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Lyapunov Exponents of the Schrödinger Equation with Quasi-Periodic Potential on a Strip

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Abstract. We prove that all the non-negative Lyapunov exponents of difference Schrödinger equation

$$-y_{n+1}+Q_n y_n - y_{n-1}=0, \qquad -\infty<n<+\infty$$

are strictly positive. Here $y_n \in R^m$ and Q_n is a symmetric $m\times m$ matrix whose off-diagonal elements do not depend on n, and the diagonal elements are quasi-periodic functions

$$q_{nj}(\theta)=\lambda f_j(e^{2\pi i(\theta+n\alpha)})-E$$

with all f_i non-constant analytic functions, λ sufficiently large, and α any irrational number.

1. Introduction and Formulation of Results

In this paper we shall study the Lyapunov exponents of the difference equation:

$$-y_{n+1}+Q_n y_n - y_{n-1}=0, \qquad -\infty<n<+\infty\,, \tag{1}$$

where $y_n \in R^m$ and Q_n is a symmetric $m\times m$ matrix whose off-diagonal elements do not depend on n, and the diagonal elements are quasi-periodic functions

$$q_{nj}(\theta)=\lambda f_j(e^{2\pi i(\theta+n\alpha)})-E$$

with $f_j(z)$ non-constant analytic on $\mathscr{A}\equiv\{z\,|\,r<|z|<1/r\}$, taking values in $[-1,1]$ for $|z|=1$, λ is a (large) parameter called coupling constant, E is the energy, and α is any irrational number. Without loss of generality we shall assume that $\max_{1\leqq i\leqq m}\sup_{|z|=1} f_i(z)=1$ and $\min_{1\leqq i\leqq m}\inf_{|z|=1} f_i(z)=-1$.

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Equation (1) becomes equivalent to the finite-difference Schrödinger equation when the off-diagonal elements are chosen properly. For example, the case $q_{ij} = -1$ for $|i-j|=1$ and $q_{ij}=0$ for $|i-j|>1$ corresponds to Schrödinger operator on the strip $\mathbb{Z} \times \{1, \ldots, m\}$. Equation (1) can be written in the form

$$\begin{pmatrix} y_{n+1} \\ y_n \end{pmatrix} = \begin{pmatrix} Q_n & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} y_n \\ y_{n-1} \end{pmatrix} \equiv A_n \begin{pmatrix} y_n \\ y_{n-1} \end{pmatrix} \tag{2}$$

and, thus, the asymptotic behavior of the solutions of Eq. (1) is determined by the asymptotic behavior of the product $S(n) := A_n \ldots A_1$.

Various problems in solid-state physics give rise to different classes of matrices Q_n. These classes are characterized by the level of randomness. The case of independent random Q_n was studied in [GM1, GM2], where it was shown that under certain algebraic conditions on the support of the corresponding measure in the space of symmetric matrices all the Lyapunov exponents are different and, therefore, the smallest non-negative exponent is, in fact, positive. When Q_n's are non-deterministic at least some of the Lyapunov exponents are strictly positive [K, S, KS]. On the other hand, if Q_n's form a periodic sequence, it is easy to show that the "interesting" Lyapunov exponents vanish.

The case of quasi-periodic potentials exhibits mixed behavior. If the coupling constant λ is small and α is poorly approximated by rationals, it is known [BLT] that at least on part of the spectrum the Lyapunov exponent is zero when $m=1$. When λ is large and $m=1$ the Lyapunov exponent is positive [Si, CS, FSW, SS].

Until now, quasi-periodic potentials have been studied only for $m=1$ [Si, CS, FSW]. Here we study the case of $m>1$ and large λ. We do not assume that α is diophantine, only that it is irrational. We shall prove that all non-negative Lyapunov exponents are positive, and, consequently, the Green's function of H decays exponentially for almost every energy E. The method we use is an extension of the one used in [SS] which grew out of the analysis of the work of Herman [H].

To describe our result let us consider the following decomposition of $S(n)$:

$$S(n) = U(n) D(n) V(n),$$

where $U, V \in O(n)$, and $D(n) = \mathrm{diag}(d_1^{(n)}, \ldots, d_{2m}^{(n)})$ with $d_1 \geqq d_2 \geqq \ldots \geqq d_{2m} > 0$. Since $A_n \in Sp(m, R)$ for all n, we have $d_k = d_{2m-k+1}^{-1}$. The k^{th} Lyapunov exponent γ_k is defined by

$$\gamma_k := \lim_{n \to \infty} \frac{1}{n} \log d_k(n). \tag{3}$$

Clearly, $\gamma_k = -\gamma_{2m-k+1}$, and $\gamma_1 \geqq \gamma_2 \geqq \ldots \geqq \gamma_m \geqq 0$. Existence of the limit in (3) for almost every θ and its independence of θ are guaranteed by the Subadditive Ergodic Theorem [Ki] and ergodicity of the underlying dynamical system $\theta \mapsto \theta + \alpha$.

We shall prove the following

Theorem. *For f_j as above, there exists λ_0 such that for all $\lambda > \lambda_0$ and all E, there exists a set $\Omega(E) \subset [0,1]$ of Lebesque measure* 1, *such that*

$$\gamma_m(E) = \gamma_m(E, \theta) > 0, \qquad \forall \theta \in \Omega(E).$$

This, together with Oseledeĉ's Multiplicative Ergodic Theorem [O, GM 2] implies existence of m solutions of Eq. (1), which decay exponentially as $n \to +\infty$ and grow exponentially as $n \to -\infty$, and m other solutions of Eq. (1), which decay exponentially as $n \to -\infty$ and grow exponentially as $n \to +\infty$. These $2m$ solutions are linearly independent and form a basis of all solutions of Eq. (1). There are two important consequences of this fact.

1. The spectrum of the operator H defined by the left-hand side of Eq. (1) is singular.
2. The Green's function of H decays exponentially for almost every energy E.

Remarks. We should point out that the spectrum of H can actually be purely singular continuous if α is a Liouville number [CFKS].

The requirement that λ is large cannot be avoided, for when λ is small (and $m=1$) KAM theory is applicable and there is absolutely continuous spectrum [BLT].

The reader will see that we very strongly use the non-triviality of f_i's in our proof. This condition is necessary, for there are examples in which the presence of constant f_i's leads to appearance of zero Lyapunov exponents.

2. Plan of the Proof

We shall obtain a lower bound on γ_m by estimating $\sum_1^m \gamma_i$ and $\sum_1^{m-1} \gamma_i$. Without loss of generality we can assume that $E \in \sigma(H)$, for otherwise Green's function decays exponentially and, consequently, $\gamma_m > 0$.

Proposition 1. *For* $E \in \sigma(H)$,

$$\sum_1^{m-1} \gamma_i \leqq (m-1) \log 2(\lambda + c),$$

where c *is the norm of the off-diagonal part of* Q_0.

Proposition 2. *For* $E \in \sigma(H)$,

$$\sum_1^m \gamma_i \geqq m \log \lambda + \text{const}.$$

Proof of Theorem. Combining Propositions 1 and 2, we have

$$\gamma_m = \sum_1^m \gamma_i - \sum_1^{m-1} \gamma_i \geqq \log \lambda + \text{const} > 0$$

for λ sufficiently large. We note that the constant above depends on m, the functions f_i, and c.

3. Proofs of the Propositions

Proof of Proposition 1. We first note that

$$\sum_1^{m-1} \gamma_i = \lim_{n \to \infty} \frac{1}{n} \sum_1^{m-1} \log d_i(n) \leqq \lim_{n \to \infty} \frac{1}{n} (m-1) \log d_1(n) = \lim_{n \to \infty} \frac{1}{n} (m-1) \log \|S(n)\|$$
$$\leqq (m-1) \sup_n \log \|A_n\| \leqq (m-1) \sup_n \log(\|Q_n\| + 2). \tag{6}$$

Now, $\|Q\| \leqq \max_{1 \leqq i \leqq m} |\lambda f_i - E| + c$, where c is the norm of the off-diagonal part of $Q(n)$. Since $E \in \sigma(H) \subset [-\lambda - c, \lambda + c]$,

$$\|Q_n\| \leqq 2(\lambda + c).$$

Consequently,

$$\sum_1^{m-1} \gamma_i \leqq (m-1)(\log\lambda + \text{const}). \quad \square \tag{7}$$

Proof of Proposition 2. To estimate $\sum_1^m \gamma_i$ from below we note the following. Let π be any m-dimensional plane in R^{2m} and let $\{v_1, \ldots, v_m\}$ be its basis. Then

$$2\sum_1^m \gamma_i \geqq \lim_{n\to\infty} \frac{1}{n} \log|G(S(n)v_1, \ldots, S(n)v_m)|, \tag{8}$$

where $G(x_1, \ldots, x_m)$ is, by definition, Gram's determinant of the vectors $x_1, \ldots, x_m$. Direct investigation of the right-hand side of (8) is rather difficult because it is hard to follow the evolution of the vectors $S(n)v_i$ since the diagonal elements of $Q(n)$ can be small. We are going to avoid this problem with the help of the following extension of Jensen's formula [SS].

Lemma 1. *Let g be meromorphic on the annulus $\mathscr{A} := \{z: r < |z| < 1\}$. Then*

$$\int_0^1 \log|g(e^{2\pi i\theta})|d\theta = \int_0^1 \log|g(re^{2\pi i\theta})|d\theta + \sum \log|p_i| + \sum \log\frac{1}{|r_j|} + \left(\log\frac{1}{r}\right) \operatorname*{Arg}_{|z|=r} g,$$

where p_i and r_j are the poles and the roots of g in $\mathscr{A}$, and

$$\operatorname*{Arg}_{|z|=r} g = \frac{1}{2\pi i} \int_{|z|=r} \frac{g'}{g} dz.$$

In our case, $g(z) = G(S(n)v_1, \ldots, S(n)v_m)$. We recall that $S(n)$ is a function of z, because our potential depends on z.

Since G is analytic on $\mathscr{A}$ and $|r_j| \leqq 1$ for all j, Jensen's formula leads to the following inequality:

$$\int_{|z|=1} \log|G|d\theta \geqq \int_{|z|=r} \log|G|d\theta + \left(\log\frac{1}{r}\right) \operatorname*{Arg}_{|z|=r} G. \tag{10}$$

In order to estimate each of the terms on the right-hand side of (10) we are going to choose $r \in (0,1)$ so that the diagonal elements $q_{ni}(z)$ of Q_n satisfy

$$|q_{ni}(z)| \geqq \min_{|z|=r} |\lambda f_i(z) - E| > \lambda\delta_0, \tag{11}$$

where $\delta_0 > 0$ depends on the functions f_i only. To see that such δ_0 exists we note that since f_i's are non-constant and analytic, for each $c \in R$ there exist $r(c)$ and $\delta(c)$ so that

$$|f_i(z) - c| > \delta(c) > 0 \quad \text{for} \quad |z| = r(c) \quad \text{and} \quad i = 1, \ldots, m$$

and the union of the ranges of values of f_i is compact.

Lemma 2. *$A_n(z)$ for $|z|=r$ preserve the set $\mathscr{F}$ of m-dimensional planes spanned by vectors of the form*

$$\begin{pmatrix} x \\ \phi x \end{pmatrix},$$

where $x \in R^m$ and the operator $\phi: R^m \to R^m$ satisfies $\|\phi\| \leqq 2/(\lambda\delta_0)$.

Proof. For any n,

$$A_n \begin{pmatrix} x \\ \phi x \end{pmatrix} = \begin{pmatrix} (Q_n - \phi)x \\ x \end{pmatrix} = (Q_n - \phi) \begin{pmatrix} x \\ (Q_n - \phi)^{-1} x \end{pmatrix}.$$

Therefore, if $\phi_{n+1} = (Q_n - \phi_n)^{-1}$ it follows from (11) that

$$\|\phi_{n+1}\| \leqq \frac{1}{\lambda\delta_0 - c - \|\phi_n\|} < \frac{2}{\lambda\delta_0} \tag{12}$$

for $\lambda > \lambda_0$ for some large λ_0. □

Let π_0 be the plane spanned by the vectors of the form

$$\begin{pmatrix} e_i \\ 0 \end{pmatrix} \in R^{2m},$$

where $\{e_1, \ldots, e_m\}$ form the standard basis of R^m and let $\pi_n = A_n \ldots A_1 \pi_0$ for $n \geqq 1$.

For any matrix A and plane π, the *dilation coefficient*

$$G(A, \pi) := \frac{G(Av_1, \ldots, Av_m)}{G(v_1, \ldots, v_m)}$$

is independent of the choice of the spanning vectors $v_1, \ldots, v_m$. Consequently,

$$G(S(n)e_1, \ldots, S(n)e_m) = \prod_{i=1}^{n} G(A_i, \pi_{i-1}). \tag{13}$$

Remark. For $v_i \in \mathbb{C}^m$ we define

$$G(v_1, \ldots, v_m) := \det((v_i, v_j))_{i,j=1}^m,$$

where

$$(v_i, v_j) = \sum_{k=1}^{m} v_i^k v_j^k \tag{14}$$

with v_i^k being the components of the vectors v_i. We note that (14) is not actually a scalar product, but defining $(\cdot, \cdot)$ in this way makes the function $G(A, \pi)$ analytic in z. We shall make essential use of this fact.

Lemma 3. *If $\|\phi_0\| \leqq 2/(\lambda\delta_0)$, then*

$$G(A_i(z), \pi_{i-1}) = \prod_{k=1}^{m} (\lambda f_k(z) - E)^2 \left(1 + O\left(\frac{1}{\lambda}\right)\right).$$

Proof. By Lemma 2, $\|\phi_i\| \leqq 2/(\lambda\delta_0)$ for all $i > 0$. Let $e_1, \ldots, e_m$ be the standard basis. Then the $2m$-dimensional vectors

$$v_k = \begin{pmatrix} e_k \\ \phi_{i-1} e_k \end{pmatrix}$$

span π_{i-1}, and the dilation coefficient

$$G(A, \pi_{i-1}) = \frac{G(Av_1, \ldots, Av_m)}{G(v_1, \ldots, v_m)}.$$

Now,

$$A_i v_k = \begin{pmatrix} (Q_i - \phi_{i-1})e_k \\ e_k \end{pmatrix} = (\lambda f_k - E) \begin{pmatrix} e_k + O\left(\frac{1}{\lambda}\right) \\ O\left(\frac{1}{\lambda}\right) \end{pmatrix} =: (\lambda f_k - E) h_k .$$

It follows that

$$G(A_i v_1, \ldots, A_i v_m) = \prod_{k=1}^{m} (\lambda f_k - E)^2 G(h_1, \ldots, h_m) = \prod_{k=1}^{m} (\lambda f_k - E)^2 \left(1 + O\left(\frac{1}{\lambda}\right)\right).$$

Since $G(v_1, \ldots, v_m) = 1 + O(\lambda^{-1})$, Lemma 3 is proved. □

Lemma 4.

$$\frac{1}{n} \int_{|z|=r} \log |G(S(n)e_1, \ldots, S(n)e_m)| \geqq 2m \log \lambda \delta_0 + O\left(\frac{1}{n}\right).$$

Proof. Lemma 3, Ineq. (11), and Eq. (13) imply

$$\log |G(S(n)e_1, \ldots, S(n)e_m)(z)| \geqq n \left(2m \log \lambda \delta_0 + O\left(\frac{1}{\lambda}\right)\right). \quad \square$$

Lemma 5.

$$\left| \operatorname*{Arg}_{|z|=r} G(S(n)e_1, \ldots, S(n)e_m) \right| \leqq \text{const}\, n .$$

Proof. By Lemma 3, Eq. (13), and Rouche's theorem

$$\operatorname*{Arg}_{|z|=r} G = \sum_{i=1}^{n} \sum_{k=1}^{m} \operatorname*{Arg}_{|z|=r} (\lambda f_k - E)^2 = O(mn). \quad \square$$

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