Commun. Math. Phys. 147, 277–294 (1992)

Communications in Mathematical Physics

Macdonald Polynomials from Sklyanin Algebras: A Conceptual Basis for the p-Adics-Quantum Group Connection*

Peter G.O. Freund[1] **and Anton V. Zabrodin**[2]

[1] Enrico Fermi Institute and Department of Physics, University of Chicago, Chicago, IL 60637, USA

[2] Institute of Chemical Physics, Kosygina Str. 4, 117334 Moscow, Russia

Received October 31, 1991

Abstract. We establish a previously conjectured connection between p-adics and quantum groups. We find in Sklyanin's two parameter elliptic quantum algebra and its generalizations, the conceptual basis for the Macdonald polynomials, which "interpolate" between the zonal spherical functions of related real and p-adic symmetric spaces. The elliptic quantum algebras underlie the Z_n-Baxter models. We show that in the $n \to \infty$ limit, the Jost function for the scattering of *first* level excitations in the $1 + 1$ dimensional field theory model associated to the Z_n-Baxter model coincides with the Harish–Chandra-like c-function constructed from the Macdonald polynomials associated to the root system A_1. The partition function of the Z_2-Baxter model itself is also expressed in terms of this Macdonald–Harish–Chandra c-function, albeit in a less simple way. We relate the two parameters q and t of the Macdonald polynomials to the anisotropy and modular parameters of the Baxter model. In particular the p-adic "regimes" in the Macdonald polynomials correspond to a discrete sequence of XXZ models. We also discuss the possibility of "q-deforming" Euler products.

1. Introduction

A connection between p-adics and quantum deformations has been noticed [1–5] in a variety of contexts over the past few years. The possibility of such a connection emerges from work on p-adic strings [6, 7] and q-strings [8, 9]; from work on scattering on real [10, 11], p-adic [4] and quantum [12] symmetric spaces; and from work on Macdonald polynomials associated to "admissible pairs" of root systems [1, 2].

All this evidence points in the direction of quantum group-like objects with *two* deformation parameters and the corresponding quantum symmetric spaces as underlying this "p-adics quantum deformation connection" [2, 3, 12]. Essentially, this is how such a connection is expected to work. Corresponding to a root system R (or more generally to an "admissible pair" of root systems), one constructs a two parameter family of quantum symmetric spaces, such that their zonal spherical

* Work supported in part by the NSF: PHY-9000386

functions (zsf's) "interpolate" between the zsf's of ordinary real and p-adic symmetric spaces in the following sense [1, 2]. If we call the two parameters q and t then

a) for $q = 0, t = 1/p, p =$ prime, these zsf's essentially reduce to the zsf's of a p-adic symmetric space whose restricted root system is $R^{\vee}$, the dual of the chosen root system R;

b) for $t = q^l \to 1$, with a certain value of l, these zsf's reduce to the zsf's of the real symmetric space with restricted root system R.

There has been some progress [12–15] in exploring property b) in the context of one-parameter quantum groups, obtained from the full two-parameter groups by imposing $t = q^l$, though not necessarily with t near one. However, none of the p-adic cases of a) above can be reached this way. We therefore have to address the full two-parameter problem.

Here we shall do just that and find that the *two parameter quantum algebra of Sklyanin and its generalizations provide the conceptual understanding of the p-adics-quantum deformation connection.* Specifically, we shall consider the Z_n-Baxter model of statistical mechanics [16–19] on a square lattice for which the underlying algebra is of the (generalized) Sklyanin type [20–24]. For this model, in a regime such that the equivalent magnetic chain is antiferromagnetic with finite gap, we will study the scattering of two *first* level excitations and will find that, in the $n \to \infty$ limit, the corresponding Jost function coincides with the Harish–Chandra-like c-function [25] obtained from Macdonald's polynomials for root system A_1 (see Eq. (5.8)). The anisotropy parameter and the modular parameter of the Baxter model, are then related with the parameters q and t according to the relations (5.9). This way of establishing the connection is like "fishing" for $SU(2)$ inside $SU(\infty)$. One can also establish a connection directly between the Z_2-Baxter model and the A_1-Macdonald–Harish–Chandra c-function, but this relation is less simple (see Eqs. (5.10)). These connections are our main results. We suspect that both the complexity of Eqs. (5.10) and the need for the $n \to \infty$ limit before the transparent Eq. (5.8) is captured, are connected with the involved coproduct situation in elliptic quantum algebras [24]. One may wonder what physics corresponds to $(q, t) = (0, 1/p)$ in which cases the Macdonald polynomials yield zsf's of p-adic symmetric spaces (case a) above). It is readily checked that the choice of parameters $(q, t) = (0, 1/p)$ corresponds to an XXZ model, with a particular value of the anisotropy parameter.

Mathematically, the most remarkable feature of our result is that Sklyanin's elliptic quantum group and its generalizations, unify the p-adic and real versions of a Lie group ($SL(2)$ in this case). Of course a unification of $SL(2, \mathbf{R})$ with the $SL(2, \mathbf{Q}_p)$'s occurs also in the adelic [26] context. But the unification which we have in mind here is of a completely different nature. It does not involve Euler products, but rather two real parameters which can be "dialed" for any Archimedean or non-Archimedean case. One can nevertheless ask the question about how this new unification relates to the adelic one. We shall therefore discuss the possibility of q-deforming Euler products!

2. Macdonald Polynomials for the Root System A_1

Starting from any "admissible pair" (R, S) of root systems [27], Macdonald [1, 2] has constructed a corresponding family of orthogonal polynomials enjoying some truly remarkable features. This construction is very general, the root system [27] R need not even be reduced, it can be of the BC_n type. For our purposes, it will suffice to describe the Macdonald construction in the simplest of all cases, where both R and S are reduced and of rank 1, so that $R = S = A_1$.

The root system A_1 has one positive root α and one negative root $-\alpha$. The root lattice is $\Lambda_r = \{n\alpha\colon n \in \mathbf{Z}\}$ and its positive "side" is $\Lambda_r^+ = \{n\alpha\colon n \in \mathbf{Z}, n > 0\}$. The weight lattice Λ of A_1 is $\Lambda = \left\{\frac{n}{2}\alpha\colon n \in \mathbf{Z}\right\}$ and the set of dominant weights is $\Lambda^+ = \left\{\frac{n}{2}\alpha\colon n \in \mathbf{Z}, n \geqq 0\right\}$.

Obviously $\Lambda_r^+ \subset \Lambda^+$. On the weight lattice Λ a partial order is defined

$$\lambda_n > \lambda_m \leftrightarrow \lambda_n - \lambda_m \in \Lambda_r^+ \tag{2.1a}$$

or more explicitly

$$\frac{n}{2}\alpha > \frac{m}{2}\alpha \leftrightarrow 0 < n - m \in 2\mathbf{Z}\,. \tag{2.1b}$$

The Weyl group W of A_1 is $W = Z_2 = \{1, \sigma\}$, with 1 the identity element and σ the reflection which takes $\pm\alpha$ into $\mp\alpha$. The weight lattice Λ is an abelian group under addition. Its group algebra A over $\mathbf{R}$ is suggestively presented in terms of formal exponentials

$$\lambda = \frac{n}{2}\alpha \in \Lambda \to e^{\lambda} = e^{\frac{n}{2}\alpha} \in A \tag{2.2}$$

so that

$$e^{\frac{m}{2}\alpha}\, e^{\frac{n}{2}\alpha} = e^{\frac{m+n}{2}\alpha} \quad \left(e^{\frac{n}{2}\alpha}\right)^{-1} = e^{-\frac{n}{2}\alpha} \quad e^0 = 1\,. \tag{2.3}$$

These $e^{\frac{n}{2}\alpha}$ form an $\mathbf{R}$ basis of A. The Weyl group action on Λ defines a W-action also on the group algebra A

$$w(e^{\lambda}) := e^{w\lambda} \quad w \in W, \lambda \in \Lambda, e^{\lambda} \in A\,. \tag{2.4}$$

The Weyl-invariant elements of A span a subalgebra

$$A^W = \{a \in A\colon wa = a, \forall w \in W\} \tag{2.5}$$

of A. Obviously the elements

$$m_n = e^{\frac{n}{2}\alpha} + e^{-\frac{n}{2}\alpha} \quad n \in \mathbf{Z}_+ \tag{2.6}$$

provide an $\mathbf{R}$-basis of A^W.

Define the Weyl characters

$$\chi_n = \frac{e^{(n+1)\frac{\alpha}{2}} - e^{-(n+1)\frac{\alpha}{2}}}{e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}}}\,. \tag{2.7}$$

Then the

$$\chi_n \quad \text{with } n \geqq 0 \tag{2.8}$$

also provide an $\mathbf{R}$-basis of A^W.

Beside these **R**-bases of A^W, there exists a much less obvious two-parameter family of **R** bases of A^W. This family of Macdonald bases comes into being due to the existence of a two parameter family of positive-definite scalar products on A. They are constructed as follows. Call the two real parameters t and q and consider the element $\Delta(t, q)$ in A defined by

$$\Delta(t, q) = \frac{(e^{\alpha};\ q)_\infty}{(te^{\alpha};\ q)_\infty} \frac{(e^{-\alpha};\ q)_\infty}{(te^{-\alpha};\ q)_\infty}\,, \tag{2.9}$$

where $e^{\pm\alpha}$ are the elements in A corresponding to the roots $\pm\alpha$. Here and throughout this paper we adopted the notation [28, 29]

$$(a;\ q)_n = (1-a)\,(1-aq)\ldots(1-aq^{n-1}) \tag{2.10a}$$

and

$$(a;\ q)_\infty = \prod_{k=0}^{\infty} (1-aq^k)\,, \tag{2.10b}$$

so that

$$(a;\ q)_n = \frac{(a;\ q)_\infty}{(aq^n;\ q)_\infty}\,. \tag{2.10c}$$

To each

$$f = \sum_{\lambda\in\Lambda} f_\lambda e^\lambda \in A \quad (f_\lambda \in \mathbf{R}) \tag{2.11a}$$

we associate its "conjugate"

$$\bar{f} = \sum_{\lambda\in\Lambda} f_\lambda e^{-\lambda} \in A\,. \tag{2.11b}$$

Now we consider the 1-torus (circle) $T = \mathbf{R}/\Lambda_r^\vee$, where $\Lambda_r^\vee$ is the root lattice of the dual root system. Obviously any $x \in \mathbf{R}$ then has an image x_T on the circle T. Each $e^\lambda \in A$ can therefore be viewed as a character of T via

$$e^\lambda(x_T) = e^{i2\pi\langle\lambda, x\rangle}\,. \tag{2.12}$$

Macdonald's two parameter family of positive-definite scalar products on A is then given by

$$\langle f, g\rangle_{t,q} = \frac{1}{2}\int_T f\bar{g}\,\Delta(t, q)\,, \tag{2.13}$$

the measure on T being the (normalized) Haar measure.

Finally, for each scalar product $\langle , \rangle_{t,q}$ in the family (2.13), we define a *Macdonald basis of* A^W by the following two requirements:

a)

$$P_m = m_m + \sum_{\substack{0\leqq n<m\\ m-n\in 2Z}} a_{mn}(q, t)\, m_n\,;$$

b) the P_m's are orthogonal under the Macdonald scalar product $\langle , \rangle_{t,q}$ on A, i.e. $\langle P_m, P_n\rangle_{t,q} = 0$ for $m \neq n$. Then all $a_{mn}(q, t)$ are rational functions of the two real parameters q and t.

These P_n' are clearly polynomials in $m_1 = e^{\alpha/2} + e^{-\alpha/2}$, or equivalently Laurent polynomials in $e^{\alpha/2}$. For the A_1 case under discussion, they are explicitly given in terms of the famous Rogers–Askey–Ismail (RAI) polynomials [30, 31] $C_n(x; t|q)$. Specifically

$$P_n(e^{\alpha/2}; t|q) = \frac{(q;\, q)_n}{(t;\, q)_n} \Phi_n(e^{\alpha/2}; t|q)\,, \tag{2.14a}$$

where

$$\Phi_n(x; t|q) = \sum_{\substack{a+b=n \\ a,b \in Z_+}} \frac{(t;\, q)_a\,(t;\, q)_b}{(q;\, q)_a\,(q;\, q)_b}\, x^{a-b} \tag{2.14b}$$

with the q-shifted factorial $(t;\, q)_a$ defined by Eq. (2.10a).

Finally, the connection between the Φ_n's and the RAI polynomials is

$$\Phi_n(e^{i\theta}; t|q) = C_n(\cos\theta; t|q)\,. \tag{2.14c}$$

For future reference we give here the expression [31] for the RAI polynomials in terms of the q-hypergeometric function ${}_3\phi_2$.

$$C_n(\cos\theta; t|q) = t^{-n} e^{-in\theta} \frac{(t^2;\, q)_n}{(q;\, q)_n}\, {}_3\phi_2(q^{-n}, t, te^{2i\theta}; t^2, 0|q, q)\,. \tag{2.14d}$$

The Macdonald scalar product (2.13) now reduces to the usual scalar product on RAI polynomials

$$\begin{aligned}\langle P_m, P_n\rangle_{t,q} &= \int_{-1}^{+1} P_m(e^{i\theta}; t|q)\, P_n(e^{i\theta}; t|q)\, w(\cos\theta; t|q)\, (\sin\theta)^{-1}\, d\cos\theta \\ &= \delta_{mn} \frac{(tq^n;\, q)_\infty\,(tq^{n+1};\, q)_\infty}{(t^2q^n;\, q)_\infty\,(q^{n+1};\, q)_\infty}\end{aligned} \tag{2.15a}$$

with the weight function w given by

$$w(\cos\theta; t|q) = \frac{1}{2\pi} \frac{(e^{2i\theta};\, q)_\infty\,(e^{-2i\theta};\, q)_\infty}{(te^{2i\theta};\, q)_\infty\,(te^{-2i\theta};\, q)_\infty}\,. \tag{2.15b}$$

With the notation (our l is Macdonald's k)

$$t = q^l \quad \text{or} \quad l = \log t/\log q\,, \tag{2.16}$$

we can also write

$$\begin{aligned}\|P_n\|^2 = \langle P_n, P_n\rangle_{t,q} &= \frac{(q^n t;\, q)_l}{(q^{n+1};\, q)_l} = \frac{(q^{n+l};\, q)_\infty\,(q^{n+l+1};\, q)_\infty}{(q^{n+2l};\, q)_\infty\,(q^{n+1};\, q)_\infty} \\ &= \frac{\Gamma_q(n+2l)}{\Gamma_q(n+l)} \frac{\Gamma_q(n+1)}{\Gamma_q(n+1+l)}\,,\end{aligned} \tag{2.17}$$

where we used Eq. (2.10c) and the definition of the q-gamma function [28, 29]

$$\Gamma_q(x) = \frac{(q;\, q)_\infty}{(q^x;\, q)_\infty} (1-q)^{1-x}\,. \tag{2.18}$$

Introducing the Harish–Chandra-like c-function of Macdonald

$$c(x;\, l|q) = \frac{\Gamma_q(x)}{\Gamma_q(x+l)} \tag{2.19}$$

we can finally recast Eq. (2.12) in the form

$$\|P_n\|^2 = \frac{c(n+1;\, l|q)}{c(n+l;\, l|q)} \,. \tag{2.20}$$

Through this formula the Macdonald polynomials P_n yield a Macdonald–Harish–Chandra c-function. We shall make extensive use of this fact in the following sections.

In Sect. 4 we shall show that this c-function as it emerges from the Macdonald scalar product via Eq. (2.20), can be obtained directly from the large n behavior of the RAI polynomials along the usual lines of Harish–Chandra theory [25].

3. Macdonald's Miracles

In spite of its relative ease, the A_1 case considered here already exhibits a number of "miracles", which as shown by Macdonald, generalize to all admissible pairs of root systems.

A) We have

$$\lim_{t=q^{1/2}\to 1} P_n(e^{i\theta};\, t|q) = \left[\frac{\Gamma(1/2)\,\Gamma(n+1)}{\Gamma(n+1/2)}\right] P_n(\cos\theta) \,, \tag{3.1}$$

where $P_n(\cos\theta)$ are the ordinary Legendre polynomials, the zsf's on the ordinary compact archimedean symmetric space $SU(2)/SO(2)$, the 2-sphere.

B) By contrast, for $q = 0$ and $t = 1/p$ with p a rational prime

$$P_n\left(x;\, \frac{1}{p}\bigg|\, 0\right) = \left[\left(1 + \frac{1}{p}\,\delta_{n0}\right)^{-1} p^{\frac{n-2}{2}}\, (1+p)\right] \rho_x(n) \tag{3.2a}$$

with

$$\rho_x(n) = \frac{x^n(px - x^{-1}) + x^{-n}(x - px^{-1})}{p^{n/2}(p+1)(x - x^{-1})} \,, \tag{3.2b}$$

the Mautner–Cartier polynomials [32, 33], the zsf's on the non-compact p-adic symmetric space $H^{(p)} = SL(2, \mathbf{Q}_p)/SL(2, \mathbf{Z}_p)$, the p-adic hyperbolic plane.

Remark. There is a big difference between the interpretations of the two "left-over" variables x and n in the two cases A) and B) above. In the Archimedean limit A), the variable $\cos\theta = (x + x^{-1})/2$ is the "radial" coordinate on the real compact symmetric space $SU(2)/SO(2)$ and the quantized (angular) momentum variable n is related to the eigenvalue of the Laplacian $n(n+1)$ ($= l(l+1)$ in more familiar notation). Things are reversed in the p-adic case B). There the discrete variable n plays the role of "radial distance" on the non-compact p-adic symmetric space

$H^{(p)}$, which is a discrete space, a Bruhat–Tits tree [34] (or Bethe lattice). Conversely, it is now the variable x which is related to the eigenvalue of the Laplacian on the tree. This switch of variables between the cases A) and B) has a counterpart for all other root systems [1, 2]. Specifically for $q = 0, t = 1/p$ (case B) one obtains the zsf's of the p-adic group G relative to a maximal compact subgroup K such that the restricted root system of this $(G/K)_{p\text{-adic}}$ is $R^\vee$, the dual of the root system R which underlies the real symmetric space $(G/K)_{\text{real}}$ whose zsf's one reproduces in the Archimedean case A) ($t = q^{1/2} \to 1$). In the case at hand, $R = R^\vee = A_1$, so that the difference between the real and p-adic cases is reflected only in the exchanged rôles of the x and n variables. This is a very important point which will be further developed in the next section.

C) For $t = 1$, the P_n's take the simple q-independent form

$$P_n(e^{\frac{\alpha}{2}}; 1|q) = e^{n\frac{\alpha}{2}} + e^{-n\frac{\alpha}{2}}, \tag{3.3}$$

in other words they reduce to the m_n's of Eq. (2.6).

D) For $q = t$ the P_n's are again q-independent and this time they reduce to the Weyl characters

$$P_n(e^{\frac{\alpha}{2}}; q|q) = \chi_n \tag{3.4}$$

with the χ_n given by Eq. (2.7).

Macdonald assumed both q and t real. If we relax this restriction, we can consider the case of q an s^{th} root of unity.

E) For $q^s = 1$, the Macdonald polynomials $P_n(y; t|s)$ are quasi-periodic in n:

$$P_{ns+k} = P_{ns}P_k, \quad n \in \mathbf{Z}_+, \quad k \in \{0, 1, 2, \ldots, s-1\}. \tag{3.5}$$

To see this, recall the recursion relation for Macdonald polynomials (see [31] Eq. (2.15))

$$P_{n+1} = (y + y^{-1})P_n - C_{n-1}P_{n-1} \tag{3.6a}$$

with

$$C_{n-1} = \frac{1 - t^2 q^{n-1}}{1 - tq^{n-1}} \frac{1 - q^n}{1 - tq^n} = \frac{\|P_n\|^2}{\|P_{n-1}\|^2}. \tag{3.6b}$$

Notice that $q^s = 1$ then implies

$$\begin{aligned} C_{ns+k} &= C_k \\ C_{ns-1} &= 0, \end{aligned} \quad n \in \mathbf{Z}_+, \quad k \in \{0, 1, 2, \ldots, s-1\}. \tag{3.7}$$

Now from Eqs. (2.14) $P_0 = 1$, $P_1 = y + y^{-1}$, so that from Eqs. (3.6), (3.7) it follows that

$$\begin{aligned} P_{ns} &= P_{ns}P_0, \\ P_{ns+1} &= P_{ns}P_1. \end{aligned} \tag{3.8}$$

Inserting (3.8) and (3.7) into the linear Eqs. (3.6a) then yields the quasi-periodicity (3.5).

F) For $t = q^{1/2} \mp 1$, the P_n's become essentially continuous q-Legendre polynomials, which can be interpreted [15] as "quasi-spherical" functions of the one-parameter quantum group $SU(2)_q$.

G) In the limit $t = q^{(m-2)/2} \to 1$ the RAI polynomials yield [35] the Gegenbauer polynomials, zsf's on the $(m-1)$-sphere $S^{m-1} = SO(m)/SO(m-1)$.

4. Further Zonal Spherical Function-like Properties of the Rogers–Askey–Ismail Polynomials

In the remark following properties A) and B) in Sect. 3, we described the remarkable interchange of (radial) coordinate and momentum variables between the Archimedean case A) and the p-adic case B). This raises the question of what the coordinate and momentum variables are for generic values of the two parameters q and t.

This question can be answered by noting a remarkable "self-duality" property of the RAI polynomials. To explain this, let us first observe that the Macdonald polynomials $P_n(x; t|q)$ yielded familiar spherical functions in the two limiting cases A) and B) above, only up to the numerical factors in square brackets in formulae (3.1) and (3.2a). These inconvenient factors can be eliminated by a change of normalization. Then, instead of the P_n's we can define

$$\Psi_n(e^{i\theta}) = \frac{\Phi_n(e^{i\theta};\ t|q)}{\Phi_n(t^{1/2};\ t|q)} . \tag{4.1}$$

These Ψ_n's, rather than the Macdonald polynomials P_n themselves, are the candidates for spherical functions of some, as yet hypothetical, quantum symmetric space. It is worth noting that

$$\Phi_n(t^{1/2};\ t|q) = t^{-n/2} \frac{(t^2;\ q)_n}{(q;\ q)_n} . \tag{4.2}$$

Define

$$v = \theta/\log q . \tag{4.3}$$

Recalling the definition of l from Eq. (2.16), and combining it with Eqs. (4.1), (4.2), (4.3) and (2.14d), we then obtain

$$\Psi_n(q^{iv}) = q^{-n(2iv+l)/2}\ {}_3\phi_2(q^{-n}, q^l, q^{2iv+l};\ q^{2l}, 0|q, q) . \tag{4.4}$$

Since the prefactor and the q-hypergeometric [28, 29] function ${}_3\phi_2$ are *both* invariant under the exchange

$$-n \leftrightarrow 2iv + l , \tag{4.5}$$

it then follows that

$$\Psi_n(q^{iv}) = \Psi_{-2iv-l}(q^{-(n+l)/2}) , \tag{4.6}$$

where the right-hand side is to be understood as obtained by analytic continuation. In Eq. (4.6) the left-hand side is relevant for the compact case, the right-hand side (an analytic continuation) applies to the non-compact case. In particular, this explains the rôle exchange of the x and n variables between the two extreme cases A) and B) above ($SU(2)/SO(2)$ is compact, $SL(2, \mathbf{Q}_p)/SL(2, \mathbf{Z}_p)$ is *not*).

We can now use Eq. (4.6) to give a conceptual definition of the Macdonald–Harish–Chandra c-function $c(x; l|q)$ of Eq. (2.19). Going to large "distance" in the non-compact case, means $n \to \infty$ in $\Psi_{-2iv-l}(q^{-(n+l)/2})$. According to Eq. (4.6) this means going to large n in $\Psi_n(e^{iv\log q})$. Using Eqs. (4.1) and (2.14c), this means going to large n in the RAI polynomials $C_n(\cos\theta; t|q)$, where $\theta = v\log q$. But this large n-asymptotics of the RAI polynomials follows from the q-integral representation of these polynomials. Specifically, for large n [35, 28]

$$C_n(\cos\theta; t|q) = (1-q)^{-l}\frac{(t;\,q)_\infty}{(q;\,q)_\infty}\left[\frac{\Gamma_q(iu(\theta))}{\Gamma_q(iu(\theta)+l)}e^{-in\theta} + \frac{\Gamma_q(-iu(\theta))}{\Gamma_q(-iu(\theta)+l)}e^{in\theta}\right] \tag{4.7a}$$

with

$$l = \log t/\log q \quad \text{and} \quad u(\theta) = \frac{2\theta}{\log q}\,. \tag{4.7b}$$

According to Harish–Chandra we expect the coefficients of $e^{\mp in\theta}$ to be $c(\pm iu; t|q)$. Comparing with Eq. (2.19), we see this is indeed the case.

We are concerned in this paper with interpreting the RAI polynomials, or more precisely the Ψ_n's (Eq. (4.1)) as zonal spherical functions (zsf's) of a quantum symmetric space. In the classical case, a complex valued function $\phi(g)$, $g\in G$ on a Lie group G is a zsf of G relative to its compact subgroup K if

i) ϕ is regular at the identity element e of G and suitably normalized there $\phi(e) = 1$;
ii) ϕ is K biinvariant, i.e., $\phi(k_1 g k_2) = \phi(g)$ for all $g\in G$ and all $k_1, k_2 \in K$;
iii) ϕ obeys the functional equation

$$\phi(g_1)\phi(g_2) = \int_K \phi(g_1 k g_2)\, d_{\mathrm{Haar}}k\,. \tag{4.8}$$

According to a classical theorem, condition iii) is tantamount to requiring that $\phi(g)$ be a pull-back to G of a function on the symmetric space G/K which is an eigenfunction of each G-invariant differential operator on G/K. As an example for Legendre polynomials $P_n(\cos\theta)$, the zsf's of $SO(3)/SO(2)$, Eq. (4.8) becomes

$$P_n(\cos\alpha)P_n(\cos\beta) = \frac{1}{2\pi}\int_0^{2\pi} P_n(\cos\alpha\cos\beta - \sin\alpha\sin\beta\cos\gamma)\,d\gamma\,. \tag{4.9}$$

Now if the RAI polynomials are zsf's of a quantum symmetric space, then we expect them to obey a relation of the type (4.8). As a matter of fact *they do* [35].

5. Macdonald Polynomials, Sklyanin Algebras and Z_n-Baxter Models

Our aim is to find the two-parameter quantum group whose zonal spherical functions are the Macdonald polynomials for the root system A_1, i.e. the RAI polynomials. To explain our way of dealing with this question, let us consider, by analogy, a more familiar problem. Suppose we are given the Mautner–Cartier polynomials Eq. (3.26) and we want to find out whether they are the zsf's of $SL(2, \mathbf{Q}_p)$ relative to $SL(2, \mathbf{Z}_p)$. The simplest way to do this would be to consider "S-wave" scattering on the p-adic hyperbolic plane $SL(2, \mathbf{Q}_p)/SL(2, \mathbf{Z}_p)$ and to find the corresponding scattering matrix element $S_p(u)$, which is expressed in terms of the Jost function $J_p(iu)$

$$S_p(u) = \frac{J_p(iu)}{J_p(-iu)} \,. \tag{5.1}$$

If the Mautner–Cartier polynomials *are* the appropriate zsf's, then the Jost function $J_p(iu)$ must coincide with the Harish–Chandra c-function derived from the large distance behavior of the Mautner–Cartier polynomials (for the chosen value of p). Similar considerations apply to the continuation to complex n of the Legendre polynomials $P_n(\cos\theta)$.

In our problem we want to see whether the RAI polynomials are spherical functions of a Sklyanin type quantum group. To this end we choose a physical system for which the underlying algebra is of the Sklyanin type. Then for this system we set up an appropriate scattering problem (of certain excitations) such that the corresponding Jost function coincides with the Macdonald–Harish–Chandra c-function (Eq. (2.19)) derived from the RAI polynomials (see Eqs. (4.7)).

The appropriate physical system is the Z_n-Baxter model ($\mathscr{B}_n$ for short) of statistical mechanics on a square lattice [16–19]. The $n^2 \times n^2$ R-matrix of this model was parametrized by Belavin [18] in terms of Jacobi theta functions. The algebra which allows a solution of the Yang–Baxter equations, thus leading to the existence of infinitely many commuting operators and therefore to the integrability of the model, was studied by Sklyanin [20, 21], Cherednik [22, 23], and by Odeskii and Feigin [24], in whose notation the algebra is $Q_{n^2, n-1}(\mathscr{E}, \gamma)$, which we shall call Q_n for short. Its data are the integer n, an elliptic curve $\mathscr{E}$ and a point γ on $\mathscr{E}$. In particular $Q_2(= Q_{4,1}(\mathscr{E}, \gamma))$ is the original Sklyanin algebra [20, 21] of the 8-vertex model [16, 17]. We do not need the detailed form of the Belavin R-matrix elements. The essential thing is that the statistical weights depend on three independent parameters: the spectral parameter z, the anisotropy parameter γ and the modular parameter τ. As is customary, we treat z as a variable and γ, τ as parameters. In fact it is convenient to treat n as a parameter on equal footing with γ and τ. Along with $\mathscr{B}_n$ we also find it useful to think in terms of the corresponding $(1+1)$-dimensional field theoretical models $\mathscr{M}_n$. The Hamiltonian of $\mathscr{M}_n$ is obtained from the transfer matrix $T(z)$ of $\mathscr{B}_n$ through logarithmic differentiation at a special point. $\mathscr{M}_n$ is known as the generalized magnetic model [36]. Note that $\mathscr{B}_2$ is Baxter's famous eight-vertex model, and $\mathscr{M}_2$ the familiar XYZ chain. We shall be interested in the antiferromagnetic regime with finite gap, so that the ground state is constructed by filling the false (ferromagnetic) vacuum with quasiparticles. The partition function $t(z)$ of the $\mathscr{B}_n$ model in the thermodynamic limit (the Perron–Frobenius dominant eigenvalue of $T(z)$) was obtained by Richey and Tracy [19]. Up to some irrelevant factors, it is of the form

$$t(z) = \vartheta\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}\left(z - \frac{i\gamma}{\pi}, \tau\right)\exp\left[-i\left(\frac{n-1}{n}\right)2\pi z - iF\left(z; \gamma; n; \frac{-i\pi\tau}{\gamma}\right)\right], \tag{5.2a}$$

where

$$F\left(z;\gamma;n,\frac{-i\pi\tau}{\gamma}\right)=2\sum_{k=1}^{\infty}\frac{1}{k}\frac{\sinh k\gamma\left(\frac{-i\pi\tau}{\gamma}-1\right)}{\sinh k\gamma\left(\frac{-i\pi\tau}{\gamma}\right)}\frac{\sinh k\gamma(n-1)}{\sinh k\gamma n}\cdot\sin 2\pi kz \tag{5.2b}$$

and $\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}$ is the standard odd theta function with modular parameter τ ($\tau\in i\mathbf{R}_+$).

Notice the remarkable *symmetry* of $F\left(z;\gamma;n,\frac{-i\pi\tau}{\gamma}\right)$ in its *last* two arguments

$$F\left(z;\gamma;n,\frac{-i\pi\tau}{\gamma}\right)=F\left(z;\gamma;\frac{-i\pi\tau}{\gamma},n\right). \tag{5.3}$$

This is our first signal to pay special attention to the variable $-i\pi\tau/\gamma$ or its inverse $i\gamma/\pi\tau$. In fact, it will turn out that precisely this combination $i\gamma/\pi\tau$ is to be identified with the parameter l in Macdonald's polynomials (Eq. (2.16)). In the next section we shall make good use of the symmetry property (5.3).

A remarkable fact [37] in quantum integrable models is that the partition function, as a function of the spectral variable z, coincides up to some simple factors and/or redefinitions of parameters with a two-particle dressed S-matrix, the spectral parameter acquiring the interpretation of relative rapidity of the scattering particles. We have to be more specific, there being $n-1$ types of dressed excitations in $\mathcal{M}_n$. We therefore briefly recall the picture of these excitations in the context of the nested Bethe ansatz (BA). The ground state, as was already mentioned is found by filling the false vacuum with $n-1$ types of quasiparticles, each type at its own "level". The momenta are distributed continuously in segments $[-\pi,+\pi]$ at each level. Excitations are viewed as "holes" in these distributions. The type of physical excitation is determined by the level at which the hole was created. In terms of the system of interacting particles on the lattice associated to the $\mathcal{M}_n$ model in the usual way, the first level corresponds to charge excitations, while the others to "isotopic" excitations. The levels are naturally ordered according to the sequence of the nested BA. It turns out that $t(z)$ of Eq. (5.2) is essentially the (scalar) S-matrix $S_1^{(n)}$ for the scattering of two *first* level dressed excitations. More precisely

$$S_1^{(n)}=\exp\left[-i\left(\frac{n-1}{n}\right)2\pi z-iF\left(z;\gamma;n,\frac{-i\pi\tau}{\gamma}\right)\right]. \tag{5.4}$$

This $S_1^{(n)}$ differs slightly from the physical S-matrix for the scattering of two first level dressed excitations, by some Blaschke-CDD pole factors and redefinitions of variables. Yet $S_1^{(n)}$ shares all the essential features of the full S matrix in question, while being much more convenient to work with. We will study this $S_1^{(n)}$. The S-matrix elements for the scattering of other types of excitations are more complicated. Therefore we restrict ourselves to the first level sector and its S-matrix $S_1^{(n)}$. At this point it pays to introduce new variables

$$l=\frac{i\gamma}{\pi\tau},\qquad q=e^{i2\pi\tau},\qquad u=-\frac{iz}{\tau} \tag{5.5}$$

(notice that l is the combination signaled already in the context of the symmetry property (5.3); our q is the usual one, i.e. the square of the one in [19]). Then we can write e^{-iF} in the form

$$\exp\left[-iF\left(z;\gamma;n,\frac{-i\pi\tau}{\gamma}\right)\right]=\frac{\sigma(iu;l;n|q)}{\sigma(-iu;l;n|q)} \tag{5.6a}$$

with

$$\sigma(iu;l;n|q)=\prod_{k=0}^{\infty}\frac{\Gamma_q(iu+nl(k+1))}{\Gamma_q(iu+nlk+l)}\frac{\Gamma_q(iu+nlk+1)}{\Gamma_q(iu+nlk+(n-1)l+1)}. \tag{5.6b}$$

Now let n go to infinity. Using the definition (2.18) and keeping in mind that as $x\to\infty$ for $q<1$, we have $(q^x,q)_\infty\to 1$, it is then readily seen that

$$\sigma(iu;l,\infty|q)=[iu]_q\,c(iu;l|q) \tag{5.7a}$$

with

$$[iu]_q=\frac{1-q^{iu}}{1-q} \tag{5.7b}$$

the "q-analogue" of iu and $c(iu;l|q)$ the Macdonald–Harish–Chandra c-function for root system A_1 Eq. (2.19)! Combining Eqs. (5.7), (5.6a) and (5.4) we find

$$S_1^{(n)}|_{n=\infty}=-\frac{c(iu;l|q)}{c(-iu;l|q)} \tag{5.8}$$

and we see the Macdonald c-function assuming the rôle of Jost function in this scattering process. This clearly establishes the connection between the $n\to\infty$ limit of the Sklyanin–Cherednik–Odeskii–Feigin algebras Q_n, which underlie the $\mathscr{B}_n$ models on the one hand, and the Macdonald polynomials for the root system A_1 on the other hand. As was mentioned above, the data for the Q_n algebra are an elliptic curve $\mathscr{E}=\mathbf{C}/\mathbf{Z}+\mathbf{Z}\tau_{\mathscr{E}}$, characterized by the modular parameter $\tau_{\mathscr{E}}$, or equivalently $q_{\mathscr{E}}=\exp(i2\pi\tau_{\mathscr{E}})$, and a point γ on $\mathscr{E}$. The data for the set of A_1-Macdonald-RAI polynomials are the two parameters t and q. The connection between the elliptic and Macdonald parameters is then

$$q=q_{\mathscr{E}},\qquad t=e^{-2\gamma}, \tag{5.9}$$

the second equation following directly from Eqs. (2.16) and (5.5). *This connection between Q_n algebras ($n\to\infty$) and Macdonald polynomials is our main result.*

At this point the question arises as to why the $n\to\infty$ limit had to be taken. On the face of it, all we should have had to deal with should have been the elliptic algebra Q_2 and the models which it underlies $\mathscr{B}_2$ and $\mathscr{M}_2$. Going to Q_n, $\mathscr{B}_n$, $\mathscr{M}_n$ and then letting $n\to\infty$ is like searching for $SU(2)$ inside $SU(\infty)$. For ordinary Lie groups this would be a detour, for elliptic quantum algebras this may be needed on account of the complicated coproduct situation [24]. But once in Q_∞, how is it we only found the Macdonald polynomials for root system A_1 and not those for higher A_n root systems? The point is that we only looked at the scattering of *two* first level excitations.

After this discussion, we would like to see what would happen, were we to choose $n=2$, as naively indicated for root system A_1, instead of letting $n\to\infty$.

From Eqs. (5.4)–(5.6) we can see that for $n = 2$ we find

$$S_1^{(n)}|_{n=2} = q^{iu/2} \prod_{k=0}^{\infty} \frac{c(iu_k - l + 1; l|q)}{c(iu_k; l|q)} \cdot \frac{c(-iu_k; l|q)}{c(-iu_k - l + 1; l|q)} \tag{5.10a}$$

with

$$iu_k = iu + l(2k + 1) . \tag{5.10b}$$

We see that the building block of $S_1^{(n)}|_{n=2}$ is again the Macdonald–Harish–Chandra c-function of Eq. (2.19), but this time in a pattern not as conceptually simple as that of Eq. (5.8). Yet we shall have more to say about this case in the next section.

Whether or not the $n \to \infty$ limit is taken, it would be nice to have a derivation of the Macdonald–Harish–Chandra c-function of Eq. (2.19) directly from symmetric spaces constructed from Q_n quantum groups, along the standard lines of Harish–Chandra theory (see e.g. [25]) and without any reference to the physics of the Z_n-Baxter models. Conversely it would be of interest to find the geometric object for which the infinite product $\sigma(iu; l; n|q)$ of Eq. (5.6b) is the c-function. To steer all this into more familiar territory, notice that in the limit $q \to 1$ the q-gamma functions in the infinite product reduce to ordinary gamma functions and the full infinite product (5.6a) becomes essentially that which appears [38] in the soliton-soliton scattering S-matrix in the sine-Gordon model, provided one sets $n = 2$ and relates our parameter l to the sine-Gordon coupling constant β via

$$l = \frac{8\pi}{\beta^2} - 1 . \tag{5.11}$$

Thus the problem of understanding the "geometric" interpretation of the infinite products has as an important special case soliton-soliton scattering in the sine-Gordon model. Conversely, we can regard the S-matrix given by Eqs. (5.4)–(5.6) as a "q-deformation" $\Gamma \to \Gamma_q$ of the sine-Gordon soliton scattering matrix of [38]. The Sklyanin algebra ($n = 2$) looks like the *further* deformation of quantum $SL(2)$ by a new parameter.

To conclude, let us mention that the expression of the Perron–Frobenius dominant eigenvalue of Baxter's zero-field 8-vertex model ($\mathscr{B}_2$) transfer matrix, has been recast in terms of the c-function (2.19).

6. Interesting Special Cases

With the just-established connection between $\mathscr{B}_n$ or $\mathscr{M}_n$ systems and Macdonald polynomials it becomes interesting to see what happens in the regime in which the polynomials, "go" p-adic, i.e., in case B) of Sect. 3. For the $n \to \infty$ situation of Eq. (5.8) this corresponds to

$$q = 0, \quad t = e^{-2\gamma} = 1/p , \tag{6.1a}$$

so that

$$\gamma = \log\sqrt{p} . \tag{6.1b}$$

Equations (5.8), (5.5), (2.19) and (2.18) then yield

$$S_1^{(n)}|_{n=\infty, q=0, t=1/p} = \frac{pe^{i2\pi z} - 1}{e^{i2\pi z} - p}, \tag{6.2}$$

which coincides with the *bare* S-matrix in the XXZ model with the same value of γ. Could this result also be obtained from a model on a Bethe lattice with $p + 1$ edges incident at each vertex?

A direct study of $\mathscr{M}_\infty$ models using the powerful quantum inverse scattering method or the Bethe ansatz is highly desirable.

The other interesting case is

$$t = q^{1/2}, \tag{6.3a}$$

so that

$$l = 1/2. \tag{6.3b}$$

This corresponds, according to Sect. 3F, to the familiar *one*-parameter quantum group $SU(2)_q$. In the limit $q \to 1$ it then yields the ordinary Lie group $SU(2)$ (Sect. 3A). For the $SU(2)_q$ case we have the direct treatment by one of us [12]. An immediate comparison with the results of [12] is not possible, since there $n = 2$, whereas for us

$$n \to \infty, \quad l^{-1} = -\frac{i\pi\tau}{\gamma} = 2. \tag{6.4}$$

It is clear from Eq. (5.4) that for large n, the S-matrix depends only on the function $F(z; \gamma; n; i\pi\tau/\gamma)$. But, as we saw in Eq. (5.3) this function is symmetric under the interchange of its last two arguments. Therefore instead of the case (6.4) we can deal with the equivalent case of

$$n = 2, \quad \frac{-i\pi\tau}{\gamma} \to \infty, \tag{6.5}$$

which is then in line with [12], provided one replaces (5.5) by

$$l = \frac{1}{n}, \quad q = e^{-2\gamma n}, \quad u = \frac{\pi z}{\gamma n}, \tag{6.6}$$

so that, yet again

$$t = q^l = e^{-2\gamma}. \tag{6.7}$$

As in [12], we find the XXZ model in this case.

We can also view the p-adic and $SU(2)_q$ cases directly on the $\mathscr{B}_2$ or 8-vertex model or on the equivalent XYZ model, à la (5.10). For instance, in terms of Baxter's parameters [16] the p-adic case corresponds to the 6-vertex model in the principal antiferroelectric regime with

$$\Gamma = 1, \quad \Delta = -\frac{p^{1/2} + p^{-1/2}}{2}. \tag{6.8}$$

In terms of the XXZ chain this corresponds to the antiferromagnetic XXZ chain ($\Gamma = 1$) with asymmetry Δ given by (6.8) (remember $J_x : J_y : J_z = 1 : \Gamma : \Delta$).

7. Applications and Generalizations

a) A large class of elliptic quantum algebras has already been brought into play in the context of the Z_n-Baxter models and the simplest Macdonald polynomials. The question then naturally arises as to a full classification of elliptic quantum symmetric spaces, in correspondence with admissible pairs of root systems.

There is one more aspect to this. The parameters q, t of Macdonald translate on the "Sklyanin side" into an elliptic curve and a point on it. Could one make the connection with elliptic curves explicit directly on the "Macdonald side?"

b) For generic t and q, the orthogonal RAI polynomials obey of course, a three term recursion relation (Eqs. (3.6)). In the p-adic regime $(q = 0, t = 1/p)$ this recursion relation becomes precisely the condition that the zsf's be eigenfunctions of the Laplacian. On the Bruhat–Tits tree, corresponding to this case, the Laplacian has a simple interpretation as a difference operator obeying the mean value theorem. It is then natural to expect that for generic t and q, the recursion relation (3.6) also corresponds to the requirement that the RAI polynomials be eigenfunctions of the Laplacian on some "non-arboreal" *discrete* space, which reduces to a tree in the p-adic regime. It would be nice to find a simple geometric description of this generic discrete space which in the p-adic case becomes a tree, whereas for $t = q^{1/2} \to 1$ becomes a (continuous) sphere of (real) dimension 2. In short then it would be interesting to have a direct geometric picture of the quantum symmetric space, not only of its zonal spherical functions.

c) The interpolation between real and p-adic symmetric spaces by varying the parameters q and t, makes one recall another real-p-adic connection, at the adelic level [26], via Euler products [39]. In fact, for $q = 0$, $t = 1/p$ the c-function is a ratio of local (p-adic) zeta functions,

$$\zeta_p(s) = \frac{1}{1 - p^{-s}}, \tag{7.1}$$

whereas for $t = q^{1/2} \to 1$, the c-function is a ratio of (real) local zeta-functions

$$\zeta_\infty(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right). \tag{7.2}$$

Taking the Euler product yields the adelic zeta-function

$$\Lambda(s) = \zeta_\infty(s)\prod_p \zeta_p(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s), \tag{7.3}$$

which involves the Riemann zeta function $\zeta(s)$ and obeys the simple functional equation $\Lambda(s) = \Lambda(1-s)$. Can this construction be q-deformed? Is there such a thing as a "q-Euler product?" To answer these questions, notice that the Euler product runs one of the Macdonald variables (namely t) over all reciprocal prime values, while the other stays fixed. As the other Macdonald variable one can choose either q or $l = -\log p/\log q$, as both of them stay fixed at zero. It will turn out that for us the sensible choice is l. So we view an Euler product as a product over $t = 1/2, 1/3, 1/5, \ldots$ while l is fixed at zero. A *deformed* Euler product then should do the same but with l fixed at some value other than zero.

At this point we have to find the deformations of the local zeta functions $\zeta_p(s)$ and $\zeta_\infty(s)$. If we call $\zeta(s; t, l)$ the (two-parameter) "deformed local zeta function," then we must impose

$$\zeta\left(s; \frac{1}{p}, 0\right) = \zeta_p(s) \tag{7.4}$$

and

$$\zeta(s; 1, 1/2) = \zeta_\infty(s) .$$

It is easy to see that ($q = t^{1/l}$)

$$\zeta(s; t, l) = \pi^{-qls}\, \Gamma_{t^{1/l}}(ls) \tag{7.5}$$

fits the bill. We have

$$\begin{aligned}\zeta(s; t, l) &= \pi^{-qls} \frac{(t^{1/l}; t^{1/l})_\infty (1 - t^{1/l})^{1-ls}}{(t^s, t^{1/l})_\infty} \\ &= \pi^{-qls}(1 - t^{1/l})^{1-ls} \frac{\prod_{n=1}^{\infty} (1 - t^{n/l})}{\prod_{n=0}^{\infty} (1 - t^{s+n/l})}\end{aligned} \tag{7.6}$$

which for $t = 1/p < 1$, $l \to 0+$, $q \to 0$ does indeed become $\dfrac{1}{1 - p^{-s}} = \zeta_p(s)$. On the other hand, for $l = 1/2$, $t \to 1$ (so that also $q \to 1$),

$$\zeta(s; 1, 1/2) = \pi^{-s/2}\, \Gamma\left(\frac{s}{2}\right) = \zeta_\infty(s) . \tag{7.7}$$

To get a meaningful q-Euler product starting from (7.5), we have to unload the π^{-qls} factor. So the q-deformed Euler factor will be $\pi^{qls}\zeta(s; t, l)|_{l=\mathrm{fixed}, t=1/p}$, and our "$q$-Euler product" or more appropriately "l-Euler product" will be ($q = t^{1/l}$)

$$\begin{aligned}E(s; l) &= \prod_p \pi^{qls} \zeta\left(s; \frac{1}{p}; l\right) \\ &= \prod_p \frac{(1 - p^{-1/l})^{1-ls} \prod_{n=1}^{\infty} (1 - p^{-n/l})}{\prod_{n=0}^{\infty} (1 - p^{-s-n/l})} = \frac{\prod_{n=0}^{\infty} \zeta(s + nl^{-1})}{\zeta(l^{-1})^{1-ls} \prod_{n=1}^{\infty} \zeta(nl^{-1})} ,\end{aligned} \tag{7.8}$$

an interesting combination.

The special role in all this of the q-gamma function $\Gamma_q(ls)$ as "interpolator" between the local zeta functions at the finite and infinite places, can be better understood by tracing it back to a remarkable property of the q-exponential and to a remarkable property of Jackson's q-integral. The point is that the q-gamma function admits a q-integral representation [29], which upon a standard change of variables turns into a $q^l(= t)$-integral representation. This is essentially a t-Mellin transform of $e_q(-x^{1/l})$. Here the q-exponential $e_q(y)$ is defined as [28, 29]

$$e_q(y) = \sum_{a=0}^{\infty} \frac{y^a}{[a]_q!} , \quad [a]_q = \frac{1 - q^a}{1 - q} , \quad [a]_q! = [a]_q [a-1]_q \ldots [1]_q . \tag{7.9}$$

Not surprisingly for $t \to 1, l \to 1/2$ (the Archimedean regime) this t-Mellin transform reduces to the ordinary real Mellin transform and $e_q(-x^{1/l})$ becomes the real Gaussian (a well known representation of the gamma function of half-argument). The p-adic regime, surprisingly allows a similar interpretation. A t-integral is a sum [29]

$$\int_0^\infty f(x)\, d_t x = \sum_{n=-\infty}^{+\infty} f(t^n)\left[(1-t)t^n\right]. \tag{7.10}$$

In the p-adic regime $q \to 0, t \to 1/p$, the factor in square brackets coincides with the volume of the "shell" $I_n = \{\xi \in \mathbf{Q}_p, |\xi|_p = p^{-n}\}$ of p-adic integration, so that the sum over n, itself amounts to an integration over $\mathbf{Q}_p$ of the complex function $f(|\xi|_p)$ of the p-adic variable ξ (f depends on ξ only through its norm $|\xi|_p$). Under the sum (7.10), $e_q(-x^{1/l})$ becomes $e_q(-t^{n/l}) = e_q(-q^n)$ (since $t = q^l$). As $q \to 0$, the q-analogues of all nonnegative integers go to one, so that $e_q(-q^n)$ behaves like the geometric sum $\sum_{a=0}^{\infty}(-q^n)^a = (1+q^n)^{-1}$ and thus equals one for $n > 0$ and zero for n negative. When the case $n = 0$ is included, this shows that the function $e_q(-x^{1/l})$, under t-integration $d_t x$, translates into the characteristic function $\chi_p(\xi)$ of the p-adic integers

$$\chi_p(\xi) = \begin{cases} 1 & \text{for} \quad |\xi|_p \geqq 1 \\ 0 & \text{for} \quad |\xi|_p > 1 \end{cases} \tag{7.11}$$

under p-adic integration. But this $\chi_p(\xi)$ is the "p-adic Gaussian," that is the Fourier self similar complex-valued function of the p-adic variable ξ.

We thus come to realize that t-integration "interpolates" between real Riemann integration and p-adic integration, while at the same time the function $e_q(-x^{1/l})$ plays the role of a "quantum Gaussian" which interpolates between the ordinary real Gaussian and the step-functions $\chi_p(\xi)$, the p-adic Gaussians. All this clearly begs for a q- and/or l-deformation of Tate's Fourier analysis on local fields.

d) Does any of this work bear on string theory? Yes, we can construct *two-parameter* deformations of string theory which for $t = 1/p, q = 0$ reproduce the known p-adic strings, for $t = q^{1/2} \to 1$, the ordinary Veneziano string, and for $t = q^{1/2} \neq 1$ involve q-strings. We shall return to this elsewhere.

Acknowledgements. We wish to thank S. Bloch, L. Chekhov, L. Mezincescu and M. Olshanetsky for valuable discussions.

References

1. Macdonald, I.G.: In: Orthogonal Polynomials: Theory and Practice. Nevai, P. (ed.). Dordrecht: Kluwer Academic 1990
2. Macdonald, I.G.: Queen Mary College preprint 1989
3. Freund, P.G.O.: In: Superstrings and Particle Theory. Clavelli, L., Harms, B. (eds.). Singapore: World Scientific 1990
4. Freund, P.G.O.: Phys. Lett. **257B**, 119 (1991)
5. Freund, P.G.O.: In: Quarks, Symmetries and Strings, A Symposium in Honor of Bunji Sakita's 60th Birthday. Kaku, M., Jevicki, A., Kikkawa, K. (eds.). Singapore: World Scientific 1991
6. Brekke, L., Freund, P.G.O., Olson, M., Witten, E.: Nucl. Phys. **B302**, 365 (1988)
7. Zabrodin, A.V.: Commun. Math. Phys. **123**, 463 (1989)

8. Coon, D.D.: Phys. Lett. **29B**, 1422 (1969)
9. Romans, L.J.: preprint USC-88/HEP014 (1988)
10. Olshanetsky, M.A., Perelomov, A.M.: Phys. Rep. **94**, 313 (1983)
11. Wehrhahn, R.F.: Phys. Rev. Lett. **65**, 1294 (1990)
12. Zabrodin, A.V.: Mod. Phys. Lett. **A7**, 441 (1992)
13. Ueno, K.: Proc. Jpn. Acad. **66A**, 42 (1990)
14. Vaksman, L.L., Korogodsky, L.I.: Moscow preprint 1990
15. Koornwider, T.: In: Orthogonal Polynomials Theory and Practice. Nevai, P. (ed.). Dordrecht: Kluwer Academic 1990
16. Baxter, R.J.: Exactly Solved Models in Statistical Mechanics. New York: Academic Press 1982
17. Gaudin, M.: La Fonction D'Onde de Bethe. Paris: Masson 1983
18. Belavin, A.A.: Nucl. Phys. **B180**, 189 (1981)
19. Richey, M.P., Tracy, C.A.: J. Stat. Phys. **42**, 311 (1986)
20. Sklyanin, E.K.: Funk. Anal. Appl. **16**, 263 (1982)
21. Sklyanin, E.K.: Funk. Anal. Appl. **17**, 273 (1983)
22. Cherednik, I.V.: Yad. Fiz. **36**, 549 (1982)
23. Cherednik, I.V.: Funk. Anal. Appl. **19**, 77 (1985)
24. Odeskii, A.V., Feigin, B.L.: Funk. Anal. Appl. **23**, 207 (1989)
25. Helgason, S.: Topics in Harmonic Analysis on Homogeneous Spaces. Basel: Birkhäuser 1981
26. Weil, A.: Adeles and Algebraic Groups. Basel: Birkhäuser 1982
27. Humphreys, J.E.: Introduction to Lie Algebras and Representation Theory. Berlin, Heidelberg, New York: Springer 1970
28. Gasper, G., Rahman, M.: Basic Hypergeometric Series. Cambridge: Cambridge University Press 1990
29. Exton, H.: q-Hypergeometric Functions and Applications. N.Y.: John Wiley 1983
30. Rogers, L.J.: Proc. Lond. Math. Soc. **26**, 318 (1895)
31. Askey, R., Ismail, R.E.H.: In: Studies in Pure Math. Erdös, P. (ed.) Basel: Birkhäuser 1983
32. Mautner, F.: Am. J. Math. **80**, 441 (1958)
33. Cartier, P.: Proc. Symp. Pure Math. Vol. **26**. Providence: A.M.S. 1973
34. Bruhat, F., Tits, J.: Publ. Math. I.H.E.S. **41**, 5 (1972)
35. Rahman, M., Verma, A.: SIAM J. Math. Anal. **17**, 1461 (1986)
36. Kulish, P.P., Reshetikhin, N.Yu.: Soviet Phys. JETP **53**, 108 (1981)
37. Zamolodchikov, A.B.: Commun. Math. Phys. **69**, 165 (1979)
38. Zamolodchikov, A.B., Zamolodchikov, Al.B.: Ann. Phys. (N.Y.) **120**, 253 (1979)
39. Langlands, R.P.: Euler Products. New Haven: Yale University Press 1971
40. Tate, J.: Thesis, Princeton 1950, reprinted In: Algebraic Number Theory. Cassels, J.W.S., Fröhlich, A. (eds.). N.Y.: Academic Press 1967
41. Gel'fand, I.M., Graev, M.I., Pyatetskii-Shapiro, I.I.: Representation Theory and Automorphic Functions. London: Saunders 1966

Communicated by N.Yu. Reshetikhin