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Non-Smoothness of Event Horizons of Robinson–Trautman Black Holes

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Abstract. It is shown that generic "small data" Robinson–Trautman space-times cannot be C^{123} extended beyond the "$r = 2m$ Schwarzschild-like" event horizon. This implies that an observer living in such a space-time can determine by local measurements whether or not he has crossed the event-horizon of the black-hole.

1. Introduction

Perhaps the two most striking predictions of Einstein's theory of gravitation are the existence of gravitational radiation and of black holes. There are known four classes of asymptotically flat space-times containing gravitational radiation, the global structure of which is reasonably well understood: the Christodoulou–Klainerman metrics [7], the Friedrichs metrics [13], the boost-rotation symmetric metrics [2] and[1] the Robinson–Trautman (RT) metrics [17]. On the other hand known examples of space-times which contain a black hole are given by the Kerr–Newman space-times, the static Einstein–Maxwell Majumdar–Papapetrou multi-black hole solutions, the Tolman–Bondi perfect fluid metrics, Christodoulou's collapsing scalar field black-holes [6] (for these last two classes of space-times the metric in the vacuum region is the Schwarzschild metric) and the RT space-times. The privileged role of the Robinson–Trautman space-times stems from the fact that they provide an arena in which both gravitational radiation and black-hole formation can be studied simultaneously, in the vacuum. These space-times were orignally

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[1] It seems that Christodoulou's scalar field space-times [5, 6] should not be considered as containing gravitational radiation, since by Birkhoff's theorem the metric is the Schwarzschild one wherever the scalar field ϕ vanishes. Moreover, the $1/r$ part of the Riemann tensor, usually thought of as the manifestation of gravitational radiation, vanishes for these metrics (D. Christodoulou, private communication)

discovered in a search for metrics containing gravitational radiation [17], and it is only recently that it has been recognized that the RT metrics can be used as building blocks for constructing black-hole space-times: in Ref. [21] it was shown that any two RT space-times can be "glued" together along a Schwarzschild-type "$r = 2m$" event horizon to form a space-time which contains both a black and a white hole[2], with global structure somewhat similar to that of the Kruskal–Szekeres extensions of the $r > 2m$ Schwarzchild space-time. Although not explicitly stated in [21], the space-times so constructed were generically expected to have a metric of C^5 but *not* C^6 differentiability class (cf. also [9, 19, 20]). It was shown in [9] that a careful choice of the space-times which were being glued together led to a space-time the metric of which was of at least C^{117} differentiability class, and the methods of proof of that paper suggested very strongly that for generic RT space-times no extensions beyond the "$r = 2m$ event horizon" with a metric of C^{118} differentiability class will exist. In this paper we show that generic RT space-times evolving from "sufficiently small" initial data admit no C^{123} extensions, vacuum or otherwise, across the "$r = 2m$" null boundary. We believe that generic RT space-times do not admit extensions with a metric of C^{118} differentiability class; thus the small-data restriction is probably not necessary, while the discrepancy between C^{123} and C^{118} is an artefact due to the inextendability criterion used.

It may be argued that a singularity which shows up in the 118'th (or 123'rd) derivatives of the metric has no physical meaning, and that anything which is C^k with $k \geqq 2$ may be considered as being smooth, as far as physical applications are concerned. We believe that this is not the case. For instance, an observer in a space-time with a smooth event-horizon has no way of detecting by local measurements whether or not he has crossed the event horizon, while an observer in a Robinson–Trautman space-time with a singular horizon can in principle keep track of the 120'th derivatives of the scalar $\nabla^\nu R^{\alpha\beta\gamma\delta}\nabla_\nu R_{\alpha\beta\gamma\delta}$ and verify, by observing their blow up, that he has entered the region from which he can no longer communicate with the outside world. This unexpected property of generic Robinson–Trautman black-holes should probably be considered as a manifestation of the naked singularity $r = 0$, since the metric in space-times evolving from smooth data on a spacelike Cauchy surface, in which a stable version of cosmic censorship holds, is necessarily smooth in a neighbourhood of the event horizon.

This paper is organized as follows: in Sect. 2 we briefly review what is known about solutions of the RT equation, and give the precise statement of our main results, Theorems 2.1 and 2.2. A "final state" characterization of those RT space-times for which the event horizon $\mathscr{H}$ is singular is presented in Sect. 2.1 when $m > 0$, ${}^2\mathscr{M} = S^2$, and in Sect. 2.2 for $m < 0$, ${}^2\mathscr{M} \neq S^2$, T^2, where S^2 is the two-dimensional sphere and T^2 is the two-dimensional torus. In Sect. 3.1 results on the linearized RT equation needed for the proofs of Theorems 2.1 and 2.2 are established; and the proofs of Theorems 2.1 and 2.2 are given in Sect. 3.2.

[2] In the "maximally extended" RT space-times, as considered in Sect. 2, the event horizon can be defined as usual as the boundary of the past of $\mathscr{I}^+$: this justifies the statement of the existence of a black hole. On the other hand the notion of the "white hole" in these space-time is only an intuitive one (cf. also [21] for a discussion), since generic RT space-times cannot be extended up to $\mathscr{I}^+$ (in the RT class of vacuum metrics; cf. [8] [Proposition 2.1])

2. A "Final State" Characterization of Robinson–Trautman Space-Times with a Singular "$r=2m$" Horizon

Let $\mathring{g}_{ab}$ be a smooth metric on a two dimensional, compact, connected, orientable manifold ${}^2\mathcal{M}$, let $f(u)$ be a u-dependent family of positive functions on ${}^2\mathcal{M}$. It has been shown by Robinson and Trautman [17] that if the u-dependent family of metrics

$$g_{ab}=f(u)^{-2}\mathring{g}_{ab} \tag{2.1}$$

satisfies the evolution equation

$$\frac{\partial g_{ab}}{\partial u}=\frac{1}{12m}\Delta_g R g_{ab}, \tag{2.2}$$

where m is a constant, $R(g)=R^{ab}{}_{ab}$ is the curvature scalar of the metric g_{ab} and Δ_g (Δ_o) denotes the Laplacian of the metric $g(\mathring{g})$, then the four-dimensional Lorentzian metric

$$ds^2=-\Phi du^2-2dudr+r^2f^{-2}\mathring{g}_{ab}dx^a dx^b, \tag{2.3}$$

$$\Phi=\frac{R}{2}+\frac{r}{12m}\Delta_g R-\frac{2m}{r},\quad R\equiv R(g),$$

will satisfy the vacuum Einstein equations. Equation (2.2) is a quasi-linear parabolic equation for f,

$$\frac{\partial f}{\partial u}=-\frac{f}{24m}\Delta_g R, \tag{2.4}$$

$$R=R(g)=f^2(R_o+2\Delta_o \ln f), \tag{2.5}$$

$$\Delta_g=f^2\Delta_o,$$

where R_o is the curvature scalar of the metric $\mathring{g}_{ab}$. Solutions of (2.4) can be found by prescribing $f(u_o)\in H_{4+k}({}^2\mathcal{M})$, $k\geqq 0$ and integrating forward in u if $m>0$ or backward in u if $m<0$ ($H_l({}^2\mathcal{M})$ is the Hillbert space of functions the derivatives of which up to order l are square integrable on ${}^2\mathcal{M}$). Local existence of solutions of this problem was first pointed out in the physical literature by Schmidt [18]; existence for all $u\geqq u_o$ with "small initial data" has been shown by Rendall [16] when ${}^2\mathcal{M}\neq S^2$ (S^2 denotes the two dimensional sphere), and in [20] when ${}^2\mathcal{M}=S^2$; existence for all $u\geqq u_o$ without restrictions on the size of the data has been shown in [8]. In that last reference it has also been shown that every solution of (2.4) immediately becomes smooth (in fact, even analytic). In [9] it has been shown that there exists a sequence $N(i)$, with $N(0)=N(1)=0$, and a strictly increasing sequence $\{v_i\}$, $v_o=0$, depending only upon the metric $\mathring{g}_{ab}$, such that every solution of (2.4) has an expansion of the form

$$\forall n\in\mathbb{N}\quad f=\sum_{i=0}^{n}\sum_{j=0}^{N} f_{i,j}u^j e^{-v_i u}+r_n \tag{2.6}$$

with some (u-independent) functions $f_{i,j}\in C^\infty({}^2\mathcal{M})$, and

$$\forall i,k\in\mathbb{N},\quad u\geqq u_o+1,\quad \left|\mathring{\nabla}^k\frac{\partial^i}{(\partial u)^i}r_n\right|\leqq C_{n,k,i}u^{N(n+1)}e^{-v_{n+1}u}, \tag{2.7}$$

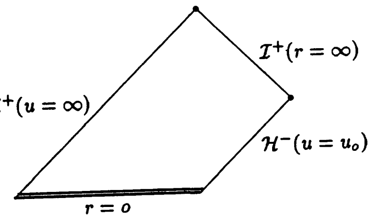

Fig. 2.1. $m>0$

where $\mathring{\nabla}$ is the covariant Riemannian derivative of the metric $\mathring{g}_{ab}$, for some constants $C_{n,k,i}$ depending upon the solution f.

2.1. $^2\mathcal{M}=S^2$. Conformally rescaling the metric $\mathring{g}_{ab}$ and redefining $f(u_o)$ if necessary we can without loss of generality assume that $\mathring{g}_{ab}$ is the standard "round metric" on the sphere, $R(\mathring{g}_{ab})=2$. In that case $f_{0,0}$ in the expansion (2.6) can be set to 1 by an appropriate conformal transformation, and we also have

$$\begin{aligned} &\nu_i=\nu\times i, && \nu=2/m,\\ &0\leqq i\leqq 14, && N(i)=0,\\ &i=15, && N(i)=1, \end{aligned}$$

(cf. [9] [Proposition 4.2]). The expansion (2.6) in the S^2 case thus takes the form

$$f=1+f_1e^{-\nu u}+f_2e^{-2\nu u}+\dots+f_{14}e^{-14\nu u}+f_{\log}ue^{-15\nu u}+f_{15}e^{-15\nu u}+\dots. \quad (2.8)$$

Given a solution of (2.4) defined on $[u_o,\infty)$ the corresponding space-time $^4\mathcal{M}$ with $m>0$ has the global structure displayed in Fig. 2.1, and it has been showed by Tod [21] that the space-time $^4\mathcal{M}$ can be extended across $\mathscr{H}^+$ in a way similar to the Kruskal–Szekeres extension of the $r>2m$ Schwarzschild space-time (cf. also [9]) (a black-hole – white-hole space-time with a metric of C^{117} differentiability can be obtained by glueing to $^4\mathcal{M}$ a time-reversed, space-inverted copy of itself along $\mathscr{H}^+$, as one does in the Kruskal-Szekeres-Schwarzschild manifold). The main result of our paper is the following:

Theorem 2.1. *There exists an open nonempty subset X of $C^\infty(^2\mathcal{M})$ such that if $f(u_o)\in X$ then the corresponding* RT *space-time cannot be extended across $\mathscr{H}^+$ in the class of manifolds with C^{123} Lorentzian, vacuum or otherwise, metrics. Moreover there exists $\varepsilon_o>0$ such that the set $B_{\varepsilon_o}\cap X$ is dense in $B_{\varepsilon_o}\cap C^\infty(^2\mathcal{M})$ equipped with a $C^\infty(^2\mathcal{M})$ topology, where $B_\varepsilon=\{f(u_o)\in H_6(^2\mathcal{M}):\|\ln f(u_o)\|_{H_6(^2\mathcal{M})}\leqq\varepsilon\}$.*

We have stated Theorem 2.1 in a C^∞ setting to emphasize the fact that the non-differentiability of the extensions across $\mathscr{H}^+$ has nothing to do with the potentially low differentiability of the initial data $f(u_o)$. In fact we also have the following stronger statement:

Theorem 2.2. *Let $k\in\mathbb{N}$, $k\geqq 4$. There exists an open nonempty subset X_k of $H_k(^2\mathcal{M})$ such that if $f(u_o)\in X_k$ then the corresponding* RT *space-time cannot be extended across $\mathscr{H}^+$ in the class of manifolds with C^{123} Lorentzian, vacuum or otherwise, metrics. Moreover for $k\geqq 6$ there exists $\varepsilon_o>0$ such that $B_{\varepsilon_o}\cap X_k$ is dense in*

$B_{\varepsilon_o} \cap H_k({}^2\mathcal{M})$ *(equipped with a* $H_k({}^2\mathcal{M})$ *topology), where* B_ε *is defined in the statement of Theorem 2.1.*

To prove Theorems 2.1 and 2.2 we shall need several auxiliary results, some of which are of independent interest; the proofs of Theorems 2.1 and 2.2 are deferred to Sect. 3. Let us start with the following statement:

Lemma 2.1. *Consider the expansion (2.8):*
1. f_1 *is a linear combination with constant coefficients of* $l=2$ *spherical harmonics* ϕ_α^+,

$$f_1 = \sum_\alpha B_\alpha^+ \phi_\alpha^+, B_\alpha^+ \in \mathbb{R}^5,$$

$$\Delta_o \phi_\alpha^+ = -6\phi_\alpha^+.$$

2. *Let* P *denote the antipodal map of the sphere into itself. For* $i=1,\dots,4$ *we have*

$$f_i \circ P = f_i. \tag{2.9}$$

3. *There exists a homogeneous polynomial* $\psi(B_\alpha^+)$ *of degree 5 in* B_α^+ *with coefficients being smooth antipodally symmetric functions on* S^2 *such that* $f_5 - \psi$ *is a linear combination with constant coefficients of* $l=3$ *spherical harmonics* ϕ_α^-,

$$f_5 = \psi(B_\alpha^+) + \sum_\alpha B_\alpha^- \phi_\alpha^-, \quad B_\alpha^- \in \mathbb{R}^7,$$

$$\Delta_o \phi_\alpha^- = -12\phi_\alpha^-.$$

If $B_\alpha^- = 0$ *then (2.9) holds for* $1 \leqq i \leqq 34$.

4. *Set* $(C_\beta) = (B_\alpha^+, (B_\alpha^-)^{1/5}) \in \mathbb{R}^{12}$. *There exist homogeneous polynomials* $E_\gamma(C_\beta)$ *of degree 15 with constant coefficients such that* $f_{\log}$ *is a linear combination of* $l=4$ *spherical harmonics* χ_γ *with coefficients* E_γ:

$$f_{\log} = \sum_\gamma E_\gamma(C_\beta)\chi_\gamma, \tag{2.10}$$

$$\Delta_o \chi_\gamma = -20\chi_\gamma.$$

Proof. Equation (2.4) can be rewritten in the form

$$\frac{\partial f}{\partial u} = -\frac{f^4}{12m}(\Delta_o^2 f + 2\Delta_o f) + \frac{f^3}{12m}[2\mathring{\nabla}^{ab} f \mathring{\nabla}_{ab} f - (\Delta_o f)^2]. \tag{2.11}$$

Inserting the expansion (2.8) in (2.11) one finds the following hierarchy of equations:

$$1 \leqq i \leqq 14, \quad L_i f_i \equiv [L + iv] f_i = g_i(f_1, \dots, f_{i-1}), \tag{2.12}$$

with

$$L = -\frac{1}{12m}(\Delta_o^2 + 2\Delta_o).$$

The g_i's are obtained by grouping together terms containing the exponent e^{-ivu} in the right-hand side of the equation

$$\sum_{i=0}^{\infty} g_i e^{-ivu} = \frac{f^4 - 1}{12m}(\Delta_o^2 f + 2\Delta_o f) - \frac{f^3}{12m}[2\mathring{\nabla}^{ab} f \mathring{\nabla}_{ab} f - (\Delta_o f)^2], \tag{2.13}$$

so that one has $g_o = g_1 = 0$. A simple REDUCE code gives

$$g_2 = \frac{1}{m}(8f_1^2 - Q(f_1, f_1)),$$

$$g_3 = -\frac{1}{m}(20f_1^3 - 24f_1f_2 + 2Q(f_1, f_2) - f_1Q(f_1, f_1)),$$

$$g_4 = \frac{1}{m}(40f_1^4 - 80f_1^2f_2 + 32f_1f_3 + 16f_2^2 - 2Q(f_1, f_3) - Q(f_2, f_2) + 2f_1Q(f_1,f_2) + f_2Q(f_1, f_1) - f_1^2Q(f_1, f_1)),$$

$$g_5 = -\frac{1}{m}(70f_1^5 - 200f_1^3f_2 + 100f_1^2f_3 - 40f_1f_4 + 100f_1f_2^2 - 40f_2f_3 + 2Q(f_1, f_4) + 2Q(f_2, f_3) - 2f_1Q(f_1, f_3) - f_3Q(f_1, f_1) - f_1Q(f_2, f_2) - 2f_2Q(f_1, f_2) + 2f_1^2Q(f_1, f_2) + 2f_1f_2Q(f_1,f_1) - f_1^3Q(f_1,f_1)),$$

etc., where

$$Q(f_i,f_j) \equiv \frac{1}{12}(2\mathring{\nabla}^{ab}f_i\mathring{\nabla}_{ab}f_i - \Delta_of_i\Delta_of_j),$$

and we have used the equations satisfied by the f_i's to somewhat simplify the expressions for the g_i's. The eigenfunctions of L are the spherical harmonics and its spectrum is given by $\left\{-\frac{(l-1)l(l+1)(l+2)}{12m}\right\}_{l\in\mathbb{N}} = \left\{-\frac{(l-1)l(l+1)(l+2)}{24}\times v\right\}_{l\in\mathbb{N}}$; it follows that

1. f_1 is in the kernel of L_1 and is thus a linear combination of $l = 2$ spherical harmonics.
2. For $2 \leqq i \leqq 14$, $i \neq 5$, the operators L_i have trivial kernels and thus Eq. (2.12) can be solved uniquely for f_i in terms of g_i.
3. Since L commutes with P, where P is the antipodal map of the sphere into itself, one can show by induction (cf. e.g. [9] [Proposition 4.2]) that for $1 \leqq i \leqq 4$ one has $g_i \circ P = g_i$, $f_i \circ P = f_i$ and also $g_5 \circ P = g_5$.
4. L_5 has a non-trivial kernel consisting of $l = 3$ spherical harmonics; this implies that Eq. (2.12) with $i = 5$ has the integrability conditions

$$\int_{S^2} g_5\psi_\sigma d\mu_o = 0,$$

where the ψ_σ are $l = 3$ spherical harmonics and $d\mu_o$ is the standard $SO(3)$ invariant measure on S^2. This is automatically satisfied because g_5 has even parity ($g_5 \circ P = g_5$) while $\psi_\sigma \circ P = -\psi_\sigma$. We can therefore solve for f_5 which is then defined up to the addition of $l = 3$ spherical harmonics.
5. For $i = 15$ one finds

$$L_{15}f_{\log} = 0, \tag{2.14}$$

$$L_{15}f_{15} \equiv [L + 15v]f_{15} = g_{15}(f_1, \cdots, f_{14}) + f_{\log}. \tag{2.15}$$

The kernel of L_{15} consists of $l = 4$ spherical harmonics χ_γ, so that $f_{\log}$ must be a linear combination of those. The integrability conditions of (2.15) read

$$\int_{S^2} (g_{15} + f_{\log}) \chi_\gamma d\mu_o = 0, \tag{2.16}$$

which determines $f_{\log}$ uniquely in terms of f_1 and f_5.
6. The functions g_i are polynomials in f_j and their derivatives by construction, which by uniqueness arguments implies in turn that the f_i's must be polynomials in B^+_α and B^-_α. The homogeneity of order i of g_i and of f_i follows easily by construction, but can also be seen by noting that if $f(u)$ solves (2.4) on $[u_o, \infty)$, then so does $f_\delta \equiv f(u+\delta)_{|[u_o,\infty)}$ for any $\delta \geqq 0$. Equation (2.8) shows that f_δ has the expansion

$$f_\delta = 1 + f_1 e^{-\nu\delta} e^{-\nu u} + f_2 e^{-2\nu\delta} e^{-2\nu u} + \cdots$$
$$+ f_{14} e^{-14\nu\delta} e^{-14\nu u} + f_{\log} e^{-15\nu\delta} u e^{-15\nu u} + (f_{15} + \delta f_{\log}) e^{-15\nu\delta} e^{-15\nu u} + \cdots,$$

and uniqueness arguments yield the result. □

Proposition 2.1. *Let P be the antipodal map of the sphere into itself and let R_φ be the rotation of the sphere around the z-axis by an angle φ, suppose that*

$$f(u_o) \circ P = f(u_o) \circ R_\varphi = f(u_o) \tag{2.17}$$

for $0 \leqq \varphi \leqq 2\pi$. We have

$$f_{\log} \neq 0,$$

unless the function f_1 in the expansion (2.8) vanishes.

Proof. Since symmetries of the initial data which are also symmetries of $\mathring{g}_{ab}$ are preserved by evolution via (2.4), it is not too difficult to show that all the expansion coefficients in (2.8) satisfy

$$f_i \circ P = f_i \circ R_\varphi = f_i.$$

It follows from Lemma 2.1 that

$$f_1 = \frac{2a}{3} \times P_2(\cos\theta), \tag{2.18}$$

where a is a real constant and P_2 is a Legendre polynomial (we use the normalization of Legendre polynomials of Ref. [1]; we have found it convenient to introduce the factor 2/3 in (2.18) to keep down the numerical value of some of the coefficients appearing in the functions f_i for large i), and that $B^-_\alpha = 0$, thus $f_{\log}$ is uniquely determined by a. We have written a REDUCE code which effectively implements the procedure described in Lemma 2.1 assuming invariance of $f(u_o)$ under rotations around the z axis. The analysis is considerably simplified by noting that the coefficients f_i must be linear combinations of spherical harmonics of order less than or equal to $2i$: this reduces the task of solving the equatiions $L_i f_i = g_i$ to algebraic operations in finite dimensional spaces. The change of variables $x = \cos\theta$ further simplifies the problem to manipulations with polynomials in x of order $2i \leqq 30$. To illustrate the results one obtains, here follow the first five functions f_i:

$$f_1 = a \times \left(x^2 - \frac{1}{3}\right),$$

$$f_2 = -a^2 \times \left(\frac{23}{78}x^4 - \frac{47}{39}x^2 + \frac{49}{234}\right),$$

$$f_3 = a^3 \times \left(\frac{997}{5226}x^6 - \frac{36697}{47034}x^4 + \frac{25309}{15678}x^2 - \frac{8899}{47034}\right),$$

$$f_4 = -a^4 \times \left(\frac{4519475}{27990456}x^8 - \frac{1636874143}{2078291358}x^6 + \frac{73857848527}{45722409876}x^4 - \frac{17112915619}{7620401646}x^2 + \frac{18150013841}{91444819752}\right),$$

$$f_5 = a^5 \times \left(\frac{646556531}{4114597032}x^{10} - \frac{5057087713397}{5567049784296}x^8 + \frac{84695216485153}{38191860727020}x^6 - \frac{1554887482454485}{511770933742068}x^4 + \frac{1086788290750781}{341180622494712}x^2 - \frac{231262717823569}{1023541867484136}\right).$$

The length of the numerators and the denominators of the coefficients tends to grow rather rapidly with i, leading to rationals involving integers of more than 100 digits for $i \geqq 11$ (up to more than 210 digits for $i = 15$), however with the normalization of (2.18) the numerical values of the coefficients of the polynomials $a^{-i}f_i$, $1 \leqq i \leqq 15$, are all of order $10^{-1} - 10^4$. It takes about two and a half hours of CPU time on a VAX 8700 to obtain[3]

$$f_{\log} \approx 1.009201657002 \times 10^{-10} \times a^{15} \times P_4(\cos\theta) \approx (0.2155750672866a)^{15} \times P_4(\cos\theta). \quad (2.19)$$

This result has been obtained using integer arithmetic, so that the only error in the first equality in (2.19) is due to round-off when translating a rational into floating point notation: the exact value of $a^{-15}P_4(\cos\theta)^{-1}f_{\log}$ is a ratio of two integers of 109 and 118 digits which we can make available to anyone interested on request. In order to minimize the risk of programming errors we have built in several checks in the code to test the consistency of the results. Because we were quite perplexed by the numerical value of $a^{-15}P_4(\cos\theta)^{-1}f_{\log}$, which is at least 9 orders of magnitude smaller[4] than the typical coefficients of the polynomials $a^{-i}f_i$, $1 \leqq i \leqq 15$, we have written a MACSYMA code[5] which checked the REDUCE results by reading the output of the REDUCE calculation and verifying whether Eq. (2.4) was satisfied up to terms decaying faster than $e^{-15\nu u}$.

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[3] We had to make various optimizations to our code to be able to obtain (2.19) without exceeding the job limit of 4 hours of CPU time on the machine we were using. The same result (to the accuracy of (2.19)) can be obtained by running the code in E-30 floating point precision in about 15 minutes of CPU time

[4] Equation (2.19) clearly shows that a change of the normalization of a by a factor ≈ 5 would lead to a coefficient of P_4 in $f_{\log}$ of order 1 – this will however not change the relative size of typical coefficients in $f_{\log}$ as compared to typical coefficients in f_{15}.

[5] The MACSYMA code was a "brute force one," without any fancy time- and memory-saving tricks; the checking run took about one and a half hours of CPU time on Sequent Symmetry. Both our codes together with all the coefficients f_i up to $i = 15$ are available on request.

Equation (2.19) shows that $f_{\log}$ does not vanish unless $a=0$, which had to be established. □

Lemma 2.1 and Proposition 2.1 imply that for "generic final states" of RT space-times the function $f_{\log}$ does not vanish:

Proposition 2.2. *There exists an open dense subset $\Omega \subset \mathbb{R}^{12}$ such that if f is a solution of (2.4) for which $(B_\alpha^+, B_\alpha^-) \in \Omega$, (B_α^+, B_α^-) as in Lemma 2.1, then*

$$f_{\log} \not\equiv 0.$$

Proof. As shown in Lemma 2.1, $f_{\log}$ is determined uniquely by (B_α^+, B_α^-). Consider the set $C\Omega$ of (B_α^+, B_α^-) for which $f_{\log} \equiv 0$. Since $f_{\log}$ is a polynomial in (B_α^+, B_α^-), $C\Omega$ is closed and thus the set Ω of (B_α^+, B_α^-) for which $f_{\log} \not\equiv 0$ is open. Suppose that Ω is not dense, therefore there exists a $p \in C\Omega$ and an open neighbourhood $\mathcal{U}$ of p such that $\mathcal{U} \subset C\Omega$, therefore $f_{\log}|\mathcal{U} = 0$. But a polynomial vanishing on an open set is identically zero, which contradicts[6] Proposition 2.1, and proves our claim. □

Let us show that the non-vanishing of $f_{\log}$ implies a form of singular behaviour of $\mathscr{H}^+$ in the corresponding RT space-time:

Proposition 2.3. *Suppose that the function $f_{\log}$ of the expansion (2.8) does not vanish. There exists no extensions of the corresponding* RT *space-time ${}^4\mathscr{M}$ across $\mathscr{H}^+$, vacuum or otherwise, with a metric of C^{123} differentiability class.*

Remark. Let us mention that all scalar functions of the form

$$C[\nabla^{\mu_1} \cdots \nabla^{\mu_i} R^{\alpha_1\beta_1\gamma_1\delta_1} \cdots \nabla_{\nu_1} \cdots \nabla_{\nu_k} R_{\alpha_j\beta_j\gamma_j\delta_j}],$$

where $C[\cdots]$ denotes a (total) contraction operation over the indices, are uniformly bounded in a neighbourhood of $\mathscr{H}^+$ in ${}^4\mathscr{M}$, which follows immediately from the fact that in the coordinate system used in (2.3) $g_{\mu\nu}$, $g^{\mu\nu}$ and all partial derivatives thereof are uniformly bounded on $\mathcal{O}_\varepsilon = \{r \geqq \varepsilon, u \geqq u_o + \varepsilon\}$, for any $\varepsilon > 0$ ($u \geqq u_o$ if $f(u_o)$ is smooth). The proof below shows that at least one entry of the tensor $\nabla_{\nu_1} \cdots \nabla_{\nu_{121}} R_{\alpha\beta\gamma\delta}$ will blow up at $\mathscr{H}^+$, whatever coordinate system one chooses, even though every scalar function constructed out of this tensor by contractions with products of $g^{\mu\nu}$, $g_{\rho\sigma}$ and $\nabla_{\mu_1} \cdots \nabla_{\mu_j} R_{\alpha_k\beta_k\gamma_k\delta_k}$ will be bounded on $\mathcal{O}_\varepsilon$.

Proof. Suppose that there exists an extension ${}^4\tilde{\mathscr{M}}$ of ${}^4\mathscr{M}$ with a metric of C^{123} differentiability class. We then have $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} \in C^{121}({}^4\tilde{\mathscr{M}})$, and a SHEEP calculations gives

$$R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} = 48m^2r^{-6} \tag{2.20}$$

on ${}^4\mathscr{M}$, therefore we can extend r to a function $\tilde{r} \in C^{121}({}^4\tilde{\mathscr{M}})$ by setting

$$\tilde{r} = \{R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}/(48m^2)\}^{-1/6}\phi,$$

---

[6] To obtain further evidence that the polynomial $f_{\log}(B_\alpha^+, B_\alpha^-)$ does not vanish identically we have also analyzed, using our REDUCE code, the case of axially symmetric initial data *without* imposing the parity condition $f(u_o) \circ P = f(u_o)$. In such a case f_5 is determined by f_1 up to the addition of $b^5 \times P_3(\cos\theta)$, where b is a real constant, and using the same normalization for f_1 as in (2.18) the REDUCE code gives $f_{\log} \approx \{(0.2155750672866a)^{15} - 0.6581020070622 \times a^5 b^{10}\} \times P_4(\cos\theta)$; this result has also been obtained using integer arithmetics, so that the $\approx$ accounts only for round-off error of the translation of a rational number into floating point notation (and subsequently taking the fifteenth root in the first factor)

where $\phi \in C^\infty({}^4\tilde{\mathscr{M}})$ is equal to 1 in ${}^4\mathscr{M}$ and in a neighborhood of $\mathscr{H}^+$, and $\phi = 0$ in a neighborhood of the points for which $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$ vanishes. According to SHEEP, on ${}^4\mathscr{M}$ the scalar $\nabla^\mu R^{\alpha\beta\gamma\delta}\nabla_\mu R_{\alpha\beta\gamma\delta}$ takes the form

$$\nabla^\mu R^{\alpha\beta\gamma\delta}\nabla_\mu R_{\alpha\beta\gamma\delta} = \frac{720}{r^8}\left(-\frac{R}{2} + 2rf^{-1}\frac{\partial f}{\partial u} + \frac{2m}{r}\right). \tag{2.21}$$

Since by hypothesis r can be extended to ${}^4\tilde{\mathscr{M}}$ in a C^{121} way and $\nabla^\mu R^{\alpha\beta\gamma\delta}\nabla_\mu R_{\alpha\beta\gamma\delta} \in C^{120}({}^4\tilde{\mathscr{M}})$ it follows from (2.21) that the function

$$\psi = -\frac{R}{2} + 2rf^{-1}\frac{\partial f}{\partial u} \tag{2.22}$$

can be extended to ${}^4\tilde{\mathscr{M}}$ as a $C^{120}({}^4\tilde{\mathscr{M}})$ function. Inserting the expansion (2.8) in (2.22) one finds

$$\psi = 1 + \psi_1 e^{-\nu u} + \cdots + \psi_{\log} u e^{-15\nu u} + \cdots,$$

with

$$\psi_{\log} = -102 f_{\log}$$

at $r = 2m$, and the argument of the proof of Theorem 4.1 of [9] shows that there exists a geodesic Γ in ${}^4\tilde{\mathscr{M}}$ on which $\dfrac{d^{120}\psi}{ds^{120}}$ blows up as Γ crosses $\mathscr{H}^+$, which contradicts $\psi \in C^{120}({}^4\tilde{\mathscr{M}})$ and proves our claim. □

It follows from the results of this section that the potentially singular character of $\mathscr{H}^+$ is controlled by the "asymptotic data" $(B_\alpha^+, B_\alpha^-) \in \mathbb{R}^{12}$. Obviously these "asymptotic data" do not determine the whole space-time, though it is tempting to conjecture that the collection of all $f_{i,0}$'s determines every RT space-time uniquely; we shall however not attempt to analyze that problem.

2.2. Other Topologies. Throughout this section we shall assume that $m < 0$ and that the genus $g({}^2\mathscr{M})$ of ${}^2\mathscr{M}$ satisfies $g({}^2\mathscr{M}) \geqq 2$ (for the remaining cases, cf. e.g. [9]). By conformally rescaling the metric $\mathring{g}_{ab}$ and redefining $f(u_o)$ if necessary we can without loss of generality assume that $R(\mathring{g}_{ab}) = -2$. Given a solution of (2.4) we can define

$$A = \{a \in \mathbb{R} : \exists C \in \mathbb{R} \text{ such that } |(f-1)\hat{u}^{-a}| \leqq C\},$$

where

$$\hat{u} = \exp\left\{-\frac{u}{4m}\right\};$$

set

$$\hat{\nu} = \limsup A.$$

From the existence of the expansion (2.6) it follows that if $\hat{\nu} \neq \infty$ then $\hat{\nu} \in A$, and if we set $\hat{\nu}_i \equiv 4|m|\nu_i$, ν_i as in (2.6), we also have

$$\hat{\nu} \geqq \hat{\nu}_1 \equiv \frac{\mu_1(\mu_1 + 2)}{3}, \tag{2.23}$$

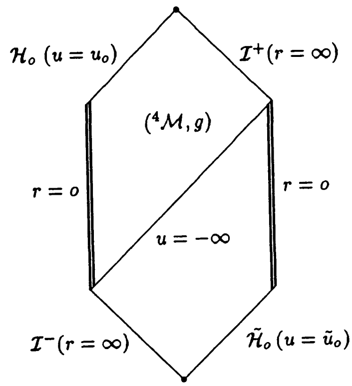

Fig. 2.2. Maximal vacuum RT extensions of $({}^4\mathcal{M}, g)$, $m < 0$, $g({}^2\mathcal{M}) > 1$

where μ_1 is the first non-trivial eigenvalue of $-\Delta_o$. For generic $f(u_o)$ one expects $\hat{v} = \hat{v}_1$: this is indeed the case if $f(u_o)$ is generic and $\ln f(u_o)$ is "small enough," as follows from Theorem 3.2. On the other hand Corollary 3.1, point 2, implies that there exist (non-generic) $f(u_o)$'s, not identically equal to 1, for which $\hat{v} \geqq \frac{\mu_2(\mu_2 + 2)}{3}$, where μ_2 is the second non-trivial eigenvalue of $-\Delta_o$ (an example in which $\hat{v} = \infty$ is given by $f = 1$, which solves (2.4) and leads to the so-called "DS-metrics" [12]). If

$$\hat{v} \geqq 2, \tag{2.24}$$

the corresponding space-time $({}^4\mathcal{M}, g)$ can be extended across the boundary $u = -\infty$ ($\hat{u} = 0$) to a space-time $({}^4\tilde{\mathcal{M}}, \tilde{g})$ with a continuous metric $\tilde{g}$ the degree of differentiability of which will depend upon $\hat{v}$ [9]. The global structure of $({}^4\tilde{\mathcal{M}}, \tilde{g})$ is displayed in Fig. 2.2. If $\hat{v} < 2$ the "hypersurface" $u = -\infty$ is expected to be singular, although we have not been able to prove such a claim (cf. also the Remark following Proposition 2.3).

As is well known, metrics $\mathring{g}_{ab}$ satisfying $R(\mathring{g}_{ab}) = -2$ may be used to parametrize the Teichmüller space $\mathcal{T}({}^2\mathcal{M})$ [11], which allows one to consider μ_1 as a function on $\mathcal{T}({}^2\mathcal{M})$. It is also known that μ_1 varies continuously over $\mathcal{T}({}^2\mathcal{M})$, is uniformly bounded from above and tends to zero as one approaches the boundaries of $\mathcal{T}({}^2\mathcal{M})$. For $g({}^2\mathcal{M}) = 2$ it has been shown[7] by Jenni [14] that there exists a metric on ${}^2\mathcal{M}$ for which

$$3.83 < \mu_1 < 3.85 \tag{2.25}$$

and also that we have the bound

$$\sup_{g({}^2\mathcal{M}) \geqq 2,\, \mathring{g}_{ab} \in \mathcal{T}({}^2\mathcal{M})} \mu_1 < 4.81 \tag{2.26}$$

[7] The following has been explained to us by C. Hodgson: it follows easily from the Gauss–Bonnet theorem that the diameter of a genus g manifold with a metric $\mathring{g}_{ab}$ for which $R(\mathring{g}_{ab}) = -2$ tends to infinity as $g \to \infty$, which together with e.g. Theorem 8 of Ref. [4] shows that $\mu_1 \leqq \hat{\mu}(g)$, with $\hat{\mu}(g) \searrow 1/4$ as $g \to \infty$. This implies that there exists g_o such that if $g({}^2\mathcal{M}) \geqq g_o$ then $\hat{v}_1 < 2$ for all metrics $\mathring{g}_{ab}$ on ${}^2\mathcal{M}$ such that $R(\mathring{g}_{ab}) = -2$; thus for $g({}^2\mathcal{M}) \geqq g_o$ the analysis that follows applies to non-generic $f(u_o)$ only.

(it is actually expected that the supremum over g is attained for $g = 2$, and that for $g(^2\mathcal{M}) = 2$ the supremum over $\mathcal{T}(^2\mathcal{M})$ is attained by the metric considered by Jenni). For Jenni's metric one obtains

$$5.44 < \hat{v} - 2 < 5.51$$

which shows that at least for $g(^2\mathcal{M}) = 2$ there will exist a large set of metrics for which $\hat{v}_1 > 2$. From the definition of $\hat{v}$ and from (2.6) if $\hat{v} \neq \infty$ we have

$$f = 1 + f_1 \hat{u}^{\hat{v}} + r_1, \quad f_1 \not\equiv 0, \tag{2.27}$$

with $r_1 = O(\hat{u}^{\hat{v}+\epsilon})$ for some $\varepsilon > 0$. From Eq. (3.3) of [9] it follows that the metric on $^4\mathcal{M}$ can be extended across the boundary $\hat{u} = 0$ in a $C^{\mathrm{Int}[\hat{v}-2]}$ way, where Int$[x]$ stands for the integer part of x, but for $\hat{v} \notin \mathbb{N}$ the metric will *not* be $C^{\mathrm{Int}[\hat{v}-1]}$ in the coordinate system used[8]. Thus for generic pairs $(\mathring{g}_{ab}, f_1)$, with $\mathring{g}$ such that $\mu_1(\mu_1 + 2) \geqq 6$, the space-time metric will have some of its derivatives blowing up in the coordinate system used in [9]; for example Jenni's metric on $^2\mathcal{M}$ will lead to C^5 but not C^6 extendible RT space-times in the coordinate system used in [9]. If (2.24) holds, an argument similar to the proof of Proposition 2.3 (note that both (2.20) and (2.21) hold irrrespective of the topology of $^2\mathcal{M}$) shows that no $C^{\mathrm{Int}[\hat{v}+4]}$ extensions of a RT space-time exist when $\hat{v} \notin \mathbb{N}$ and the function f_1 in the expansion (2.27) does not vanish (by Theorem 3.2 this will be the case if e.g. $\ln f(u_o)$ is small enough and generic) – no details will be given.

3. Generic "Final States" Versus Generic Initial Data

The results presented in the previous sections show that RT space-times with "generic asymptotic data" in the sense of Proposition 2.2 are inextendible across $\mathcal{H}^+$ in the class of manifolds with C^∞ metrics. This leads immediately to the question, do generic Cauchy data for Eq. (2.4) lead to generic asymptotic parameters (B^+_α, B^-_α)? We believe that this is indeed the case, a partial answer to this question will be given in Theorem 3.2, Sect. 3.2 below. Before addressing this problem we shall need some results concerning the linearization of the RT equation, which are derived in the next section:

3.1. The Linearized Problem. Throughout this and the next section, the letter C denotes a constant the value of which may vary from line to line; by $\mathring{g}_{ab}$ we will denote a metric of constant scalar curvature $R_o = 2$ for $^2\mathcal{M} = S^2$, $R_o = 0$ for $^2\mathcal{M} = T^2$ and $R_o = -2$ otherwise. In the arguments that follow we shall assume that the reader is familiar with the methods and the results of [8] and [9], and we shall skip the non-essential details which may be filled in using either the results or methods of [8] and [9]. Let us simply recall here that from what has been proved in [8] it follows that for $f(u_o) \in H_k(^2\mathcal{M})$, $k \geqq 4$, there exists a solution of the RT equation satisfying $f \in C([u_o, \infty), H_k(^2\mathcal{M})) \cap C^1([u_o, \infty), H_{k-4}(^2\mathcal{M})) \cap C^\infty((u_o, \infty) \times {^2\mathcal{M}})$.

[8] In the S^2 case a sufficient and necessary condition for a singular $\mathcal{H}^+$ is the occurrence of log terms in the expansion (2.6), because for $(S^2, \mathring{g}_{ab})$, with $\mathring{g}_{ab}$ – the standard round metric, the spectrum of $-\Delta_o$ consists of integers; this will certainly not be the case for a generic $(^2\mathcal{M}, \mathring{g}_{ab})$ with $R(\mathring{g}_{ab}) = -2$. Whenever $\hat{v} = \mu_1 \notin \mathbb{N}$ and f_1 in (2.27) does not vanish, the log terms become irrelevant

Let f_t be a one-parameter family of solutions of the modified RT equation (cf. [9, 20] for details), with $f_t|_{t=0} = f, \left.\frac{df_t}{dt}\right|_{t=0} = \varphi$:

$$\frac{\partial f_t}{\partial u} = -\frac{f_t^3}{24m}\Delta_o[f_t^2(R_o + 2\Delta_o \ln f_t)] - f_t \sum_i \alpha_i(f_t)\phi_i - X^a \mathring{\nabla}_a f_t, \tag{3.1}$$

where

$$\alpha_i(f) = c \oint \phi_i f \, d\mu_o \equiv \frac{c}{A} \int_{{}^2\mathcal{M}} \phi_i f \, d\mu_o, \tag{3.2}$$

$$A = \int_{{}^2\mathcal{M}} d\mu_o,$$

$$X^a = \sum_i \alpha_i(f) \mathring{\nabla}^a \phi_i,$$

with $c = 0$ unless ${}^2\mathcal{M} = S^2$; in this last case the ϕ_i's form an L^2-orthonormal basis (with respect to $\oint$) of the space of $l = 1$ spherical harmonics ($\Delta_o \phi_i = -2\phi_i$), and c is a constant which we shall choose to satisfy

$$c > \frac{\mu_4(\mu_4 - R_o)}{12m} \tag{3.3}$$

(if ${}^2\mathcal{M} = S^2$, then $\mu_l = l(l+1)$). We shall assume that the f_t's are normalized in such a way that

$$\oint f_t^{-2}(u_o) d\mu_o = 1.$$

It follows from (3.1) that φ satisfies the equation

$$\frac{\partial \varphi}{\partial u} = L\varphi + L_1 \varphi, \tag{3.4}$$

$$L\varphi = -\frac{1}{12m}[\Delta_o^2 + 2\Delta_o]\varphi - \sum_i \alpha_i(\varphi)\phi_i, \tag{3.5}$$

$$L_1 \varphi = -\frac{f^3}{12m}\Delta_o\left[f^2 \Delta_o\left(\frac{\varphi}{f}\right)\right] - \frac{f^3}{12m}\Delta_o\left(\frac{R\varphi}{f}\right) - \frac{\varphi f^2}{8m}\Delta_o R$$
$$- \sum_i \{\alpha_i(f)[\phi_i \varphi + \mathring{\nabla}^a \phi_i \varphi_{,a}] + \alpha_i(\varphi)[\phi_i f + \mathring{\nabla}^a \phi_i f_{,a}]\} - L\varphi. \tag{3.6}$$

If $f(u_o) \in H_k({}^2\mathcal{M})$, $k \geqq 4$, it is simple to show by straightforward energy estimates that for $\varphi(u_o) \in H_k({}^2\mathcal{M})$ there exists a solution of (3.4) satisfying $\varphi \in C([u_o, \infty), H_{k-2}({}^2\mathcal{M})) \cap C^\infty((u_o, \infty) \times {}^2\mathcal{M})$; using the methods of Appendix B of [8] one can then show that moreover $\varphi \in C([u_o, \infty), H_k({}^2\mathcal{M})) \cap C^1([u_o, \infty), H_{k-4}({}^2\mathcal{M}))$. Using the methods of [9] it can be shown that φ has an expansion as in (2.6).

Define

$$f^o \equiv P^o f \equiv \oint f \, d\mu_o \equiv \int_{{}^2\mathcal{M}} f \, d\mu_o \Big/ \int_{{}^2\mathcal{M}} d\mu_o. \tag{3.7}$$

1. If ${}^2\mathcal{M} = S^2$, let $P: S^2 \to S^2$ be the antipodal map, set

$$f^+ \equiv \tfrac{1}{2}(f \circ P + f); \quad f^- \equiv \tfrac{1}{2}(f \circ P - f),$$

let P^+, respectively P^-, denote the L^2-orthogonal projection operator onto the second, respectively the third, non-trivial eigenspace, $\mathscr{E}^+$, respectively $\mathscr{E}^-$, of $-\Delta_o$. If $\{\phi_\alpha^\pm\}$ are L^2-orthonormal bases of $\mathscr{E}^\pm$:

$$\Delta_o \phi_\alpha^\pm = -\mu^\pm \phi_\alpha^\pm;$$

set $(\phi_\alpha) = (\phi_\alpha^+, \phi_\alpha^-)$, and

$$\bar{P} = P^+ + P^-, \quad \tilde{P} = 1 - \bar{P} - P^o.$$

2. If ${}^2\mathscr{M} \neq S^2$, set $f^+ = f$, $f^- = 0$, let P^+ denote the L^2-orthogonal projection operator onto the first non-trivial eigenspace, $\mathscr{E}^+$, of $-\Delta_o$, let $\{\phi_\alpha^+\}$ be an L^2-orthonormal basis of $\mathscr{E}^+$:

$$\Delta_o \phi_\alpha^+ = -\mu^+ \phi_\alpha^+, \quad \mu^+ = \mu_1 > 0;$$

we also define

$$\bar{P}f = P^+ f, \quad P^- f = 0, \quad \tilde{P} = 1 - \bar{P} - P^o, \quad \mu^- = 0,$$

and $\{\phi_\alpha\} = \{\phi_\alpha^+\}$, $\{\phi_\alpha^-\} = \{0\}$.

Whatever the topology of ${}^2\mathscr{M}$, we thus have

$$P^\pm \varphi = \sum_\alpha A_\alpha^\pm \phi_\alpha^\pm, \quad A_\alpha^\pm = \oint \phi_\alpha^\pm \varphi \, d\mu_o, \tag{3.8}$$

$$\bar{P}\varphi = \sum_\alpha A_\alpha \phi_\alpha, \quad (A_\alpha) = (A_\alpha^+, A_\alpha^-), \tag{3.9}$$

and we define

$$(v =) v^+ \equiv \frac{1}{24m} \mu^+ (\mu^+ - R_o),$$

$$v^- \equiv \frac{1}{24m} \mu^- (\mu^- - R_o).$$

Since the modified RT equation is area-preserving,[9]

$$\oint f_t^{-2}(u) \, d\mu_o = 1, \tag{3.10}$$

it follows that for all $u \geq u_o$ we have

$$\oint f^{-3} \varphi \, d\mu_o = 0 \Rightarrow \varphi^o = -\frac{1}{\oint f^{-3} d\mu_o} \oint f^{-3} (\bar{\varphi} + \tilde{\varphi}) \, d\mu_o, \tag{3.11}$$

where $\bar{\varphi} = \bar{P}\varphi$ and $\tilde{\varphi} = \tilde{P}\varphi$. From (3.4) and (3.11) one obtains

$$\frac{dA_\alpha^\pm}{du} = -v^\pm A_\alpha^\pm + \sum_{\sigma = \pm, \beta} \Psi_{\alpha\beta}^{\pm\sigma} A_\beta^\sigma + \Xi_\alpha^\pm [\tilde{\varphi}], \tag{3.12}$$

$$\Psi_{\alpha\beta}^{\pm\pm} = \oint \phi_\alpha^\pm L_1 \phi_\beta^\pm \, d\mu_o - \frac{1}{\oint f^{-3} d\mu_o} \oint L_1^* \phi_\alpha^\pm \, d\mu_o \oint f^{-3} \phi_\beta^\pm \, d\mu_o, \tag{3.13}$$

---

[9] As pointed out in [20], this follows from the fact that the RT equation is area-preserving, and that the solutions of the modified RT equation differ from solutions of the RT equation only by a pull-back by a diffeomorphism

$$\Xi_\alpha^\pm = \oint \tilde{\varphi} L_1^* \phi_\alpha^\pm d\mu_o - \frac{1}{\oint f^{-3} d\mu_o} \oint L_1^* \phi_\alpha^\pm d\mu_o \oint f^{-3} \tilde{\varphi} d\mu_o, \tag{3.14}$$

where L_1^* is the formal adjoint of L_1, and

$$\frac{\partial \tilde{\varphi}}{\partial u} = L\tilde{\varphi} + \tilde{L}_1 \tilde{\varphi} + \tilde{\Xi}[A_\alpha^\pm], \tag{3.15}$$

$$\tilde{L}_1 \tilde{\varphi} = L_1 \tilde{\varphi} - \oint \tilde{\varphi} L_1^* 1 d\mu_o - \sum_\alpha \Xi_\alpha[\tilde{\varphi}]\phi_\alpha + \Gamma(\tilde{\varphi}), \tag{3.16}$$

$$\tilde{\Xi}[A_\alpha^\pm] = \sum_\alpha \left(L_1 \phi_\alpha - \oint L_1 \phi_\alpha d\mu_o - \sum_\beta \Psi_{\beta\alpha} \phi_\beta + \Gamma(\phi_\alpha) \right) A_\alpha, \tag{3.17}$$

where

$$\Gamma(\theta) = \frac{\oint f^{-3} \theta d\mu_o}{\oint f^{-3} d\mu_o} \left(\oint L_1 1 d\mu_o - L_1 1 \right).$$

The sums in (3.16) and (3.17) (and in the matrix equations below) are implicitly over both the index and their "associated $\pm$." We shall be interested in solutions of (3.4) for which

$$A_\alpha^\pm = e^{-\nu^\pm u} (\tilde{B}_\alpha^\pm + F_\alpha^\pm), \tag{3.18}$$

where $\tilde{B}_\alpha^\pm$ are prescribed constants, and $F_\alpha^\pm \xrightarrow{u \to \infty} 0$. In order to prove existence of such solutions it turns out to be necessary to keep track separately of the even and odd parity parts[10], $\tilde{\varphi}^\pm$, of $\tilde{\varphi}$; if we set

$$\zeta^\pm = e^{\nu^\pm u} \tilde{\varphi}^\pm, \tag{3.19}$$

one finally obtains the following system of equations:

$$\frac{dF}{du} = \Psi(F + \tilde{B}) + \xi[\zeta^+, \zeta^-] \tag{3.20}$$

$$\frac{\partial \zeta^\pm}{\partial u} = (L + \nu^\pm)\zeta^\pm + \sum_{\sigma = \pm} L^{\pm\sigma} \zeta^\sigma + \rho^\pm [F + \tilde{B}], \tag{3.21}$$

with

$$F = \begin{pmatrix} F_\alpha^+ \\ F_\alpha^- \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} \tilde{B}_\alpha^+ \\ \tilde{B}_\alpha^- \end{pmatrix},$$

$$\Psi = \begin{pmatrix} \Psi_{\alpha\beta}^{++} & e^{(\nu^+ - \nu^-)u} \Psi_{\alpha\beta}^{+-} \\ e^{(\nu^- - \nu^+)u} \Psi_{\alpha\beta}^{-+} & \Psi_{\alpha\beta}^{--} \end{pmatrix},$$

$$\xi[\zeta^+, \zeta^-] = \begin{pmatrix} e^{\nu^+ u} \Xi_\alpha^+ [e^{-\nu^+ u} \zeta^+ + e^{-\nu^- u} \zeta^-] \\ e^{\nu^- u} \Xi_\alpha^- [e^{-\nu^+ u} \zeta^+ + e^{-\nu^- u} \zeta^-] \end{pmatrix}, \tag{3.22}$$

$$\sum_{\sigma = \pm} L^{\pm\sigma} \zeta^\sigma \equiv e^{\nu^\pm u} [\tilde{L}_1 (e^{-\nu^+ u} \zeta^+ + e^{-\nu^- u} \zeta^-)]^\pm,$$

$$\rho^\pm [F + \tilde{B}] \equiv e^{\nu^\pm u} [\tilde{\Xi}[e^{-\nu^\pm u} (F_\alpha^\pm + \tilde{B}_\alpha^\pm)]]^\pm,$$

[10] The idea of separating the even parity terms from the odd parity terms in the RT equation has also been considered by Rendall [16]

$\Psi^{\pm\pm}_{\alpha\beta}$, $\Xi^{\pm}_{\alpha}$, $\tilde{L}_1$ and $\tilde{\Xi}$ being as in (3.13), (3.14), (3.16) and (3.17) respectively. With the choice of c given by (3.3), it follows from the results of [9] by the same arguments as in the proof of Lemma 2.1 that any solution of the modified RT equation (3.1) on S^2 has the asymptotic expansion

$$f = 1 + f_1^+ e^{-\nu u} + \cdots + (f_5^+ + f_5^-)e^{-\nu^- u} + O(e^{-6\nu u}), \tag{3.23}$$

with $f_1^+ \in \mathscr{E}^+$, $f_5^- \in \mathscr{E}^-$,

$$f_1^+ = \sum \tilde{B}_\alpha^+ \phi_\alpha^+, \quad f_5^- = \sum \tilde{B}_\alpha^- \phi_\alpha^-, \quad \Delta_o \phi_\alpha^\pm = -\mu^\pm \phi_\alpha^\pm.$$

Moreover, there exists a constant C_f depending only upon $\|\ln f(u_o)\|_{H_6(^2\mathscr{M})}$ such that

$$\begin{aligned} &\|\ln f\|_{C^4(^2\mathscr{M})} + e^{\nu^+ u}\|f^+ - 1\|_{C^4(^2\mathscr{M})} + e^{\nu^- u}\|f^-\|_{C^4(^2\mathscr{M})} \\ &\quad + e^{(\nu^+ + \nu^-)u}\|f^- - f_5^- e^{-\nu^- u}\|_{C^4(^2\mathscr{M})} \leqq C_f \end{aligned} \tag{3.25}$$

and it follows from Proposition 3.1, point 1, and the results of [8], that for any $\varepsilon > 0$ there exists a $\delta > 0$ such that if $\|\ln f(u_o)\|_{H_6(^2\mathscr{M})} \leqq \delta \Rightarrow C_f \leqq \varepsilon$. A straightforward analysis, which requires somewhat tedious parity considerations if $^2\mathscr{M} = S^2$, shows the following key lemma:

Lemma 3.1. *Let C_f be the constant defined by (3.25). There exists a constant $C(K)$ such that if $C_f \leqq K$, then*

$$|\alpha_i(f)| \leqq CC_f e^{-(\nu^+ + \nu^-)u}, \tag{3.26}$$

$$|\Psi| \leqq CC_f e^{-\nu^+ u}, \tag{3.27}$$

$$|\xi| \leqq CC_f e^{-\nu^+ u}(\|\zeta^+\|_{L^1(^2\mathscr{M})} + \|\zeta^-\|_{L^1(^2\mathscr{M})}), \tag{3.28}$$

$$|\rho^\pm(X)| \leqq CC_f e^{-\nu^+ u}\|X\|_{\mathbb{R}^{12}}, \tag{3.29}$$

and the operators $L^{\pm\pm}$ can be written in the form

$$\sum_{i=0}^{4} A^{\pm\pm}_{\alpha_1\cdots\alpha_i} \mathring{\nabla}^{\alpha_1}\cdots\mathring{\nabla}^{\alpha_i}, \tag{3.30}$$

for some tensors $A^{\pm\pm}_{\alpha_1\cdots\alpha_i}$ satisfying

$$|A^{\pm\pm}_{\alpha_1\cdots\alpha_i}| \leqq CC_f e^{-\nu^+ u}, \tag{3.31}$$

$A^{\pm\pm}_{\alpha_1\cdots\alpha_i}$ smooth for all $u > u_o$ and uniformly C^2 for $u \geqq u_o$.

Let us start by analysing Eq. (3.20) assuming that ξ is a given function of $(u, p) \in [u_o, \infty) \times {}^2\mathscr{M}$:

Lemma 3.2. *For $\sigma > 0$ and $k, m \in \mathbb{N}$, let $X_k^\sigma(\mathbb{R}^m) = \{F = e^{-\sigma u}G : G \in C^k([u_o, \infty), \mathbb{R}^m)\}$, define*

$$\|F\|_{X_k^\sigma} = \|e^{\sigma u}F\|_{C^k([u_o,\infty),\mathbb{R}^m)},$$

suppose that $\mathscr{L}: X_{k+1}^\sigma \to X_k^\sigma$ is defined by

$$\mathscr{L}F = \frac{dF}{du} - \Psi F, \tag{3.32}$$

$$\Psi \in X_k^\nu(GL(\mathbb{R}^m, \mathbb{R}^m)), \quad \nu > 0. \tag{3.33}$$

Then $\mathscr{L}$ is an isomorphism, in particular there exists a constant C such that for every $\xi \in X_k^\sigma$, there exists a unique solution of the equation

$$\frac{dF}{du} = \Psi F + \xi \tag{3.34}$$

satisfying

$$\| F \|_{X_{k+1}^\sigma} \leqq C \| \xi \|_{X_k^\sigma}. \tag{3.35}$$

Proof. 1. Surjectivity: it is sufficient to show surjectivity on $[u_1, \infty)$, with some u_1 large enough, since any solution defined on $[u_1, \infty)$ can be uniquely continued backwards in u by standard theorems on solutions of linear ODE's on compact intervals. Consider the problem

$$\frac{dF_{i+1}}{du} = \Psi F_i + \xi, \tag{3.36}$$

$F_0 = 0$, thus

$$F_{i+1} = - \int_u^\infty (\Psi F_i + \xi)(s) ds, \tag{3.37}$$

and

$$|(F_{i+1} - F_i)(u)| \leqq \int_u^\infty |\Psi(F_i - F_{i-1})(s)| ds$$
$$\leqq \frac{e^{-(\sigma+\nu)u}}{\sigma+\nu} \| \Psi \|_{X_o^\nu} \| F_i - F_{i-1} \|_{X_0^\sigma},$$

so that for $u \in [u_1, \infty)$ we get

$$\| F_{i+1} - F_i \|_{X_0^\sigma} \leqq \frac{e^{-\nu u_1}}{\sigma+\nu} \| \Psi \|_{X_o^\nu} \| F_i - F_{i-1} \|_{X_0^\sigma}$$

and if $u_i > \frac{1}{\nu} \ln \left(\frac{\| \Psi \|_{X_o^\nu}}{\sigma+\nu} \right)$ the contraction mapping principle shows the existence of a fixed point F for the problem (3.37), which solves (3.34) and is in X_{k+1}^σ.

2. Injectivity: Let F_1, F_2 satisfy (3.34), then we have

$$\frac{d(F_1 - F_2)}{du} = \Psi(F_1 - F_2), \quad F_1 - F_2 \xrightarrow{u \to \infty} 0. \tag{3.38}$$

Suppose that $\| F_1 - F_2 \|_{\mathbb{R}^m} > 0$, then $\ln \| F_1 - F_2 \|_{\mathbb{R}^m}$ is differentiable and from (3.38) one obtains

$$\frac{d(\ln \| F_1 - F_2 \|_{\mathbb{R}^m})}{du} \geqq - \| \Psi \|_{X_o^\nu} e^{-\nu u}$$

$$\Rightarrow \| (F_1 - F_2)(u_1) \|_{\mathbb{R}^m} \leqq \exp \left\{ \frac{\| \Psi \|_{X_o^\nu} e^{-\nu u_1}}{\nu} \right\} \| (F_1 - F_2)(u_2) \|_{\mathbb{R}^m} \quad \text{for} \quad u_2 \geqq u_1.$$

Letting $u_2 \to \infty$ one obtains $\| (F_1 - F_2)(u_1) \| = 0$ which contradicts the assumption that $\| F_1 - F_2 \|_{\mathbb{R}^m} > 0$, i.e., there exists $\tilde{u}$ such that $(F_1 - F_2)(\tilde{u}) = 0$. In this case, $F_1 \equiv F_2$ follows from standard results for first order ODE's.

3. To prove (3.35), note that $\mathscr{L}$ is a bounded linear bijection from X^σ_{k+1} to X^σ_k thus, by the open mapping theorem, $\mathscr{L}^{-1}$ is a continuous linear operator from X^σ_k to X^σ_{k+1}, which implies (3.35). □

Lemma 3.2 shows that Eq. (3.20) can be solved for F in terms of $\tilde{B}$ and $\zeta^\pm$. By well known results, there exists functions $R^{\pm\pm}_{\alpha\beta}$ such that

$$F^\pm_\alpha = \int_u^\infty \sum_{\sigma=\pm,\beta} R^{\pm\sigma}_{\alpha\beta}(u,s)\{\xi^\sigma_\beta[\zeta^\pm(s)] + \sum_{\rho=\pm,\gamma} \Psi^{\sigma\rho}_{\beta\gamma}\tilde{B}^\rho_\gamma\} ds. \tag{3.39}$$

Inserting (3.39) into (3.21), one obtains an equation of the form

$$\frac{\partial \zeta^\pm}{\partial u}(u,x) = \left[(L + v^\pm)\zeta^\pm + \sum_{\sigma=\pm} L^{\pm\sigma}\zeta^\sigma + \tilde{\rho}^\pm(\tilde{B}) \right](u,x)$$
$$+ \int_u^\infty ds \int_{{}^2\mathscr{M}} \sum_{\sigma=\pm} R^{\pm\sigma}(u,s,x,x')\xi^\sigma(s,x')d\mu_o(x'), \tag{3.40}$$

which has the amusing property that the derivative $\frac{\partial \zeta^\pm}{\partial u}$ at time u depends on $\xi^\pm(v)$ for all $v \geqq u$. To avoid the supplementary step of proving estimates on $R^{\pm\pm}$, rather than analysing (3.40) we shall consider the system (3.20), (3.21) directly. We shall need the following Lemma, which gives information about solutions of Eq. (3.21) when $\rho^\pm$ are considered as being given functions of (u, p):

Lemma 3.3. *Let $\zeta = (\zeta^+, \zeta^-)$, $\rho = (\rho^+, \rho^-)$, set*

$$\| \zeta \|^2_{H_k({}^2\mathscr{M})} = \| \zeta^+ \|^2_{H_k({}^2\mathscr{M})} + \| \zeta^- \|^2_{H_k({}^2\mathscr{M})}, \tag{3.41}$$

$$\| \rho \|^2_{H_k({}^2\mathscr{M})} = \| \rho^+ \|^2_{H_k({}^2\mathscr{M})} + \| \rho^- \|^2_{H_k({}^2\mathscr{M})}. \tag{3.42}$$

There exists constants C_1, C_2 such that for all u_1 satisfying

$$u_1 \geqq \mathring{u}_1 = \frac{1}{v}\ln(C_1 C_f), \tag{3.43}$$

where C_f has been defined in (3.25), all solutions $\zeta^\pm \in C^1([u_1, \infty), L^2({}^2\mathscr{M})) \cap C([u_1, \infty), H_4({}^2\mathscr{M}))$ of (3.21) satisfy, for $u \geqq u_1$,

$$\| \zeta(u) \|^2_{L^2({}^2\mathscr{M})} \leqq \| \zeta(u_1) \|^2_{L^2({}^2\mathscr{M})} + C_2 e^{-2vu} \int_{u_1}^u e^{2vs} \| \rho(s) \|^2_{L^2({}^2\mathscr{M})} ds.$$

Proof. In what follows we shall assume ${}^2\mathscr{M} = S^2$; for other topologies the proof is obtained by similar arguments. By expanding $\tilde{\varphi}^\pm$ in spherical harmonics and using straightforward approximation arguments (cf. e.g. [9] [Lemma 4.1]), one obtains

$$\oint \zeta^+ L \zeta^+ d\mu_o \leqq -15v \oint (\zeta^+)^2 d\mu_o, \tag{3.44}$$

$$\oint \zeta^- L \zeta^- d\mu_o \leqq -35v \oint (\zeta^-)^2 d\mu_o, \tag{3.45}$$

and similarly one proves that there exists a constant C_o such that

$$\oint \varphi d\mu_o = 0 \ \Rightarrow \ \oint \varphi L \varphi d\mu_o \leqq -C_o \| \varphi \|^2_{H_2({}^2\mathscr{M})}. \tag{3.46}$$

Set

$$E(u)=\oint[(\zeta^+)^2+(\zeta^-)^2]d\mu_o,$$

let $\alpha,\beta>0$, $\alpha+\beta=1$; we have, by (3.44)–(3.46),

$$\begin{aligned}\frac{dE}{du}&=2\oint\zeta^+(L+v)\zeta^+d\mu_o+2\oint\zeta^-(L+v^-)\zeta^-d\mu_o\\&\quad+2\sum_{\rho=\pm,\sigma=\pm}\oint\zeta^\rho L^{\rho\sigma}\zeta^\sigma d\mu_o+2\sum_\pm\oint\zeta^\pm\rho^\pm d\mu_o\\&\leqq-(30\alpha-2)v\oint(\zeta^+)^2d\mu_o-(70\alpha-10)v\oint(\zeta^-)^2d\mu_0\\&\quad-2\beta C_o(\|\zeta^+\|^2_{H_2(^2\mathcal{M})}+\|\zeta^-\|^2_{H_2(^2\mathcal{M})})\\&\quad+2\sum_{\rho=\pm,\sigma=\pm}\oint\zeta^\rho L^{\rho\sigma}\zeta^\sigma d\mu_0+\varepsilon E(u)+\frac{1}{\varepsilon}\sum_\pm\|\rho^\pm\|^2_{L^2(^2\mathcal{M})},\end{aligned}$$

where we have used $2ab\leqq\frac{1}{\varepsilon}a^2+\varepsilon b^2$. Some integration by parts in the terms containing $\zeta^\pm L^{\pm\pm}\zeta^\pm$ gives

$$\sum_{\rho=\pm,\sigma=\pm}\oint\zeta^\rho L^{\rho\sigma}\zeta^\sigma d\mu_o\leqq CC_fe^{-vu}(\|\zeta^+\|^2_{H_2(^2\mathcal{M})}+\|\zeta^-\|^2_{H_2(^2\mathcal{M})})\tag{3.47}$$

(cf. (3.30), (3.31)), so that with (3.41) and (3.42), one finally obtains

$$\frac{dE}{du}\leqq-(30\alpha v-10v-\varepsilon)E(u)-(2C_o\beta-CC_fe^{-vu})\|\zeta\|^2_{H_2(^2\mathcal{M})}+\frac{1}{\varepsilon}\|\rho\|^2_{H_2(^2\mathcal{M})}.$$

Choosing

$$\alpha=\beta=\frac{1}{2},\quad\varepsilon=3v,\quad e^{-v\mathring{u}_1}=\frac{C_o}{CC_f},$$

one gets

$$u\geqq u_1\geqq\mathring{u}_1\Rightarrow E(u)\leqq e^{2v(u_1-u)}E(u_1)+C_2\int_{u_1}^{u}e^{2v(s-u)}\|\rho(s)\|^2_{H_2(^2\mathcal{M})}ds,$$

for some constant C_2. □

Theorem 3.1. *Let $f\in C^0([u_0,\infty),H_6(^2\mathcal{M}))\cap C^1([u_o,\infty),H_2(^2\mathcal{M}))$ be a solution of the modified* RT *equation. There exists $\mathring{u}_1\geqq u_o$ depending only upon $\|\ln f(u_o)\|_{H_6(^2\mathcal{M})}$ such that for all $u_1\geqq\mathring{u}_1$, $\tilde{B}^\pm_\alpha\in\mathbb{R}^{12}$ and $\tilde{\varphi}_0\in\tilde{P}H_4(^2\mathcal{M})$ there exists $\bar{\varphi}_0\in\bar{P}H_4(^2\mathcal{M})\equiv(1-\tilde{P}-P^o)H_4(^2\mathcal{M})$ with the property that the unique solution of (3.4) satisfying*

$$(\tilde{P}\varphi)(u_1)=\tilde{\varphi}_0,\quad\bar{P}\varphi(u_1)=\bar{\varphi}_0,$$

with $\varphi^o(u_1)=(P^o\varphi)(u_1)$ given by (3.11), satisfies, for $u\to\infty$,

$$\varphi^\pm=\sum_\alpha\tilde{B}^\pm_\alpha\phi^\pm_\alpha e^{-v\pm u}+O(e^{-(v+v^\pm)u}),$$

$$\Delta_o\phi^+_\alpha=-6\phi^+_\alpha,\quad\Delta_o\phi^-_\alpha=-12\phi^-_\alpha.$$

Moreover there exists $\varepsilon_0>0$ such that if $\|\ln f(u_o)\|_{H_6(^2\mathcal{M})}\leqq\varepsilon_0$, then $\mathring{u}_1=u_o$.

Proof. Set $F_0 = 0$, and for $i \geqq 1$ let $F_i, \zeta_i^{\pm}$ be solutions of the equations

$$\frac{dF_i}{du} = \Psi(F_i + \tilde{B}) + \xi[\zeta_i^+, \zeta_i^-],$$

$$\frac{\partial \zeta_{i+1}^{\pm}}{\partial u} = (L + v^{\pm})\zeta_{i+1}^{\pm} + \sum_{\sigma = \pm} L^{\pm\sigma}\zeta_{i+1}^{\sigma} + \rho^{\pm}[F_i + \tilde{B}],$$

with $\zeta_{i+1}^{\pm}(u_1) = \zeta^{o\,\pm}(u_1)$. For $i \geqq 1$, we have

$$\frac{d(F_{i+1} - F_i)}{du} = \Psi(F_{i+1} - F_i) + \xi[\zeta_{i+1}^+ - \zeta_i^+, \zeta_{i+1}^- - \zeta_i^-], \tag{3.48}$$

$$\frac{\partial(\zeta_{i+1}^{\pm} - \zeta_i^{\pm})}{\partial u} = (L + v^{\pm})(\zeta_{i+1}^{\pm} - \zeta_i^{\pm}) + \sum_{\sigma = \pm} L^{\sigma\pm}(\zeta_{i+1}^{\sigma} - \zeta_i^{\sigma}) + \rho^{\pm}[F_i - F_{i-1}], \tag{3.49}$$

with

$$(\zeta_{i+1}^{\pm} - \zeta_i^{\pm})(u_1) = 0 \quad (\Rightarrow \|\zeta_{i+1}^{\pm} - \zeta_i^{\pm})(u_1)\|_{L^2({}^2\mathcal{M})} = 0), \tag{3.50}$$

and Lemma 3.2 with $\sigma = v$ and (3.28) gives, for $u \geqq u_1$,

$$\begin{aligned} \|(F_{i+1} - F_i)(u)\|_{\mathbb{R}^{12}} &\leqq Ce^{-vu} \sup_{s \geqq u_1} e^{vs} \|\xi[\zeta_{i+1}^+ - \zeta_i^+, \zeta_{i+1}^- - \zeta_i^-](s)\|_{\mathbb{R}^{12}} \\ &\leqq CC_f e^{-vu} \sup_{s \geqq u_1} \sum_{\pm} \|(\zeta_{i+1}^{\pm} - \zeta_i^{\pm})(s)\|_{L^2({}^2\mathcal{M})}. \end{aligned}$$

This together with (3.29), (3.49), (3.50) and Lemma 3.3 implies, for $u \geqq u_1, u_1$ large enough,

$$\left[\sum_{\pm} \|(\zeta_{i+2}^{\pm} - \zeta_{i+1}^{\pm})(u)\|_{L^2({}^2\mathcal{M})}\right]^2 \leqq CC_f^4 e^{-2v(u_1+u)} \left[\sup_{s \geqq u_1} \sum_{\pm} \|(\zeta_{i+1}^{\pm} - \zeta_i^{\pm})(s)\|_{L^2({}^2\mathcal{M})}\right]^2,$$

so that if

$$C^{1/4} C_f e^{-vu_1} < 1$$

the sequence $\zeta_i^{\pm}(u)$ converges in $L^2({}^2\mathcal{M})$ for each u to a function $\zeta_{\infty}^{\pm}(u)$. It is straightforward to show that in fact $\zeta_{\infty}^{\pm}(u) \in C([u_1, \infty), H_4({}^2\mathcal{M})) \cap C^{\infty}((u_1, \infty) \times {}^2\mathcal{M})$, and the remaining claims follow by methods similar to those of [8, 9]. Note that it follows from the methods of [8, 9] that the constant C_f in (3.25) depends only upon $\|\ln f(u_o)\|_{H_6({}^2\mathcal{M})}$, and in fact for $\|\ln f(u_o)\|_{H_6({}^2\mathcal{M})} \leqq 1$ it follows from Proposition 3.1, point 2, that we have

$$C_f \leqq C \|\ln f(u_o)\|_{H_6({}^2\mathcal{M})}, \tag{3.51}$$

which shows that for $\|\ln f(u_o)\|_{H_6({}^2\mathcal{M})}$ small enough we can set $u_1 = u_o$. □

3.2. The Nonlinear Equation. In this section we shall show how the genericity problem can be reduced to the linearised problem analysed in Sect. 3.1. Let us note the following:

Proposition 3.1. *Consider the map* $\mathcal{B}_{u_o,k} : H_k({}^2\mathcal{M}) \to \mathbb{R}^N$ *(respectively* $\bar{\mathcal{B}}_{u_o,k} : H_k(S^2) \to \mathbb{R}^N$*)* $(N = 12$ *if* ${}^2\mathcal{M} = S^2)$ *which to an initial datum* $f(u_o) \in H_k({}^2\mathcal{M})$ *assigns*

1. *the coefficients* $(B_\alpha^{\pm})$ *of the expansion* (2.8) *if* ${}^2\mathcal{M} = S^2$ *(respectively* $(\bar{B}_\alpha^{\pm})$ *of expansion* (3.23)*), or*
2. *the function* $f_{1,0}$ *of the expansion* (2.6) *otherwise*[11].

Then

1. *for* $k \geqq 4$ *the map* $\mathcal{B}_{u_o,k}(f(u_o))$ *(respectively* $\bar{\mathcal{B}}_{u_o,k}(f(u_o))$*) is a* C^0 *function of* $f(u_o)$*, and*
2. *for* $k \geqq 6$ *the map* $\mathcal{B}_{u_o,k}(f(u_o))$ *(respectively* $\bar{\mathcal{B}}_{u_o,k}(f(u_o))$*) is a* C^1 *function of* $f(u_o)$*.*

The proof of Proposition 3.1 is a rather lengthy and straightforward application of the techniques developed in [8] together with parity considerations similar to those of the previous section; no details will be given. It is rather likely that the thresholds $k \geqq 4$ for continuity and $k \geqq 6$ for differentiability are not optimal; we have not attempted to analyse this question. With these thresholds it is easy to prove continuity and/or differentiability in an $L^2({}^2\mathcal{M})$ norm, and use interpolation to get the result for higher norms as well.

Suppose that $\mathring{f}(u_o)$ is such that $\mathcal{B}_{u_o,k}(\mathring{f}(u_0)) \in \Omega$ where, for ${}^2\mathcal{M} = S^2, \Omega$ is the set of $(B_\alpha^{\pm})$ for which $f_{\log} \not\equiv 0$ (cf. Proposition 2.4), while for ${}^2\mathcal{M} \neq S^2$, we set $\Omega = \mathbb{R}^N \backslash \{0\}$. As discussed in Sect. 2, for ${}^2\mathcal{M} \neq T^2$ the corresponding space-time will be geometrically singular at the null boundary $\mathcal{H} = \{mu = \infty\}$ ($m > 0$ for ${}^2\mathcal{M} = S^2, m < 0$ otherwise, and if ${}^2\mathcal{M} \neq S^2$ then we assume $\mu_1 \neq \mathbb{N}$). It follows from Proposition 3.1, point 1, that for $k \geqq 4$ the map $\mathcal{B}_{u_o,k}$ is continuous, so that for *all* $f(u_o)$'s in a sufficiently small neighborhood of $\mathring{f}(u_o)$ in $H_4({}^2\mathcal{M})$ the corresponding space-times will be singular at $\mathcal{H}$. Thus to prove genericity of the set of $f(u_o)$'s which lead to singular $\mathcal{H}$'s we only need to analyse what happens for $f(u_o)$'s such that $\mathcal{B}_{u_o,k}(f(u_o)) \in C\Omega \equiv \mathbb{R}^N \backslash \Omega$.

Proposition 3.2. *Suppose* $k \geqq 6$*, let* $f(u)$ *be a solution of the* RT *equation,* $f(u_o) \in H_k({}^2\mathcal{M})$*. There exists* $\mathring{u}_1 \geqq u_o$ *depending only upon* $\|\ln f(u_o)\|_{H_6({}^2\mathcal{M})}$ *such that for all* $u_1 \geqq \mathring{u}_1$ *the map* $\mathcal{B}'_{u_1,k}(f(u_1)) \equiv \left.\frac{\delta \mathcal{B}_{u_1,k}(\varphi)}{\delta\varphi}\right|_{\varphi = f(u_1)}$ *is surjective. Moreover there exists* $\varepsilon_0 > 0$ *such that if* $\|\ln f(u_o)\|_{H_6({}^2\mathcal{M})} \leqq \varepsilon_0$*, then* $\mathring{u}_1$ *can be chosen to be equal to* u_0*.*

Proof. If ${}^2\mathcal{M} \neq S^2$, this is Theorem 3.1, let us thus consider the case ${}^2\mathcal{M} = S^2$. Let $\tilde{f}(u)$ be the solution of the modified RT equation (3.1), with $\tilde{f}(u_0) = f(u_0)$. Recall that f can be obtained from $\tilde{f}$ by the following procedure [9, 20]: let $M(u) \in SL(2, \mathbb{C})$ satisfy the equation

$$\frac{dM}{du} = MA(\alpha), \quad M(u_o) = \mathrm{id}, \tag{3.52}$$

with

$$[A(\alpha)]^a{}_b = \sum_b A^a{}_{bi}\alpha_i,$$

for some constants $A^a{}_{bi}$, where

$$\alpha_i = c\oint \phi_i \tilde{f}\, d\mu_o$$

(cf. (3.2)). There is a natural identification between $SL(2, \mathbb{C})$ and the group of

[11] Recall that $f_{1,0}$ satisfies $\Delta_o f_{1,0} = -\mu_1 f_{1,0}$ and is thus determined by a finite number of parameters $(B_\alpha) \in \mathbb{R}^{m_1}$, where m_1 is the dimension of the first non-trivial eigenspace of Δ_o

conformal transformations of S^2, define Φ_M as the conformal map of S^2 into itself corresponding to $M \in SL(2,\mathbb{C})$, let Ψ^2_M be the corresponding conformal factor,

$$\Phi^*_M \mathring{g}_{ab} = \Psi^2_M \mathring{g}_{ab}.$$

We then have

$$f(u) = [\Psi_{M(u)} \tilde{f}(u)] \circ \Phi^{-1}_{M(u)}. \tag{3.53}$$

Equation (3.53) and $\lim_{u\to\infty} \tilde{f}(u) = 1$ show that

$$f(u) \circ \Phi_{M(\infty)} \times \Psi^{-1}_{M(\infty)} \xrightarrow{u\to\infty} 1,$$

thus the solution of the RT equation which has the expansion (2.8) is $f(u)\circ\Phi_{M(\infty)} \times \Psi^{-1}_{M(\infty)}$; (3.53) gives

$$f(u)\circ\Phi_{M(\infty)} \times \Psi^{-1}_{M(\infty)} = [\Psi_{M(u)} \tilde{f}(u)] \circ \Phi^{-1}_{M(u)} \circ \Phi_{M(\infty)} \times \Psi^{-1}_{M(\infty)}.$$

Since $\alpha_i = O(e^{-(v^+ + v^-)u})$ (cf. (3.26)) it follows from (3.52) and from Lemma 2.1 that

$$|M(u) - M(\infty)| \leqq Ce^{-(v^+ + v^-)u} \Rightarrow |\tilde{f}(u) - \tilde{f}(u)\circ\Phi^{-1}_{M(u)}\circ\Phi_{M(\infty)}| \leqq Ce^{-(v^+ + v^-)u},$$

$$|(\Psi_{M(u)}\circ\Phi^{-1}_{M(u)}\circ\Phi_{M(\infty)}) \times \Psi^{-1}_{M(\infty)} - 1| \leqq Ce^{-(v^+ + v^-)u},$$

which shows that

$$B^\pm_\alpha = \bar{B}^\pm_\alpha \Rightarrow \mathscr{B}_{u_o,k} = \bar{\mathscr{B}}_{u_o,k}.$$

It follows that for $u_1 \geqq u_o$, the derivative $\frac{\delta \mathscr{B}_{u_1,k}}{\delta f}$ acting on $\varphi(u_1)$ is the map which to $\varphi(u_1)$ assigns the coefficients $\bar{B}^\pm_\alpha$ of the expansion analogous to (3.23) of solutions of the linearised equation (3.1), which is surjective by Theorem 3.1. □

Theorem 3.2. *Let $k \geqq 6$, suppose that $f(u), u \geqq u_o$ is a solution of the* RT *equation with $f(u_o) \in H_k(^2\mathscr{M})$, let $\mathscr{B}_{u_o,k}(f(u_o)) = (\mathring{B}^\pm_\alpha) \in \mathbb{R}^N$.*

1. *There exists $u_1 \geqq u_0$ and a neighborhood $\mathscr{O}_{u_1,k}$ of $\mathring{B}^\pm_\alpha$ such that for all $B^\pm_\alpha \in \mathscr{O}_{u_1,k}$ there exists a solution $f_{B^\pm_\alpha} \in C([u_1,\infty), H_k(^2\mathscr{M}))$ of the RT equation such that $\mathscr{B}_{u_1,k}(f_{B^\pm_\alpha}(u_1)) = B^\pm_\alpha$.*
2. *For any $\delta_1 > 0$ there exists $\delta_2 > 0$ such that for $\| B^\pm_\alpha - \mathring{B}^\pm_\alpha \|_{\mathbb{R}^N} < \delta_2$ we can choose $f_{B^\pm_\alpha}(u_1)$ so that $\| f_{B^\pm_\alpha}(u_1) - f(u_1) \|_{H_k(^2\mathscr{M})} < \delta_1$.*
3. *There exists $\varepsilon > 0$ such that if $\| \ln f(u_o) \|_{H_6(^2\mathscr{M})} \leqq \varepsilon$, then u_1 can be chosen to be equal to u_o.*
4. *There exists $\delta_3 > 0$ such that for all $B^\pm_\alpha \in \mathscr{O}_{u_1,k}$, the set of $f_{B^\pm_\alpha}(u_1)$'s satisfying $\| f_{B^\pm_\alpha}(u_1) - f(u_1) \|_{H_k(^2\mathscr{M})} < \delta_3$ is a C^1 submanifold of finite codimension N ($N = 12$ if $^2M = S^2$) of $H_k(^2\mathscr{M})$.*

Proof. It follows from Proposition 3.1 that for $u_1 \geqq u_o$, $\mathscr{B}_{u_1,k}: H_k(^2\mathscr{M}) \to \mathbb{R}^N$ is differentiable, thus both $\operatorname{Ker} \mathscr{B}'_{u_1,k}$ and $(\operatorname{Ker} \mathscr{B}'_{u_1,k})^\perp$ are (closed) Banach spaces, and the map $\mathscr{F}: (\operatorname{Ker} \mathscr{B}'_{u_1,k})^\perp \to \mathbb{R}^N$ defined by $\mathscr{F}(\varphi) = \mathscr{B}_{u_1,k}(\varphi)$ is differentiable. We have $\mathscr{F}' \equiv \frac{\delta \mathscr{F}}{\delta \varphi} = \mathscr{B}'_{u_1,k}|_{(\operatorname{Ker} \mathscr{B}'_{u_1,k})^\perp}$, and Proposition 3.2 shows that $\mathscr{F}'$ is an isomorphism. By the implicit function theorem, $\mathscr{F}$ is an isomorphism from $\mathscr{U}_{u_1,k} \subset (\operatorname{Ker} \mathscr{B}'_{u_1,k})^\perp$

to an open neighbourhood $\mathcal{O}_{u_1,k}$ of $(\mathring{B}^{\pm}_{\alpha})$, which proves points 1–3. Point 4 follows from the implicit mapping theorem, cf. e.g. [15 Chapter I, §5]. □

Corollary 3.1. 1. *Let* $^2\mathcal{M} = S^2$. *There exist non-trivial* (i.e. $f \not\equiv f_\infty$, *where* f_∞ *is a conformal factor for a conformal transformation*) RT *metrics such that* $|f - f_\infty| \leqq Ce^{-15v+u} = Ce^{-30u/m}$. *Every such metric can be* C^{557} *extended across* $\mathcal{H}$.
2. *Let* $^2\mathcal{M} \neq S^2$. *There exist non-trivial* (i.e. $f \neq 1$) RT *metrics such that* $|f-1| \leqq Ce^{-\lambda_2 u}$, $\lambda_2 = \mu_2(\mu_2 + R_o)/(12m)$, *where* μ_2 *is the second non-trivial eigenvalue of* $-\Delta_o$.

Proof. We shall consider the case $^2\mathcal{M} = S^2$ only, the remaining cases follow in a similar manner. Let $u_o \in \mathbb{R}$ be arbitrary, let $f(u_o) = 1$, thus the corresponding RT space-time is the Schwarzschild space-time; for any $k \geqq 6$ it follows from Theorem 3.2, point 4, that there exists a submanifold of codimension 12 of $f(u_o)$'s in $H_k(^2\mathcal{M})$ for which $\mathcal{B}_{u_o,k}(f(u_o)) = 0$, and the arguments of the proof of Lemma 2.1 show that for such initial data we will have $|f - f_\infty| \leqq Ce^{-15v+u}$, for some $f_\infty = \lim_{u\to\infty} f(u)$. The C^{557} extendability follows from Lemma 2.1, point 3, by parity considerations.

Proof of Theorem 2.2. Let $X_k = \{\mathring{f} \in H_k(^2\mathcal{M})$: the horizon of the RT space-time with f such that $f(u_0) = \mathring{f}$ is singular$\}$. For $k \geqq 4$ openness of X_k follows from Proposition 3.1, point 1. To prove density, consider a solution $f(u)$ of the RT equation such that $\|\ln f(u_o)\|_{H_6(^2\mathcal{M})} < \varepsilon$, ε given by Theorem 3.2, point 3, and assume that $f(u_o) \notin X_k$. Let $(\mathring{B}^{\pm}_{\alpha}) = \mathcal{B}_{u_o,k}(f(u_o))$, thus $(\mathring{B}^{\pm}_{\alpha}) \in C\Omega$, where $\Omega = \{B^{\pm}_{\alpha} : f_{\log} \not\equiv 0\}$ if $^2\mathcal{M} = S^2$, and $\Omega = \mathbb{R}^N \backslash \{0\}$ otherwise. Since Ω is dense in $\mathbb{R}^N$ (cf. Proposition 2.2 if $^2\mathcal{M} = S^2$), it follows that there exists a sequence $B^{\pm}_{\alpha,i} \xrightarrow{i\to\infty} \mathring{B}^{\pm}_{\alpha}$, such that $B^{\pm}_{\alpha,i} \in \Omega$; for $k \geqq 6$ it follows from Theorem 3.2, point 2, that there exists a sequence $f_i(u_o) \in H_k(^2\mathcal{M})$ converging to $f(u_o)$ in $H_k(^2\mathcal{M})$ norm such that $\mathcal{B}_{u_o,k}(f_o(u_o)) = B^{\pm}_{\alpha,i}$, and thus the corresponding space-times are singular at $\mathcal{H}$. □

Proof of Theorem 2.1. Let $X = \{\mathring{f} \in C^\infty(^2\mathcal{M})$: the horizon $\mathcal{H}$ of the RT space-time with f such that $f(u_o) = \mathring{f}$ is singular$\}$. Openness follows as in the proof of Theorem 2.2; to show density consider a solution $f(u)$ of the RT equation such that $f(u_o) \notin X$, and $\|\ln f(u_o)\|_{H_6(^2\mathcal{M})} < \varepsilon$, ε given by Theorem 3.2, point 3. For any $k \in \mathbb{N}$ by Theorem 3.2 point 2 we can find a sequence $f^k_i \in H_{\max(k+2,6)}(^2\mathcal{M})$ converging to $f(u_o)$ in $H_{\max(k+2,6)}(^2\mathcal{M})$ norm such that $f^k_i \in X$. By density of $C^\infty(^2\mathcal{M})$ in $H_{\max(k+2,6)}(^2\mathcal{M})$ there exists a sequence $f^k_{i,j} \in C^\infty(^2\mathcal{M})$ such that $f^k_{i,j} \xrightarrow{j\to\infty} f^k_i$ in $H_{\max(k+2,6)}(^2\mathcal{M})$ norm; by continuity of $\mathcal{B}_{u_o,\max(k+2,6)}$ it follows that for $j \geqq j(i)$, $\mathcal{B}_{u_o,\max(k+2,6)}(f^k_{i,j}) \in \Omega$, and thus for $j \geqq j(i)$ we have $f^k_{i,j} \in X$. We can find a sequence $j_i \geqq j(i)$ such that $f^k_{i,j_i} \xrightarrow{i\to\infty} f(u_o)$ in $H_{\max(k+2,6)}(^2\mathcal{M})$ norm, thus $f^k_{i,j_i} \xrightarrow{i\to\infty} f(u_o)$ in $C^k(^2\mathcal{M})$ norm by Sobolev embedding; redefine f^k_i to be f^k_{i,j_i}. Define a sub-sequence $f^k_{i_l}$ of f^k_i by the condition $\|f^k_{i_l} - f(u_o)\|_{C^k(^2\mathcal{M})} \leqq 2^{-(l+k)}$. The sequence $f^l_{i_l}$ converges to $f(u_o)$ in C^∞ topology as l tends to infinity, and the solutions $f_l(u)$ of the RT equation satisfying $f_l(u_o) = f^l_{i_l}$ are singular on the event horizon $\mathcal{H}$. □

4. The Robinson–Trautman Equation as a Dynamical System on the Space of Metrics

Let Riemm$(^2\mathcal{M})$ be the space of smooth metrics on a compact, connected, orientable two dimensional manifold $^2\mathcal{M}$, let Diff$_o(^2\mathcal{M})$ be the connected component of the space of smooth diffeomorphisms of ${}^2\mathcal{M}$ which contains the identity, both equipped with the C^∞ topology. The RT equation with, say, 12m = 1, may be used to define[12] a flow $\Phi_u : \mathrm{Riemm}({}^2\mathcal{M})/\mathrm{Diff}_o({}^2\mathcal{M}) \to \mathrm{Riemm}({}^2\mathcal{M})/\mathrm{Diff}_o({}^2\mathcal{M})$ as follows: let $[g] \in \mathrm{Riemm}({}^2\mathcal{M})/\mathrm{Diff}_o({}^2\mathcal{M})$, let $g(u)$ be the solution of the RT equation such that $g(0) = g$, then $\Phi_u([g]) = [g(u)]$ (throughout this section we use $[g]$ to denote the equivalence class of g in $\mathrm{Riemm}({}^2\mathcal{M})/\mathrm{Diff}_o({}^2\mathcal{M})$). If we define $\mathrm{Riemm}_o({}^2\mathcal{M})$ as the set of metrics $\mathring{g}_{ab}$ for which $R_o \equiv R(\mathring{g}_{ab}) \in \{-2, 0, 2\}$, then the results of [8] show that $\Phi_\infty = \lim_{u \to \infty} \Phi(u)$ exists, and that $\mathcal{T}({}^2\mathcal{M}) \equiv \mathrm{Riemm}_o({}^2\mathcal{M})/\mathrm{Diff}_o({}^2\mathcal{M})$ is an attractor for Φ_u, with basin of attraction equal to the whole of $\mathrm{Riemm}({}^2\mathcal{M})/\mathrm{Diff}_o({}^2\mathcal{M})$. Given $[g_o] \in \mathcal{T}({}^2\mathcal{M})$ let $\mathcal{A}([g_o], {}^2\mathcal{M})$ be the basin of attraction of $[g_o]$:

$$\mathcal{A}([g_o], {}^2\mathcal{M}) = \{[g] \in \mathrm{Riemm}({}^2\mathcal{M})/\mathrm{Diff}_o({}^2\mathcal{M}) : \Phi_\infty([g]) = [g_o]\}.$$

We have

$$\bigcup_{[g_o] \in \mathcal{T}({}^2\mathcal{M})} \mathcal{A}([g_o], {}^2\mathcal{M}) = \mathrm{Riemm}({}^2\mathcal{M})/\mathrm{Diff}_o({}^2\mathcal{M}), \tag{4.1}$$

$$\forall [g_o] \in \mathcal{T}({}^2\mathcal{M}) \quad \mathcal{A}([g_o], {}^2\mathcal{M}) \neq \emptyset. \tag{4.2}$$

We wish to point out that because of the asymptotic expansion (2.6) one can naturally associate to $\mathcal{T}({}^2\mathcal{M})$ an infinite sequence of "blow-up" structures, as follows: let $[g_o] \in \mathcal{T}({}^2\mathcal{M})$, let $\{\mu_i\}_{i \geqq 1}$ be the increasingly ordered spectrum of $-\Delta_o$, $\mu_1 > 0$, let $\mathcal{H}_i \subset L^2({}^2\mathcal{M})$ be the i^{th} eigenspace; we have $\mathcal{H}_i \approx \mathbb{R}^{m_i}$ for some m_i. Define $\{\lambda_i\} = \{\mu_i(\mu_i - 2)\}_{i \geqq 2}$ if ${}^2\mathcal{M} = S^2$; and $\{\lambda_i\} = \{\mu_i(\mu_i + R_o)\}_{i \geqq 1}$ otherwise (thus $\{\lambda_i\}$ is the spectrum of the operator which appears at the right-hand side of the linearization at a metric $g_o \in \mathrm{Riemm}_o({}^2\mathcal{M})$ of the RT (${}^2\mathcal{M} \neq S^2$) or of the modified RT (${}^2\mathcal{M} = S^2$) equation). Let $\{v_i\}_{i \geqq 1}$ be as described in Sect. 2 (cf. [9] for more details). We have $\{\lambda_i\} \subset \{v_i\}$, and along the lines of the proof of Lemma 2.1 one shows, that if we write

$$f \sim 1 + \sum_{i=1}^{\infty} \sum_{j=0}^{N(i)} f_{i,j} u^j e^{-v_i u}, \tag{4.3}$$

where "$\sim$" stands for "asymptotic to," in the sense of Eqs. (2.6), (2.7), then

1. if $v_i \notin \{\lambda_l\}_{l \in \mathbb{N}}$, then the functions $f_{i,j}$, $j = 0, \ldots, N(i)$ are defined uniquely by the functions $f_{k,0}$, $1 \leqq k \leqq i-1$,
2. if $v_i = \lambda_l$ for some l, then the functions $f_{i,j}$, $j = 1, \ldots, N(i)$ are defined uniquely by the functions $f_{k,0}$, $1 \leqq k \leqq i-1$, and there exists a function ϕ_i determined uniquely by $f_{k,0}$, $1 \leqq k \leqq i-1$, such that $f_i - \phi_i \in \mathcal{H}_i$.

The coefficients $X_i = (X_{i,k})_{k=1}^{m_i} \in \mathbb{R}^{m_i}$ of the decomposition $f_i - \phi_i = \sum_{k=1}^{m_i} X_{i,k} \phi_{i,k}$, where the functions $\phi_{i,k}$ form a basis of $\mathcal{H}_i$, will be called the free coefficients of the expansion (4.3).

---

[12] Since $\mathrm{Riemm}({}^2\mathcal{M})/\mathrm{Diff}_o({}^2\mathcal{M})$ is a stratified Inverse–Limit–Hilbert manifold [3] with singularities occurring on sets of metrics for which the isometry group jumps, and since the symmetries of a metric are preserved by the RT equation, it seems likely that one can define the RT equation on $\mathrm{Riemm}({}^2\mathcal{M})/\mathrm{Diff}_o({}^2\mathcal{M})$; a simpler way, which avoids some technicalities, is to proceed as above

Let $k \in \mathbb{N}$, we have $\bigoplus_{i=1}^{k} \mathscr{H}_i = \mathbb{R}^{M_k}, M_k = \sum_{i=1}^{k} m_i$. For any $X \in \mathbb{R}^{M_k}$ we define

$$\mathscr{A}_k(X, [g_o], {}^2\mathscr{M}) = \{[g] \in \mathscr{A}([g_o], {}^2\mathscr{M}): \text{the free coefficients of the expansion (4.3) are equal to } X\}.$$

We have

$$\bigcup_{X \in \mathbb{R}^{M_k}} \mathscr{A}_k(X, [g_o], {}^2\mathscr{M}) = \mathscr{A}([g_o], {}^2\mathscr{M}),$$

and more generally, if we set $\bigoplus_{i=1}^{l} \mathscr{H}_i = \mathbb{R}^{M_k} \oplus \mathbb{R}^{M_l - M_k}$, where $\mathbb{R}^{M_l - M_k} = \bigoplus_{i=k+1}^{l} \mathscr{H}_i$, then

$$\bigcup_{X_2 \in \mathbb{R}^{M_l - M_k}} \mathscr{A}_l((X_1, X_2), [g_o], {}^2\mathscr{M}) = \mathscr{A}_k(X_1, [g_o], {}^2\mathscr{M}).$$

Note that if $[g] \in \mathscr{A}_l((0, X_2), [g_o], {}^2\mathscr{M}), X_2 \in \mathbb{R}^{M_l - M_k}$, then $\Phi_u([g])]$ converges to $\Phi_\infty([g])$ exponentially fast with decay rate larger than or equal to λ_{k+1}.

The terminology introduced above allows us to restate what has been proved in Sect. 3 as follows:

Proposition 4.1. *For any $g_o \in \mathrm{Riemm}_o({}^2\mathscr{M})$ there exists $\varepsilon_o > 0$ such that if $X \in \mathbb{R}^{M_1}$, $\|X\|_{\mathbb{R}^{M_1}} < \varepsilon_o$, then*

$$\mathscr{A}_1(X, [g_o], {}^2\mathscr{M}) \neq \emptyset.$$

Proposition 4.2. *Let ${}^2\mathscr{M} = S^2$. For any $g_o \in \mathrm{Riemm}_o({}^2\mathscr{M})$ there exists $\varepsilon_o > 0$ such that if $X \in \mathbb{R}^{M_2}$, $\|X\|_{\mathbb{R}^{M_2}} < \varepsilon_o$,* then

$$\mathscr{A}_2(X, [g_o], {}^2\mathscr{M}) \neq \emptyset.$$

If rather than considering smooth metrics and diffeomorphisms we consider H_l metrics and H_{l+1} diffeomorphisms, the slice theorem [10] and Theorem 3.2 give:

Theorem 4.1. *Let $g_o \in \mathrm{Riemm}_o({}^2\mathscr{M})$ and suppose that the isometry group of g_o is trivial. For $l \geqq 6$ there exists a neighbourhood $\mathscr{O}_l$ of $[g_o]$ (in the H_l quotient topology) such that $\mathscr{A}_1(X, [g_o], {}^2\mathscr{M}) \cap \mathscr{O}_l$ is a C^1 submanifold of finite codimension.*

It seems natural to ask the question:

$$\{X : \mathscr{A}_k(X, [g_o], {}^2\mathscr{M}) = \emptyset\} \stackrel{?}{=} \emptyset.$$

Define X_∞ "$\in \mathbb{R}^\infty$" as a sequence $X_i \in \mathbb{R}^{M_i}$ such that $P_{\mathbb{R}^{M_j}} X_i = X_j$ for $j \leqq i$, where $P_{\mathbb{R}^{M_j}}$ is the coordinate projection on $\mathbb{R}^{M_j} = \mathbb{R}^{M_j} \oplus \{0\} \subset \mathbb{R}^{M_j} \oplus \mathbb{R}^{M_i - M_j} = \mathbb{R}^{M_i}$; set

$$\mathscr{A}_\infty(X_\infty, [g_o], {}^2\mathscr{M}) = \bigcap_k \mathscr{A}_k(X_k, [g_o], {}^2\mathscr{M}).$$

The conjecture mentioned at the end of Sect. 2.1 can be formulated as follows:

$$\mathscr{A}_\infty(X_\infty, [g_o], {}^2\mathscr{M}) \stackrel{?}{=} \{\text{a one dimensional submanifold, unless empty}\}.$$

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