

Monstrous Moonshine from Orbifolds

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Received July 17, 1990

Abstract. We consider the Monster Module of Frenkel, Lepowsky, and Meurman as a Z_2 orbifold of a bosonic string compactified by the Leech lattice. We show that the main Conway and Norton Monstrous Moonshine properties, stating that the Thompson series for each Monster group conjugacy class has a modular invariance group of genus zero, follow from an orbifold construction based on an orbifold group composed of Monster group elements. It is shown that a conjectured vacuum structure for the orbifold twisted sectors is sufficient to specify the modular group and the genus zero property for each Thompson series. It is also shown that the Power Map formula of Conway and Norton follows from the same vacuum structure. Finally, we demonstrate the validity of the vacuum conjectures for sectors twisted by Leech lattice automorphisms in many cases.

Introduction

One of the most intriguing examples of a conformal field theory (CFT) is the Monster Module described by Frenkel, Lepowsky, and Meurman [1]. They construct a meromorphic bosonic string theory on which the largest sporadic finite group M known as the Fischer-Griess Monster [2] acts as a natural automorphism with the states at each Virasoro level forming a representation for M . In this paper we consider the famous Moonshine properties conjectured by Conway and Norton [3] which relate each conjugacy class of M to a modular group whose fundamental region is always of genus zero. In a recent paper, Borcherds [4] has demonstrated that this genus zero property is correct using the theory of generalised Kac-Moody algebras [5]. In this paper we will show that these properties can also be understood in terms of orbifolds [6, 7] where the orbifold group is constructed from Monster group elements. We demonstrate that the vacuum structure of the corresponding twisted spaces is sufficient to ensure the genus zero property and the independent Power Map formula of Conway and Norton.

We begin in Sect. 2 with a review of the Monster Module construction [1] as a bosonic string compactified by the Leech lattice which is then Z_2 orbifolded [8]. The resulting theory has no dimension 1 (massless) fields and the Monster group arises as the automorphism group of a closed algebra of dimension 2 fields. Section 2 also serves to illustrate some general features of orbifold constructions that we exploit later. In Sect. 3 we describe the main Conway and Norton Moonshine properties [3]. The section begins with a discussion on the meromorphic properties of a function invariant under a modular group with fundamental region of genus zero. For any such group, a unique invariant function with a single simple pole, known as a hauptmodul, always exists. The main Moonshine property is that for each $g \in M$ one can define a modular function T_g (known as the Thompson series which depends only on the conjugacy class of g) which is a hauptmodul for some genus zero modular group. In Sect. 4 we interpret T_g as an orbifold contribution and introduce the corresponding g twisted sector. The actual method of construction of such a sector is not known but we can nevertheless describe some of the expected relevant properties on physical grounds. We discuss the origin of the modular invariance group for T_g and show that the singularity structure of T_g is determined by the vacuum properties of related twisted sectors. We introduce a set of vacuum conjectures which determine the full modular invariance group of T_g and explain the genus zero Moonshine properties. In addition, we demonstrate that the vacuum conjectures imply the Power Map formula observed by Conway and Norton relating the invariance group of T_g to that of T_{g^d} for any power d . In Sect. 5 we discuss an important subgroup of M related to Leech lattice automorphisms for which T_g has a closed form. We discuss the validity of the vacuum conjectures of Sect. 4 for the strictly Leech lattice automorphism elements of this subgroup for which a standard twisted sector construction is possible. Section 6 concludes with a brief description of generalised Moonshine properties that we expect to follow also from the vacuum conjectures. We also briefly discuss the connection between the conjectured uniqueness of the Monster Module and the Moonshine properties. In the Appendix we give a description of a convenient set of inequivalent parabolic cusp points of the fundamental region for the basic modular group appearing in the Moonshine properties.

2. The Monster Module

In this section we review some features of the construction of the two-dimensional CFT associated with the Monster simple finite group known as the FLM Monster Module [1]. Most of the language adopted here follows the string theory exposition of the work of FLM in [8]. We will emphasise the orbifold aspects of this construction and highlight a few important general features that we will return to later.

The starting point is to define a Euclidean bosonic closed string compactified to a 24-dimensional torus T^{24} [9]. The torus we choose is defined by quotienting R^{24} with the Leech lattice \mathcal{A} , the unique even self-dual lattice in 24 dimensions without vectors of length squared 2 [10, 11]. The left-moving bosonic string variables $x^i(z)$ obey $x^i(e^{2\pi i}z) = x^i(z) + 2\pi\beta$ with $z = e^{2\pi iw}$, where $\text{Re}w$ and $\text{Im}w$ are “space” and “time” world-sheet coordinates and where $\beta \in \mathcal{A}$. The mode expansion for $x^i(z)$ is

$$x^i(z) = q^i - ip^i \ln z + i \sum_{n \neq 0} \frac{\alpha_n^i}{n} z^{-n} \quad (2.1)$$

with

$$\begin{aligned}
 [q^i, p^j] &= i\delta^{ij}, \\
 [\alpha_m^i, \alpha_n^j] &= m\delta^{ij}\delta_{m+n, 0}.
 \end{aligned}
 \tag{2.2}$$

A similar expression holds for $x^i(\bar{z})$ the right-moving part of the string. Since \mathcal{A} is even self-dual [11], the 1-loop partition function with world-sheet torus $w \sim w + 1 \sim w + \tau$, takes the modular invariant form $Z(\tau)Z(\bar{\tau})$, where

$$\begin{aligned}
 Z(\tau) &= \text{Tr}(q^{L_0}) = \frac{\Theta_{\mathcal{A}}}{\eta^{24}}, \\
 \Theta_{\mathcal{A}} &= \sum_{\beta \in \mathcal{A}} q^{\beta^2/2}
 \end{aligned}
 \tag{2.3}$$

with $q = e^{2\pi i\tau}$ and where L_0 is the normal ordered Virasoro Hamiltonian operator

$$L_0 = \frac{1}{2}p^2 + \sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i - 1.
 \tag{2.4}$$

The normal ordering constant $-C/24$ is given by the tachyonic vacuum energy $24 \times \frac{1}{2} \sum n = -1$, where $C=24$ is the usual central charge. η is the Dedekind eta function $\eta = q^{\frac{1}{24}} \prod_n (1 - q^n)$ arising from the oscillator modes. $\Theta_{\mathcal{A}}$ is the theta function associated with the Leech lattice \mathcal{A} and is a modular form of weight 12 [11, 12]. The Hilbert space of states for this theory also factorises into meromorphic/antimeromorphic (in z) pieces. We may therefore consistently regard the left-moving string as a meromorphic CFT [8, 13] and ignore the right-moving part from now on. The Hilbert space for this meromorphic CFT has the following Fock space representation

$$H_{(+)} = \{ \alpha_{-n_1}^{i_1} \dots \alpha_{-n_k}^{i_k} | \beta \rangle \}, \quad n_i \in \mathbb{Z}, \quad \beta \in \mathcal{A},
 \tag{2.5}$$

where $|\beta\rangle$ is a highest weight state so that $\alpha_n^i |\beta\rangle = 0$ for $n > 0$ and $p^i |\beta\rangle = \beta^i |\beta\rangle$ [11]. The Fock space is graded by L_0 with integer level $\frac{1}{2} \beta^2 + \sum n_i$. The subscript (+) denotes the periodicity of states under $z \rightarrow e^{2\pi i} z$ which implies for a typical state that

$$e^{2\pi i L_0} |\psi\rangle = |\psi\rangle
 \tag{2.6}$$

since L_0 generates such a transformation. The integral grading of L_0 ensures that (2.6) is correct.

Returning to the partition function for these states we note that $Z(\tau)$ is a meromorphic and modular invariant function of τ . $Z(\tau)$ may be found immediately in terms of the modular invariant function $j(\tau)$

$$j(\tau) = \frac{E_2^3}{\eta^{24}} = \frac{1}{q} + 744 + 196884q + \dots,
 \tag{2.7}$$

where $E_2(\tau)$ is the Eisenstein modular form of weight 4 [12] corresponding to the Θ function for the E_8 root lattice. The j function is the unique modular invariant function with a single simple pole at $q=0$ (up to a constant). This uniqueness property is discussed further in Sect. 3. It is a generalised version of this property for other modular groups that is the central issue in Monstrous Moonshine.

Since $Z(\tau)$ also has a simple pole at $q=0$, due to the negative vacuum energy for L_0 , we find that

$$Z(\tau) = J(\tau) + 24, \tag{2.8}$$

where we define $J(\tau) = j(\tau) - 744$. The constant 24 reflects the existence of only 24 level one states $|\alpha_{-1}^i\rangle$ in this theory. Since the Leech lattice contains no length squared 2 vectors only these $U(1)$ bosonic states contribute.

The next stage of the Monster Module construction is the introduction of a Z_2 twist operation $g : X^i \rightarrow -X^i$ which acts as an automorphism on the states (2.5) of the theory [1]. We can construct a new CFT corresponding to string propagation on the quotient of T^{24} by g . Geometrically, the manifold T^{24} becomes an orbifold T^{24}/Z_2 which is a manifold except at the conical singularities due to any fixed points of g [6, 7]. In this case, these are labelled by $A/2A$ and have multiplicity 2^{24} . The states of this new orbifold or “twisted” theory are composed partly of the g invariant projection of the original Hilbert space $H_{(+)}$ and new twisted states which must be introduced for modular consistency.

The original Hilbert space $H_{(+)}$ can be decomposed into ± 1 eigenstates of g as follows

$$H_{(+)}^+ = \{ \alpha_{-n_1}^{i_1} \dots \alpha_{-n_{2k}}^{i_{2k}} (|\beta\rangle + |-\beta\rangle) \} \cup \{ \alpha_{-n_1}^{i_1} \dots \alpha_{-n_{2k+1}}^{i_{2k+1}} (|\beta\rangle - |-\beta\rangle) \}, \tag{2.9a}$$

$$H_{(+)}^- = \{ \alpha_{-n_1}^{i_1} \dots \alpha_{-n_{2k}}^{i_{2k}} (|\beta\rangle - |-\beta\rangle) \} \cup \{ \alpha_{-n_1}^{i_1} \dots \alpha_{-n_{2k+1}}^{i_{2k+1}} (|\beta\rangle + |-\beta\rangle) \}, \tag{2.9b}$$

where defining the projection operator $\mathcal{P} = \frac{1}{2}(1 + g)$ we have $H_{(+)}^+ = \mathcal{P}H_{(+)}$.

Consider now the partition function for the g invariant states $H_{(+)}^+$

$$\text{Tr}_{H_{(+)}}(\mathcal{P}q^{L_0}) = \frac{1}{2} \text{Tr}_{H_{(+)}}(q^{L_0}) + \frac{1}{2} \text{Tr}_{H_{(+)}}(gq^{L_0}). \tag{2.10}$$

The first trace is the original partition function $Z(\tau)$ of (2.3). The second can be interpreted as a closed string contribution periodic in the “space” direction but twisted by g in the “time” direction. We denote this contribution in the standard way by $-\square_+$. Calculating this trace from (2.9b) we have [8]

$$-\square_+ = q^{-1} \prod_n (1 + q^n)^{-24} = 2^{12} \left[\frac{\eta}{\Theta_2} \right]^{12}. \tag{2.11}$$

Similarly, we can denote the original trace of (2.3) by $+\square_+$. Performing the modular transformation $S : \tau \rightarrow -1/\tau$ on $-\square_+$ the space and time boundary conditions are interchanged to give a contribution $+\square_+$ corresponding to states twisted by g in the space direction but periodic in the time direction. Such a contribution arises from a new twisted sector with Hilbert space $H_{(-)}$ which is introduced below. In general, we can consider a Hilbert space H_h of states twisted by some group element h . Then performing the trace $\text{Tr}_{H_h}(gq^{L_0}) = g \square_+^h$, with some appropriate L_0 , the boundary conditions transform under a general modular transformation $\tau \rightarrow (a\tau + b)/(c\tau + d)$ to give $g^a h^{-b} \square_+^{c h^a}$ for h, g commuting (up to phase factors) [6, 14].

To define the twisted Hilbert space we consider a closed string field $x^i(z)$ on T^{24}/Z_2 with boundary conditions $x^i(e^{2\pi i}z) = -x^i(z) + \beta^i, \beta^i \in A$. The mode expansion

sion is now

$$x^i(z) = \tilde{q}^i + i \sum_{n \in \mathbb{Z} + \frac{1}{2}} \frac{\alpha_n^i}{n} z^{-n}, \tag{2.12}$$

where the oscillator modes obey the same commutation relations given in (2.1). There is a similar expansion for $x^i(\bar{z})$ involving the same centre of mass coordinates \tilde{q}^i which take values in the 2^{24} -dimensional fixed point space $A/2A$ of g . Each fixed point corresponds to one Fock vacuum $|\tilde{q}_a\rangle$ ($a=1 \dots 2^{24}$), where

$$\tilde{q}|\tilde{q}_a\rangle = \tilde{q}_a|\tilde{q}_a\rangle, \quad \tilde{q}_a \in A/2A. \tag{2.13}$$

Twisted states are then constructed by acting on these vacua with left and right creation operators. Since we are interested in meromorphic twisted states only 2^{12} of these Fock vacua are actually employed. These vacua are denoted by $|\sigma_a\rangle$, $a=1 \dots 2^{12}$, and form a 2^{12} -dimensional representation space T of a Z_2 central extension of $A/2A$. This in turn is induced by a Z_2 central extension of A with a two-cycle $s(\alpha, \beta)$ for $\alpha, \beta \in A$ familiar from the vertex construction for the states $|\beta\rangle$ of (2.5). Some of the details of this construction are discussed in Sect. 5.

The twisted Hilbert space $H_{(-)}$ is now defined by the Fock space representation

$$H_{(-)} = \{ \alpha_{-n_1}^{i_1} \dots \alpha_{-n_k}^{i_k} |0\rangle \otimes |\sigma_a\rangle \}, \quad n_i \in \mathbb{Z} + \frac{1}{2}, \quad a=1 \dots 2^{12}, \tag{2.14}$$

where $\alpha_n^i|0\rangle = 0, n > 0$. These states are graded by the Virasoro Hamiltonian

$$L_0 = \sum_{n = \frac{1}{2}}^{\infty} \alpha_{-n}^i \alpha_n^i + \frac{1}{2} \tag{2.15}$$

with half integer level $\frac{1}{2} + \sum n_i$. The appropriate normal ordering constant for L_0 can be found by zeta function regularisation to be $24 \times \frac{1}{2} \sum (n + \frac{1}{2}) = \frac{1}{2}$. (In general we have $\sum (n+r) = \frac{1}{12}(-1 + 6r(1-r))$ [7].) The automorphism g also induces a group operation on $H_{(-)}$, which can be decomposed into ± 1 eigenstates as before

$$H_{(-)}^+ = \{ \alpha_{-n_1}^{i_1} \dots \alpha_{-n_{2k+1}}^{i_{2k+1}} |0\rangle \otimes |\sigma_a\rangle \}, \tag{2.16a}$$

$$H_{(-)}^- = \{ \alpha_{-n_1}^{i_1} \dots \alpha_{-n_{2k}}^{i_{2k}} |0\rangle \otimes |\sigma_a\rangle \}, \tag{2.16b}$$

where $\mathcal{P}H_{(-)} = H_{(-)}^+$ and the twisting of a typical state of $H_{(-)}$ is expressed by

$$g e^{2\pi i L_0} |\psi\rangle = |\psi\rangle, \tag{2.17}$$

which is easily verified by noting that the states of $H_{(-)}^+$ ($H_{(-)}^-$) have levels in \mathbb{Z} ($\mathbb{Z} + \frac{1}{2}$).

We may now compute the partition function for $H_{(-)}^+$ to be

$$\text{Tr}_{H_{(-)}^+}(q^{L_0}) = \frac{1}{2} (+ \square + - \square), \tag{2.18}$$

where we find [8]

$$\begin{aligned} + \square &= \text{Tr}_{H_{(-)}^+}(q^{L_0}) \\ &= 2^{12} q^{1/2} \prod_n^{\infty} (1 - q^{n - \frac{1}{2}})^{-24} = 2^{12} \left[\frac{\eta}{\Theta_4} \right]^{12}, \end{aligned} \tag{2.19a}$$

$$\begin{aligned} - \square &= \text{Tr}_{H_{(-)}^-}(g q^{L_0}) \\ &= -2^{12} q^{1/2} \prod_n^{\infty} (1 + q^{n - \frac{1}{2}})^{-24} = -2^{12} \left[\frac{\eta}{\Theta_3} \right]^{12}, \end{aligned} \tag{2.19b}$$

where the vacuum degeneracy contributes 2^{12} . The boundary conditions associated with these traces (2.11), (2.19) transform amongst themselves under modular transformation. Therefore, the full partition function is the modular invariant

$$Z_{rw}(\tau) = \frac{1}{2} \left(\begin{matrix} + \square + - \square \\ + \quad \quad + \end{matrix} \right) + \frac{1}{2} \left(\begin{matrix} + \square + - \square \\ - \quad \quad - \end{matrix} \right) \tag{2.20}$$

which gives $Z_{rw}(\tau) = J(\tau)$ on substitution.

In general, to ensure that (2.20) is correct, we must check the consistency of any phases that may arise under modular transformations. Specifically, we must ensure that a modular transformation which fixes boundary conditions leaves the appropriate trace invariant [14]. Thus $\begin{matrix} + \square \\ + \end{matrix}$ (and $\begin{matrix} - \square \\ - \end{matrix}$) must be invariant under

$T^2: \tau \rightarrow \tau + 2$. We find therefore that the stability group of fixing transformations for $\begin{matrix} - \square \\ + \end{matrix}$ is generated by T and ST^2S [which generate the modular group $\Gamma_0(2)$].

The necessary existence of such a stability group for modular consistency is a general feature of orbifold constructions. In general, this leads to constraints on the automorphisms that can be employed to form a modular consistent theory [14, 15]. In Sect. 4 we will return to this feature of orbifold constructions. In the present case it is easy to check that these conditions are satisfied, i.e. the partition functions $\begin{matrix} + \square \\ + \end{matrix}$ and $\begin{matrix} - \square \\ - \end{matrix}$ are T^2 invariant since the twisted sector levels are half integrally graded.

The orbifold theory now constructed is the Monster Module of FLM [1]. The states of the theory are given by the g invariant Hilbert space $H_{(+)}^+ \oplus H_{(-)}^+$ denoted as V^h by FLM. It is the states of this theory on which the Monster group M has its action.

One of the motivations of FLM for looking at such a structure was the observation by McKay and Thompson [16] that the coefficients of the modular invariant function $j(\tau)$ are sums of dimensions of representations of M . Thus the coefficient of q^1 in (2.7) is the sum of the trivial and lowest dimensional representations $196884 = 1 + 196883$. Likewise, all higher coefficients of $j(\tau)$ can be expanded as sums of dimensions of M representations [16]. The only exception is the constant term 744 of j . By construction the Monster Module has no states of level 1 and so this exceptional case is excluded. The Monster group M now arises naturally out of the commutator algebra of the 196884 level 2 operators which includes the Energy-Momentum operator $T(z) = \frac{1}{2} \partial x \cdot \partial x$ [8]. Taking Fourier components of these level 2 operators, denoted ϕ_m^i , a symmetric non-associative cross-bracket operation can be defined as follows:

$$\phi_m^i \times \phi_n^j = \frac{1}{2} [\phi_{m+1}^i, \phi_{n-1}^j] + \frac{1}{2} [\phi_{n+1}^j, \phi_{m-1}^i]. \tag{2.21}$$

The purpose of this cross-bracket is to remove any level 3 operators which necessarily arise in the commutation of two level 2 operators. Thus the cross-bracket algebra for level 2 operators closes to form the following algebra [1, 8]:

$$\phi_m^i \times \phi_n^j = b^{ijk} \phi_{m+n}^k + \frac{1}{6} \delta^{ij} L_{m+n} + \frac{m^2}{2} \delta^{ij} \delta_{m+n,0}, \tag{2.22a}$$

$$L_n \times \phi_m^i = 2\phi_{m+n}^i, \tag{2.22b}$$

$$L_m \times L_n = 2L_{m+n} + 6m^2 \delta_{m+n,0}, \tag{2.22c}$$

where b^{ijk} are some coefficients symmetric in i, j and L_n is the Virasoro generator $\frac{1}{2} \sum \alpha_{-n} \alpha_{n+m}$. Equations (2.22b) and (2.22c) are symmetrized versions of the usual level 2 conformal properties of ϕ_n^i and the Virasoro algebra.

The closed algebra (2.22a) is an affine version of the Griess algebra [2] which can be recovered by setting $m = n = 0$. The automorphism group of this algebra was shown by Griess to be the largest finite simple sporadic group known as the Fischer-Griess Monster group M [2]. The level 2 operators $\{\phi_n^i\}$ form the lowest dimensional 196883 adjoint representation of M . The remaining level 2 operator, L_0 , is a singlet under M . Therefore, the states of the theory form representations for M at each level given by L_0 . This is the basis for the original observations of McKay and Thompson about the coefficients of $j(\tau)$.

For much of the discussion in the following two sections, the explicit form of the elements of M will not be required. What is important, for our immediate purposes, is that the states at each level form representations of M . We can, however, identify here one important element: the involution (order two element) i under which all untwisted states have eigenvalue $+1$ and twisted states have eigenvalue -1 . i can be thought of as a “fermion number” operator. The correct identification of the centralizer C of i in M (all elements of M that commute with i) forms a crucial part of the FLM construction. The monster group is characterized by the fact that C together with another involution σ generate M [2, 1]. In Sect. 5 we will consider C in more detail.

3. Hauptmoduls and Monstrous Moonshine

We will now describe the famous Moonshine properties of Conway and Norton [3] in terms of the Monster Module V^h of the last section. In order to do this we will return initially to the j modular function of (2.6). We will prove the uniqueness property of j in sufficiently general terms to be of use in both stating and understanding Monstrous Moonshine.

The j modular function is a meromorphic function of τ invariant under all modular transformations $\tau \rightarrow (a\tau + b)/(c\tau + d)$, $ad - bc = 1$ for $a, b, c, d \in \mathbb{Z}$, generated by the full modular group $\Gamma = PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/\{\pm 1\}$. The τ variable is defined over the upper half plane H which is in general invariant under the larger class of Möbius transformations generated by $PSL(2, \mathbb{R})$ of which Γ is a discrete subgroup. We may therefore factor H by Γ to obtain the usual fundamental region $\mathcal{F} = H/\Gamma$ conventionally taken as $|\tau| \geq 1$, $|\text{Im} \tau| \leq \frac{1}{2}$ as shown in Fig. 1. Adding in the parabolic cusp point at $i\infty$ ($q=0$) and identifying edges under Γ we obtain the compactified fundamental region $\overline{\mathcal{F}}$. This is clearly the Riemann sphere of genus zero. The modular invariance of $j(\tau)$ under Γ means that j is single-valued on $\overline{\mathcal{F}}$ (including the elliptic fixed points $\rho = e^{i\pi/3} = ST\rho$ and $i = Si$ [17]). Therefore, j provides a map from $\overline{\mathcal{F}}$ to $C \cup \{i\infty\}$. In fact, this map is an isomorphism and explicitly realizes the spherical topology of $\overline{\mathcal{F}}$. To see this, we will introduce a standard general theorem which will be exploited greatly later on.

Consider a more general situation as follows. Let Γ_1 be some subgroup of $PSL(2, \mathbb{R})$ with fundamental region \mathcal{F}_1 . We will assume that Γ_1 is commensurable with Γ , i.e. the index of $\Gamma \cup \Gamma_1$ in Γ and Γ_1 is finite [the dimensions of $\Gamma/(\Gamma \cup \Gamma_1)$ and $\Gamma_1/(\Gamma \cup \Gamma_1)$ are finite]. This implies that Γ_1 is also discrete like Γ . We may compactify \mathcal{F}_1 by adding in any parabolic cusp points and identifying edges under Γ_1 to obtain a compact Riemann surface $\overline{\mathcal{F}}_1$ of some genus [17]. (A parabolic cusp point is any

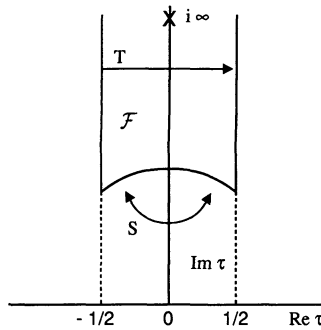


Fig. 1. The fundamental region \mathcal{F} for the full modular group Γ is equivalent to the Riemann sphere on identifying edges and adding in the point at infinity

point of $R \cup \{i\infty\}$ which is fixed by an element of Γ_1 .) Consider next some meromorphic Γ_1 invariant function $f(\tau)$. This induces a meromorphic function $\bar{f}(\tau)$ on $\bar{\mathcal{F}}_1$. For a regular point τ of \mathcal{F}_1 this is just $f(\tau)$ whereas for elliptic and parabolic fixed points a little more care is required to define a local coordinate patch on $\bar{\mathcal{F}}_1$ (see [17] for details). We can now state the following general theorem:

Hauptmodul Theorem. *Let $j_1(\tau)$ be a meromorphic Γ_1 invariant function with induced function $\bar{j}_1(\tau)$ on $\bar{\mathcal{F}}_1$. Then $\bar{j}_1(\tau)$ has a unique simple pole on $\bar{\mathcal{F}}_1$ if and only if $\bar{j}_1: \bar{\mathcal{F}}_1 \rightarrow C \cup \{i\infty\}$ is an isomorphism, i.e. $\bar{\mathcal{F}}_1$ is the Riemann sphere of genus zero.*

The proof of this theorem relies on the property that the number of zeros equals the number of poles for a meromorphic function on a compact Riemann surface. Suppose first that $\bar{j}_1(\tau)$ has a unique simple pole, $\tau = \tau_\infty$ on $\bar{\mathcal{F}}_1$. Then choosing a contour γ about the pole but sufficiently small to exclude any zeros we find by Cauchy's theorem that

$$\frac{1}{2\pi i} \oint_{\gamma} \frac{d\bar{j}_1}{\bar{j}_1} = -1. \tag{3.1}$$

Since $\bar{\mathcal{F}}_1$ is compact, we may also evaluate (3.1) as a sum of residues due to the poles and zeros of $\bar{j}_1(\tau)$ outside of γ . Therefore, $\bar{j}_1(\tau)$ has a unique zero of order 1 on $\bar{\mathcal{F}}_1$.

Consider next any $c \in C$. Then $\bar{f}_c = \bar{j}_1 - c$ is Γ_1 invariant with a unique simple pole at τ_∞ and therefore also vanishes at some unique point τ_c , i.e. $\bar{j}_1(\tau_c) = c$. Finally, $\bar{j}_1(\tau)$ is infinite at one point τ_∞ of $\bar{\mathcal{F}}_1$. Therefore, $\bar{j}_1: \bar{\mathcal{F}}_1 \rightarrow C \cup \{i\infty\}$ is an isomorphism and hence $\bar{\mathcal{F}}_1$ is the Riemann sphere.

Likewise, if $\bar{j}_1(\tau)$ is an isomorphism then it vanishes at only one point τ_0 of $\bar{\mathcal{F}}_1$. If the zero is of order n (implying that the unique pole is of order n also) then expanding we find

$$\bar{j}_1(\tau) = (\tau - \tau_0)^n (a_n + a_{n+1}(\tau - \tau_0) + \dots) \tag{3.2}$$

for $|\tau - \tau_0|$ sufficiently small. It is easy to see that $\bar{j}_1(\tau)$ cannot be single valued near τ_0 if $n > 1$. This follows by noticing that a closed contour about τ_0 in $\bar{\mathcal{F}}_1$ encircles the origin of the complex plane n times and therefore must self-intersect at least $n - 1$ times implying multi-valuedness. Therefore, $n = 1$ and $\bar{j}_1(\tau)$ has a unique pole of order 1.

The power of this theorem is that we may express any meromorphic Γ_1 invariant function $f(\tau)$ as a rational function of $j_1(\tau)$:

$$f(\tau) = \frac{\sum_i^m a_i j_1^i}{\sum_k^n b_k j_1^k}, \tag{3.3}$$

where the coefficients a_i, b_k are chosen such that the residues for the zeros and poles of $f(\tau)$ are matched. The set of all such Γ_1 invariant meromorphic functions is called a *genus zero function field* and Γ_1 is called a *genus zero modular group*. The special function $j_1(\tau)$ is called a *hauptmodul* or a *generator* for this genus zero function field.

As an example of the property (3.3) one can show that any two hauptmoduls j_1, j'_1 for a genus zero group Γ_1 must be related by

$$j'_1(\tau) = \frac{Aj_1 + B}{Cj_1 + D} \tag{3.4}$$

for some complex numbers A, B, C, D chosen to give the correct pole and zero for $j'_1(\tau)$. This relation is a $PSL(2, C)$ transformation on j_1 demonstrating the well known result that the automorphism group of the Riemann sphere is $PSL(2, C)$.

Returning to the case of the j function for Γ , the full modular group, we can see from Eq. (2.6) that j has a unique simple pole at $q=0$. This corresponds to a simple pole on \mathcal{F} since the invariance of j under T ensures that for $\text{Re } \tau = \text{const}$ in \mathcal{F} , one circuit of the contour $|q| = \text{const}$ is executed on \mathcal{F} . Therefore, the hauptmodul $j(\tau)$ is the unique Γ invariant function with a pole at $q=0$ as earlier stated. To emphasize the importance of these ideas for Γ we can consider the example of the modular invariant function $f(\tau) = (\theta_2^4 - \theta_3^4 + \theta_4^4) / \eta^{24}$. This has no poles and vanishes at $q=0$ on \mathcal{F} and therefore must vanish everywhere giving the Jacobi triple identity. Likewise, the fact that all Γ modular forms can be expressed as polynomials in the basic Eisenstein series E_2 and E_3 follows from the uniqueness of $j(\tau)$ [12].

Let us return now to the Z_2 orbifold Monster Module V^h of Sect. 2 and define the Thompson series for an element g of M . The states of V^h transform under (reducible) representations of M . Therefore, we may define the following trace:

$$T_g(\tau) = \text{Tr}_{V^h}(gq^{L_0}) \tag{3.5}$$

which is known as the Thompson series for $g \in M$ [16]. As an example, we can calculate immediately the series for the involution i defined in Sect. 2,

$$T_i(\tau) = \text{Tr}_{H^{\dot{+}}}(q^{L_0}) - \text{Tr}_{H^{\dot{-}}}(q^{L_0}), \tag{3.6}$$

which on substituting (2.8), (2.12), and (2.19) gives

$$T_i(\tau) = 2^{12} \left[\frac{\eta}{\Theta_2} \right]^{12} + 24 = \left[\frac{\eta(\tau)}{\eta(2\tau)} \right]^{24} + 24. \tag{3.7}$$

In general, expanding in q we find

$$T_g(\tau) = \frac{1}{q} + 0 + (1 + \chi_1(g))q + \dots, \tag{3.8}$$

where $\chi_1(g)$ is the character for g in the 196883 representation. Likewise, the higher coefficients are sums of characters of g in the various representations occurring at

each level. For each of the 172 conjugacy classes of M there is such a Thompson series. Conway and Norton [3] calculated the coefficients for each of these series up to q^{10} and formulated their Moonshine properties as a result. They conjectured that each Thompson series is a hauptmodul for a genus zero modular group Γ_g commensurable with Γ , i.e. each $T_g(\tau)$ is invariant under some fixing modular group Γ_g and that $\mathcal{F}_g = \overline{H}/\overline{\Gamma}_g$ is of genus zero where T_g is the hauptmodul with a simple pole at $q=0$. [All the states of V^h are of integer level so that T_g is T invariant. Therefore, as for the Γ invariant function $j(\tau)$, the pole at $q=0$ is of order 1 on \mathcal{F}_g .] The uniqueness of the hauptmodul allowed Conway and Norton to list all the modular functions $T_g(\tau)$ and to find many non-trivial formulae amongst these functions.

To describe the Γ_g groups relevant to the Moonshine properties we must first define a number of modular groups of interest.

$\Gamma_0(N)$: The group of matrices $\gamma \in \Gamma$ of the form

$$\gamma = \begin{pmatrix} a & b \\ cN & d \end{pmatrix}, \quad \det \gamma = 1, \tag{3.9}$$

where $a, b, c, d \in \mathbb{Z}$.

The full structure of the Normalizer group of $\Gamma_0(N)$ in $PSL(2, R)$, $\mathcal{N}(N) = \{\varrho \in PSL(2, R) \mid \varrho \Gamma_0(N) \varrho^{-1} = \Gamma_0(N)\}$, is also required to describe Γ_g . Here we will quote from the elegant description of $\mathcal{N}(N)$ given by Conway and Norton [3].

Let $h \in \mathbb{Z}$ where $h^2 \mid N$ (h^2 divides N) and let $N = nh$. Then we define the following sets of matrices.

$\Gamma_0(n|h)$: The group of matrices of the form

$$\begin{pmatrix} a & b \\ cn & d \end{pmatrix}, \quad \det = 1, \tag{3.10}$$

where $a, b, c, d \in \mathbb{Z}$. For h the largest divisor of 24 for which $h^2 \mid N$, $\Gamma_0(n|h)$ forms a subgroup of $\mathcal{N}(N)$. The group is generated by the cosets $C_1 = \{\gamma \in \Gamma_0(n|h) \mid a = b = d = 1 \pmod{h}, c = 0 \pmod{h}\}$ and $C_2 = \{\gamma \in \Gamma_0(n|h) \mid a = c = d = 1 \pmod{h}, b = 0 \pmod{h}\}$ [3].

Note that each element of C_1 (or C_2) can be expressed as $T^{\frac{1}{h}}: \tau \rightarrow \tau + \frac{1}{h}$ (or respectively ST^hS) times an element of $\Gamma_0(N)$.

W_e : The set of matrices for a given integer e

$$\begin{pmatrix} ae & b \\ cN & de \end{pmatrix}, \quad \det = e, \quad e \parallel N, \tag{3.11}$$

where $a, b, c, d \in \mathbb{Z}$. $e \parallel N$ denotes the property that $e \mid N$ and the greatest common divisor $\left(e, \frac{N}{e}\right) = 1$. The set W_e forms a single coset of $\Gamma_0(N)$ in $\mathcal{N}(N)$ with $W_1 = \Gamma_0(N)$. It is straightforward to show that (up to scale factors)

$$W_e^2 = 1 \pmod{\Gamma_0(N)}, \tag{3.12}$$

$$W_{e_1} W_{e_2} = W_{e_2} W_{e_1} = W_{e_3} \pmod{\Gamma_0(N)}, \quad e_3 = \frac{e_1 e_2}{(e_1, e_2)^2}.$$

The coset W_e is referred to as an Atkin-Lehner (AL) involution for $\Gamma_0(N)$. The simplest example is the Fricke involution W_N with coset representative $\begin{pmatrix} 0 & 1 \\ -N & 0 \end{pmatrix}$ which generates $\tau \rightarrow -1/N\tau$. This transformation is already familiar in another string theory context [18, 19].

w_e : The set of matrices for a given integer e of the form

$$\begin{pmatrix} a & b \\ ae & \frac{b}{h} \\ cn & de \end{pmatrix}, \quad \det = e, \quad e \parallel \begin{vmatrix} n \\ h \end{vmatrix}, \quad (3.13)$$

where $a, b, c, d \in \mathbb{Z}$. The set w_e is called an Atkin-Lehner (AL) involution for $\Gamma_0(n|h)$. The properties (3.12) are similarly obeyed by w_e with $\Gamma_0(N)$ replaced by $\Gamma_0(n|h)$.

$\mathcal{N}(N)$: The Normalizer of $\Gamma_0(N)$ in $PSL(2, R)$ is generated by adjoining to $\Gamma_0(n|h)$ all its AL involutions w_{e_1}, w_{e_2}, \dots , where h is the largest divisor of 24 and $h^2|N, N = nh$.

$\Gamma_0(n|h) + e_1, e_2, \dots$: This denotes the group obtained by adjoining to $\Gamma_0(n|h)$ a particular subset of AL involutions w_{e_1}, w_{e_2}, \dots . Clearly, this forms a subgroup of $\mathcal{N}(N)$.

All of the groups defined above are commensurable with the full modular group Γ . We are now in a position to state the main Moonshine properties of Conway and Norton [3, 20].

Moonshine Properties. Let $g \in M$ of order n .

- (a) The Thompson series $T_g(\tau)$ is invariant up to a phase of order h under a subgroup of $\mathcal{N}(N)$ of the form $\Gamma_0(n|h) + e_1, e_2, \dots$, where $h|24, h|n$, and $N = nh$.
- (b) The subgroup Γ_g of these transformations which fixes T_g (and contains $\Gamma_0(N)$) is of genus zero where T_g is the hauptmodul with a simple pole at $q=0$.

Let us examine how these properties apply to an element of prime order p . Part (a) tells us that $h = 1$ (excepting only one class of M with $n = 3$ and $h = 3$ [3] which we exclude from the following remarks). Part (b) tells us that the fixing group Γ_g contains $\Gamma_0(p)$ and is contained in $\mathcal{N}(p)$. However, $\mathcal{N}(p)$ contains only one non-trivial element, the Fricke involution $W_p: \tau \rightarrow -1/p\tau$. $\mathcal{N}(p)$ is therefore found by adjoining W_p to $\Gamma_0(p)$ which is denoted by $\Gamma_0(p) +$. Therefore, we find that either $\Gamma_g = \Gamma_0(p)$ or $\Gamma_0(p) +$. It is well known that $\Gamma_0(p)$ is of genus zero only when $(p - 1)|24$ for p prime [21]. The corresponding hauptmodul is just

$$\left[\frac{\eta(\tau)}{\eta(p\tau)} \right]^r + r, \quad (3.14)$$

where $(p - 1)r = 24$. Invariance under $\Gamma_0(p)$ is ensured for r an integer. The constant in (3.14) is chosen so that the q^0 coefficient is zero. For each p , with $(p - 1)|24$ there is indeed a conjugacy class of M , labelled $p -$, with the Thompson series (3.14). Recalling our earlier example with the involution element i we obtain (3.7) from (3.14) for $r = 24$ and $p = 2$. In Sect. 5 we shall explicitly reproduce the remaining Thompson series of this form from Leech lattice automorphisms.

The primes for which $\Gamma_0(p) +$ is of genus zero have been found by Ogg [22] to be precisely the primes that divide the order of M . These are all primes p with $2 \leq p \leq 31$ and $p = 41, 47, 59, 71$. For each of these primes there is a conjugacy class, labelled $p +$, whose Thompson series is the hauptmodul for this group. A subset of these classes with $(p + 1)|24$ associated with Leech lattice automorphisms will be explicitly calculated in Sect. 5.

In general, there are 174 possible discrete subgroups of genus zero which contain some $\Gamma_0(N)$ and are contained in $\Gamma_0(N) + e_1, e_2, \dots$ [3]. There are 171 classes for M , each of which has a unique Thompson series corresponding to one of these discrete groups (except for two classes of order 27 which share the same Thompson series). The remaining 3 genus zero groups ($N = 25, 49, 50$) correspond to the so-called “ghost elements” of M [3].

There are many remarkable relationships found by Conway and Norton between different Thompson series many of which owe their origin to the hauptmodul property (3.2). In particular, the so-called replication formulae reveal that all coefficients of any Thompson series T_g can be expressed in terms of the first five of $T_g, T_{g^2}, T_{g^4}, \dots$ by recursion formulae. Borcherds has recently shown that these properties can also be deduced from determinant formulae based on generalised Kac-Moody algebras and hence the hauptmodul properties follow [4].

Another interesting and important consequence of the hauptmodul property is the behaviour of T_g under a transformation generated by an element ρ of the normalizer of Γ_g in $PSL(2, R)$. The modular function $T_g^\rho = T_g(\rho(\tau))$ is meromorphic on H with a simple pole at $\rho^{-1}(i\infty)$. In addition, T_g^ρ is Γ_g invariant since $\rho\Gamma_g\rho^{-1} = \Gamma_g$. Therefore, by the Hauptmodul Theorem, T_g is also a hauptmodul for Γ_g . By (3.4) we know that

$$T_g^\rho(\tau) = \frac{AT_g + B}{CT_g + D} \tag{3.15}$$

for some complex numbers A, B, C, D . As an example of this, consider the class p – with $(p - 1) \mid 24$, which has Thompson series (3.14). The only non-trivial element of $\mathcal{N}(p)$ is W_p . Defining $\phi_g = T_g - r$ we find that ϕ_g is inverted by W_p :

$$W_p(\phi_g(\tau)) = \left[\frac{\eta(-1/p\tau)}{\eta(-1/\tau)} \right]^r = \frac{p^{\frac{r}{2}}}{\phi_g(\tau)}. \tag{3.16}$$

In general, the normalizer of Γ_g in $PSL(2, R)$ contains the Fricke involution W_N . Because $W_N^2 = 1 \pmod{\Gamma_0(N)}$ we find from (3.15) that either T_g is W_N invariant (i.e. $W_N \in \Gamma_g$) or is inverted according to

$$W_N(\phi_g(\tau)) = \frac{c}{\phi_g(\tau)}, \quad W_N \notin \Gamma_g, \tag{3.17}$$

where $\phi_g(\tau) = T_g(\tau) - T_g(0)$ and $c \in C$.

Equations (3.16) and (3.17) allow us to determine the phase referred to in part (a) of the Moonshine properties. Assuming that $T_g(\tau)$ is $\Gamma_0(N)$ invariant, it follows from our remarks about the cosets C_1 and C_2 that it is sufficient to describe the phase associated with the transformations $T^{\frac{1}{h}}$ and $ST^N S$. The leading term of $T_g(\tau)$ is always q^{-1} . Hence $T^{\frac{1}{h}}$ can only act invariably up to a phase by

$$T^{\frac{1}{h}}(T_g(\tau)) = e^{-2\pi i/h} T_g(\tau). \tag{3.18}$$

Suppose that T_g is W_N invariant. Then we have

$$\begin{aligned} ST^N S(T_g(\tau)) &= ST^N(T_g(\tau/N)) \\ &= e^{-2\pi i/h} T_g(\tau), \quad W_N \in \Gamma_g, \end{aligned} \tag{3.19}$$

using (3.18). Alternatively, from (3.17), ϕ_g is inverted by W_N . Assuming T_g is ST^nS invariant up to a phase implies that for $h \neq 1$ we must have $T_g(0) = 0$ and

$$\begin{aligned} ST^nS(T_g(\tau)) &= ST^n\left(\frac{c}{T_g(\tau/N)}\right) \\ &= e^{2\pi i/h} T_g(\tau), \quad W_N \notin \Gamma_g. \end{aligned} \tag{3.20}$$

The phase for any element of $\Gamma_0(n|h)$ is generated by these phases¹. The phase associated with any AL transformation w_e for $\Gamma_0(n|h)$ is discussed in the next section.

4. Moonshine from Orbifolds

Having outlined the Moonshine properties in Sect. 3 we will now describe a new physical interpretation and explanation for this phenomenon. The correct physical setting turns out to be an orbifold model constructed from the Monster Module V^\natural by factoring with a subgroup of M . This was suggested in [8] where it was observed that the Thompson series $T_g(\tau)$ can be interpreted as an orbifold contribution from untwisted states in V^\natural but twisted in the “time direction” by g . This suggestion was motivated by Norton’s generalised Moonshine formulation [20] which we will return to briefly in Sect. 6. In the notation of Sect. 2 we have

$$T_g(\tau) = \text{Tr}_{V^\natural}(gq^{L_0}) = g \square_1. \tag{4.1}$$

This contributes to the partition function for an orbifold based on any group containing g .

As we reviewed in Sect. 2, the modular invariance of an orbifold partition function requires the introduction of twisted Hilbert spaces. Thus an S transformation on (4.1) interchanges world-sheet space and time boundary conditions to give

$$S: g \square_1 = 1 \square_g, \tag{4.2}$$

i.e. the partition function for the twisted Hilbert space H_g of states periodic in the space direction up to g . We will assume that the states are graded by some appropriate Virasoro Hamiltonian L_0 (according to the usual assumptions of CFT). An explicit general construction of H_g is still lacking although we can expect that generalised Kac-Moody algebras have an important role here [5]. Orbifold constructions [6, 7] are usually based on an automorphism group of the embedding space of the string (e.g. the Monster Module itself is a Z_2 quotient of the torus $T^{24} = R^{24}/A$). This embedding allows one to construct a twisted Hilbert space in a natural way. In addition, there is a natural automorphism group induced on this Hilbert space. However, in the present case an element g of M is not in general induced by a geometrical action on the orbifold T^{24}/Z_2 . Therefore, the usual route to constructing twisted Hilbert spaces cannot be taken. An important exception to this is the subgroup of M given by Leech lattice automorphisms which we discuss in Sect. 5.

¹ In [3] and [23] the phases (3.19) and (3.20) appear to have been mistakenly interchanged

In the absence of a general construction, we will postulate the existence and properties of H_g that can be expected from physical arguments. Let $|E\rangle_g^i \in H_g$, where E is the (energy) level of L_0 , the Virasoro Hamiltonian for H_g and i labels states within each level. L_0 contains some normal ordering constant giving the vacuum energy of H_g . We will assume that for each E , the set of states $\{|E\rangle_g^i\}$ form a reducible, unitary representation for $C(g) = \{h \in M \mid gh = hg\}$. The states of H_g will then contribute to an orbifold based on any abelian subgroup of M containing g .

Suppose that g is of order n . Then the twisting of a typical state $|E\rangle_g^i$ of H_g is expressed by

$$g^{ij} e^{2\pi i L_0} |E\rangle_g^j = e^{2\pi i \phi_g/n} |E\rangle_g^i, \tag{4.3}$$

where g^{ij} is shorthand for the appropriate representation of M at level E and $2\pi\phi_g/n$ is a phase which can in general arise as discussed below. Note that by acting with $h \in M$ on (4.3) we see that the twisted space H_g is isomorphic to $H_{g'}$, where $g' = hgh^{-1}$, i.e. g' is conjugate to g .

The phase in (4.3) is responsible for any global anomalies associated with the modular stability group of an orbifold partition function discussed previously in Sect. 2 [14]. This is clear on considering a T transformation on the partition function $\text{Tr}_{H_g}(hq^{L_0})$ for $h \in M$

$$T: h \square_g = e^{2\pi i \phi_g/n} h g^{-1} \square_g. \tag{4.4}$$

Assuming that the action of g in H_g is also of order n then transforming by T^n leaves the boundary conditions fixed but introduces a global phase anomaly $e^{2\pi i \phi_g}$. Therefore, the partition function $h \square_g$ does not have a well-defined phase associated with a given pair of boundary conditions. (Note that for a left-right symmetric bosonic string such phase anomalies are irrelevant since the opposite phase appears from the right-hand part. Only in the context of an asymmetric bosonic or heterotic orbifold do such considerations place constraints on the orbifold automorphism group [14, 15]). This anomalous behaviour can be directly seen in (4.3) in terms of the levels E . Applying the twisting operation n times we find

$$nE = \phi_g \pmod{1}. \tag{4.5}$$

When the global phase vanishes (mod 1) we get the usual consistency condition on the levels of L_0 [14, 7, 15] i.e. $E \in \mathbb{Z}/n$. In general, we can express ϕ_g from (4.5) in terms of the lowest vacuum energy $E_0 = \phi_g/n$.

In [14] the phase associated with a modular transformation of an orbifold trace contribution is discussed. One can show that under a general modular transformation T_g becomes

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}: g \square_1 = e^{2\pi i c d E_0} g^{-d} \square_{g^c}. \tag{4.6}$$

The phase computed in [14] is for a lattice automorphism. We will assume here that (4.6) is also valid for our orbifold construction.

Let us now consider initially a group element g of order n without any phase anomaly so that $E_0 \in \mathbb{Z}/n$. The more general situation will be discussed later on. The associated Thompson series T_g is therefore $\Gamma_0(n)$ invariant without phases as follows

$$\begin{pmatrix} a & b \\ cn & d \end{pmatrix}: g \square_1 = g^d \square_1 = g \square_1, \tag{4.7}$$

since for $(d, n) = 1$ we have $T_g = T_{g^d}$. This follows from the reality of the characteristic equation for each unitary representation of g . The invariance of T_g under $\Gamma_0(n)$ corresponds to part (a) of the Moonshine properties when $h = 1$ and $N = n$. Of the 171 Thompson series, 120 are of this form. The remaining series with $h \neq 1$ which do have a global phase anomaly are dealt with later on.

Let us now turn to the reasons for any AL invariance of T_g and the hauptmodul property for the full invariance group $\Gamma_g = \Gamma_0(n) + e_1, e_2, \dots$ as given in part (b) of the Moonshine properties. We will show that this will follow from the specific nature of any tachyonic singularities present in the twisted sectors H_{g^k} . Consider $T_g(\tau)$ as a function on the fundamental region $\mathcal{F}_n = H/\Gamma_0(n)$. The definition (3.5) implies that T_g can be singular only at the parabolic cusps of \mathcal{F}_n contained in $Q \cup \{i\infty\}$ which are described in the Appendix. There we state and prove a lemma in which a convenient set of inequivalent parabolic cusps for $\Gamma_0(n)$ is determined. Consider next the action of $\rho \in \mathcal{N}(n)$ on \mathcal{F}_n . Let $\tau_1 \sim \tau_2$ denote the equivalence relation: $\tau_1 = \gamma\tau_2$ for $\gamma \in \Gamma_0(n)$. Then $\rho\tau_1 \sim \rho\tau_2$ so that an automorphism is induced on \mathcal{F}_n . In particular, all the parabolic cusps are mapped into each other by ρ . Of particular importance is the cusp point τ_e generated by the action of an AL transform W_e of $\Gamma_0(n)$ on $i\infty$:

$$\begin{aligned} \tau_e = W_e(i\infty) &= \frac{ae}{cn} \sim \infty, & e = 1, \\ &\sim 0, & e = n, \\ &\sim \frac{1}{f}, & e \neq 1, n, \end{aligned} \tag{4.8}$$

where $f = n/e$, $(e, f) = 1$. We refer to τ_e as an AL cusp from now on. From the Appendix we see that τ_e of (4.8) is amongst the list given. The full set of AL cusps $\{i\infty, 0, \tau_{e_1}, \tau_{e_2}, \dots\}$ is mapped into itself under any AL transformation for $\Gamma_0(n)$ by the closure property (3.12). Thus

$$W_{e_1}(\tau_{e_2}) = \tau_{e_3}, \quad e_3 = \frac{e_1 e_2}{(e_1, e_2)^2}. \tag{4.9}$$

This closure property is important in proving AL invariance for T_g . In general, the complete set of parabolic cusps of \mathcal{F}_n can be generated by acting on $i\infty$ with the remaining elements of $\mathcal{N}(n)$.

Physically, we see from (4.6) that the behaviour of T_g at these cusp points is determined by the presence of tachyonic states in the twisted sectors. Consider any rational point a/c with $(a, c) = 1$. Then we can find integers b, d such that $ad - bc = 1$. These integers form the components of an element γ of Γ which maps $\tau = i\infty$ to a/c . Therefore, the behaviour of T_g at a/c is given by the behaviour of the trace over the twisted sector H_{g^c} at $\tau = i\infty$ according to (4.6). This is singular when L_0 has negative vacuum energy, i.e. H_{g^c} contains a tachyonic state. The behaviour of T_g at one of the AL cusps τ_e of (4.8) is therefore determined by the presence of a tachyon in the sector twisted by g^f of order e [which must also present in the sector twisted by g^{cf} , where $(cf, e) = 1$]. The behaviour at the remaining cusps, associated with the other normalizers, is determined by sectors twisted by g^k of order $n' = n/(n, k)$, where $(n', k) \neq 1$. An explicit construction of the twisted sectors would reveal whether or not the vacuum energy is tachyonic. We will postulate the nature of the vacuum energy below and confirm these postulates for some elements of M given

by Leech lattice automorphisms in Sect. 5. Specifying these singularities will allow us to prove the hauptmodul property for an appropriate fixing group Γ_g .

Consider next the action of an AL transformation W_e on $T_g(\tau)$. We can relate this to a twisted sector trace by observing that

$$W_e = \begin{pmatrix} ae & b \\ cn & de \end{pmatrix} = \gamma \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix}, \tag{4.10a}$$

$$\gamma = \begin{pmatrix} a & b \\ cf & de \end{pmatrix}, \tag{4.10b}$$

where $\gamma \in \Gamma$ because $\det W_e = e$. Acting with W_e on T_g and applying (4.6) we find that the phase factor is one with

$$T_g(W_e(\tau)) = g^{-de} \underset{g^{cf}}{\square}(e\tau), \tag{4.11}$$

where the right-hand side denotes a trace over $H_{g^{cf}}$ followed by a rescaling of τ by e . (This relation is a generalisation of the fact that the Fricke involution $W_n: \tau \rightarrow -1/n\tau$ is an S transform followed by a rescaling with n .) Note that the right-hand side is also $\Gamma_0(n)$ invariant since W_e is in $\mathcal{N}(n)$.

The set of matrices W_e for a given e form a single coset of $\Gamma_0(n)$ in $\mathcal{N}(n)$. Therefore, (4.11) is the same for all c, d provided $\det W_e = e$. This is equivalent to the condition

$$(de, cf) = 1. \tag{4.12}$$

We can choose for convenience the W_e coset representative with $a = c = 1$ for $e \neq n$ and $a = 0, c = 1$ for $e = n$ in (4.10). These choices correspond to the AL cusp representatives in (4.8). Therefore, W_e acting on T_g is equivalent to a twisted sector H_{g^f} contribution with τ rescaled by e . This in turn is equivalent to a $H_{g^{cf}}$ trace where $(c, e) = 1$. From (4.12) we note that g^{cf} is of order e so that the unscaled trace in (4.11) should be T^e invariant. This is indeed the case since (4.11) is T invariant.

The property expressed by (4.11) for AL transformations is unique to these elements of the $\mathcal{N}(n)$ in the following sense. If $\varrho \in \mathcal{N}(n)$ can be expressed as a Γ transformation followed by a general linear transformation ($\tau \rightarrow A\tau + B$) then one can show that the cusp τ_q must be an AL cusp. Thus the AL cusps have a special relation to twisted sector contributions.

The final stage of the argument is to specify the nature of any tachyons in the twisted sectors and thereby prove the hauptmodul property. We will demonstrate this initially for T_g a $\Gamma_0(p)$ invariant for p prime. The fundamental region \mathcal{F}_p has only two inequivalent cusps at $i\infty$ and 0 which are interchanged by the unique normaliser W_p . Applying (4.11) we find

$$W_p: T_g(\tau) = 1 \underset{g}{\square}(p\tau). \tag{4.13}$$

As discussed in Sect. 3 part (b) of the Moonshine properties implies that either (a) $T_g(\tau)$ is inverted by W_p in the sense of (3.17) or (b) $T_g(\tau)$ is W_p invariant. Expanding (4.13) in q we find this implies

$$1 \underset{g}{\square}(p\tau) = T_g(0) + cq + \dots, \quad W_p \notin \Gamma_g, \tag{4.14a}$$

$$= \frac{1}{q} + 0 + \dots, \quad W_p \in \Gamma_g, \tag{4.14b}$$

where c is the constant in (3.17). Therefore, the twisted sector H_g either has (a) non-negative vacuum energy E_0 or (b) negative vacuum energy $E_0 = -1/p$ with degeneracy 1. The second singular case corresponds to a tachyon in H_g . We now show that these vacuum energy conditions are sufficient to ensure that T_g is a hauptmodul for either (a) $\Gamma_0(p)$ or (b) $\Gamma_0(p)+$. For case (a) this is immediate since $T_g(\tau)$ is regular at the cusp $\tau=0$. Therefore, T_g is $\Gamma_0(p)$ invariant with a unique simple pole at $q=0$. Applying the Hauptmodul Theorem of Sect. 2 implies that T_g is a hauptmodul for $\Gamma_0(p)$.

Consider now case (b) where H_g contains a tachyon of energy $-1/p$. Then by (4.14b) we have $T_g(W_p\tau) = q^{-1} + \dots$ and so the function

$$\psi(\tau) = T_g(\tau) - T_g(W_p(\tau)) \tag{4.15}$$

is regular at both $\tau=0$ and $\tau=i\infty$. The vacuum energy and degeneracy must be as given to ensure this. The function ψ is $\Gamma_0(p)$ invariant defining a holomorphic function on \mathcal{F}_p and therefore must be constant. Acting with W_p again shows that $\psi=0$. Therefore, $T_g(\tau)$ is invariant under $\Gamma_0(p)+$. The cusps at 0 and $i\infty$ are identified under W_p and so $T_g(\tau)$ has a unique simple pole on $H/\Gamma_0(p)+$. By the Hauptmodul Theorem we again find that T_g is a hauptmodul for $\Gamma_0(p)+$.

These arguments easily generalise to the case of T_g a $\Gamma_0(n)$ invariant for n not prime. We must specify the vacuum energy of each twisted sector H_{g^k} to reproduce the Moonshine properties as follows:

Vacuum Conjectures. *Suppose that H_g has no global phase anomalies so that T_g is $\Gamma_0(n)$ invariant with g of order n . Let $n' = n/(n, k)$ denote the order of g^k .*

I. *The vacuum energy E_0 for H_{g^k} is non-negative for $(n', k) \neq 1$. If $(n', k) = 1$ then E_0 is either (a) non-negative or (b) negative (tachyonic) with energy $-1/n'$ and degeneracy one.*

II. *(Closure) Let $e_1 \parallel n$ and $e_2 \parallel n$. If both sectors $H_{g^{f_1}}$ and $H_{g^{f_2}}$ are tachyonic then the sector $H_{g^{f_3}}$ is also tachyonic where $e_3 = e_1 e_2 / (e_1, e_2)^2$ and $f_3 = n/e_3$.*

We now show that these properties are sufficient to ensure that T_g is a hauptmodul for an appropriate group Γ_g . Applying I of the Vacuum Conjectures to (4.11) we find that either (a) $T_g(\tau)$ is regular at the AL cusp τ_e or (b) $T_g(W_e(\tau)) \sim q^{-1} + \dots$. These conditions coincide with Norton's observation that τ_e is a singularity of $T_g(\tau)$ if and only if $T_{g^f}(\tau)$ is invariant under the Fricke involution $\tau \rightarrow -1/e\tau$ [20]. The closure condition II is required to give invariance under more than one AL transformation. Thus if T_g is singular at τ_{e_1} and τ_{e_2} it is also singular at τ_{e_3} . In general, the singularities of T_g are given by $\{\tau_{e_1}, \tau_{e_2}, \dots\}$ a subset of all AL cusps which closes under the corresponding AL transformations W_{e_1}, W_{e_2}, \dots by (4.9).

We may now repeat the argument above for $n=p$. Suppose that T_g is singular at τ_{e_1} and consider $\psi(\tau) = T_g(\tau) - T_g(W_{e_1}(\tau))$. Clearly, ψ is regular at $\tau = i\infty$ and τ_{e_1} by I(a) of the Vacuum Conjectures. Likewise, for any other singular AL cusp τ_{e_2} we have $W_{e_1}(\tau_{e_2}) = \tau_{e_3}$ which by II is also a singular cusp. Therefore, $\psi(\tau)$ is regular also at τ_{e_2} . Notice that both the degeneracy and order of the poles must be as given to ensure this. Hence ψ is holomorphic on \mathcal{F}_n and is therefore a constant. Applying W_{e_1} again to ψ shows that $\psi=0$ and so $T_g(\tau)$ is W_{e_1} invariant. This is similarly true for any other singular AL cusp. Therefore, $T_g(\tau)$ is invariant under $\Gamma_g = \Gamma_0(n) + e_1, e_2, \dots$ if it is singular at $\{\tau_{e_1}, \tau_{e_2}, \dots\}$. This concurs with part (a) of the Moonshine properties for $h=1, N=n$. The associated fundamental region \mathcal{F}_g has

at least one cusp at $i\infty$ which is identified with $\{\tau_{e_1}, \tau_{e_2}, \dots\}$ under the corresponding AL transformations. T_g has a unique simple pole at this cusp and therefore, T_g is a hauptmodul for the genus zero group $\Gamma_0(n) + e_1, e_2, \dots$ by the Hauptmodul Theorem in agreement with part (b) of the Moonshine properties as required.

Let us return to the remaining Thompson series for which the twisted sector H_g has a global phase anomaly as in (4.3). These series are expected to be invariant up to a phase of order h under $\Gamma_0(n|h) + e_1, e_2, \dots$ with $h|24, h|n$ according to the Moonshine properties. We find it necessary to introduce further assumptions to reproduce this result. We will assume that a global phase anomaly can only occur when T_g is invariant under $T^{\frac{1}{h}}$ as in (3.18). Therefore, only every h^{th} level of V^a contributes. We will also assume that $h|n$ and $h|24$. Lastly, we will assume that the global phase anomaly for H_g is $\phi_g = \frac{l}{h}$ for some integer l . Therefore, from (4.4) we have

$$ST^n S : g \begin{matrix} \square \\ 1 \end{matrix} = e^{2\pi i l/h} g \begin{matrix} \square \\ 1 \end{matrix}. \tag{4.16}$$

The assumption that $h|24$ is not unreasonable when one considers that the normal ordering constant for L_0 is typically $-C/24$ for a twisted space associated with a lattice automorphism for integral central charge C . From (4.6) it is also clear that T_g is $\Gamma_0(N)$ invariant without any phases for $N = nh$. Altogether, with the above assumptions we see that T_g is $\Gamma_0(n|h)$ invariant up to phases of order h . The invariance group without phases contained in this will be denoted by Γ'_g [which contains $\Gamma_0(N)$].

We will now reinterpret $T_g(\tau)$ as an orbifold contribution arising from a theory defined on a new world-sheet torus: $w \sim w + 1 \sim w + \tau'$, where $\tau' = h\tau$. On this new torus we may apply the methods employed above to obtain AL invariance and the hauptmodul property. Since $T_g(\tau)$ is $T^{\frac{1}{h}}$ invariant up to a phase $e^{-2\pi i/h}$ only the states at level $E = kh - 1$ contribute for $k = 0, 1, 2, \dots$. Denoting these states by $|k'\rangle_1^i$ we find that

$$e^{2\pi i L_0} |k'\rangle_1^i = e^{-2\pi i/h} |k'\rangle_1^i, \tag{4.17}$$

where $L'_0 = L_0/h$. We may thus regard $|k'\rangle_1^i$ as an untwisted state with a global phase anomaly $e^{-2\pi i/h}$ and with Virasoro Hamiltonian L'_0 . The transformation (4.17) represents the periodicity of $|k'\rangle_1^i$ under $T' : \tau' \rightarrow \tau' + 1$, i.e. under $T^{\frac{1}{h}}$. Referring to Fig. 2 we can see that this corresponds to a modular transformation

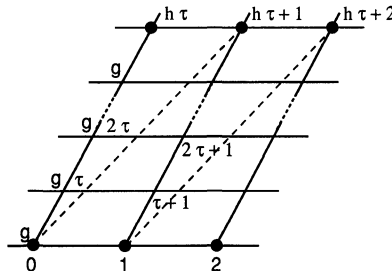


Fig. 2. The world sheet torus is determined by the lattice generated by 1 and τ . The new lattice generated by 1 and τ' is invariant under $\tau' = h\tau \rightarrow \tau' + 1$

on a new torus $w \sim w + 1 \sim w + \tau'$. In addition, the boundary condition in the space direction is now g^h for a boundary condition of g on the original torus. Therefore, the trace $T_g(\tau)$ is equivalent to a trace of g^h over $|k'\rangle_1^i$ with L'_0 as above on the new torus:

$$\begin{aligned} T_g(\tau)g\Box_1 &= g^h\Box_1(\tau') \equiv T_{g^h}(\tau'), \end{aligned} \tag{4.18}$$

where the prime denotes the new torus trace. Applying (4.16) we find that $T_{g^h}(\tau')$ is $S'T'^n S'$ invariant up to the phase $e^{2\pi i l/h}$. In general, T_{g^h} is $\Gamma_0\left(\frac{n}{h}\right)$ invariant up to phases.

On the primed torus the sector twisted by g^h has a global phase anomaly $\phi'_{g^h} = l/h$. From (4.5) we see that the levels E' of L'_0 in this sector therefore obey $E' = (l + hk)/n$ for $k \in \mathbb{Z}$. Each state of L'_0 corresponds to a state on the original torus with $L_0 = hL'_0$ and level

$$E = hE' = (l + hk)\frac{h}{n}. \tag{4.19}$$

Therefore, from (4.5), since g^h is of order n/h , these twisted states have no global phase anomaly on the original torus. This observation will allow us to apply the Vacuum Conjectures to sectors twisted by g^h and related sectors. This will be sufficient to prove that T_g is a hauptmodul for an appropriate fixing group Γ_g .

Consider the action of an AL transformation w_e of $\Gamma_0(n|h)$ as in (3.13) on $T_g(\tau)$. Then $\tau' = h\tau$ becomes

$$hw_e(\tau) = \frac{ae(h\tau) + b}{c\frac{n}{h}(h\tau) + de} = W'_e(\tau'), \tag{4.20}$$

where W'_e is an AL transformation for $\Gamma_0\left(\frac{n}{h}\right)$. Now we can apply the decomposition (4.10) to W'_e to find that

$$\begin{aligned} T_g(w_e(\tau)) &= T_{g^h}(W'_e(\tau')) \\ &= e^{i\phi} g^{-hde} \Box_{g^{hcf}}'(e\tau'), \end{aligned} \tag{4.21}$$

where $f = n/eh$, $(e, f) = 1$ and $e^{i\phi}$ is some phase of order h that depends on the components of w_e . The trace is over a sector twisted by $g^{hcf} = g^{nc/e}$ of order e with $(de, cf) = 1$ as before in (4.12). Since W'_e is in the normaliser of $\Gamma_0\left(\frac{n}{h}\right)$ we expect invariance up to phase under T' which is consistent with the order of the twisting being e . Likewise, $w_e \in \mathcal{N}(N)$ and T_g is $\Gamma_0(N)$ invariant so that we expect $T = T'^h$ invariance without a phase. This confirms that $e^{i\phi}$ is of order h .

We may now apply a similar argument as given previously. Any singular behaviour of T_g at the AL cusp $w_e(i\infty)$ is due to the presence of a tachyon in the primed torus sector twisted by $g^{n/e}$ [and therefore also $g^{nc/e}$ for $(c, e) = 1$]. From (4.21) it is clear that T_g can only be w_e invariant up to a phase if the vacuum energy for the primed torus sector twisted by $g^{n/e}$ is $-1/eh$. Therefore, from (4.19) we see that the sector twisted by $g^{n/e}$ on the original torus contains a tachyon of energy

$-1/e$. This concurs with the Vacuum Conjectures applied to g^h , which we have argued has no global phase anomaly on the original torus, where we consider the sector twisted by $(g^h)^{n/eh} = g^{n/e}$ of order e with $e \parallel n/h$. The singularities of T_g are therefore supplied also by the Vacuum Conjectures via (4.19) and (4.21). We now show that T_g is a hauptmodul for some appropriate group.

It is convenient to consider the modular function

$$\psi_g(\tau) = [T_g(\tau/h)]^h. \tag{4.22}$$

This is $\Gamma_0\left(\frac{n}{h}\right)$ invariant without phases. Applying the Vacuum Conjectures for g^h we have that either (a) $\psi_g(W_e(\tau)) = c + \dots$ or (b) $\psi_g(W_e(\tau)) = q^{-1} + \dots$, where W_e is a $\Gamma_0\left(\frac{n}{h}\right)$ AL transformation. We may then argue that ψ_g is W_e invariant for every singular AL cusp of $T_g(\tau)$ as before. However, the invariance group for ψ_g is exactly that for the hauptmodul $T_{g^h}(\tau)$ because the singularities arise from the same twisted sectors. Since they both have a simple pole at $q=0$ we have

$$[T_g(\tau/h)]^h = T_{g^h}(\tau) + c \tag{4.23}$$

for some constant c . This is the so-called Harmonic Formula of Conway and Norton [3] which they derive from the Power Map Formula discussed below.

The invariance of ψ_g above implies that $\Gamma_0(n|h) + e_1, e_2, \dots$ is the invariance group of T_g up to phases of order h . This corresponds to part (a) of the Moonshine properties for $h \neq 1$. The invariance group Γ_g of T_g is a subgroup of this generated by Γ'_g [where $\Gamma_0(N) \subseteq \Gamma'_g \subseteq \Gamma_0(n|h)$] and any AL transformation w_e which leaves T_g invariant. From (3.13) we know that w_e is a single coset for $\Gamma_0(n|h)$ but not for Γ'_g .

Thus for $\gamma \in \Gamma_0\left(\frac{n}{h}\right)$ we can define $w'_e = w_e \gamma$ such that w'_e and w_e act on T_g with a different phase, e.g. choose $\gamma = T^{k/h}$ for $k = 0, 1, 2, \dots$. Nevertheless, the corresponding W_e action on ψ_g in (4.22) is independent of γ . Likewise, the singular cusp of T_g associated with w_e is unique as the choice of $\gamma = T^{k/h}$ indicates. We may therefore choose a representative, denoted by \hat{w}_e , from the set of matrices w_e which has no phase. Then $T_g(\tau)$ is invariant under Γ_g given by adjoining to Γ'_g the appropriate AL transformations $\hat{w}_{e_1}, \hat{w}_{e_2}, \dots$. As before, T_g is only singular at the corresponding AL cusps which are now identified under Γ_g so that T_g is a hauptmodul for the genus zero group Γ_g .

The Vacuum Conjectures can be further exploited to prove the Power Map Formula of Conway and Norton [3]. This result does not follow from the Moonshine properties but is an independent empirical observation.

Power Map Formula. *Suppose T_g is invariant up to phases under $\Gamma_0(n|h) + e_1, e_2, \dots$. Then for any d , T_{g^d} is invariant up to phases under $\Gamma_0(n'|h') + e'_1, e'_2, \dots$, where $n' = n/(n, d)$, $h' = h/(h, d)$ and e'_1, e'_2, \dots are the divisors of n'/h' amongst the numbers e_1, e_2, \dots .*

Let us prove this initially for T_g without a global phase anomaly so that $h = h' = 1$ above. The general result will then follow from the Harmonic Formula (4.23). Consider $g' = g^d$ of order $n' = n/(n, d)$. Then $T_{g'}$ is $\Gamma_0(n')$ invariant since no global phase anomaly occurs for any g^k sector. If $T_{g'}$ is $W_{e'}$ invariant ($e' \parallel n'$) then the vacuum energy is $-1/e'$ for the sector twisted by $(g')^{n'/e'} = (g^{n/e})^{d'}$ of order e' , where $d' = d/(n, d)$ and $(d', e') = 1$. But T_g is singular only at its AL cusps by the Vacuum

Conjectures and so the sector twisted by $(g^{n/e'})^{d'}$ can only be tachyonic provided $e' \parallel n$ by (4.6). Therefore, $W_{e'}$ is an AL symmetry of T_g also. On the other hand, if W_e is an AL symmetry for T_g then the sector twisted by $(g^{n/e})^c$ of order e , $(c, e) = 1$, has vacuum energy $-1/e$. This sector can only influence the behaviour of $T_{g'}$ provided $e \parallel n'$ (and hence $e \parallel n'$ implying that $T_{g'}$ is W_e invariant). If $e \not\parallel n'$ then $(e, d) \neq 1$ and so $(g')^k = g^{dk}$ cannot be of order e for any k . Therefore, no modular transformation of $T_{g'}$ can introduce this tachyonic sector and hence $T_{g'}$ cannot be W_e invariant. To summarise, we find that if T_g is $\Gamma_0(n) + e_1, e_2, \dots$ invariant then T_{g^d} is $\Gamma_0(n') + e'_1, e'_2, \dots$ invariant where e'_1, e'_2, \dots are the divisors of $n' = n/(n, d)$ amongst e_1, e_2, \dots . This corresponds to the Power Map Formula for $h = h' = 1$.

The general result now follows by using the Harmonic Formula (4.23). We apply the global phase anomaly free Power Map Formula to T_{g^h} with power $d' = d/(d, h)$. Then $T_{g^{d'h}}$ is $\Gamma_0(m) + e'_1, e'_2, \dots$ invariant where

$$m = \frac{\frac{n}{h}}{\left(\frac{n}{h}, d'\right)} = \frac{n'}{h'} \tag{4.24}$$

with $n' = n/(n, h)$ and $h' = h/(h, d)$ and where e'_1, e'_2, \dots are the divisors of m amongst e_1, e_2, \dots . The relation (4.24) follows from the identity $(n, d) = (h, d)(n/h, d')$. Finally, noting that $g^{d'h} = (g^d)^{h'}$ and applying the Harmonic Formula (4.23) we find that T_{g^d} is invariant up to phases under $\Gamma_0(n'|h') + e'_1, e'_2, \dots$ as claimed.

5. Leech Lattice Automorphisms

In the last section we derived the Moonshine properties for a Thompson series T_g by assuming the Vacuum Conjectures for the associated twisted spaces. In this section we will describe the construction of such spaces for a subgroup of M based on Leech lattice automorphisms and demonstrate the validity of the Vacuum Conjectures in many such cases.

We begin by returning to the untwisted Hilbert space $H_{(+)}$ of (2.5). The highest weight state $|\beta\rangle$, $\beta \in \mathcal{A}$ can be constructed by introducing the vertex operator $V(\beta, z)$ as follows [1, 11]

$$V(\beta, z) = :e^{i\beta \cdot x(z)}: c(\beta), \tag{5.1a}$$

$$|\beta\rangle = V(\beta, 0)|0\rangle, \tag{5.1b}$$

where $c(\beta)$ is an element of the central extension $\hat{\mathcal{A}}$ of \mathcal{A} by $\{\pm 1\}$ which is required to ensure the associativity of the vertex operator algebra. The coefficients $c(\beta)$ obey the cocycle conditions

$$\begin{aligned} c(\alpha)c(\beta) &= \varepsilon(\alpha, \beta)c(\alpha + \beta), \\ \varepsilon(\alpha, \beta) &= (-1)^{\alpha \cdot \beta} \varepsilon(\beta, \alpha). \end{aligned} \tag{5.2}$$

Consider \bar{a} an element of the automorphism group of the Leech lattice known as the Conway group $\cdot 0$, i.e. $\bar{a}: \mathcal{A} \rightarrow \mathcal{A}$ and $\bar{a}\alpha \cdot \bar{a}\beta = \alpha \cdot \beta$ for all $\alpha, \beta \in \mathcal{A}$. Each such automorphism induces a natural automorphism on $H_{(+)}$ by

$$\bar{a}: |\beta\rangle \rightarrow |\bar{a}\beta\rangle \tag{5.3}$$

with the remaining Fock states (2.5) transforming in an obvious way. This transformation preserves L_0 and hence the states at each level transform amongst themselves.

From the point of view of the vertex operator (5.1 a) there is a larger class of related automorphisms available. These are the automorphisms of $\hat{\lambda}$, denoted by C_0 , that preserves (5.2). C_0 is a central extension by Z_2^{24} of $\cdot 0$ denoted by $C_0 = 2^{24}(\cdot 0)$. (This situation is familiar for a Lie algebra of rank l where the group of Weyl reflections W is extended by Z_2^l to form the discrete automorphism group $Z_2^l(W)$ [24]). For $a \in C_0$ we have [1, 20]

$$a : c(\beta) \rightarrow (-1)^{f_a(\beta)} c(\bar{a}\beta), \tag{5.4a}$$

$$a : |\beta\rangle \rightarrow (-1)^{f_a(\beta)} |\bar{a}\beta\rangle, \tag{5.4b}$$

where $\bar{a} \in \cdot 0$ is associated with a and where $f_a(\beta)$ is a Z_2 central extension of \bar{a} obeying $f_a(\alpha + \beta) = f_a(\alpha) + f_a(\beta)$. It is useful to realise $f_a(\alpha)$ as follows. Let $\{e^{(r)}\}$ be a basis for \mathcal{A} and $\{e_{(s)}\}$ a dual basis with $e^{(r)} \cdot e_{(s)} = \delta_s^r$. Then for $\alpha = \alpha_e e^{(r)}$ we have $f_a(\alpha) = \alpha_e m^r$ with $m^r = f_a(e^{(r)}) \in Z_2$. Defining the lattice vector $m = m^r e_{(r)}$ we have for any $\alpha \in \mathcal{A}$

$$f_a(\alpha) = e^{i\pi\alpha \cdot m}. \tag{5.5}$$

The set of inequivalent choices of m for a given \bar{a} is given by $\mathcal{A}/2\mathcal{A}$ of dimension 2^{24} giving $C_0 = 2^{24}(\cdot 0)$. The transformation (5.3) then corresponds to the trivial extension $m=0$.

We turn now to the Monster Module V^h and describe the role played by such automorphisms. We first describe the automorphism group \hat{C}_0 of the full set of states $H_{(+)} \oplus H_{(-)}$ where no mixing between untwisted and twisted sectors occurs. A Z_2 quotient of \hat{C}_0 then provides the appropriate group for V^h .

The automorphism group C_0 for $H_{(+)}$ induces an automorphism group on $H_{(-)}$ of (2.14) as follows. Recall that the vacuum states of $H_{(-)}$ are associated with the fixed points $\mathcal{A}/2\mathcal{A}$ of $g : x \rightarrow -x$. Let $K = \{c(2\alpha) | \alpha \in \mathcal{A}\}$ which is a subgroup of $\hat{\lambda}$ isomorphic to $2\mathcal{A}$ [1, 20]. From (5.5) we have $f_a(2\alpha) = 1$ so K is also invariant under C_0 . Therefore, C_0 induces an automorphism group on $\hat{\lambda}/K \cong 2_+^{1+24}$ the ‘‘extra special’’ group of order 2^{25} [1, 2]. This automorphism group has a unique 2^{12} -dimensional representation acting on a vector space denoted by T . The vacuum states $\{|\sigma_a\rangle\}$ of $H_{(-)}$ introduced in Sect. 2 form a set of basis vectors for T . The representation acting on T is in fact a projective representation of C_0 with a two-fold covering [1, 20] where defining

$$\hat{C}_0 = \{(a, a_T) \in C_0 | a_T : T \rightarrow T\} \tag{5.6}$$

we have $C_0 = \hat{C}_0 / \{I, \pm I_T\}$ where $I(I_T)$ is the identity of $C_0(T)$. The group \hat{C}_0 is then the automorphism group of $H_{(+)} \oplus H_{(-)}$ where no mixing between untwisted and twisted states occurs, i.e. all elements of \hat{C}_0 commute with $i = (I, -I_T)$ which is the ‘‘fermion number’’ introduced in Sect. 2.

Consider next the g invariant projection V^h of $H_{(+)} \oplus H_{(-)}$ and notice that the element $(-I, -I_T)$ has eigenvalue $+1$ for all states of V^h from (2.9 a) and (2.16a). Therefore, the automorphism group of V^h that commutes with i is

$$C = \hat{C}_0 / \{\pm(I, I_T)\} = 2_+^{1+24}(\cdot 1), \tag{5.7}$$

where $\cdot 1 = \cdot 0 / \{\pm 1\}$ is the Conway simple group. The group C is the centraliser of i in M . The correct identification of C with $2_+^{1+24}(\cdot 1)$ is a crucial part of the Monster

group construction since Griess has shown that C together with only one other involution element σ generates all of M [2]. FLM [1] constructed σ (which mixes $H_{(+)}$ and $H_{(-)}$) from a triality symmetry [25] inherent in the Monster Module construction to show that V^{\natural} has M as its automorphism group.

FLM have also shown how to compute the Thompson series T_g for any $g \in C$. Let $g = (a, a_T) \in C$ corresponding to $\pm(a, a_T) \in \hat{C}_0$. We will define the basic modular functions that arise in calculating $T_g(\tau)$. The matrix $\bar{a} \in \cdot 0$ associated with a has characteristic polynomial

$$\det(x - \bar{a}) = \prod_{k|n} (x^k - 1)^{a_k}, \tag{5.8}$$

where $\sum ka_k = 24$ and where each a_k is a (not necessarily positive) integer with \bar{a} of order n . For any such \bar{a} we can define the generalised eta function

$$\eta_{\bar{a}} = \prod_{k|n} (\eta(k\tau))^{a_k}. \tag{5.9}$$

Likewise, we can define the theta function for \bar{a}

$$\Theta_{\bar{a}}(\tau) = \sum_{\beta \in \Lambda_{\bar{a}}} q^{\beta^2/2}, \tag{5.10}$$

where $\Lambda_{\bar{a}} = \{\beta \in \Lambda \mid \bar{a}\beta = \beta\}$ is the invariant lattice for \bar{a} of dimension $d = \sum a_k$, the number of unit eigenvalues of (5.8). We may also define a modified theta function for a by

$$\begin{aligned} \Theta'_a(\tau) &= \sum_{\beta \in \Lambda_{\bar{a}}} (-1)^{f_a(\beta)} q^{\beta^2/2} \\ &= \sum_{\beta \in \Lambda_{\bar{a}}} e^{i\pi\beta \cdot m_{\parallel}} q^{\beta^2/2}, \end{aligned} \tag{5.11}$$

where m_{\parallel} is the projection of m in (5.5) parallel to Λ_a , i.e. defining $\mathcal{P}_a = (1 + \bar{a} + \dots + \bar{a}^{n-1})/n$ we have $m_{\parallel} = \mathcal{P}_a m$ and $\beta \cdot m = \beta \cdot m_{\parallel}$ for $\beta \in \Lambda_a$.

We may now compute the Thompson series for g by expanding the partition function as

$$\begin{aligned} g_{\square} &= \frac{1}{2} \left[\underset{+}{a\square} + \underset{+}{-a\square} \right] \\ &\quad + \frac{1}{2} \left[\underset{-}{(a, a_T)\square} + \underset{-}{(-a, -a_T)\square} \right], \end{aligned} \tag{5.12}$$

where the right-hand side corresponds to traces over $H_{(+)}$ and $H_{(-)}$. Each of these traces can be separately computed from the Fock spaces (2.5) and (2.14) to obtain the closed form [1]

$$\begin{aligned} T_g(\tau) &= \frac{1}{2} \left[\frac{\Theta'_a(\tau)}{\eta_{\bar{a}}(\tau)} + \frac{\Theta'_{-a}(\tau)}{\eta_{-\bar{a}}(\tau)} \right] \\ &\quad + \frac{1}{2} \left[\text{Tr}(a_T) \frac{\eta_{\bar{a}}(\tau)}{\eta_{\bar{a}}(\tau/2)} + \text{Tr}(-a_T) \frac{\eta_{-\bar{a}}(\tau)}{\eta_{-\bar{a}}(\tau/2)} \right], \end{aligned} \tag{5.13}$$

where $\text{Tr}(\pm a_T)$ is a trace over the representation space T .

As described in Sect. 4, we expect $T_g(\tau)$ to be $\Gamma_0(n)$ invariant up to a global phase. Notice that we can also consider the first trace of (5.12) as a new Thompson series for the untwisted sector $H_{(+)}$ which we denote by $T_a^{(+)} = \underset{+}{a\square}$. Naively, it would appear that $T_a^{(+)}$ is also $\Gamma_0(n)$ invariant up to a phase. However, it is necessary to check that in the sector twisted by a , the automorphism \hat{a} induced by a is also of

order n . This is in general not the case as we shall see below. To understand this point it is useful to consider the modular properties of η_a and Θ'_a before describing the twisted sector construction.

Suppose that $h|k$ for all $a_k \neq 0$ then η_a is $T^{\frac{1}{h}}$ invariant up to a phase $e^{2\pi i/h}$ with $h|n, h|24$. Under S we find

$$\eta_a(-1/\tau) = \frac{(-i\tau)^{d/2}}{D^{1/2}} \prod_{k|n} (\eta(\tau/k))^{a_k}, \tag{5.14}$$

where

$$D = \prod_{k|n} k^{a_k}. \tag{5.15}$$

The modular function (5.14) is covariant under T^n up to a phase $e^{-2\pi i n E_0}$, where

$$E_0 = -\frac{1}{24} \sum_{k|n} \frac{a_k}{k} \tag{5.16}$$

which follows directly from the leading behaviour q^{-E_0} of the product of eta functions. These properties are clearly reproducing some of the features of the global phase anomaly $\phi_g = nE_0$ of (4.3) for vacuum energy E_0 . In particular, we note that $\phi_g = l/h'$ for integers l, h' where $h'|24$. At this stage h and h' are not yet related.

Consider next the theta functions Θ_a and Θ'_a . In many cases these are actually equivalent. Thus for n odd we have $f_a(\beta) = 0 \pmod{2}$ for all $\beta \in A_a$ and $\Theta_a = \Theta'_a$. Alternatively, if n is even and $a = b^2$, $b \in C_0$ then $f_a(\beta) = f_b(\beta) \pm f_b(\beta) = 0 \pmod{2}$ for $\beta \in A_a$, $b\beta = \pm \beta$ and again $\Theta_a = \Theta'_a$. In general, the modular properties of Θ'_a are as follows. If $h|\beta^2/2$ for all $\beta \in A_a$ then Θ'_a is $T^{1/h}$ invariant. Under S we have

$$\Theta'_a(-1/\tau) = \frac{(-i\tau)^{d/2}}{V_a} \Theta'^*_a(\tau) \tag{5.17}$$

using Poisson resummation [11, 10], where V_a is the lattice volume $|A^*_a/A_a|^{1/2}$ and Θ'^*_a is the modified theta function for the dual lattice A^*_a

$$\Theta'^*_a(\tau) = \sum_{\beta \in A^*_a} q^{(\beta^* + m_{||}/2)^2/2}. \tag{5.18}$$

The dual lattice is in fact equivalent to the parallel projection of A with $A^*_a = A_{||} \equiv \mathcal{P}_a A$ [15]. Thus (5.18) is invariant under $m_{||} \rightarrow m_{||} + 2\beta_{||}$ for $\beta_{||} \in A_{||}$ confirming that the inequivalent choices for m are given by $A/2A$. In general, the norm of $\beta_{||} \in A_{||}$ obeys

$$n\beta_{||}^2 = 0 \pmod{2}, \quad n \text{ odd}, \tag{5.19a}$$

$$n\beta_{||}^2 = 0 \pmod{1}, \quad n \text{ even}, \tag{5.19b}$$

which follows from the fact that $n\beta_{||} = n\mathcal{P}_a \beta \in A_a$.

Consider now the cases for which $m_{||} = 0$ and so $\Theta_a = \Theta'_a$. Considering (5.14) and (5.17) we find the S transform of $T_a^{(+)}$ to be

$$1 \square_{\bar{a}} = \frac{D^{1/2}}{V_a} \frac{\Theta^*_a(\tau)}{\prod (\eta(\tau/k))^{a_k}} = \frac{D^{1/2}}{V_a} q^{E_0} + \dots \tag{5.20}$$

Then (5.19) implies that $T_a^{(+)}$ is ST^nS invariant for n odd but $ST^{2n}S$ invariant for n even (up to a phase $e^{2\pi i n E_0}$ and $e^{4\pi i n E_0}$, respectively). This latter property reflects the

fact that in general, the automorphism \bar{a} induced by \bar{a} on the sector twisted by \bar{a} , can be of order $2n$ leading to a $Z/2n$ grading rather than the naively expected Z/n grading [26, 15]. Fortunately, for Leech lattice automorphisms, this problem does not arise for the following reasons. For n even, in general, we have using $\beta \cdot \bar{a}^k \cdot \beta = \beta \cdot \bar{a}^{n-k} \cdot \beta$ that [15]

$$n\beta_{\parallel}^2 = \beta \cdot \bar{b} \cdot \beta \pmod{2}, \quad (5.21)$$

where $\bar{b} = \bar{a}^{n/2}$ is of order 2. The Conway group contains three conjugacy classes of order 2 [23]. However, two of these classes are members of the same conjugacy class in M which leads, according to [3], to two independent ST^2S invariant Thompson series. Hence (5.21) implies that $n\beta_{\parallel}^2 = 0 \pmod{2}$ and therefore, $T_{\bar{a}}^{(+)}$ is ST^nS invariant up to the phase $e^{2\pi i n E_0}$ for all n . Similarly, the sum of remaining three traces of (5.12) is ST^nS invariant up to a phase with $-\bar{a} \square$ separately invariant for n even.

The situation for $m_{\parallel} \neq 0$ is more complicated. Then we find that $n\beta_{\parallel} \cdot m_{\parallel} \in Z$ so that $\mathcal{O}'_{\bar{a}}$ is $ST^{2n}S$ covariant up to the phase $e^{\pi i n m_{\parallel}^2/4}$. Therefore, the sector twisted by \bar{a} has $Z/2n$ grading. This implies that the induced automorphism \bar{a} in this sector is of order $2n$ in general. Nevertheless, the combination of traces in (5.14) is ST^nS invariant up to phase which presumably results from a transformation of the form

$$\begin{aligned} ST^nS[a \square] &= ST^n[+ \square] \\ &= e^{i\phi} S[- \square] = e^{i\phi} a \square \end{aligned} \quad (5.22)$$

for some phase $e^{i\phi}$ and similarly for the trace $-\bar{a} \square$.

Let us restrict ourselves from now on to the case where $m_{\parallel} = 0$ and construct the \bar{a} twisted sector [6, 27, 26]. The starting point is to consider the mode expansion for $x^i(z)$ satisfying the twisted boundary condition $x^i(z) = \bar{a}^{ij} x^j(e^{2\pi i} z) + \beta^i$, $\beta \in \mathcal{A}$ as in (2.12). (For $m_{\parallel} \neq 0$ this geometrical approach of quotienting T^{24} by a lattice automorphism is no longer valid and some different method of construction is required.) It is convenient to express \bar{a} in a diagonal basis with $\bar{a} = \begin{pmatrix} I_d & 0 \\ 0 & b \end{pmatrix}$, where I_d is the d -dimensional identity matrix and $b = \text{diag}(e^{2\pi i r_{d+1}}, \dots, e^{2\pi i r_{24}})$ for $0 < nr_i < n$. Defining the projections of x by $x_{\parallel} = \mathcal{P}_{\bar{a}} x$ and $x_T = (1 - \mathcal{P}_{\bar{a}})x$ we obtain the mode expansion

$$x_{\parallel}^i = q_{\parallel}^i - i p_{\parallel}^i \ln z + i \sum_{n \in Z} \frac{\alpha_n^i}{n} z^{-n}, \quad i = 1, \dots, d, \quad (5.23a)$$

$$x_T^i = q_T^i + i \sum_{n \in Z+r_i} \frac{\alpha_n^i}{n} z^{-n}, \quad i = d+1, \dots, 24, \quad (5.23b)$$

where q_{\parallel} , p_{\parallel} , and α_n obey the usual commutation relations (2.1). The vector q_T denotes any fixed point of T^{24} under \bar{a} . The mode expansion obeys the \bar{a} twisting condition provided

$$p_{\parallel} \in \mathcal{A}_{\parallel}, \quad q_T \in (1-b)^{-1} \mathcal{A}_{\bar{a}}^T, \quad (5.24)$$

where $\mathcal{A}_{\bar{a}}^T = \{\beta \in \mathcal{A} \mid \mathcal{P}_{\bar{a}} \beta = 0\}$ is the set of lattice vectors perpendicular to $\mathcal{A}_{\bar{a}}$. The set of inequivalent fixed points q_T is determined by the coset $\mathcal{A}_{\bar{a}}^T / (1-b) \mathcal{A}_T$, where $\mathcal{A}_T = (1 - \mathcal{P}_{\bar{a}}) \mathcal{A}$ [15]. In addition, just as $\mathcal{A}_{\parallel} = \mathcal{A}_{\bar{a}}^*$ so $\mathcal{A}_T = \mathcal{A}_{\bar{a}}^{T*}$. It is straightforward

to show also that $A_a^*/A_a = A_a^{T*}/A_a^T$ so that the lattice volume V_a is given by $|A_T/A_a^T|^{1/2}$. Therefore, the number of fixed points n_a of \bar{a} is [15]

$$\begin{aligned} n_a &= |A_a^T/(1-b)A_T| \\ &= V_a^{-2} \det(1-b) = \frac{D}{V_a^2}, \end{aligned} \tag{5.25}$$

where extracting a factor of $(x-1)^d$ from (5.9) we obtain $\det(1-b) = \prod k^{a_k} = D$ (5.15). The n_a fixed points determine the vacuum states $|q_T\rangle$ for the left and right moving theory. The left moving sector is expected from (5.20) to have a vacuum degeneracy of $n_a^{1/2} = D^{1/2}/V_a$ which is always an integer [15, 26]. Thus there must exist a set of left moving vacuum states $\{|\sigma_r^T\rangle\}$ for $r = 1, 2, \dots, n_a^{1/2}$. The full twisted Hilbert space is then constructed as a Fock space by acting with α_n^i on the highest weight states $|\beta_{||}\rangle \otimes |\sigma_r^T\rangle$, $\beta_{||} \in A_{||}$ generalising (2.24). The states are graded by the Virasoro Hamiltonian $L_0 = p_{||}^2/2 + \sum \alpha_{-n}^i \alpha_n^i + E_0$, where E_0 is the normal ordering constant $\frac{1}{2} \sum_i \sum_n (n+r_i)$ which provides the vacuum energy. By zeta function regularisation or otherwise we find [7]

$$E_0 = -1 + \frac{1}{4} \sum_{i=1}^{24} r_i(1-r_i). \tag{5.26}$$

This sum can be performed using the parametrisation (5.8) to obtain

$$E_0 = -1 + \frac{1}{4} \sum_{k|n} a_k \sum_{j=1}^{k-1} \frac{j}{k} \left(1 - \frac{j}{k}\right) = -1 + \frac{1}{24} \sum_{k|n} a_k \left(k - \frac{1}{k}\right) \tag{5.27}$$

which agrees with (5.16) on using $\sum ka_k = 24$. We may now compute the twisted sector partition function $1_{\bar{a}}^{\square}$ to obtain the expected result (5.20).

Let us next discuss the validity of the Vacuum Conjectures for the sectors twisted by \bar{a} . We will consider $T_{\bar{a}}^{(+)}$ for $\bar{a} \in \cdot 0$ and show that $T_{\bar{a}}^{(+)}$ is a hauptmodul for an appropriate modular group given some specific properties of the parameters a_k and the invariant lattice A_a . The remaining terms of (5.12) then sum to form the same hauptmodul differing only by a constant. It was originally proposed by Conway and Norton [3] that $T_{\bar{a}}^{(+)}$ is a hauptmodul for all $\bar{a} \in \cdot 0$. However, this conjecture turns out to be false in some cases². The specific reason for this failure in these cases is not known to us.

Consider the parameters a_k of (5.8) and suppose that for some h we have that $h|k$ for all $a_k \neq 0$. Then $\eta_{\bar{a}}$ is $T^{\frac{1}{h}}$ invariant up to a phase $e^{2\pi i/h}$. Let $e^{\left\| \frac{n}{h} \right\|}$ with $f = n/eh$ and decompose k uniquely for $a_k \neq 0$ as follows

$$k = hk_e k_f, \quad k_e = \left(\frac{k}{h}, e\right), \quad k_f = \left(\frac{k}{h}, f\right), \tag{5.28}$$

² In [20] it is stated that the first counter-example was found by Koike and that all counter-examples have been found by K. Harada and collaborator

where $(k_e, k_f) = 1$. The action of the associated AL transformation w_e of (3.13) on η_a can now be computed using

$$\begin{aligned}
 hk_e k_f w_e(\tau) &= \gamma \left(h \frac{e}{k_e} k_f \tau \right), \\
 \gamma &= \begin{pmatrix} ak_e & bk_f \\ cn/ehk_f & de/k_e \end{pmatrix},
 \end{aligned}
 \tag{5.29}$$

where $\gamma \in \Gamma$. Therefore, η_a is transformed into itself (up to phases and $c\tau + d$ factors) if the parameters a_k obey for all k

$$a_{hk_e k_f} = a_{h(e/k_e)k_f}. \tag{5.30}$$

On the other hand, η_a is inverted by w_e if

$$a_{hk_e k_f} = -a_{h(e/k_e)k_f}. \tag{5.31}$$

This second relation implies that $d=0$ so that $A_a=0$ and $T_a^{(+)}=1/\eta_a$ in this case.

We can now restate Conway and Norton's original conjecture about $T_a^{(+)}$ as follows: For most conjugacy classes of $\cdot 0$ in M , a representative \bar{a} can be chosen for each $e \parallel \frac{n}{h}$ such that either (5.30) or (5.31) is obeyed.

Notice that the w_e inversion property is stronger than the general Moonshine properties. However, in general, within each conjugacy class of $\cdot 0$ in M there are a number of conjugacy classes of $\cdot 0$ in itself, each of which has the same Thompson series up to a constant. The claim is that for some representative choice of \bar{a} within these classes, $T_a^{(+)}$ is w_e inverted for an appropriate constant. This claim is strengthened by the observation that all Thompson series listed in Table 3 of [3], which are not invariant under at least one AL transformation, have the form $T_a^{(+)}=1/\eta_a$ corresponding to $d=0$.

Assuming that (5.30) or (5.31) is true for a choice of \bar{a} , we can immediately calculate E_0 from (5.16) by choosing $e=n/h, f=1$. Then $a_k = \pm a_{k'}$ for $k'=nh/k$ and

$$E_0 = \mp \frac{1}{24nh} \sum_{k'|n} k' a_{k'} = \mp \frac{1}{nh} \tag{5.32}$$

which agrees with the vacuum energy of Sect. 4 for a global phase anomaly $e^{\pm 2\pi i/h}$.

We may also calculate the vacuum energy for powers of \bar{a} . Let $\bar{b} = \bar{a}^r$ which is of order $n' = n/(n, r)$ and which has characteristic equation parameters

$$b_{k'} = \sum_{k=k'(k, r)} (k, r) a_k, \tag{5.33}$$

where $k'|n'$. If $(n, r) = 1$ then $b_k = a_k$ and the same vacuum energy results. Choosing $r=h$ we obtain $b_k = ha_{hk'}$ for all $k'|n', n'=n/h$ and vacuum energy $E'_0 = \mp 1/n'$. This agrees with the discussion on the modified torus construction in Sect. 4, i.e. the sector twisted by g^h of order n' is without a global phase anomaly. It is therefore necessary to only consider \bar{a} with $h=1$ from now on.

Suppose next that $h=1$ and $\bar{b} = \bar{a}^r$ with $(n', r) = 1$, i.e. $n' = e$, where $e \parallel n$ and $r = cf$ with $f = n/e, (c, e) = 1$. Then we find that $(k, r) = k_f$ and $b_{k'} = \pm b_{e/k'}$ according to (5.30) or (5.31). Thus the \bar{b} sector vacuum energy is $\mp 1/e$ in agreement with the Vacuum Conjectures. Likewise, one can show that for $(n', r) \neq 1$, (5.30) can never be satisfied and therefore, the \bar{b} sector cannot be tachyonic. The correct tachyonic

singularity is therefore provided by (5.30) in each case. The residue of this singularity giving the vacuum degeneracy $n_a^{1/2}$ is discussed below.

We can also demonstrate the closure property of the Vacuum Conjectures from the symmetry relation (5.30). Thus if (5.30) is obeyed for $e_1 \parallel n$ and $e_2 \parallel n$ then a_k obeys (5.30) for $e_3 = e_1 e_2 / (e_1, e_2)^2$. This can be checked directly by applying (5.30) for e_1 followed by e_2 and showing that the resulting symmetry relation is that for e_3 . Alternatively, $W_{e_3} = W_{e_1} W_{e_2}$ preserves η_a covariantly and so (5.30) is obeyed for e_3 .

Let us turn now to the vacuum degeneracy $n_a^{1/2} = D^{1/2} / V_a$. For a tachyonic twisted sector with $h=1$ we expect (5.30) to be obeyed for $e=n$ to obtain (5.32). Substituting into (5.15) we obtain $D = n^{d/2}$ in these cases. Therefore, the tachyon is unique ($n_a = 1$) provided the invariant lattice volume obeys

$$V_a = n^{d/4}. \tag{5.34}$$

If $d=0$ then this is trivially true and $T_a^{(+)} = 1/\eta_a$ is W_n invariant (but is inverted by some W_e). For $d \neq 0$, (5.34) is a non-trivial condition on A_a . The pole residues are now correctly specified by (5.34) and therefore, $T_a^{(+)}$ is a hauptmodul for an appropriate fixing group $\Gamma_0(n) + e_1, e_2, \dots$. If $T_a^{(+)}$ is invariant under all AL involutions then (5.30) is obeyed for all $e \parallel n$, $h=1$ and $\Theta_a \neq 1$. In particular, W_n invariance implies that $\Theta_a^*(n\tau) = \Theta_a(\tau)$. Therefore, A_a must be \sqrt{n} self-dual, i.e. $A_a \equiv \sqrt{n} A_a^*$ for which (5.34) is naturally satisfied. This is a non-trivial property of such invariant sublattices of the Leech lattice.

Finally, let us consider as an example, the form of $T_a^{(+)}$ for \bar{a} of prime order p . All prime divisors of the Conway group $\cdot 0$ satisfy $(p \pm 1) | 24$ [23] which corresponds to W_p invariance (inversion) as follows. Suppose that $d=0$ and so \bar{a} has no unit eigenvalues. Then the characteristic polynomial (5.8) is completely determined with $a_p = -a_1 = r$, where $(p-1)r = 24$. Therefore, (5.31) is obeyed for $h=1$, $e=p$ giving $T_a^{(+)} = (\eta(\tau)/\eta(p\tau))^r$ which is inverted by W_p and is the hauptmodul for $\Gamma_0(p)$ as stated in (3.14).

If $d \neq 0$ then applying (5.30) for $e=p$ gives $a_p = a_1 = s$ with $(p+1)s = 24$. Then $T_a^{(+)} = \Theta_a / (\eta(\tau)\eta(p\tau))^s$ is a hauptmodul for $\Gamma_0(p) +$, where A_a is a $2s$ -dimensional \sqrt{p} self-dual lattice. For $p=2$ we have $\Theta_a = (\theta_3 \theta_4)^8 + 2^{-4} \theta_2^{16}$ [18] whereas for $p > 2$ a more suitable closed form can be computed from the $\Gamma_0(p)$ invariant sum of the three remaining traces in (5.14). Then we find

$$\psi(\tau) + \frac{2^s}{\psi(\tau/2)} - \frac{2^s \psi(\tau/2)}{\psi(\tau)} \tag{5.35}$$

with $\psi(\tau) = (\eta(\tau)\eta(p\tau))/\eta(2\tau)\eta(2p\tau)^s$ and where $\text{Tr}(\pm a_\tau) = \pm 2^s$ is found by $\Gamma_0(p)$ invariance. It is easy to check that (5.35) is $\Gamma_0(p) +$ invariant for $(p+1)s = 24$ with a unique pole at $q=0$ and is therefore a hauptmodul for $\Gamma_0(p) +$.

6. General Remarks and Conclusions

We have discussed the relationship between the Moonshine properties and an orbifold construction based on Monster group elements. We have shown that the correct vacuum structure for the corresponding twisted sectors guarantees the hauptmodul property for any Thompson series and the Power Law Formula observed by Conway and Norton. We will now briefly discuss a number of other features of this orbifold construction that we will expand on elsewhere [28].

In [8] it was noted that the remarks of Norton in the appendix of [20] about Generalised Moonshine also fit into an orbifold picture very naturally. Generalised Moonshine refers to the hauptmodul properties of a generalised Thompson series $T(g_1, g_2)$ given by the orbifold contribution

$$T(g_1, g_2)(\tau) = g_1 \square_{g_2}, \tag{6.1}$$

where g_1, g_2 are two elements of M that commute. Then, as stated in [20], $T(g_1, g_2)$ is unchanged under simultaneous conjugation of g_1 and g_2 . In addition, $T(g_1, g_2)$ transforms in the expected way under a modular transformation $\tau \rightarrow (a\tau + b)/(c\tau + d)$ with $T(g_1, g_2) \rightarrow T(g_1^a g_2^{-b}, g_1^{-c} g_2^d)$. We can also give the modular invariance group for $T(g_1, g_2)$ as follows. Let $n_1(n_2)$ be the order of $g_1(g_2)$ and let $h_1(h_2)$ be the corresponding divisor that describes the global phase anomaly for the sector twisted by $g_1(g_2)$. Then $T(g_1, g_2)$ is T^{n_2} invariant up to a phase $e^{\pm 2\pi i/h_2}$ and $ST^{n_1}S$ invariant up to a phase $e^{\pm 2\pi i/h_1}$. These phases are determined by the vacuum energy for the sector twisted by g_2 and g_1 , respectively. In general, we expect $T(g_1, g_2)$ to be invariant up to phases under

$$\Gamma(n_2, n_1) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 \bmod n_2 & 0 \bmod n_2 \\ 0 \bmod n_1 & 1 \bmod n_1 \end{pmatrix} \right\}.$$

Finally, any singular behaviour of $T(g_1, g_2)$ at a rational cusp point a/c is again determined by the presence of a tachyon in the sector twisted by $g_1^{-c} g_2^a$. The hauptmodul property for some appropriate group is then expected to follow from a similar argument as before.

In all our considerations thus far we have not examined the nature of the Monster partition function but have only concentrated on particular trace contributions to the full theory. Let us consider the example of the orbifold found by quotienting with respect to a Z_p subgroup of M generated by an element g of prime order p . The full modular invariant partition function $Z_{\text{orb}}(\tau)$ for this theory is given by [6, 7]

$$Z_{\text{orb}}(\tau) = \frac{1}{p} \sum_{l, k=0}^{p-1} g^k \square_{g^l}. \tag{6.2}$$

With p prime we can determine all of these traces from the Thompson series for g since $T_g(\tau) = T_{g^l}(\tau)$ for $l = 1, 2, \dots, p-1$ which implies that $g^k \square_{g^l} = T^k S(T_g(\tau))$ and so

$$Z_{\text{orb}}(\tau) = \frac{1}{p} J(\tau) + \frac{p-1}{p} \left(1 + \sum_{k=0}^{p-1} T^k S \right) T_g(\tau), \tag{6.3}$$

where $J(\tau)$ is the modular invariant partition function $1 \square_1$ for the Monster Module V^{\natural} . The remaining terms sum to form a modular invariant function with leading coefficient $(p-1)/p$ for the q^{-1} pole and therefore the sum gives $\frac{p-1}{p} J(\tau) + C$ for some constant C according to the Hauptmodul Theorem again. p

Suppose that the Thompson series T_g is inverted by the Fricke involution W_p . Then the g twisted space has no tachyon and T_g is given by (3.14), where $(p-1)|24$ and g corresponds to a Leech lattice automorphism without any fixed points. We then find that the partition function (6.3) becomes $Z_{\text{orb}}(\tau) = J(\tau) + 24$, where the constant is given by $C = (p-1)r = 24$. Thus the partition function coincides with that of the original theory defined on $T^{24} = R^{24}/\Lambda$. On the other hand, if the

Thompson series is invariant under W_p (so that the g twisted space contains a unique tachyon of energy $-1/p$) then (6.3) becomes $Z_{\text{orb}}(\tau) = J(\tau)$, i.e. the partition function is unchanged by this orbifolding procedure.

FLM have conjectured that the Monster Module is the unique (bosonic) CFT with light-cone central charge 24 (giving the tachyon of mass -1) without massless states [1]. This characterisation is analogous to the unique determination of the Leech lattice as the only even self-dual lattice in 24 dimensions without any norm 2 vectors. On the basis of the prime ordered examples above and this conjecture we therefore make the following three suggestions (in an increasing order of speculation):

- (a) The Monster Module can be alternatively constructed from the original untwisted theory on $T^{24} = R^{24}/\Lambda$ by Z_p orbifolding with respect to prime ordered Leech lattice automorphisms without any fixed points. The resulting partition function is easily computed from the untwisted Thompson series $T_g^{(+)}$ as above to give $J(\tau)$. A further Z_p orbifolding of this theory then returns us to the untwisted theory on T^{24} again explaining our result above.
- (b) Orbifolding the Monster Module either returns the same Monster Module theory or brings us back to the original untwisted theory again (at least for prime ordered automorphisms). Whether this property can be extended to all orbifoldings of the Monster Module is now under investigation.
- (c) Given that (b) is true for all orbifoldings, we are led to conjecture that the Moonshine properties are in fact due to the uniqueness of the Monster Module, i.e. the constraints placed on a Thompson series by (b) are sufficient to ensure the Vacuum Conjectures and hence the Moonshine properties follow. This would certainly offer the most satisfactory explanation for Monstrous Moonshine.

Apart from these more general considerations, the most immediate unanswered question is how do we construct the twisted sector associated with a general Monster group element in order to verify the Vacuum Conjectures? The most promising starting point is surely a vertex operator approach beginning with a centrally extended lattice automorphism as discussed in Sect. 5 (which does not appear to be known) before considering a general Monster element. The theory of generalised Kac-Moody algebras can be expected to play a central role here [5].

In conclusion, we would like to remark that, according to the uniqueness conjecture of FLM above, the Monster Module is the unique string theory without any massless states and therefore possesses no gauge or gravitational symmetry. We can speculate that this theory describes the state of the early universe before some phase transition sets in producing the necessary gauge bosons, graviton, etc. Leaving such a possibility aside, it is somewhat ironic that the string theory with the least degree of physical content has such an interesting mathematical one.

Appendix. The Parabolic Cusps Points of \mathcal{F}_N

In this appendix we will discuss the inequivalent parabolic cusps of $\mathcal{F}_N = H/\Gamma_0(N)$. In general, every rational point is a parabolic cusp point of some copy of \mathcal{F}_N in H . The following lemma describes a convenient set of inequivalent cusps [29].

Lemma. *Let r/S be a rational number with $(r, S) = 1$. For a given N define*

$$k = (N, S), \quad N = kn, \quad S = ks. \quad (\text{A.1})$$

Then under $\Gamma_0(N)$ we have

$$\frac{r}{ks} \sim \infty, \quad k=N, \quad (\text{A.2a})$$

$$\sim 0, \quad k=1, \quad (\text{A.2b})$$

$$\sim \frac{l}{k}, \quad k \neq 1, N, \quad (\text{A.2c})$$

where $0 < l < k$, $(l, k) = 1$ and l is a solution of

$$l \bmod m = rs \bmod m, \quad m = (n, k). \quad (\text{A.3})$$

The proof of this lemma is as follows. If $k=N$ then $(r, Ns) = 1$ implies that for some $b, d \in \mathbb{Z}$ we have $rd - Nsb = 1$. Therefore, $\gamma = \begin{pmatrix} r & b \\ Ns & d \end{pmatrix} \in \Gamma_0(N)$ and $\gamma(\infty) = r/Ns$ as in (A.2a).

If $k \neq 1$ then for $\gamma = \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \in \Gamma_0(N)$ we have $\gamma(r/ks) = r'/ks'$ for $r' = ar + bks$ and $s' = ncr + ds$ with $(r', s'k) = 1$. It follows that

$$r's' \bmod m = rs \bmod m \quad (\text{A.4})$$

for $m = (n, k)$. We will now construct γ such that $s' = 1$ and $r' = l$ with $0 \leq l < k$, i.e. a, b, c, d obey

$$a = s + nlc, \quad (\text{A.5a})$$

$$kb = -r + ld. \quad (\text{A.5b})$$

If $k=1$ then (A.5) is satisfied for $l=0$ giving (A.2b). If $k \neq 1$ then we will show that (A.5b) can always be solved. We begin by noting that $(nr, s) = 1$ which implies that there exists c, d such that $cnr + ds = 1$. If c_0, d_0 are solutions of this then the general solution is $c = c_0 + js$, $d = d_0 - jnr$ for any integer j . We will show that d satisfies

$$ld \bmod k = r \bmod k \quad (\text{A.6})$$

for some $0 < l < k$ so that (A.5b) can be solved for b .

Let $k = pq$ with $(n, p) = 1$ and $(n, q) \neq 1$, where p is maximal in the sense that all prime divisors of q divide n . Since $(r, k) = 1$ and $(n, p) = 1$ we have that $(nr, p) = 1$. Therefore, $j(nr) \bmod p$ takes on all possible values for $0 \leq j < p-1$. In particular, choose j such that $jnr \bmod p = (d_0 - 1) \bmod p$, i.e. $d = 1 \bmod p$ for $d = d_0 - jnr$ which implies that $(d, p) = 1$.

We also know that $(d, n) = 1$ implying that $(d, q) = 1$. Thus $(d, k) = 1$ for the choice of d above. Therefore, $ld \bmod k$ takes on all possible values in $\{1, \dots, k-1\}$ for $0 < l < k$. In particular, choose l such that (A.6) is satisfied implying that $(l, k) = 1$. Hence we can solve (A.5b) for b giving $\gamma \in \Gamma_0(N)$ with $\gamma(r/ks) = l/k$ with l as in (A.3). This concludes the proof.

The inequivalent cusps of \mathcal{F}_N are therefore determined by all $k|N$ and all $0 < l < k$ satisfying (A.3). We will illustrate this with a number of examples.

N prime: If N is prime then $k=1$ or N giving only two inequivalent cusps ∞ and 0 of (A.2a) and (A.2b) as is well known.

AL cusps: If $e|N$ [i.e. $(e, f) = 1$ for $f = N/e$] then for $k=f$ we have $m=1$ and (A.3) implies that $r/fs \sim l/f \sim 1/f$ for all $0 < l < k$. Such a cusp point is defined in Sect. 4 as the Atkin-Lehner cusp for the AL transformation W_e .

As an example of a general situation we can calculate the inequivalent cusps for $N=36$. The divisors of N are $k=(1, 2, 3, 4, 6, 9, 12, 18, 36)$ with corresponding values of $m=(1, 2, 3, 1, 6, 1, 3, 2, 1)$. There are 12 inequivalent cusps determined by (A.2) and (A.3) as follows

$$0, \frac{1}{2}, \left\{ \frac{1}{3}, \frac{2}{3} \right\}, \frac{1}{4}, \left\{ \frac{1}{6}, \frac{5}{6} \right\}, \frac{1}{9}, \left\{ \frac{1}{12}, \frac{5}{12} \right\}, \frac{1}{18}, \infty \quad (\text{A.7})$$

respectively where $0, \frac{1}{4}$, and $\frac{1}{9}$ are the AL cusps for $e=1, 9, 4$. Likewise, the action of the elements of $\mathcal{N}(36)$ on the cusps can be computed. Thus the Fricke involution $W_{36}: \tau \rightarrow -1/36\tau$ acts as follows: $(0, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{1}{6}) \rightarrow (\infty, \frac{1}{18}, \frac{5}{12}, \frac{1}{12}, \frac{1}{9}, \frac{5}{6})$ where in particular $\frac{1}{3} \rightarrow -\frac{1}{12} \sim \frac{5}{12}$ ($m=3, -1=5 \pmod{3}$), $\frac{2}{3} \rightarrow -\frac{1}{12} \sim \frac{1}{12}$ ($m=3, -1.2=1 \pmod{3}$) and $\frac{1}{6} \rightarrow -\frac{1}{6} \sim \frac{5}{6}$ ($m=6, -1=5 \pmod{6}$). Therefore, $\Gamma_0(36)+36$ has 6 cusps.

Acknowledgements. The author would like to thank Janos Balog and Noel Gorman for a number of useful discussions and Richard Borcherds for his helpful comments.

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Communicated by N. Yu. Reshetikhin

