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The Induced Action of W_3 Gravity

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Abstract. We obtain the induced action $\Gamma[h, b]$ for chiral W_3 gravity in the $c \to \pm\infty$ limit from the induced action of a gauged $Sl(3, \mathbf{R})$ Wess-Zumino-Witten model by imposing constraints on the currents of the latter. In the process we find a closed gauge algebra for the gauge sector of W_3 gravity in which the currents T and W become auxiliary fields. An explicit realization of T and W in terms of the gauge fields is given. In terms of new fields r and s, which are a generalization of Polyakov's f variable for ordinary gravity, the complete induced action $\Gamma[h, b; c \to \pm\infty]$ becomes local.

1. Introduction

Gravity in two dimensions has been extensively studied. Surprisingly, it was found that in the weakly coupled regime ($c < 1$), three equivalent descriptions exist for $d = 2$ gravity. There is the direct approach which starts from the induced action for $d = 2$ gravity:

$$\Gamma = \frac{c}{96\pi} \int d^2x \sqrt{g} \left[R \frac{1}{\Box} R + \Lambda \right]. \tag{1.1}$$

This action has been studied both in the conformal gauge, where it reduces to the Liouville action, and in the light cone gauge where it becomes the Polyakov action [1]. In both cases Eq. (1.1) becomes local.

An alternative approach relies on a discretization of the two-dimensional space and leads to the study of matrix models. A third formulation of $d = 2$ gravity theory is through a topological quantum field theory.

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In two dimensions, there exist higher spin extensions of gravity. These theories are based upon an algebra which is of W type. Reviews of the recent activity in this field, which so far mainly focussed on the classical theory, can be found in [2]. In this paper, we focus on quantum W_3 gravity in the chiral light cone gauge, with gauge fields h_{--} and b_{---}. Our main result, which we present in Sect. 4, will be an all order result for the induced action of W_3 gravity, which is exact in the limit of large c. We will derive this action by using the hidden $Sl(3,\mathbf{R})$ symmetry in the theory.

Before we come to this, we will first, in this introductory section, review some algebraic aspects of classical and quantum gravity theories and indicate how they generalize to the case of $d=2$ W_3 gravity. In Sect. 2, we will then review the treatment of $d=2$ induced gravity as a reduced $Sl(2,\mathbf{R})$ Wess-Zumino-Witten (WZW) model. In Sect. 3, we discuss induced W_3 gravity, both for infinite and for finite central charge c. After the presentation of our main results in Sect. 4, we will, in Sect. 5, discuss some ideas about the geometry of W_3 gravity, which we base on a "W_3 superspace." We will also comment on the construction of the covariant induced action and on the description of W_3 gravity coupled to minimal W_3 matter systems through matrix models and topological quantum field theories.

The results of this paper for the *induced* action of quantum W_3 gravity extend the results of [3], where the lowest terms (through 3-loop, but without restricting c to be large) of the induced action were computed explicitly. We will argue that the full *effective* action, which includes the effects of fluctuations in the quantum fields h_{--} and b_{---}, is obtained from the action constructed in this paper, by renormalizing some constants, which are the level $k=c/24+\ldots$ of the $Sl(3,\mathbf{R})$ algebra and z-factors for the fields h_{--} and b_{---} (see [4], for a detailed discussion).

Let us now briefly elucidate the algebraic structure underlying classical and quantum (induced) gravity. The general starting point is the construction of a gauge theory for some algebra, which is then supplied with constraints on the curvatures. One uses here the observation that a general coordinate transformation on a gauge field can be written as a field dependent gauge transformation modulo terms proportional to the Yang-Mills curvature. Indeed, consider an infinitesimal general coordinate transformation with parameter ξ, on a gauge field A:

$$\begin{aligned}\delta_{\mathrm{gct}}A_\mu &= \xi^\nu\partial_\nu A_\mu+\partial_\mu\xi^\nu A_\nu \\ &= \partial_\mu(\xi\cdot A)+[\xi\cdot A, A_\mu]+\xi^\nu R_{\nu\mu}\,. \end{aligned} \tag{1.2}$$

By putting certain curvature tensors to zero, general coordinate transformations become equal to gauge transformations [5]. In gravity one puts the curvature tensors corresponding to the translations (which are among the gauge transformations) to zero. This has two implications:

i) The spin connection (the gauge field associated with local Lorentz transformations) can be solved in terms of the vielbeins (the gauge fields associated with translations).
ii) Local translations are identified with general coordinate transformations.

In the two-dimensional case one starts with the group $Sl(2,\mathbf{R})$ with its Lie algebra generated by $T_\pm$ and T_0,

$$\begin{aligned}[T_0, T_\pm] &= \pm 2T_\pm\,, \\ [T_+, T_-] &= \lambda T_0\,, \end{aligned} \tag{1.3}$$

where λ is a real constant. For $\lambda \to 0$, this algebra reduces the Poincaré algebra in two dimensions, ISO(2). The gauge fields $A^{\pm}$ and A^0 transform as usual

$$\begin{aligned} \delta A^{\pm} &= d\eta^{\pm} \pm 2\eta^0 A^{\pm} \mp 2\eta^{\pm} A^0, \\ \delta A^0 &= d\eta^0 + \lambda \eta^+ A^- - \lambda \eta^- A^+, \end{aligned} \tag{1.4}$$

and one has the curvature tensors

$$\begin{aligned} R^{\pm} &= dA^{\pm} \mp 2A^0 \wedge A^{\pm}, \\ R^0 &= dA^0 - \lambda A^+ \wedge A^-. \end{aligned} \tag{1.5}$$

According to the previous discussion, one now puts $R^{\pm}=0$ and solves this constraint for A^0. The resulting theory describes an anti-De Sitter gravity theory with zweibeins $A^{\pm}$. One could wonder whether an additional constraint $R^0=0$ (constant Riemann curvature) makes sense. This condition is dynamical and can be viewed as the equation of motion for the effective gravity theory. Indeed, parametrizing the zweibeins as

$$A^+ = e^{\varphi_+}(dz + h_{--}d\bar{z}), \qquad A^- = e^{\varphi_-}(d\bar{z} + h_{++}dz), \tag{1.6}$$

one finds that in the chiral gauge, $h_{++}=0$ and $\varphi_+ = \varphi_- = 0$, the constraint is $\partial^2 h_{--} = -2\lambda$ and in the conformal gauge, $\varphi_+ = \varphi_-$ and $h_{--} = h_{++} = 0$, one gets $\partial\bar{\partial}\varphi = \lambda \exp(2\varphi)$. One sees that in the conformal gauge, one obtains the Liouville equation, which is indeed the equation of motion for the induced action of $d=2$ gravity. The interpretation in the chiral gauge is not completely clear. Upon taking an extra derivative one obtains $\partial^3 h_{--}=0$, which is indeed the equation of motion of the induced gravity theory in the chiral gauge. (Use Eq. (2.24) and the on-shell condition $u=0$.)

In [6, 7], it was shown that gauge fixing the symmetries generated by T_+ and T_0 by putting $A_z^0 = 0$ and $A_z^- =$ constant, results in the fact that A_z^- can be viewed as the light-cone component h_{--} of the metric. Surprisingly, A_z^+ transforms then as the effective energy-momentum tensor under the remaining T_- transformation. Two of the curvature constraints turn out to be algebraic again, while the third one reproduces the Ward identity of induced gravity. Starting from the observation that the curvature constraints can be seen as the Ward identities for a gauged Wess-Zumino-Witten theory in the light-cone gauge, one can solve the gravitational Ward identity (i.e. construct the induced action) using the known induced action of the gauged Wess-Zumino-Witten model. This will be shown in the next section. Though at first this looks rather arbitrary, a more systematic derivation can be given, along the lines of [6], by starting from an $Sl(2, \mathbf{R})$ Chern-Simons theory in $2+1$ dimensions.

Finally, let us give some comments on the situation for W_3 gravity [2]. The quantum W_3 algebra [8] is generated by $\{L_m, W_n; m, n \in \mathbf{Z}\}$ with commutation relations:

$$\begin{aligned} [L_m, L_n] &= \frac{c}{12} m(m^2-1)\delta_{m+n,0} + (m-n)L_{m+n}, \\ [L_m, W_n] &= (2m-n)W_{m+n}, \\ [W_m, W_n] &= \frac{c}{360} m(m^2-1)(m^2-4)\delta_{m+n,0} \\ &\quad + (m-n)\left\{\frac{1}{15}(m+n+3)(m+n+2) - \frac{1}{6}(m+2)(n+2)\right\} L_{m+n} \\ &\quad + \beta(m-n)\Lambda_{m+n}, \end{aligned} \tag{1.7}$$

where

$$\beta = \frac{16}{22+5c}, \tag{1.8}$$

and

$$\Lambda_m = \sum_{n\in\mathbf{Z}} :L_{m-n}L_n: - \frac{3}{10}(m+3)(m+2)L_m, \tag{1.9}$$

and the normal ordering prescription is given by

$$\begin{aligned} :L_mL_n: &= L_mL_n \quad \text{if} \quad m \leqq -2 \\ &= L_nL_m \quad \text{if} \quad m > -2. \end{aligned} \tag{1.10}$$

As can be seen from Eq. (1.7), the novel feature of W type algebras is the appearance of composite terms at the right-hand side of the commutators. The W_3 transformations globally defined on the sphere are $w = \{L_{\pm 1}, L_0, W_{\pm 2}, W_{\pm 1}, W_0\}$. From Eq. (1.7) it follows that, due to the presence of the non-linear terms, these generators do not form a subalgebra. One might expect that in the $c \to \infty$ (classical) limit, the W_3 algebra linearizes. However, the relation

$$[L_m, \beta\Lambda_n] = (3m-n)\beta\Lambda_{m+n} + \frac{8}{15}(m^3-m)L_{m+n}, \tag{1.11}$$

shows that simply dropping the nonlinear terms in the limit $c \to \infty$ is not a consistent procedure. Nevertheless, the previous equation (take $m = \pm 1$ and $m = 0$) does imply that only in the subalgebra w, the non-linear terms can consistently be put to zero. The resulting algebra, which can be seen as the on-shell version of the projective subalgebra, is isomorphic to $su(2,1)$. Indeed identifying

$$\begin{aligned} &T_1 \equiv \frac{1}{4}\tilde{W}_0 + \frac{1}{2}L_0, \qquad T_2 \equiv -\frac{1}{4}\tilde{W}_0 + \frac{1}{2}L_0, \\ &T_{\pm 1} \equiv \pm\frac{1}{\sqrt{8}}(\tilde{W}_{\mp 1} + L_{\mp 1}), \qquad T_{\pm 2} \equiv \mp\frac{1}{\sqrt{8}}(\tilde{W}_{\mp 1} - L_{\mp 1}), \\ &T_{\pm 3} \equiv \frac{1}{4}\tilde{W}_{\mp 2}, \end{aligned} \tag{1.12}$$

where

$$\tilde{W} = \sqrt{-10}W, \tag{1.13}$$

one finds that $\{T_1, T_2, T_{\pm 1}, T_{\pm 2}, T_{\pm 3}\}$ satisfy the $Sl(3, \mathbf{R})$ commutation rules. Taking into account the factors i in Eq. (1.13), one has that over $\mathbf{R}$, the algebra $\{L_{\pm 1}, L_0, W_{\pm 2}, W_{\pm 1}, W_0\}$ is isomorphic to $SU(2,1)$.

The previous analysis suggests a natural generalization of the Poincaré algebra to the W_3 case. For pure gravity, $\{L_{-1}, \bar{L}_{-1}, L_0 - \bar{L}_0\}$ (the unbarred generators are left movers while the barred generators are right movers, left and right movers mutually commute) generate the Poincaré algebra, which, as we mentioned above, is a contraction of $Sl(2, \mathbf{R})$. For W_3, it is natural to define as a Poincaré-like algebra, the algebra generated by $\{L_0 - \bar{L}_0, W_0 - \bar{W}_0, L_{-1}, \bar{L}_{-1}, W_{-1}, \bar{W}_{-1}, W_{-2}, \bar{W}_{-2}\}$. This algebra is precisely the contraction of the $Sl(3, \mathbf{R})$ algebra used by Li in [9].

As such, it is to be expected that $Sl(3, \mathbf{R})$ will play a role in W_3-gravity, similar to the role played by $Sl(2, \mathbf{R})$ in $d=2$ gravity. This connection will be made precise in Sect. 4, where we will show how the W_3 gravity Ward identities (in the chiral gauge) arise from the $Sl(3, \mathbf{R})$ structure. The covariant formulation of induced W_3 gravity will be treated elsewhere [10].

2. Gauged Wess-Zumino-Witten Models

In this section we review some basic results on WZW models [11, 12]. As an application we shall rederive the effective action for induced gravity from an $Sl(2, \mathbf{R})$ theory (see also [13, 14]).

An affine Lie algebra is determined by the following OPE[1],

$$J_a(x)J_b(y) = -\frac{k}{2}g_{ab}(x-y)^{-2} + (x-y)^{-1}f_{ab}^c J_c(y) + \dots . \tag{2.1}$$

The generating functional for current correlation functions $\Gamma[A]$ is defined by

$$e^{-\Gamma[A]} = \left\langle \exp -\frac{1}{\pi x}\int d^2x \,\mathrm{tr}\{J(x)A(x)\} \right\rangle \tag{2.2}$$

and transforms as

$$\delta\Gamma[A] = -\frac{k}{2\pi x}\int d^2x \,\mathrm{tr}\{\eta\partial A\} \tag{2.3}$$

under

$$\delta A = \bar{\partial}\eta + [\eta, A]. \tag{2.4}$$

The relation (2.3) states that the anomaly comes only from the lowest order (2-point) diagram.

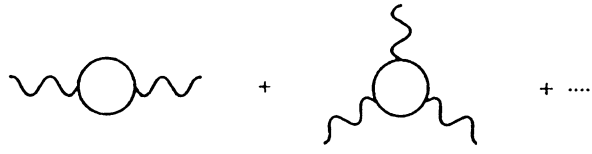
From (2.3) and (2.4) we derive the following Ward identity

$$\bar{\partial}u - [A, u] = \partial A, \tag{2.5}$$

where

$$u_a(x) = -\frac{2\pi}{k}\frac{\delta\Gamma[A]}{\delta A^a(x)}. \tag{2.6}$$

This Ward identity can be solved for $\Gamma[A]$ yielding

$\Gamma[A] =$  $+ \dots$

[1] We normalize such that if $[T_a, T_b] = f_{ab}^c T_c$ then $f_{ac}^d f_{bd}^c = -\tilde{h}g_{ab}$, where $\tilde{h}$ is the dual Coxeter number. In a representation R we have $\mathrm{tr}(T_a T_b) = -x g_{ab}$, where x is the index of the representation ($x = \tilde{h}$ for the adjoint representation). For $Sl(n, \mathbf{R})$ one has $\tilde{h} = n$ and $x = \frac{1}{2}$ for the vector (defining) representation. Finally, we always work in a two-dimensional Euclidean space. We will use complex coordinates and denote them by x and $\bar{x}$ (or z and $\bar{z}$) instead of x^- and x^+

$$= \frac{k}{4\pi x}\int d^2x \,\mathrm{tr}\left\{A\frac{\partial}{\bar\partial}A + \frac{2}{3}A\left[\frac{1}{\bar\partial}A, \frac{\partial}{\bar\partial}A\right] + \ldots\right\}$$

$$= \frac{k}{2\pi x}\int d^2x \,\mathrm{tr}\left\{A\sum_{n\geqq 0}\frac{1}{n+2}\left(\left[\frac{1}{\bar\partial}A, \cdot\right]\right)^n \frac{\partial}{\bar\partial}A\right\}. \tag{2.7}$$

Polyakov and Wiegemann [12] found a very elegant alternative formulation for $\Gamma[A]$. Parametrizing A as $A \equiv \bar\partial g g^{-1}$, one finds that $u = \partial g g^{-1}$ because Eq. (2.5) states that the curvature for the Yang-Mills fields $\{A, u\}$ vanishes. In this parametrization one has

$$\delta\Gamma = -\frac{k}{2\pi x}\int d^2x \,\mathrm{tr}\{\partial(\bar\partial g g^{-1})\delta g g^{-1}\}, \tag{2.8}$$

where $(\exp\eta)g = g + \delta g$, which is recognized as the equation of motion for the Wess-Zumino-Witten action

$$\Gamma[g] = -\frac{k}{4\pi x}\int d^2x \,\mathrm{tr}\{\partial g^{-1}\bar\partial g\} - \frac{k}{12\pi x}\int d^3x \varepsilon^{\alpha\beta\gamma}\,\mathrm{tr}\{g_{,\alpha}g^{-1}g_{,\beta}g^{-1}g_{,\gamma}g^{-1}\}, \tag{2.9}$$

with $d^3x = dx^3 dx^+ dx^-$ and $\varepsilon^{3+-} = -1$ and where $\partial = \partial_z$ and $\bar\partial = \partial_{\bar z}$.

It is also easy to find the covariant action. Indeed

$$\Gamma(A, \bar A) = \Gamma(A) + \Gamma(\bar A) - \frac{k}{2\pi x}\int d^2x \,\mathrm{tr}\{A(x)\bar A(x)\} \tag{2.10}$$

is invariant under Eq. (2.4) and

$$\delta\bar A = \partial\eta + [\eta, \bar A]. \tag{2.11}$$

The covariant action (2.10) can be viewed as the induced action of a gauged Wess-Zumino-Witten model. Indeed, let us consider a WZW action $\Gamma[h]$, which is invariant under

$$h(x, \bar x) \to \bar\gamma(\bar x) h(x, \bar x)\gamma(x). \tag{2.12}$$

The currents associated with these symmetries are (use Eq. (2.8))

$$J(x) = -\frac{k}{2}h^{-1}\partial h, \qquad \bar J(\bar x) = \frac{k}{2}\bar\partial h h^{-1}, \tag{2.13}$$

and $J(x)$ satisfies the OPE Eq. (2.1) with the same OPE for $\bar J(\bar x)$. We now consider the action $\Gamma(h, A, \bar A]$:

$$\Gamma[h, A, \bar A] = \Gamma[h] - \frac{1}{\pi x}\int d^2x \,\mathrm{tr}\left[JA + \bar J\bar A - \frac{k}{2}A\bar A + \frac{k}{2}\bar A h A h^{-1}\right]. \tag{2.14}$$

Parametrizing A and $\bar A$ as $A = \bar\partial g g^{-1}$ and $\bar A = \partial g' g'^{-1}$ one finds using the identity

$$\Gamma[hg] = \Gamma[h] + \Gamma[g] + \frac{k}{2\pi x}\int d^2x \,\mathrm{tr}[\bar\partial g g^{-1}h^{-1}\partial h], \tag{2.15}$$

which is obtained through direct computation, that

$$\Gamma[h, A, \bar A] = \Gamma[g'^{-1}hg] - \Gamma[g'^{-1}g]. \tag{2.16}$$

From this we immediately read off that the action has a vectorial gauge invariance

$$h \to \gamma h \gamma^{-1}, \quad g' \to \gamma g', \quad g \to \gamma g \tag{2.17}$$

while the axial transformations

$$h \to \gamma^{-1} h \gamma^{-1}, \quad g' \to \gamma^{-1} g', \quad g \to \gamma g \tag{2.18}$$

are not symmetries. In [15], it was shown that after integrating out matter i.e., the fields $h(x)$, the induced action of the gauge WZW model is indeed given by Eq. (2.10). Upon choosing a chiral gauge $\bar{A}=0$, we retrieve our point of departure Eq. (2.2).

We now restrict our discussion to $Sl(2, \mathbf{R})$. We choose as basis

$$T_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad T_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad T_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \tag{2.19}$$

with metric

$$g_{ab} = \begin{pmatrix} -4 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & -2 & 0 \end{pmatrix}. \tag{2.20}$$

We impose the following constraints on the currents u

$$u = \begin{bmatrix} 0 & u^+ \\ a & 0 \end{bmatrix}, \tag{2.21}$$

where a is a real constant. The reason for these constraints will become clear. Using (2.21) and (2.5) we can eliminate A^+ and A^0 as independent variables, giving

$$A = \begin{bmatrix} \frac{1}{2a}\partial A^- & -\frac{1}{a}\left(\frac{1}{2a}\partial^2 A^- - u^+ A^-\right) \\ A^- & -\frac{1}{2a}\partial A^- \end{bmatrix}, \tag{2.22}$$

and the Ward identities reduce to a single equation:

$$\left(\bar{\partial} - \frac{2}{a}\partial A^- - \frac{1}{a}A^- \partial\right) u^+ = -\frac{1}{2a^2}\partial^3 A^-. \tag{2.23}$$

Compare this now with the Ward identity for induced gravity [1]:

$$(\bar{\partial} - 2\partial h - h\partial) u = \partial^3 h, \tag{2.24}$$

where

$$u(x) = \frac{12\pi}{c}\frac{\delta \Gamma[h]}{\delta h(x)} \tag{2.25}$$

and

$$e^{-\Gamma[h]} = \left\langle \exp -\frac{1}{\pi}\int d^2x T(x) h(x) \right\rangle. \tag{2.26}$$

We note that upon identifying

$$\begin{aligned} h &\equiv \frac{1}{a}A^-, \\ u &\equiv -2au^+, \end{aligned} \tag{2.27}$$

Eqs. (2.23) and (2.24) coincide. This observation implies that one can obtain $\Gamma[h]$ from $\Gamma[A]$ as follows.

On the one hand, u^+ is defined by

$$u^+ = \frac{\pi}{k}\frac{\delta\Gamma[A]}{\delta A^-} \quad \text{at} \quad A^0 = A^0(A^-), \quad A^+ = A^+(A^-). \tag{2.28}$$

On the other hand, the object $u(x)$ in pure gravity is obtained by varying an effective action $\Gamma[h]$. This suggests that $\Gamma[h]$ is related to $\Gamma[A]$ in which the constraints have already been imposed on A. Therefore, we reverse the order in which we differentiate with respect to A^- and impose constraints, and find from the chain rule

$$u^+ = \frac{\pi}{k}\frac{\delta}{\delta A^-}\left\{\Gamma[A^-, A^+(A^-), A^0(A^-)] - \frac{k}{\pi}\int d^2x u^+ A^-\right\}, \tag{2.29}$$

where we used

$$\int d^2x\left(u_+\frac{\delta A^+}{\delta A^-} + u_0\frac{\delta A^0}{\delta A^-}\right) = -2\frac{\delta}{\delta A^-}\int d^2x u^+ A^- . \tag{2.30}$$

From (2.27) and (2.25) we have

$$u^+ = -\frac{1}{2a}u = -\frac{6\pi}{ac}\frac{\delta\Gamma[h]}{\delta h}. \tag{2.31}$$

Combining Eqs. (2.29) and (2.31) yields

$$\Gamma[h] = -\frac{c}{6k}\left\{\Gamma[A^-, A^+(A^-), A^0(A^-)] - \frac{k}{\pi}\int d^2x u^+ A^-\right\}. \tag{2.32}$$

The leading or classical term in the KPZ-formula [16, 17, 14], is $k = c/6$.

Before deriving a more manageable form for $\Gamma[A]$ we first reduce the transformation rules. From $\delta u = \partial\eta + [\eta, u]$ and the constraints, one obtains

$$\eta = \begin{bmatrix} \frac{1}{2a}\partial\eta^- & -\frac{1}{a}\left(\frac{1}{2a}\partial^2\eta^- - u^+\eta^-\right) \\ \eta^- & -\frac{1}{2a}\partial\eta^- \end{bmatrix}. \tag{2.33}$$

From $\delta A = \bar{\partial}\eta + [\eta, A]$ one finds then that

$$\delta h = \bar{\partial}\varepsilon + \varepsilon\partial h - \partial\varepsilon h, \tag{2.34}$$

where $\varepsilon = \eta^-/a$. The stress tensor transforms according to

$$\delta u = \partial^3\varepsilon + \varepsilon\partial u + 2\partial\varepsilon u. \tag{2.35}$$

A local expression for $\Gamma[h]$ is obtained by using a Gaussian parametrization for $Sl(2, \mathbf{R})$

$$g = \begin{pmatrix} 1 & \varphi \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \phi & 1 \end{pmatrix}. \tag{2.36}$$

The constraints in (2.21) can now be solved explicitly, giving

$$\varphi = -\frac{1}{2a}\frac{\partial^2\phi}{\partial\phi}, \qquad \lambda^2 = \frac{a}{\partial\phi} \tag{2.37}$$

and

$$u = (\partial\phi)^{-2}\left(\partial^3\phi\partial\phi - \frac{3}{2}\partial^2\phi\partial^2\phi\right), \qquad h = \frac{\bar{\partial}\phi}{\partial\phi}, \tag{2.38}$$

where we used that $A = \bar{\partial}gg^{-1}$ and $u = \partial gg^{-1}$ and Eq. (2.27).

Substituting the Gaussian decomposition into the action in (2.32) one obtains

$$\begin{aligned}\Gamma[h] &= \left(\frac{k}{12\pi}\right)\int d^3x\varepsilon^{\alpha\beta\gamma}[-6\lambda\partial_\alpha\varphi\partial_\beta\phi\partial_\gamma\lambda] \\ &= \frac{c}{24\pi}\int d^2x\left\{\frac{(\partial^2\phi)^2\bar{\partial}\phi}{(\partial\phi)^3} - \frac{\partial^2\phi\partial\bar{\partial}\phi}{(\partial\phi)^2}\right\} \\ &= \frac{c}{24\pi}\int d^2x\frac{\bar{\partial}\phi}{\partial\phi}\partial^2(\ln\partial\phi)\end{aligned} \tag{2.39}$$

with $k = c/6$, which is indeed the action for induced gravity in the light-cone gauge. It is amusing to note that the correction term in Eq. (2.32) precisely cancels the kinetic term of the WZW-action. As such the action for induced gravity arises from the WZW-term.

Note that the previous construction provides us with a Lax pair [20, 19] for the Virasoro Ward identity. Indeed consider $(\partial - u)\tilde{\psi} = 0$ and $(\bar{\partial} - A)\tilde{\psi} = 0$, these have the WZW Ward identity as integrability condition. Upon imposing the constraints, these equations reduce to

$$\left(\partial^2 + \frac{1}{2}u\right)\psi = 0, \qquad \left(\bar{\partial} - h\partial + \frac{1}{2}\partial h\right)\psi = 0, \tag{2.40}$$

which indeed have the Virasoro Ward identity as integrability condition. Consider the two independent solutions to the Lax pair, ψ_1 and ψ_2. From the second equation in Eq. (2.40) it follows that we can identify $\phi = \psi_1/\psi_2$ since it yields $h = \bar{\partial}\phi/\partial\phi$ and from the first equation in (2.40) one immediately gets the explicit form for ψ_1 and ψ_2 in terms of ϕ while it also gives u as the Schwarzian derivative, Eq. (2.38).

If we compare this analysis to the work of Bershadsky and Ooguri [14] we see that the main difference lies in the constraint imposed on u. While in the present work we impose the constraints $u^- = \text{constant}$ and $u^0 = 0$, in [14], one imposed $u^- = \text{constant}$ and $\varphi = 0$. It is interesting to note that while our constraints identify ϕ with the coordinate transformation f, the choice of [14] (see also [13]) resulted in the identification of ϕ with the inverse transformation F defined through

$$F(f(x, \bar{x}), \bar{x}) = x. \tag{2.41}$$

In the work of [14], the induced action for gravity in terms of the F variable arose completely from the kinetic term of the Wess-Zumino-Witten action, while here it arises from the Wess-Zumino term.

Finally, the previous construction explains the residual $Sl(2,\mathbf{R})$ symmetry of effective gravity theories [1]. The A fields are the Noether currents associated with this symmetry. As can be seen by combining Eq. (2.33) with $\delta\Gamma[h] = c/12\pi \int d^2x \partial^3 \varepsilon h$ and $\varepsilon = \eta^-/a$, the induced action has indeed a residual affine $Sl(2,\mathbf{R})$ symmetry.

3. Induced W_3 Gravity

Before we study the relation between $Sl(3,\mathbf{R})$ WZW models and induced W_3 gravity, we first review some properties of the latter. We restrict ourselves throughout this article to "pure W_3 gravity" as given by its abstract algebra. For an alternative approach based on a realization of the currents in terms of n scalar matter fields φ^i, see [3].

The W_3-algebra is generated by currents $T(x)$ and $W(x)$ satisfying the operator product expansions

$$\begin{aligned} T(x)T(y) &= \frac{c}{2}(x-y)^{-4} + 2(x-y)^{-2}T(y) + (x-y)^{-1}\partial T(y), \\ T(x)W(y) &= 3(x-y)^{-2}W(y) + (x-y)^{-1}\partial W(y), \\ W(x)W(y) &= \frac{c}{3}(x-y)^{-6} + 2(x-y)^{-4}T(y) + (x-y)^{-3}\partial T(y) \\ &\quad + (x-y)^{-2}\left[2\beta\Lambda(y) + \frac{3}{10}\partial^2 T(y)\right] \\ &\quad + (x-y)^{-1}\left[\beta\partial\Lambda(y) + \frac{1}{15}\partial^3 T(y)\right], \end{aligned} \tag{3.1}$$

where

$$\Lambda(x) = (TT)(x) - \frac{3}{10}\partial^2 T(x) \tag{3.2}$$

and β was defined in Eq. (1.8). These OPE's are equivalent to the commutation relations Eq. (1.7).

We again consider the generating functional for current correlation functions

$$e^{-\Gamma[h,b]} = \left\langle \exp -\frac{1}{\pi}\int d^2x [h(x)T(x) + b(x)W(x)] \right\rangle. \tag{3.3}$$

Under the variations

$$\begin{aligned} \delta h &= \bar{\partial}\varepsilon + \varepsilon\partial h - \partial\varepsilon h, \\ \delta b &= \varepsilon\partial b - 2\partial\varepsilon b, \end{aligned} \tag{3.4}$$

the induced action $\Gamma[h,b]$ transforms as

$$\delta_\varepsilon \Gamma[h,b] = -\frac{c}{12\pi}\int d^2x \varepsilon \partial^3 h, \tag{3.5}$$

while under

$$\begin{aligned}\delta h &= \frac{1}{15}\lambda\partial^3 b - \frac{1}{10}\partial\lambda\partial^2 b + \frac{1}{10}\partial^2\lambda\partial b - \frac{1}{15}\partial^3\lambda b,\\ \delta b &= \bar{\partial}\lambda + 2\lambda\partial h - \partial\lambda h,\end{aligned} \tag{3.6}$$

one has that

$$\delta_\lambda \Gamma[h,b] = -\frac{c}{360\pi}\int d^2x\lambda\partial^5 b - \frac{\beta}{\pi}\int d^2x\lambda(x)(2\partial b + b\partial)\Lambda_{\text{eff}}(x). \tag{3.7}$$

Here

$$\begin{aligned}\Lambda_{\text{eff}}(x) &= \left\langle \Lambda(x)\exp -\frac{1}{\pi}\int d^2y[h(y)T(y)+b(y)W(y)]\right\rangle \Big/ e^{-\Gamma[h,b]}\\ &= \lim_{y\to x}\left\langle\left(T(y)T(x) - \frac{c}{2}(y-x)^{-4} - 2(y-x)^{-2}T(x)\right.\right.\\ &\quad \left.\left. -(y-x)^{-1}\partial T(x) - \frac{3}{10}\partial^2 T(x)\right)\exp\left(-\frac{1}{\pi}\int[hT+bW]\right)\right\rangle \Big/ e^{-\Gamma[h,b]}\\ &= \frac{c^2}{144}u(x)u(x) - \frac{c\pi}{12}\lim_{y\to x}\left\{\frac{\delta u(x)}{\delta h(y)} - \frac{\partial_x^3}{\bar{\partial}_x}\delta(x-y)\right.\\ &\quad \left. -2\frac{\partial_x}{\bar{\partial}_x}\delta(x-y)u(x) - \frac{1}{\bar{\partial}_x}\delta(x-y)\partial u(x)\right\} - \frac{c}{40}\partial^2 u(x),\end{aligned} \tag{3.8}$$

where $u(x) = \dfrac{12\pi}{c}\dfrac{\delta\Gamma[h,b]}{\delta h(x)} = \dfrac{12}{c}T_{\text{eff}}$. We have used the identity $\bar{\partial}x^{-1} = \pi\delta^2(x)$. The explicit form for $\delta_\lambda h$ in (3.6) follows by requiring that *all* β-independent T^{eff} terms cancel in the right-hand side of the Ward identity. In a different context we found this same $\delta_\lambda h$ rule in [3]. Part of these results were also found in [18], though there the incorrect assumption was made that the non-linear terms decouple in the large c limit.

In the limit $c\to\pm\infty$ one obtains $c^{-2}\Lambda_{\text{eff}} = \dfrac{1}{144}u^2$, and Eq. (3.7) becomes $\delta\Gamma[h,b] = -\dfrac{c}{360\pi}\int d^2x\lambda\partial^5 b - \dfrac{c}{45\pi}\int d^2x\lambda(2\partial b + b\partial)uu$. In this limit we can also reduce the λ-anomaly to the minimal one by adding an extra term to the h transformation rule Eq. (3.6):

$$\delta h = \frac{4}{15}(\lambda\partial b - b\partial\lambda)u. \tag{3.9}$$

However, it turns out to be more advantageous to make a different choice for $\delta_{\text{extra}}h$:

$$\delta_{\text{extra}}h = \frac{8}{15}(\lambda\partial b - b\partial\lambda)u. \tag{3.10}$$

Indeed, for this choice we have that u and v transform according to the operator product expansion in the limit $c \to \pm\infty$,

$$\begin{aligned}\delta u &= \partial^3\varepsilon + \varepsilon\partial u + 2\partial\varepsilon u + \frac{1}{15}\lambda\partial v + \frac{1}{10}\partial\lambda v,\\ \delta v &= \varepsilon\partial v + 3\partial\varepsilon v + \partial^5\lambda + (2\lambda\partial^3 + 9\partial\lambda\partial^2 + 15\partial^2\lambda\partial + 10\partial^3\lambda)u\\ &\quad + 16(\partial\lambda u u + \lambda u\partial u),\end{aligned} \tag{3.11}$$

where

$$v = \frac{360\pi}{c}\frac{\delta\Gamma[h,b]}{\delta b(x)} = \frac{360}{c}W_{\text{eff}}(x). \tag{3.12}$$

The algebra becomes then

$$\begin{aligned}&[\delta(\varepsilon_1), \delta(\varepsilon_2)] = \delta(\varepsilon_3 = \varepsilon_2\partial\varepsilon_1 - \varepsilon_1\partial\varepsilon_2),\\ &[\delta(\varepsilon_1), \delta(\lambda_2)] = \delta(\lambda_3 = 2\lambda_2\partial\varepsilon_1 - \varepsilon_1\partial\lambda_2),\\ &[\delta(\lambda_1), \delta(\lambda_2)] = \delta\Big(\varepsilon_3 = \frac{1}{30}(2\partial^3\lambda_1\lambda_2 - 3\partial^2\lambda_1\partial\lambda_2 + 3\partial\lambda_1\partial^2\lambda_2 - 2\lambda_1\partial^3\lambda_2)\\ &\qquad + \frac{8}{15}(\lambda_2\partial\lambda_1 - \lambda_1\partial\lambda_2)u\Big).\end{aligned} \tag{3.13}$$

As we will see later on, it is precisely this choice for δh which will emerge from a constrained $Sl(3,\mathbf{R})$ theory. The drawback of this choice is that the λ-anomaly is *not* the minimal one:

$$\delta_\lambda\Gamma[h,b] = -\frac{c}{360\pi}\int d^2x\lambda\partial^5 b + \frac{c}{45\pi}\int d^2x\lambda(2\partial b + b\partial)uu. \tag{3.14}$$

A useful check of this result is the analysis of the Wess-Zumino conditions for consistent anomalies, which are indeed satisfied (compare with our analysis in [3]). Using the chain rule for $\delta\Gamma$ and Eqs. (3.4)–(3.6), (3.10), and (3.14) we find the final form of the Ward identities in the $c \to \pm\infty$ limit:

$$\begin{aligned}(\bar\partial - 2\partial h - h\partial)u - \left(\frac{1}{10}\partial b + \frac{1}{15}b\partial\right)v &= \partial^3 h,\\ (\bar\partial - 3\partial h - h\partial)v - (10\partial^3 b + 15\partial^2 b\partial + 9\partial b\partial^2 + 2b\partial^3)u - 8(2\partial b + b\partial)uu &= \partial^5 b.\end{aligned} \tag{3.15}$$

In fact, Eq. (3.15) and the consistency of the anomalies hold whether or not we impose Eq. (3.10).

Finally, let us briefly comment upon the situation for finite c. For this purpose, we first define a reference functional, denoted by $\Gamma_L[h,b]$, which is defined by the property that

$$u \equiv +\pi\frac{\delta\Gamma_L}{\delta h}, \qquad v \equiv +30\pi\frac{\delta\Gamma_L}{\delta b} \tag{3.16}$$

satisfy Eqs. (3.15). Similarly, we define $W_L[u,v]$ by the property that

$$h \equiv -\pi\frac{\delta W_L}{\delta u}, \qquad b \equiv -30\pi\frac{\delta W_L}{\delta v} \tag{3.17}$$

satisfy the same Eqs. (3.15). Obviously, Γ_L and W_L are related by a simple Legendre transformation. We now consider the generating functional $W[t,w]$ of connected

Green's functions of quantum h and b fields, defined by

$$e^{-W[t,w]}=\int DhDbe^{-\Gamma[h,b]+\frac{1}{\pi}\int(ht+bw)}. \tag{3.18}$$

The above results for the $c\to\infty$ limit of the induced action $\Gamma[h,b]$ can now be stated as follows

$$\begin{aligned}\Gamma[h,b]&\overset{c\to\infty}{=}\frac{c}{12}\Gamma_L[h,b],\\ W[t,w]&\overset{c\to\infty}{=}\frac{c}{12}W_L\left[\frac{12}{c}t,\frac{360}{c}w\right].\end{aligned} \tag{3.19}$$

If we now look at finite c, we should consider $1/c$ corrections to the formulas (3.19). Such corrections were first obtained in [3]. Recently, we found strong evidence that the full result for $W[t,w]$ can be written as

$$W[t,w]=2k_cW_L[z_c^{(t)}t,z_c^{(w)}w], \tag{3.20}$$

where k_c, $z_c^{(t)}$, and $z_c^{(w)}$ are c-dependent factors. The leading $1/c$ corrections are given by

$$\begin{aligned}k_c&=\frac{c}{24}\left(1-\frac{122}{c}+O\left(\left(\frac{1}{c}\right)^2\right)\right),\\ z_c^{(t)}&=\frac{12}{c}\left(1+\frac{50}{c}+O\left(\left(\frac{1}{c}\right)^2\right)\right),\\ z_c^{(w)}&=\frac{360}{c}\left(1+\frac{386}{5c}+O\left(\left(\frac{1}{c}\right)^2\right)\right).\end{aligned} \tag{3.21}$$

The result for k_c, which has the interpretation of the renormalized level of a $Sl(3,\mathbf{R})$ current algebra, is consistent with the all order formula first proposed in [14, 18],

$$-48(k+3)=50-c+\sqrt{(c-2)(c-98)}, \tag{3.22}$$

which is the *conjectured* outcome of a KPZ-type analysis of constraints in a covariant formulation of W_3 gravity. We finally remark that the validity of Eq. (3.20) crucially depends on the cancellation of certain non-local terms in the Ward identity, coming from i) the induced action itself, ii) the determinant factors for taking into account fluctuations around the saddle point of the path integral Eq. (3.18). This clearly shows that it is $W[t,w]$ or, equivalently, the full *effective* action and not the *induced* action Eq. (3.3) which can be directly related to the constrained $Sl(3,\mathbf{R})$ WZW model. Details of these new results for finite c will be published elsewhere [4].

4. From $Sl(3,\mathbf{R})$ to W_3

We now extend the analysis of $Sl(2,\mathbf{R})$, which reproduced pure gravity, to the case of $Sl(3,\mathbf{R})$. Some earlier work in this direction was presented in [7, 19, 20]. Our purpose is to reproduce the Ward identities, transformation laws and action of W_3 gravity, and then to express all objects (h,b as well as u,v) as *local* expressions in terms of new variables r and s.

We choose the following basis for $Sl(3,\mathbf{R})$:

$$\begin{aligned} &T_1 \equiv e_{11}-e_{22}, \qquad T_2 \equiv e_{22}-e_{33}, \\ &T_{+1} \equiv e_{12}, \qquad T_{+2} \equiv e_{23}, \qquad T_{+3} \equiv e_{13}, \\ &T_{-1} \equiv e_{21}, \qquad T_{-2} \equiv e_{32}, \qquad T_{-3} \equiv e_{31}, \end{aligned} \tag{4.1}$$

where e_{ij} are 3×3 matrices, $(e_{ij})_{kl}=\delta_{ik}\delta_{jl}$. The metric g_{ab} is given by $g_{+i,-i}=-2$ for $i=1,2$ or 3 while $g_{11}=g_{22}=-4$ and $g_{12}=+2$. We impose the following constraints[2]:

$$u \equiv \begin{pmatrix} 0 & u^{+1} & u^{+3} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{4.2}$$

Again the Ward identities $\bar{\partial}u-[A,u]=\partial A$ can be reduced to two independent equations. First the fields A^1, A^2, $A^{\pm 2}$, A^{+1}, and A^{+3} are expressed in terms of A^{-1}, A^{-3} and their conjugates u^{+1}, u^{+3}

$$\begin{aligned} A^{-2} &= -\partial A^{-3}+A^{-1}, \\ A^1 &= -\frac{1}{3}(\partial^2 A^{-3}-3\partial A^{-1}-u^{+1}A^{-3}), \\ A^2 &= -\frac{2}{3}\left(\partial^2 A^{-3}-\frac{3}{2}\partial A^{-1}-u^{+1}A^{-3}\right), \\ A^{+1} &= \frac{1}{3}(\partial^3 A^{-3}-3\partial^2 A^{-1}-\partial(u^{+1}A^{-3}))+u^{+1}A^{-1}+u^{+3}A^{-3}, \\ A^{+2} &= \frac{2}{3}\left(\partial^3 A^{-3}-\frac{3}{2}\partial^2 A^{-1}-\partial(u^{+1}A^{-3})\right)+u^{+3}A^{-3}, \\ A^{+3} &= \frac{2}{3}\left(\partial^4 A^{-3}-\frac{3}{2}\partial^3 A^{-1}-\partial^2(u^{+1}A^{-3})\right)+\partial(u^{+3}A^{-3})+u^{+3}A^{-1}, \end{aligned} \tag{4.3}$$

and then these results are used to obtain the two Ward identities

$$\begin{aligned} -2\partial^3\tilde{A}^{-1} &= (\bar{\partial}-2\partial\tilde{A}^{-1}-\tilde{A}^{-1}\partial)u^{+1}-(2A^{-3}\partial+3\partial A^{-3})\tilde{u}^{+3}, \\ \frac{1}{6}\partial^5 A^{-3} &= \frac{1}{12}(2A^{-3}\partial^3+9\partial A^{-3}\partial^2+15\partial^2 A^{-3}\partial+10\partial^3 A^{-3})u^{+1} \\ &\quad -\frac{1}{3}\partial(u^{+1}u^{+1})A^{-3}-\frac{2}{3}u^{+1}u^{+1}\partial A^{-3} \\ &\quad +(\bar{\partial}-\tilde{A}^{-1}\partial-3\partial\tilde{A}^{-1})\tilde{u}^{+3}, \end{aligned} \tag{4.4}$$

where

$$\tilde{A}^{-1}=A^{-1}-\frac{1}{2}\partial A^{-3}; \qquad \tilde{u}^{+3}=u^{+3}-\frac{1}{2}\partial u^{+1}. \tag{4.5}$$

[2] Instead of $u^{-1}=u^{-2}=1$, one can choose arbitrary real constants without changing any of the final results

Comparing Eq. (4.4) with the W_3 Ward identities in Eq. (3.15), one finds that they are the same if one identifies

$$\begin{aligned} u &= -\frac{1}{2}u^{+1}, \\ v &= -15\gamma\tilde{u}^{+3}, \\ b &= \gamma^{-1}A^{-3}, \\ h &= \tilde{A}^{-1}, \end{aligned} \tag{4.6}$$

where $\gamma^2 = -2/5$.

From $\delta u = \partial\eta + [\eta, u]$, we can express $\eta^1, \eta^2, \eta^{\pm 2}, \eta^{+1}$, and η^{+3} in terms of η^{-1}, η^{-3}, u^{+1}, and u^{+3}. The result can, of course, immediately be read off from the fact that $\bar{\partial}u - [A, u] = \partial A$ and $\delta u - [\eta, u] = \partial\eta$ have a similar structure,

$$\begin{aligned} \eta^{-2} &= -\partial\eta^{-3} + \eta^{-1}, \\ \eta^{1} &= -\frac{1}{3}(\partial^2\eta^{-3} - 3\partial\eta^{-1} - u^{+1}\eta^{-3}), \\ \eta^{2} &= -\frac{2}{3}\left(\partial^2\eta^{-3} - \frac{3}{2}\partial\eta^{-1} - u^{+1}\eta^{-3}\right), \\ \eta^{+1} &= \frac{1}{3}(\partial^3\eta^{-3} - 3\partial^2\eta^{-1} - \partial(u^{+1}\eta^{-3})) + u^{+1}\eta^{-1} + u^{+3}\eta^{-3}, \\ \eta^{+2} &= \frac{2}{3}\left(\partial^3\eta^{-3} - \frac{3}{2}\partial^2\eta^{-1} - \partial(u^{+1}\eta^{-3})\right) + u^{+3}\eta^{-3}, \\ \eta^{+3} &= \frac{2}{3}\left(\partial^4\eta^{-3} - \frac{3}{2}\partial^3\eta^{-1} - \partial^2(u^{+1}\eta^{-3})\right) + \partial(u^{+3}\eta^{-3}) + u^{+3}\eta^{-1}. \end{aligned} \tag{4.7}$$

For the transformation rules of u and v we find

$$\begin{aligned} \delta u &= \partial^3\varepsilon + \varepsilon\partial u + 2\partial\varepsilon u + \frac{1}{15}\lambda\partial v + \frac{1}{10}\partial\lambda v, \\ \delta v &= \varepsilon\partial v + 3\partial\varepsilon v + \partial^5\lambda + (2\lambda\partial^3 + 9\partial\lambda\partial^2 + 15\partial^2\lambda\partial + 10\partial^2\lambda)u \\ &\quad + 8(2\partial\lambda + \lambda\partial)uu, \end{aligned} \tag{4.8}$$

where

$$\begin{aligned} \varepsilon &= \eta^{-1} - \frac{1}{2}\partial\eta^{-3}, \\ \lambda &= \gamma^{-1}\eta^{-3}, \end{aligned} \tag{4.9}$$

which again agrees with W_3 gravity, Eqs. (3.11). Combining this with $\delta A = \bar{\partial}\eta + [\eta, A]$ and Eqs. (4.6) we obtain

$$\begin{aligned} \delta h &= \bar{\partial}\varepsilon + \varepsilon\partial h - \partial\varepsilon h + \frac{1}{30}(2\lambda\partial^3 - 3\partial\lambda\partial^2 + 3\partial^2\lambda\partial - 2\partial^3\lambda)b \\ &\quad + \frac{8}{15}(\lambda\partial b - b\partial\lambda)u, \\ \delta b &= \varepsilon\partial b - 2\partial\varepsilon b + \bar{\partial}\lambda + 2\lambda\partial h - \partial\lambda h, \end{aligned} \tag{4.10}$$

in agreement with (3.6) and (3.10).

The action $\Gamma[h,b]$ for induced W_3 gravity can now be obtained from the $Sl(3,\mathbf{R})$ action in exactly the same way as we obtained the action $\Gamma[h]$ for induced pure gravity from $Sl(2,\mathbf{R})$. We find

$$\Gamma[h,b]=-\Gamma_{\mathrm{WZW}}[A^{-1},A^{-3}]+\frac{k}{\pi}\int d^2x(u^{+1}A^{-1}+2u^{+3}A^{-3}), \tag{4.11}$$

where we should put

$$k=\frac{c}{24}. \tag{4.12}$$

The latter identification is made such as to agree with the leading term of a W_3 KPZ formula (this indeed agrees with conjectured formulae in [14, 18] and can also directly be checked using the results of the preceding section).

Let us now choose a Gaussian parametrization for $Sl(3,\mathbf{R})$:

$$g=\begin{pmatrix}1&\varphi_1&\varphi_3\\0&1&\varphi_2\\0&0&1\end{pmatrix}\begin{pmatrix}\lambda_1^{-1}&0&0\\0&\lambda_2^{-1}&0\\0&0&\lambda_1\lambda_2\end{pmatrix}\begin{pmatrix}1&0&0\\\phi_2&1&0\\\phi_3&\phi_1&1\end{pmatrix}. \tag{4.13}$$

The constraints express all variables in terms of two independent variables. The mere fact that in general only two nonlocal expressions occur, guarantees that one can introduce two new coordinates, in terms of which all results become local. It turns out that $r\equiv\phi_1$ and $s\equiv\phi_3$ are such a set of coordinates. The solution of the constraints, Eq. (4.2), reads

$$\begin{aligned}
\phi_2&=\frac{\partial s}{\partial r},\\
\varphi_1&=-\frac{1}{3}(\partial r)^{-1}\partial^2 r-\frac{2}{3}\left(\partial\left(\frac{\partial s}{\partial r}\right)\right)^{-1}\partial^2\left(\frac{\partial s}{\partial r}\right),\\
\varphi_2&=-\frac{1}{3}\,\frac{\partial^3 s\partial r-\partial s\partial^3 r}{\partial^2 s\partial r-\partial s\partial^2 r},\\
\varphi_3&=\partial\varphi_2+\varphi_2^2,\\
\lambda_1^3&=\left(\partial\left(\frac{\partial s}{\partial r}\right)\right)^{-2}(\partial r)^{-1},\\
\lambda_2^3&=\partial\left(\frac{\partial s}{\partial r}\right)(\partial r)^{-1}.
\end{aligned} \tag{4.14}$$

From $A=\bar{\partial}gg^{-1}$ and Eq. (4.6) we obtain

$$\begin{aligned}
h&=\frac{\bar{\partial}\left(\frac{\partial s}{\partial r}\right)}{\partial\left(\frac{\partial s}{\partial r}\right)}-\frac{\gamma}{3}\,\frac{\partial^3 s\partial r-\partial s\partial^3 r}{\partial^2 s\partial r-\partial s\partial^2 r}b-\frac{\gamma}{2}\partial b,\\
b&=\gamma^{-1}\frac{(\bar{\partial}s\partial r-\bar{\partial}r\partial s)}{(\partial^2 s\partial r-\partial^2 r\partial s)}.
\end{aligned} \tag{4.15}$$

The effective currents u and v are

$$\begin{aligned} u &= -\frac{1}{2}\left[\partial\left(2\frac{\partial\lambda_1}{\lambda_1}+\frac{\partial\lambda_2}{\lambda_2}\right)+\left(\frac{\partial\lambda_1}{\lambda_1}\right)^2+\left(\frac{\partial\lambda_2}{\lambda_2}\right)^2+\left(\frac{\partial\lambda_1}{\lambda_1}\right)\left(\frac{\partial\lambda_2}{\lambda_2}\right)\right], \\ v &= -15\gamma\left[\frac{1}{2}\partial^2\left(\frac{\partial\lambda_2}{\lambda_2}\right)+\left(\frac{3}{2}\frac{\partial\lambda_1}{\lambda_1}+\frac{\partial\lambda_2}{\lambda_2}\right)\partial\left(\frac{\partial\lambda_2}{\lambda_2}\right)\right. \\ &\quad \left.+\frac{1}{2}\left(\frac{\partial\lambda_2}{\lambda_2}\right)\partial\left(\frac{\partial\lambda_1}{\lambda_1}\right)+\left(\frac{\partial\lambda_1}{\lambda_1}+\frac{\partial\lambda_2}{\lambda_2}\right)\left(\frac{\partial\lambda_1}{\lambda_1}\right)\left(\frac{\partial\lambda_2}{\lambda_2}\right)\right]. \end{aligned} \tag{4.16}$$

Combining the solution of the constraints with the fact that $(\delta g g^{-1})=\eta$ yields

$$\begin{aligned} \delta r &= \varepsilon\partial r+\gamma\lambda\partial^2 r-\frac{\gamma}{2}\partial\lambda\partial r-\frac{2\gamma}{3}\lambda\partial r\partial\ln[(\partial^2 s\partial r-\partial^2 r\partial s)], \\ \delta s &= \varepsilon\partial s+\gamma\lambda\partial^2 s-\frac{\gamma}{2}\partial\lambda\partial s-\frac{2\gamma}{3}\lambda\partial s\partial\ln[(\partial^2 s\partial r-\partial^2 r\partial s)]. \end{aligned} \tag{4.17}$$

A different parametrization, which stays closer to the Polyakov parametrization, is given by

$$\begin{aligned} r &= f, \\ s &= \frac{1}{2}f^2+g. \end{aligned} \tag{4.18}$$

In linearized form, this parametrization was already found in [3]. In these variables one has that

$$\begin{aligned} h &= (1+\bar{\partial}^2 g)^{-1}\left(\frac{\bar{\partial}f}{\partial f}+\frac{1}{\partial f}\bar{\partial}\bar{\partial}g\right) \\ &\quad -\frac{\gamma}{3}\partial(\ln[(\partial f)^3(1+\bar{\partial}^2 g)])b-\frac{\gamma}{2}\partial b, \\ b &= \frac{1}{\gamma}\frac{\left(\bar{\partial}-\dfrac{\bar{\partial}f}{\partial f}\partial\right)g}{(\partial f)^2(1+\bar{\partial}^2 g)}, \end{aligned} \tag{4.19}$$

where

$$\bar{\partial}=(\partial f)^{-1}\partial. \tag{4.20}$$

The ε and λ transformations of these variables read

$$\begin{aligned} \delta f &= \varepsilon\partial f-\gamma\lambda\partial^2 f-\frac{1}{2}\gamma\partial\lambda\partial f-\frac{2}{3}\gamma\lambda\frac{\bar{\partial}^3 g}{1+\bar{\partial}^2 g}, \\ \delta g &= \varepsilon\partial g+\gamma\lambda(\partial f)^2+\gamma\lambda\partial^2 g-\frac{1}{2}\gamma\partial\lambda\partial g \\ &\quad -\frac{2}{3}\gamma\lambda\partial g\left(3\frac{\partial^2 f}{\partial f}+\frac{\partial\bar{\partial}^2 g}{1+\bar{\partial}^2 g}\right). \end{aligned} \tag{4.21}$$

In these variables, the reduction from W_3 gravity to W_2 gravity becomes transparent; it is simply given by putting $g=0$.

We can now substitute the Gaussian decomposition into the action in (4.11). There is now no cancellation between the kinetic term of the WZW model and the uA correction term and one finds the following surprisingly simple expression for the induced action:

$$\Gamma = -\frac{k}{2\pi}\int d^2x \left[\frac{\bar\partial\left(\frac{\partial s}{\partial r}\right)}{\partial\left(\frac{\partial s}{\partial r}\right)}\partial\left(\frac{\partial\lambda_1}{\lambda_1}\right) + \frac{\bar\partial r}{\partial r}\partial\left(\frac{\partial\lambda_1}{\lambda_1}+\frac{\partial\lambda_2}{\lambda_2}\right) \right.$$
$$+\frac{\bar\partial s\partial r-\bar\partial r\partial s}{\partial^2 s\partial r-\partial^2 r\partial s}\left[\frac{\partial\lambda_2}{\lambda_2}\partial\left(\frac{\partial\lambda_1}{\lambda_1}\right)-\frac{\partial\lambda_1}{\lambda_1}\partial\left(\frac{\partial\lambda_2}{\lambda_2}\right)\right.$$
$$\left.\left.-\left(\frac{\partial\lambda_1}{\lambda_1}\right)\left(\frac{\partial\lambda_2}{\lambda_2}\right)\left(\frac{\partial\lambda_1}{\lambda_1}+\frac{\partial\lambda_2}{\lambda_2}\right)\right]\right]. \tag{4.22}$$

By using the expressions in Eq. (4.14), this can be further reduced to an expression in terms of r and s only. An expression of Γ in terms of h and b seems hard to obtain, as it is not clear to us how the relations (4.15) or (4.19) can be inverted explicitly.

We finally draw the reader's attention to the following variables

$$\omega_1 = \ln(\partial f), \quad \omega_2 = \ln(1+\phi^2 g), \tag{4.23}$$

which play the role of "connections" in the theory. They obey the following differential equations,

$$\begin{aligned}
\bar\partial\omega_1 &= \partial h - \frac{\gamma}{2}\partial^2 b + \partial\omega_1 h - \frac{3\gamma}{2}\partial\omega_1\partial b \\
&\quad -\gamma(\partial^2\omega_1+(\partial\omega_1)^2)b - \frac{2\gamma}{3}(\partial^2\omega_2+\partial\omega_1\partial\omega_2)b - \frac{2\gamma}{3}\partial\omega_2\partial b, \\
\bar\partial\omega_2 &= \gamma\partial^2 b + 3\gamma\partial\omega_1\partial b + 2\gamma((\partial\omega_1)^2+\partial^2\omega_1)b \\
&\quad + h\partial\omega_2 + \frac{3\gamma}{2}\partial\omega_2\partial b + \gamma\left(\frac{1}{3}(\partial\omega_2)^2 + 2\partial\omega_2\partial\omega_1 + \partial^2\omega_2\right)b.
\end{aligned} \tag{4.24}$$

Using

$$\frac{\partial\lambda_1}{\lambda_1} = -\partial\omega_1 - \frac{2}{3}\partial\omega_2 \quad \text{and} \quad \frac{\partial\lambda_2}{\lambda_2} = \frac{1}{3}\partial\omega_2 \tag{4.25}$$

and Eq. (4.16), one expresses the effective currents u and v in terms of the ω-variables. The relations (4.24) then reduce to the fundamental Ward identities (3.15). Under ε and λ transformations ω_1 and ω_2, like u and v, transform (non-linearly) into themselves, but with inhomogeneous terms proportional to $\partial\varepsilon$ and $\partial^2\lambda$ rather than to $\partial^3\varepsilon$ and $\partial^5\lambda$ as in (3.11). These observations suggest that (ω_1, ω_2), rather than (u, v) should be considered as the fundamental W_3 multiplet at the quantum level.

Finally, the previous construction provides us with a Lax pair for the W_3 Ward identities. Indeed a similar reasoning as in the $Sl(2, \mathbf{R})$ case yields the following Lax pair:

$$(\partial^3 - u^{+1}\partial - u^{+3})\psi = 0,$$
$$\left(\bar{\partial} - A^{-1}\partial + \partial A^{-1} - A^{-3}\partial^2 + \partial A^{-3}\partial - \frac{2}{3}\partial^2 A^{-3} + \frac{2}{3}u^{+1}A^{-3}\right)\psi = 0, \tag{4.26}$$

and one can easily check that Eqs. (4.4) are reproduced as its integrability condition.

5. Concluding Remarks

It is clear that one of the major open problems in the study of W_3 gravity is the understanding of its geometry. We expect that it will be possible to understand this geometry in a "W_3 superspace," which will be similar to the chiral superspace used for $d=2$ supergravity. For this reason, we first take a closer look at induced $N=1$ supergravity (in the chiral gauge). In [21] and [19], the Ward identities for induced $N=1$ supergravity in x-space were derived from a reduction of an $OSp(1|2)$ WZW model. However, in this formulation the geometry of supergravity is obscure. A natural framework to study supergravity is in superspace. Indeed the Neveu-Schwarz algebra has a natural realization as analytic reparametrizations of the superplane. In the following we will show that the reduction of $OSp(1|2)$ in x-space already suggests the structure of chiral $N=1$ superspace.

The derivation of the supergravity Ward identities in x-space goes very similar as before. Consider the superalgebra $OSp(1|2)$. It is defined by the following vector representation

$$T_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad T_{\mp} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad T_{=} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{5.1}$$

$$T_{+} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \quad T_{-} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}. \tag{5.2}$$

From this we immediately deduce the abstract commutation rules:

$$\begin{aligned} [T_0, T_{\mp}] &= +2T_{\mp}, & [T_0, T_{=}] &= -2T_{=}, \\ [T_0, T_{+}] &= +T_{+}, & [T_0, T_{-}] &= -T_{-}, \\ [T_{+}, T_{+}]_{+} &= -2T_{\mp}, & [T_{-}, T_{-}]_{+} &= +2T_{=}, \\ [T_{-}, T_{\mp}] &= -T_{+}, & [T_{+}, T_{=}] &= -T_{-}, \\ [T_{\mp}, T_{=}] &= T_0, & [T_{+}, T_{-}]_{+} &= T_0. \end{aligned} \tag{5.3}$$

We consider again Lie algebra valued gauge fields A and effective currents u which satisfy the Ward identity Eq. (2.5) and we impose the constraints

$$u=\begin{bmatrix} 0 & u^{\mp} & u^{+} \\ 1 & 0 & 0 \\ 0 & -u^{+} & 0 \end{bmatrix}. \tag{5.4}$$

We can proceed in exactly the same way as before and use the Ward identity, Eq. (2.5), to express $A^{\mp}$, A^{0}, and A^{+} in terms of $A^{=}$, A^{-}, $u^{\mp}$, and u^{+}. We are then left with two independent Ward identities:

$$\begin{aligned} \partial^3 h &= (\bar{\partial}-2\partial h-h\partial)u-\left(\frac{3}{2}\partial\psi+\frac{1}{2}\psi\partial\right)v, \\ \partial^2\psi &= \left(\bar{\partial}-\frac{3}{2}\partial h-h\partial\right)v-\frac{1}{2}\psi u, \end{aligned} \tag{5.5}$$

where

$$\begin{aligned} h &\equiv A^{=}, & \psi &\equiv 2iA^{-}, \\ u &\equiv -2u^{\mp}, & v &\equiv 2iu^{+}, \end{aligned} \tag{5.6}$$

and these are precisely the Ward identities for induced supergravity [21]. We can view the constraints as a gauge fixing of the subalgebra of $OSp(1|2)$ generated by $\{T_{\mp}, T_0, T_{+}\}$. From $\delta u=\partial\eta+[\eta,u]$ and Eq. (5.4) one finds that $\eta^{\mp}$, η^{0}, and η^{+} are given as functions of $\eta^{=}$, η^{-}, $u^{\mp}$, and u^{+}. The fields h, ψ, u, and v transform as

$$\begin{aligned} \delta h &= \bar{\partial}\varepsilon+\varepsilon\partial h-\partial\varepsilon h+\frac{1}{2}\kappa\psi, \\ \delta\psi &= \bar{\partial}\kappa+\frac{1}{2}\kappa\partial h-\partial\kappa h+\varepsilon\partial\psi-\frac{1}{2}\partial\varepsilon\psi, \\ \delta u &= \partial^3\varepsilon+\varepsilon\partial u+2\partial\varepsilon u+\frac{3}{2}\partial\kappa v+\frac{1}{2}\kappa\partial v, \\ \delta v &= \varepsilon\partial v+\frac{3}{2}\partial\varepsilon v+\frac{1}{2}\kappa u, \end{aligned} \tag{5.7}$$

where

$$\varepsilon\equiv\eta^{=}, \qquad \kappa\equiv 2i\eta^{-}. \tag{5.8}$$

The induced supergravity action can now be obtained starting from an $OSp(2|1)$ WZW-model and using the same techniques as in Sects. 2 and 3.

This reduction procedure suggests a natural coset in which to formulate the supergravity theory. Indeed consider $\mathcal{G}=OSp(1|2)$ and its subgroup $\mathcal{H}$ generated by $\{T_{\mp}, T_0, T_{+}\}$. The reduction procedure looks somewhat like a modding out of $\mathcal{H}$. We parametrize the elements of the non-reductive coset $\mathcal{G}/\mathcal{H}$ by

$$k=\exp(zT_{=}+\theta T_{-}). \tag{5.9}$$

Using standard methods (for a review, see [22]), we can construct the isometries of this coset space [i.e. the action of $OSp(1|2)$ on the coset]:

$$\begin{aligned} T_{\mp} &= -z^2\partial - \theta z D, & T_{=} &= \partial, \\ T_{+} &= zD, & T_{-} &= D, \\ T_0 &= 2z\partial + \theta D, \end{aligned} \tag{5.10}$$

where $\partial = \partial/\partial z$ and $D = \partial/\partial\theta + \theta\partial/\partial z$. Compare this now with the algebra of regular Neveu-Schwarz transformations:

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n}, \\ [L_m, G_r] &= \left(\frac{1}{2}m - r\right) G_{m+r}, \\ [G_r, G_s] &= 2L_{r+s}, \end{aligned} \tag{5.11}$$

where $m, n \in \mathbf{Z}$, $m, n \leqq +1$, $r, s \in \mathbf{Z} + 1/2$, and $r, s \leqq +1/2$. One sees that upon identifying $L_{+1} \equiv T_{=}$, $L_0 \equiv 1/2 T_0$, $L_{-1} \equiv -T_{\mp}$, $G_{+1/2} \equiv T_{-}$, and $G_{-1/2} \equiv T_{+}$, one obtains a realization of the projective subalgebra on the cosetspace. In order to recover the whole of Eq. (5.11), one takes the group $\mathcal{G}$ of Neveu-Schwarz transformations regular at the origin, generated by $\{L_m, G_r | m \leqq +1, r \leqq +1/2\}$ and its subgroup $\mathcal{H}$ generated by $\{L_m, G_r | m \leqq 0, r \leqq -1/2\}$. We consider the coset space $\mathcal{G}/\mathcal{H}$ with representant k:

$$k = e^{zL_{+1} + \theta G_{+1/2}}. \tag{5.12}$$

Again, using coset space techniques, we obtain the Killing vectors:

$$\begin{aligned} L_m &= z^{-m+1}\partial + \frac{1}{2}(1-m) z^{-m} \theta D, \\ G_r &= z^{-r+1/2} D. \end{aligned} \tag{5.13}$$

From this, one sees that the super conformal transformations can be rewritten through the introduction of a superfield $E(z, \theta)$:

$$E(z, \theta) = \varepsilon(z) + 2\theta\kappa(z), \tag{5.14}$$

where

$$\varepsilon(z) = \sum_{m \leqq +1} \varepsilon^m z^{1-m}, \qquad \kappa(z) = \sum_{r \leqq +1/2} \kappa^r z^{1/2-r}, \tag{5.15}$$

and we have that

$$\begin{aligned} \delta\theta &= \frac{1}{2} DE, \\ \delta z &= E - \frac{1}{2}\theta DE. \end{aligned} \tag{5.16}$$

The finite transformations are then given by

$$z \to z'(z, \theta), \qquad \theta \to \theta'(z, \theta), \tag{5.17}$$

where

$$Dz' = \theta' D\theta'. \tag{5.18}$$

A further application of the theory of induced representations leads immediately to the definition of $N=1$ primary fields $\Phi(Z)$ which transform as

$$\delta\Phi(Z)=E\partial\Phi(Z)+\frac{1}{2}DED\Phi(Z)+h\partial E\Phi(Z). \tag{5.19}$$

Now that we constructed the superplane, the question arises whether we can derive the Ward identities for induced supergravity in superspace.

Consider a chiral superspace, i.e. the left-movers are parametrized by the coordinates z and θ while the right-movers are parametrized with the coordinate $\bar{z}$. The corresponding gauge fields are $H_M=\{A_{\bar{z}}, u_z, u_\theta\}$, the Yang-Mills curvatures $R_{z\bar{z}}$, $R_{\theta\bar{z}}$, $R_{\theta z}$, and $R_{\theta\theta}$ are defined

$$R_{MN}=D_MH_N-(-)^{MN}D_NH_M-H_MH_N+(-)^{MN}H_NH_M-T^P_{MN}H_P. \tag{5.20}$$

The torsions T^P_{MN} are defined by

$$D_MD_N-(-)^{MN}D_ND_M=T^P_{MN}D_P, \tag{5.21}$$

where only the torsion component $T^z_{\theta\theta}=2$ is non-vanishing. We impose the following constraints on the two lowest dimensional curvatures

$$R_{\theta\theta}=R_{\theta\bar{z}}=0. \tag{5.22}$$

The Bianchi identities imply then that also $R_{\theta z}=R_{z\bar{z}}=0$. The constraint $R_{\theta\theta}=0$ is easily solved and yields

$$H_z=D_\theta H_\theta-H_\theta H_\theta. \tag{5.23}$$

We now take the gauge group to be the supergroup $OSp(1|2)$ and we partially fix the gauge by

$$u_\theta=\begin{bmatrix}0 & u^{\mp} & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{bmatrix}. \tag{5.24}$$

In much the same way as before we can solve the constraint $R_{\theta\bar{z}}=0$ such that $A^{\mp}_{\bar{z}}$, $A^0_{\bar{z}}$, $A^+_{\bar{z}}$, and $A^-_{\bar{z}}$ are expressed in terms of $A^{=}_{\bar{z}}$ and $u^{\mp}_\theta$. One of the components of $R_{\theta\bar{z}}=0$ remains and expresses $u^{\mp}_\theta$ as a function of $A^{=}_{\bar{z}}$:

$$\partial^2DH=\left(\bar{\partial}-\frac{3}{2}\partial H-\frac{1}{2}DHD-H\partial\right)U, \tag{5.25}$$

where $H\equiv -A^{=}_{\bar{z}}$ and $U\equiv 2u^{\mp}_\theta$. Equation (5.25) is recognized as the Ward identity for induced $N=1$ supergravity [23, 24]. Indeed consider the generating functional:

$$e^{-\Gamma[H]}=\left\langle \exp-\frac{1}{\pi}\int d^2xd\theta HQ\right\rangle \tag{5.26}$$

with x-space expansions $H=h+\theta\psi$, where h is the graviton and ψ the gravitino in the lightcone gauge, and $Q=G+\theta T$, where T is the energy-momentum tensor and G the dimension 3/2 supercurrent. Under the transformation

$$\delta H=\bar{\partial}E+E\partial H+\frac{1}{2}DEDH-\partial EH \tag{5.27}$$

one has that

$$\delta\Gamma[H] = -\frac{c}{12\pi}\int d^2x d\theta E\partial^2 DH\,. \tag{5.28}$$

Defining U through

$$\delta\Gamma[H] = \frac{c}{12\pi}\int d^2x d\theta U\delta H\,, \tag{5.29}$$

one finds Eq. (5.25) back by combining Eqs. (5.27) and (5.28). In components, Eq. (5.25) reduces to Eqs. (5.5).

From previous remarks, one expects that starting from an $N=1$ $OSp(1|2)$ WZW model in a chiral superspace [25], one can construct the action for induced supergravity.

From the reduction procedure in Sect. 4 and the remarks above, one again expects a cosetspace structure for a "W_3 plane." The groups involved are $\mathscr{G} = Sl(3, \mathbf{R})$ and $\mathscr{H}$ generated by $\{T_1, T_2, T_{+1}, T_{\pm 2}, T_{+3}\}$. The local structure of the W_3 plane should be given by the coset $\mathscr{G}/\mathscr{H}$. We choose to represent an element k of $\mathscr{G}/\mathscr{H}$ by

$$k = e^{xT_{-1}} e^{yT_{-3}}\,. \tag{5.30}$$

It is not hard to find the Killing vectors:

$$\begin{aligned} T_{-1} &= \partial_x\,, \quad T_{-2} = -x\partial_y\,, \quad T_{-3} = \partial_y\,, \\ T_1 &= 2x\partial_x + y\partial_y\,, \quad T_2 = y\partial_y - x\partial_x\,, \\ T_{+1} &= -x^2\partial_x - xy\partial_y\,, \quad T_{+2} = -y\partial_x\,, \quad T_{+3} = -y^2\partial_y - xy\partial_x\,. \end{aligned} \tag{5.31}$$

"Superfields" in this space will in general be $Sl(2)$ multiplets as $\mathscr{H}$ consists of an $Sl(2)$ algebra and a vector representation of it. However, if we want to recover the whole conformal structure, it looks more natural to consider a 3-dimensional space.

Indeed, consider the group $\mathscr{G}$ of regular W_3 transformations in the $c\to\infty$ limit. This group is generated by $\{L_m, W_n | m \leqq +1, n \leqq +2\}$ with commutation relations

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n}\,, \\ [L_m, W_n] &= (2m-n)W_{m+n}\,, \\ [W_m, W_n] &= (m-n)\left[\frac{1}{15}(m+n+3)(m+n+2) - \frac{1}{6}(m+2)(n+2)\right] \\ &\quad \times L_{m+n} + \frac{16}{5c}(m-n)(LL)_{m+n}\,. \end{aligned} \tag{5.32}$$

As was explained in the first section, one cannot drop the non-linear terms. This can easily be seen from the $[LWW]$ Jacobi identity. Working in the limit $c\to\infty$ has the advantage that the non-linear terms do not need to be regularized. Precisely because of the presence of the non-linear terms, the algebra given above is not a subalgebra of the W_3 algebra as the Jacobi identities require that the sum in $\Lambda_m = \sum_{n\in\mathbf{Z}} L_{m-n}L_n$ runs over the whole of $\mathbf{Z}$. However, the algebra realized on fields, given by Eq. (3.13) with the restriction that $\varepsilon(z)$ and $\lambda(z)$ are analytic, is closed provided we introduce field dependent structure functions, which depend on auxiliary fields $u(z)$ and $v(z)$ (which themselves are also analytic), defined by their

transformation rules Eq. (3.11)[3]. Explicit realizations of these auxiliary fields as W_3 Schwarzian derivatives can be found in Eqs. (4.16). Starting from the algebra Eq. (3.13), it is not hard to see that its maximal subalgebra is generated by analytic parameters $\varepsilon(z)$ and $\lambda(z)$ with $\varepsilon(0)=\lambda(0)=\partial\lambda(0)=0$. Given these observations, it is natural to anticipate that the full W_3 conformal structure can be most easily formulated in the 3-dimensional coset $\mathscr{G}/\mathscr{H}$, where $\mathscr{G}$ is the algebra Eq. (3.13) and $\mathscr{H}$ is its maximal subalgebra. A further analysis of this requires a generalization of the theory of induced representations to algebras with field dependent structure functions. Work in this direction is in progress.

In [6], Verlinde gave a beautiful account of induced gravity in an $Sl(2,\mathbf{R})$ Chern-Simons formulation. Starting from an $Sl(2,\mathbf{R})$ Chern-Simons theory in $2+1$ dimensions in the temporal gauge, an $Sl(2,\mathbf{R})$ breaking polarization was chosen. The coordinates are A_z^-, $A_{\bar{z}}^-$, and A_z^0, while the conjugate momenta are the remaining gauge fields. Parametrizing the gauge fields as $A^- \equiv e^{\varphi}(dz+hd\bar{z})$, $A_z^0=\omega$ and imposing the Gauss law on a state $\Psi[\omega,\varphi,h]$: $R^{\pm}\Psi=R^0\Psi=0$ results in

$$\Psi[\omega,\varphi,h] = \exp\left\{-\frac{k}{\pi}\int\left[\frac{1}{4}\partial\varphi\bar{\partial}\varphi+\omega\bar{\partial}\varphi-h\left(\left(\frac{1}{2}\partial\varphi+\omega\right)^2-\frac{1}{2}\partial^2\varphi-\partial\omega\right)\right]+\Gamma[h]\right\} \tag{5.33}$$

where $\Gamma[h]$ is the induced action for gravity in the light-cone gauge. The norm of this state,

$$\begin{aligned}\|\Psi[\omega,\varphi,h]\|^2 &= \int[d\varphi][dh][d\omega]e^{-\frac{k}{\pi}\int d^2(2\omega\bar{\omega}+e^{\varphi+\bar{\varphi}}(1-h\bar{h}))}|\Psi[\omega,\varphi,h]|^2 \\ &= \int[d\varphi][dh]e^{S[h,\bar{h},\varphi,\bar{\varphi}]}\end{aligned} \tag{5.34}$$

is such that $S[h,\bar{h},\varphi,\bar{\varphi}]$ precisely gives the *covariant* induced action. Presently, we are investigating whether this approach can be generalized to the case of W_3 gravity [10] such as to give both the covariant action for induced W_3 gravity as well as the full set of W_3 gauge transformations.

Finally, one wonders whether in the weakly coupled regime, $c<2$, there exist equivalent formulations of W_3 gravity in terms of topological quantum field theories or matrix models. The former question seems to be readily accessible through the study of twisted $N=2$ supersymmetric W_3 conformal field theories. Recently, the $N=2$ W_3 algebra has been constructed [27] (it consists of a dimension 2 and a dimension 1 $N=2$ superfield) and in view of the previous motivation, it would be very interesting to work out its representation theory. At present it is not clear how to construct a matrix model formulation of W_3 gravity. It might happen that the final answer to this question will only come after the construction of "W_3 Riemann surfaces." However, one might speculate that, as the matrix chains presently studied have incidence relations determined by the weight lattice of affine $Sl(2)$, the W_3 matrix models could be based on incidence relations determined by the weight lattice of affine $Sl(3)$.

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[3] This situation is very similar to the one encountered in [26], where a gauge theory of the regular W_3 transformations was constructed

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