:

$$\begin{aligned} &\int \mathscr{D}^{\alpha\hat{e}} \tilde{x} (\det{}_D \mathscr{L}_{g_1^t})^{-13} \exp(-W[g_1^t, \sigma_1 \mid (\tilde{x}_i)_{i=1}^{b'-1}, \tilde{x}(-.)]) \\ &\qquad \times (\det{}_D \mathscr{L}_{g_2^t})^{-13} \exp(-W[g_2^t, \sigma_2 \mid \tilde{x}, (x_i)_{i=b'}^{b}]) \\ &\quad = (\det{}_D \mathscr{L}_{g^t})^{-13} \exp(-W[g^t, \sigma \mid (\tilde{x}_i)_{i=1}^{b}]) \,, \end{aligned}$$

where $g_1^t$, $g_2^t$ denote the restrictions of the metric $g^t$ on $M^1_{h,b}$ and $M^2_{h,b}$ respectively and $\sigma = (\sigma_{11} \ldots \sigma_{1b'-1}, \sigma_{22} \ldots \sigma_{2b''})$. The relation above is just the formula for sewing at a fixed conformal structure derived in [12]. Using a similar formula for ghost fields we finally have the following result:

$$\begin{aligned} I_{12} = &\int_{T_{12}} d\hat{\omega}^{\mathrm{WP}} \prod_{i=1}^{b} (2\pi)^{-1} \alpha_i \hat{\ell} \int_{-\pi}^{\pi} d\theta_i \, (\det{}_A P^+_{g^t} P_{g^t})^{1/2} (\det{}_D \mathscr{L}_{g^t})^{-13} \\ &\times \exp(-W[g^t, \sigma \mid \tilde{x}_i(. + \theta_i)_{i=1}^{b}] \\ &\quad W_{gh}[g^t, \sigma \mid (\tilde{\eta}_i(. + \theta_i), \tilde{\xi}_i(. + \theta_i))_{i=1}^{b}]) \,. \end{aligned} \tag{5.11}$$

## 6. Towards the Covariant Closed String Field Theory

In this section we will briefly discuss the problem of reconstructing a CCSFT from off-shell amplitudes. Let us note that the passage from off-shell to on-shell amplitudes consists in filling each "hole" of an off-shell amplitude by a disc with an appropriate vertex insertion which corresponds to a physical string state. Since the physical string states are supposed to respect the residual gauge invariance this procedure is gauge independent. The choice of gauge however becomes crucial if we try to interpret off-shell amplitudes as a set (possibly consisting of one element) of Feynman diagrams arising from a perturbation expansion of some CCSFT. In fact the problem of reconstruction can be posed as a problem of constructing a gauge with some properties ensuring the Feynman diagram interpretation. Such properties were recently formulated within the framework of nonpolynomial CCSFT [16]. Here we propose a slightly different formulation motivated by the path integral approach.

Within our framework the process of reconstruction consists in constructing the following objects:
a) off-shell amplitudes;
b) off-shell amplitudes with "cut propagators" in external legs (ACP);
c) string Feynman diagrams;
d) a sewing procedure;
e) interaction vertices.

a) In the previous sections the functional integral method was used to derive expressions for off-shell string amplitudes in the constant curvature gauge. This method can be applied in an arbitrary gauge yielding for each topological type $(h, b)$ a well defined volume form $\Omega_{h,b}$ on the moduli space $m_{h,b}$ which is a functional of boundary values of $x$- and ghost variables. For a detailed discussion of the gauge fixing in the path integral over bordered surfaces we refer to our previous papers [1, 2]. Let us note that for the present discussion the third stage of a gauge fixing considered in [1, 2] is relevant. Since all further stages of the reconstruction process crucially depend on the definition of off-shell amplitudes, the existence of objects b)–d) can be regarded as an implicit form of very restrictive requirements for an admissible gauge.

b) The construction of ACP involves two choices. Firstly for every topological type $(h, b)$ one has to identify $b$-real modular parameters $(\tau_1, \ldots, \tau_b) \in \mathbb{R}^b_+$ corresponding to "times" of cut propagators in external legs. Secondly, this identification should be supplemented by a prescription how to reduce the volume form $\Omega_{h,b}$ on $m_{h,b}$ to volume forms $\Omega_{h,b}(\tau_1, \ldots, \tau_b)$ on the restricted moduli spaces $m_{h,b}(\tau_1, \ldots, \tau_b) \subset m_{h,b}$ determined by each set of fixed values of "time" parameters. Both choices are not canonical and should be regarded as a part of a gauge fixing.

c) For each restricted moduli space $m_{h,b}(\tau_1, \ldots, \tau_b)$ we introduce a family $m^i_{h,b}(\tau_1, \ldots, \tau_b)$ of open subsets of $m_{h,b}(\tau_1, \ldots, \tau_b)$ such that

$$\overline{\bigcup_i m^i_{h,b}(\tau_1, \ldots, \tau_b)} \supset m_{h,b}(\tau_1, \ldots, \tau_b), \tag{6.1}$$

and for any $i = j$

$$m^i_{h,b}(\tau_1, \ldots, \tau_b) \cap m^j_{h,b}(\tau_1, \ldots, \tau_b) = \emptyset\,. \tag{6.2}$$

The string diagram with $h$-loops and $b$-external legs is defined as the integral

$$F^i_{h,b}(\tau_1, \ldots, \tau_b) = \int\limits_{m^i_{h,b}(\tau_1, \ldots, \tau_b)} \Omega_{h,b}(\tau_1, \ldots, \tau_b)\,.$$

d) The sewing procedure consists of two basic operations on string diagrams:
A – sewing two external legs of different string diagrams,
B – sewing two external legs of the same diagram.

Both operations are supposed to produce a string diagram of an appropriate topological type and involve integrations over common boundary values and a common time parameter of sewing legs. In addition we require the locality of the sewing procedure which means that the measure of this integration as well as its range are independent of the global structure of sewing diagrams.

e) Having constructed the objects a)–d) satisfying the properties listed above one can define the quantum interaction vertices of CCSFT as a minimal set of string diagrams generating all diagrams by the sewing procedure. Restricting oneself throughout all constructions to Riemannian surfaces of the topological type $(0, b)$, $b \geq 3$ one gets interaction vertices corresponding to a classical CCSFT.

To find a gauge for which all objects above exist and satisfy the required properties is a very difficult problem. There are in fact only two approaches: the covariantized light cone CCSFT [19, 20] and the recently developed nonpolynomial one [14–18]. In the rest of this section we will concentrate on the comparison of the constant curvature gauge with the minimal area one underlying the nonpolynomial approach.

Let us look how the constant curvature gauge fits into the reconstruction scheme sketched above. This gauge was defined in [2] by the choice of the subspace of metrics with the constant scalar curvature equal to $-1$. For the identification of "time" parameters one uses the Frenchel-Nielson coordinates on the Teichmüller space. The "time" $\tau_i$ of a cut propagator in an external leg can be identified with the inverse of the length $\ell_i$ of a corresponding boundary component. With this choice the restriction of the volume form $\Omega_{h,b}$ is determined by neglecting the term $d\ell_1 \wedge \cdots \wedge d\ell_b$ in the Weil-Petersson volume form [2]. The resulting expression for ACP is given by the formula (4.13). The structure of string diagrams in this gauge is very simple – for each topological type $(h, b)$ there is only one string diagram coinciding with ACP. As it was pointed out in the previous section the sewing procedure defined by means of the overlap path integral does not produce a string diagram. Since the volume forms are sewn up perfectly the only problem is the range of integration over a common "time" parameter. One can try to improve this sewing procedure just by taking a range of integration yielding a unique cover of the restricted moduli space (clearly it always exists). Such a range however crucially depends on the global structure of sewing objects. Moreover, it is extremely difficult to calculate it even in the simplest cases. If one accepts this improved sewing which is essentially nonlocal then the only interaction vertex is the cubic one. One also has the manifest modular invariance as well as a factorization property in each channel.

Within the nonpolynomial approach [14–18] the off-shell amplitudes are defined in terms of punctured surfaces with prescribed coordinates around each puncture. The choice of these coordinates corresponds to the choice of a gauge in the functional approach under consideration. In order to make the comparison of both formulations more clear let us consider the punctured spheres for which the prescription of coordinates around punctures is fully established. (Note that there are strong indications that a generalization of this prescription works for arbitrary surfaces as well [15]). The corresponding tree off-shell amplitudes are completely determined by the minimal area metric which is a unique solution of the minimal (reduced) area problem (under the condition that all noncontractible closed paths on a surface have lengths greater than or equal to $2\pi$) [15]. Around each puncture this metric determines a semi-infinite tube with the constant circumference equal to $2\pi$. The "time" parameter of an external leg is identified with the length parameter of a corresponding tube. The ACP in the minimal area gauge is defined by cutting external legs along constant time lines. In comparison with the constant curvature gauge the structure of string diagrams is much more complicated. The strategy to determine this structure is to start with the symmetric Witten vertex and with a completely local sewing procedure and then to analyse missing regions in moduli spaces of higher order ACP's. It turned out to be an infinite procedure yielding recursion relations for missing regions and required interaction vertices [15, 16]. As a result one gets (at least in the classical theory) a finite number of string diagrams at every order and an infinite set of interaction vertices with an increasing number of external legs. The unique cover of the moduli space (the properties (6.1), (6.2) in our formulation) was recently proved

[14–16, 18]. Also some lower order calculations show that the correct measure on the moduli space is reproduced [17]. Due to the properties of the minimal area metric the modular invariance and the factorization in every channel are manifest. Summing up, the minimal area gauge is in a way complementary to the constant curvature one. One gets a formulation of CCSFT with the completely local sewing procedure but with the nonpolynomial (and in fact nonlocal) interaction.

The nonlocality of both approaches leads to serious calculation problems. The explicit calculation of a correct range in the constant curvature gauge is, however, much more hopeless. Moreover, it is very hard to realise how such nonlocal sewing could follow from a perturbation expansion of a cubic CCSFT action. We will finish this section with some speculation about a possible "localization" of the sewing procedure in the constant curvature gauge.

From the mathematical point of view the origin of the overcounting in the overlap formula (5.11) is clear – the modular group of a surface is essentially larger than the direct product of modular groups related to its component. For this reason the problem of determining a fundamental domain of the modular group in the Frenchel-Nielsen coordinates is very difficult. On the other hand it is interesting to indicate a "physical" origin of the breakdown of the overlap formula which is motivated by the well known factorization property of the Feynman path integral. It can be done by a careful examination of the passage from the Minkowski to the Euclidean space. In fact the factorization of a functional integral over trajectories is deeply related with the time evolution of a system. In the case of the interacting bosonic string it can be easily seen in the light cone gauge where the global time parameter enters explicitly. In this gauge one can check the factorization for any light cone diagram cut along a constant time line. As elementary building blocks of light cone diagrams one can take pairs of pants with a flat metric singular at the point of interaction and with geodesic boundaries. Note that this metric determines a global internal time on the pair of pants in such a way that the boundary components are lines of constant time. Gluing several pairs of pants of this type together one gets a light cone diagram if and only if all internal times give rise to a global internal time on a resulting surface. This special feature of the light cone diagrams ensures in fact a unique cover of the moduli space [29]. One can expect that a similar pattern could be applied to cure the overcounting in the constant curvature gauge. It requires some additional structure on a bordered surface playing the role of an internal time and uniquely characterized by some data on boundary components. The realisation of this idea requires some existence and uniqueness theorems and is beyond the scope of the present paper. Let us only mention that the resulting theory should be very similar to the covariantized light cone approach [19, 20].

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