Commun. Math. Phys. 145, 149–179 (1992)

Communications in
Mathematical
Physics

Vortex Scattering

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Received June 27, 1991

Abstract. The geodesic approximation to vortex dynamics in the critically coupled abelian Higgs model is studied. The metric on vortex moduli space is shown to be Kähler and a scheme for its numerical computation described. The scheme is applied to the 2-vortex system and the geodesic scattering compared with previous simulations of the full field theory. The quantum scattering is also analysed.

1. Introduction

Describing the dynamics of field theory solitons is in general a difficult problem. Classically, it requires that one solve the initial value problem for a set of non-linear hyperbolic partial differential equations. Although there are some very special (exactly-integrable) systems for which explicit time-dependent multisoliton solutions can be constructed – for instance, the sine-Gordon model – no such systems enjoying Lorentz-invariance have been found in more than one space dimension. In more physically interesting cases one must resort to numerical simulation or work within some kind of approximation scheme.

One possibility, at low energies, is that most of the degrees of freedom of the fields remain unexcited and the field theory can be well approximated by a finite-dimensional system. Truncating the field theory in this way is usual in the collective coordinate description of a single soliton. That it might be appropriate to the description of several strongly interacting solitons was first proposed by Manton [1] in connection with the scattering of Bogomol'nyi-Prasad-Sommerfield (BPS) monopoles. This theory is one of a class admitting static multisoliton solutions corresponding to arbitrary configurations of solitons at rest. The existence of the solutions may be understood physically as due to the absence of static forces between separated solitons. Mathematically, the essential property appears to be that they saturate a topological lower bound on the field energy and as a consequence satisfy a first order field equation (Bogomol'nyi equation) [2]. Manton's idea is that in such theories the low energy dynamics of several solitons may – just as for a single soliton – be approximated by motion on the space of the corresponding static solutions.

Let $\mathscr{C}$ be the field configuration space of the theory, and $L = T - V$ its Lagrangian, with T and V the kinetic and potential energies respectively. The n-soliton static solutions form a submanifold M_n (the moduli space) of $\mathscr{C}$ on which (in the charge-n sector) V takes its absolute minimum. Now consider intitial conditions corresponding to a slow motion tangent to M_n. Imparting small velocities to n widely separated solitons would, for instance, be described by such conditions. In the subsequent evolution, the trajectory of the system will be constrained by V to lie close to M_n. V will thus remain approximately constant, and the field evolution described by a geodesic motion on M_n, the metric being that induced by the kinetic energy T. The problem of describing the soliton dynamics is thus reduced to finding the metric and solving the ordinary differential geodesic equations on M_n. One may also obtain an approximate quantization by considering wavefunctions over M_n, and taking a Hamiltonian equal to (minus) the covariant Laplacian. A proposal to generalize the prescription to the case where a Bogomol'nyi bound is only approximately attained and there are weak forces between the solitons has been made in [3].

In general, to find the metric one must calculate the zero modes about each of the static solutions and evaluate the kinetic energy functional T on them. Perhaps the best studied example is the theory of BPS monopoles. Here, in principle, one may construct the static multisoliton solutions explicitly. However, to calculate the metric directly from the zero modes would in practice be very difficult. Instead, the 2-monopole metric has been found indirectly by Atiyah and Hitchin [4]. They showed that the metric on the n-monopole space is hyper-Kähler. When $n = 2$, this property of the metric, together with its symmetries and the requirement that it be complete, determines it uniquely. This has allowed the low energy dynamics of two monopoles, both classical and quantum mechanical, to be studied in some detail [4–6].

In some other cases where explicit multisoliton solutions are known, a direct calculation of the metric has proved possible. Examples include Kaluza-Klein monopoles [7], maximally-charged black holes [8], and the "lumps" of the $\mathbb{C}P_1$ sigma model in $(2+1)$ dimensions [9]. In the last case the metric on the moduli space is (formally) Kähler, and this result has been generalized to $\mathbb{C}P_N$ models with $N > 1$ [10], and also to sigma models with arbitrary Kähler target [11]. The scattering of lumps has been studied in both the geodesic approximation [12] and in the full field theory, but numerical difficulties in the latter case have prevented a proper comparison from being made (reported in [12]). A final example is that of vortices in the critically coupled abelian Higgs model in hyperbolic 2-space, studied in [13]. When the hyperbolic space has special curvature exact solutions are available – the $SO(3)$-invariant instantons – and the metric is again found to be Kähler.

Here we are concerned with the solitons of another model in $(2+1)$-dimensions for which, by contrast, no explicit construction has been found. These are the vortex solitons of the critical coupled abelian Higgs model in flat space. The geodesic approximation was first applied to vortex scattering in [14]. The 2-vortex metric was determined, on grounds of symmetry, up to two unknown functions, and it was shown that two vortices in head-on collision scatter through 90°. Some understanding of the vortex metric was obtained in [15, 16], though not enough to further specify its form. There have also been numerical simulations of the true scattering, governed by the full equations of motion [17, 18]. These confirmed the 90° scattering, which persists up to high energies. In [17] the scattering data was found to be roughly velocity independent at low energies, as the geodesic picture would predict, and this data was used to make an approximate determination of the functions in the 2-vortex metric.

Here the problem is studied further. The work is presented as follows.

In Sect. 2 the abelian Higgs model and its static solutions at critical coupling are reviewed.

In Sect. 3 we review the geodesic description of low energy vortex scattering, and discuss the non-singularity of the metric on moduli space.

In Sect. 4 the form of the metric is investigated. It is shown to be Kähler in a similar way to Strachan in [13], and a scheme for computing it in terms of the properties of the static solutions is presented. The centre of mass motion is also discussed. A different way of showing the Kähler property, due to Ruback, is sketched in Appendix B[1].

In Sect. 5 these ideas are applied to the 2-vortex system. The metric is shown to depend on a single function of the vortex separation, and an integral constraint on this function obtained. The metric, and the geodesic motion in this metric, are computed numerically. The geodesic scattering is compared with previous simulations of the true scattering of vortices, and good agreement is found, even for quite high impact speeds.

In Sect. 6 we try to unerstand the integral constraint on the 2-vortex metric from a more general point of view.

In Sect. 7 the 2-vortex quantum scattering problem is examined and the cross-section in the long-wavelength limit found explicitly.

Some of this work has appeared in a less developed form in a previous publication by the author [19].

2. Vortices in the Abelian Higgs Model

2.1. Background

The abelian Higgs model [20] is one of the simplest theories exhibiting the Higgs mechanism. As well as its interest in the general context of field theory, it has relevance to the study of cosmic strings [21] and to the phenomenological description of superconducting materials [22]. In $(3+1)$ dimensions it admits topologically stable soliton solutions in which the energy density of the fields is concentrated in tubes of definite width. The tubes are threaded by magnetic flux, quantized, together with the energy per unit length, in integer multiples of a basic unit. The dynamics of these flux tubes is in general rather complicated; its study has relied largely on computer simulations [23]. A simpler special case, which nevertheless exhibits interesting dynamics, is obtained by imposing translational symmetry along a particular direction. The flux tubes, or "vortices," are then all parallel to this direction and the problem is essentially $(2+1)$-dimensional. This is the case of interest here.

The model comprises a complex scalar Higgs field $\phi = \phi_1 + i\phi_2$ coupled to a $U(1)$ gauge field A_μ, together with a symmetry breaking potential. The Lagrangian density is

$$\mathscr{L} = \tfrac{1}{2}\, D_\mu\phi\overline{D^\mu\phi} - \tfrac{1}{4}\, F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{8}\,\lambda(|\phi|^2 - 1)^2, \tag{2.1}$$

where $D_\mu\phi = (\partial_\mu - iA_\mu)\phi$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ $(\mu, \nu = 0, 1, 2)$, and the metric is taken to have signature $(1, -1, -1)$. As a result of symmetry breaking the gauge field acquires a mass. Units have been chosen so that this mass, and also the gauge field coupling, are both equal to one. The free parameter λ which remains is the (square of the) Higgs mass.

[1] I am very grateful to Peter Ruback for allowing me to reproduce his argument here

The value of λ determines the relative strengths of the attractive scalar force and repulsive magnetostatic force between vortices. Since the fields are massive, these forces are short range. Roughly speaking, when $\lambda < 1$, the scalar forces prevail and the only static solutions are those representing coincident vortices. Conversely, when $\lambda > 1$, the vortices repel each other; again only coincident solutions exist, but these are now unstable. In the Ginzburg-Landau theory of superconductivity these two cases correspond to Type I and Type II materials respectively. At the critical value $\lambda = 1$, the forces exactly cancel, and static solutions exist corresponding to arbitrary configurations of vortices. We now describe the vortex solutions in this case.

2.2. *Vortex Solutions*

It is convenient to work in the gauge $A_0 = 0$. The equation of motion associated with A_0 must be imposed as a constraint (Gauss' law),

$$\partial_i \dot{A}_i + \dot{\phi}_a \varepsilon_{ab} \phi_b = 0 \tag{2.2}$$

and the Lagrangian is then $L = T - V$, where T and V are the kinetic and potential energies respectively:

$$T = \tfrac{1}{2} \int d^2x (\dot{\phi}_a \dot{\phi}_a + \dot{A}_i \dot{A}_i) \qquad (i = 1, 2)\,, \tag{2.3}$$

$$V = \tfrac{1}{2} \int d^2x (D_i \phi \overline{D_i \phi} + F_{12}^2 + \tfrac{1}{4} (|\phi|^2 - 1)^2)\,. \tag{2.4}$$

The total conserved energy is $E = T + V$. Finiteness of E implies the boundary conditions

$$\left.\begin{array}{l} |\phi| \to 1 \\ D_i \phi \to 0 \end{array}\right\} \quad \text{as} \quad |x| \to \infty \tag{2.5}$$

so that on the circle at infinity ϕ is a pure phase. It follows that the space of all finite energy fields decomposes into topologically distinct sectors labelled by the winding number n of the map

$$\phi|_{|x| \to \infty} : S^1_\infty \to U(1)\,. \tag{2.6}$$

In the n^{th} sector the total magnetic flux through the plane is (using Stokes' theorem)

$$\int d^2x F_{12} = 2\pi n\,. \tag{2.7}$$

Note that if $n \neq 0$, then by continuity ϕ must have zeros somewhere in the plane.

Now consider static fields, $\dot{A}_i = 0$, $\dot{\phi} = 0$. Gauss' law (2.2) is then satisfied and the kinetic energy T vanishes. The static energy $E = V$ may be written as follows:

$$\begin{aligned} E = \tfrac{1}{2} \int d^2x \big[&(D_1 \pm i D_2)\phi \overline{(D_1 \pm i D_2)\phi} + \big\{ F_{12} \pm \tfrac{1}{2} (|\phi|^2 - 1) \big\}^2 \\ &\pm i \{ \partial_2 (\bar{\phi} D_1 \phi) - \partial_1 (\bar{\phi} D_2 \phi) \} \pm F_{12} \big]\,. \end{aligned} \tag{2.8}$$

The boundary conditions (2.5) ensure that the total derivative terms vanish. Hence, using (2.7),

$$E \geq \pi |n| \tag{2.9}$$

with equality if and only if

$$(D_1 \pm i D_2)\phi = 0\,, \tag{2.10}$$

$$F_{12} \pm \tfrac{1}{2} (|\phi|^2 - 1) = 0\,. \tag{2.11}$$

Equations (2.10) and (2.11) are called the Bogomol'nyi equations. The upper and lower signs correspond to $n > 0$ and $n < 0$ respectively. Their solutions minimise the static energy, so automatically satisfy the full second order static equations following from (2.4). In fact, it has been shown that all solutions of the full static equations are solutions of (2.10), (2.11) [24]. Hence to study the static theory it suffices to consider just these first order equations. In all the work which follows we assume $n > 0$ and take the upper signs. In this case the solutions are called vortices; for $n < 0$ they are called anti-vortices.

It is straightforward to find rotationally symmetric solutions of (2.10), (2.11). In polar coordinates (r, θ), the ansatz

$$\begin{aligned} \phi &= \mathrm{e}^{in\theta} \varrho(r)\,, \\ A_r = 0, &\qquad A_\theta = na(r) \end{aligned} \tag{2.12}$$

gives the equations

$$\begin{aligned} r\,\frac{d\varrho}{dr} - n(1-a)\varrho &= 0\,, \\ \frac{2n}{r}\,\frac{da}{dr} + (\varrho^2 - 1) &= 0\,, \end{aligned} \tag{2.13}$$

where the appropriate boundary conditions are $\varrho(0) = a(0) = 0$, $\varrho(\infty) = a(\infty) = 1$. The asymptotic behaviour of ϱ is given by

$$\begin{aligned} \varrho &\sim Ar^n & r &\to 0\,, \\ \varrho &\sim 1 - BK_0(r) & r &\to \infty\,, \end{aligned} \tag{2.14}$$

where K_0 is the zero[th] order modified Bessel function. These solutions are interpreted as describing n coincident vortices. The profile for the single vortex ($n = 1$) is shown in Fig. 1; in this case $A \cong 0.603$.

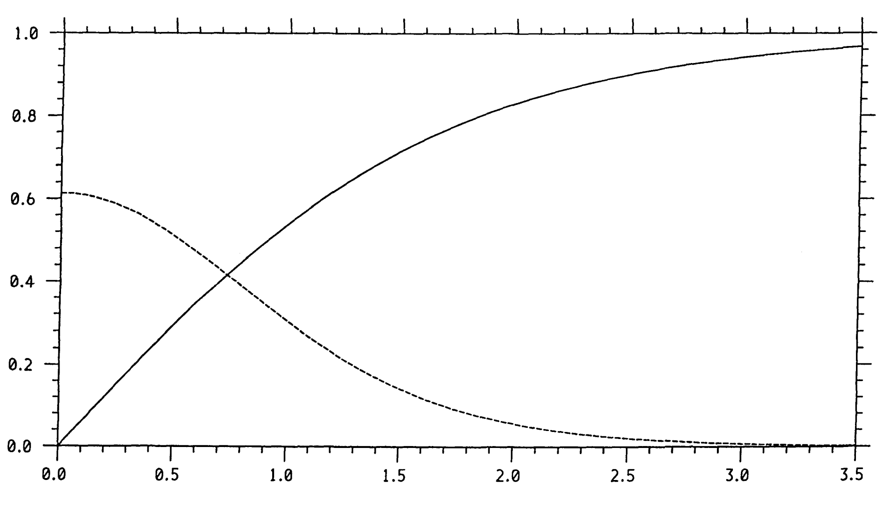


Fig. 1. The profile of the Higgs field $|\phi|$ (solid line) and energy density (broken line) of a single vortex

The coincident vortices where the first solutions obtained [25]. Subsequently a numerical analysis of the 2-vortex case indicated that the forces between critically coupled vortices vanish [26]. Furthermore, an index theorem argument of Weinberg showed that about any solution of charge n there are $2n$ (square integrable) zero modes [27]. These results suggested the existence of a $2n$-parameter family of n-vortex solutions, finally established with the rigorous work of Taubes [24].

In the n^{th} topological sector the space of smooth solutions (modulo gauge transformations) is a $2n$-dimensional manifold M_n. Each solution is uniquely specified by choosing n unordered points (counted with multiplicity) in $\mathbb{R}^2$, where the Higgs field is zero (though it should be noted that no explicit form for the solutions is known). It is useful to make the identification $\mathbb{R}^2 \cong \mathbb{C}$ and write the position of a point (x_1, x_2) in $\mathbb{R}^2$ as $z = x_1 + ix_2$. We shall denote the positions of the zeros by z_r $(r = 1, \ldots, n)$. The $\{z_r\}$ provide good local coordinates on most of M_n but, because they assume an ordering, break down on the $(2n-2)$-dimensional subspace Δ_n, where two or more zeros coincide. Good global coordinates on M_n are provided by the coefficients of the complex polynomial with roots z_r [15]:

$$\begin{aligned} P_n(z) &\equiv w_0 + w_1 z + \ldots + w_{n-1} z^{n-1} + z^n \\ &\equiv \prod_{r=1}^{n} (z - z_r) . \end{aligned} \tag{2.15}$$

Thus M_n is the topologically trivial space $\mathbb{C}^n = \{w_k\}$, inheriting a natural complex structure from the complex structure on the plane. Note that the space of *ordered* points, $\mathbb{C}^n = \{z_r\}$, is a branched covering of M_n.

Taubes' general results also relate the properties of the solutions closely to the positions of the zeros. It follows from the first Bogomol'nyi equation (2.10) that in a neighbourhood of a zero at z_r of multiplicity n_r

$$\phi(x) = (z - z_r)^{n_r} h_r(x) , \tag{2.16}$$

where h_r is a smooth, non-vanishing function of x. Away from the zeros, the fields, being massive, rapidly approach their asymptotic values. In particular, the Higgs field has the following decay property: for any $\delta > 0$ there exists a $M(\delta) > 0$ such that

$$0 \leq 1 - |\phi|^2 < M(\delta) \mathrm{e}^{-(1-\delta)|x|} . \tag{2.17}$$

The rapid decay means that solutions corresponding to well-separated vortices are approximated by a superposition of 1-vortex solutions with errors only exponentially small in the separations (see Appendix A). The positions of the zeros then correspond to the locations of the vortices – i.e. to where the energy density (and magnetic flux) of the fields is concentrated – and we may regard the vortices as independent particles of mass π carrying flux 2π. On the other hand, when the vortices are close together, it is no longer proper to think of them as distinct objects, and the zeros of the Higgs no longer correspond in a direct way to the energy distribution of the fields.

This completes our review of the static vortex solutions. We now turn to the consideration of the low energy scattering.

3. Geodesic Description of Low Energy Scattering

3.1. The Geodesic Approximation

To motivate the geodesic approximation of low energy vortex scattering we reinterpret equations (2.2), (2.3), (2.4) in terms of the true configuration space of the theory, following [1, 14]. Let $\mathscr{A}$ be the space of finite energy fields $a = (A_i, \phi)$, and $\mathscr{G}$ the group of gauge transformations over $\mathbb{R}^2$. The true configuration space is the quotient $\mathscr{C} = \mathscr{A}/\mathscr{G}$ obtained by identifying gauge equivalent fields[2]. There is a natural metric h on $\mathscr{A}$ given by the standard L^2-norm

$$h(\dot{a}, \dot{a}) = \tfrac{1}{2}\int d^2x(\dot{A}_i\dot{A}_i + \dot{\phi}_a\dot{\phi}_a)\,. \tag{3.1}$$

The inner product of $\dot{a}$ with an infinitesimal gauge transformation $\dot{\lambda} = (\partial_i \Lambda, i\Lambda\phi)$ is then

$$\begin{aligned} h(\dot{a}, \dot{\lambda}) &= \tfrac{1}{2}\int d^2x(\dot{A}_i\partial_i\Lambda - \dot{\phi}_a\Lambda\varepsilon_{ab}\phi_b) \\ &= -\tfrac{1}{2}\int d^2x(\partial_i\dot{A}_i + \dot{\phi}_a\varepsilon_{ab}\phi_b)\Lambda \end{aligned} \tag{3.2}$$

Thus Gauss' law (2.2) is just the condition that $\dot{a}$ be orthogonal to the gauge orbits through a. If we represent tangent vectors $\dot{c}$ on $\mathscr{C}$ by tangent vectors $\dot{a}$ on $\mathscr{A}$ satisfying Gauss, then the metric h is well-defined on $\mathscr{C}$ – it is just the kinetic energy (2.3). Furthermore, the potential energy (2.4) is gauge-invariant so automatically well-defined on $\mathscr{C}$. We may therefore interpret the dynamics following from (2.1) as motion on $\mathscr{C}$ with metric defined by T, and potential energy function V. See [14] for a presentation via the Hamiltonian formalism.

In the geodesic approximation of the low energy scattering the theory is truncated to M_n, and the evolution given by geodesic motion with respect to the metric induced by T. The approximation will be good provided the amount of energy transferred to field oscillations orthogonal to M_n remains small. No rigorous field theory analysis of the problem exists. However, since there are no massless fields, the frequency of the transverse oscillations is bounded below by a positive constant $\omega_0 \sim 1$. Investigations of finite dimensional systems suggest that the energy transfer in a scattering process should then be of order $e^{-\frac{1}{v}}$, where v is a typical vortex speed, and so rather strongly suppressed [29]. Indeed, in the numerical simulations in [17, 18], the energy transfer in a head-on collision of two vortices is found to be negligible up to impact speeds of 0.4 (of the speed of light). We would thus expect the approximation to be rather robust. We shall test it directly in Sect. 6.

3.2. The Metric on Moduli Space

The metric on M_n is the restriction of (3.1) to vectors $\dot{\tilde{a}} = (\dot{A}_i, \dot{\phi})$ satisfying both Gauss' law and the linearization of the Bogomol'nyi equations. The result of Weinberg mentioned in Sect. 2.2 – that at each static solution the space of square-integrable zero modes is $2n$-dimensional – means (assuming it can be made rigorous) that this metric is well-defined (i.e. finite) everywhere. It is worth remarking that Weinberg did not use Gauss' law to fix the gauge, but rather $\partial_i \dot{A}_i = 0$. However, projecting his vectors so that they are orthogonal to gauge orbits can only reduce their length (in the metric h), so the space of vectors $\dot{\tilde{a}}$ of finite length (i.e. finite kinetic energy) is also $2n$-dimensional.

If the metric is everywhere well-defined, the map at each point of M_n between the space of vectors $\dot{\tilde{a}}$ of finite length, and the tangent space at the corresponding

[2] Note that for $\mathscr{C}$ to be a manifold, $\mathscr{G}$ must be restricted to gauge transformations which act freely on $\mathscr{A}$ [27]. In particular, the gauge transformations must tend to the identity at infinity, otherwise $\mathscr{C}$ will have a singularity at the point $a = 0$

point of $\mathbb{C}^n = \{w_k\}$, is non-singular. It is instructive to try to understand this a little more explicitly. This work may be regarded as a generalization of Ruback's analysis in [14] of the 2-vortex system.

Consider $z \neq z_r$ (all r) so that $\phi \neq 0$. The first Bogomol'nyi equation (2.10) may then be rewritten as

$$A = i\partial_z \ln \bar{\phi}\,, \tag{3.3}$$

where we have introduced the complex notation $A = \frac{1}{2}(A_1 - iA_2)$. Linearising (3.3) gives

$$\dot{A} = i\partial_z \bar{\eta}\,, \tag{3.4}$$

where η is defined by

$$\dot{\phi} = \phi\eta\,. \tag{3.5}$$

These equations may be used to eliminate A from the second Bogomol'nyi equation (2.11) and Gauss' law (2.2). Writing $f = \ln|\phi|^2$ the second Bogomol'nyi equation becomes

$$\nabla^2 f + 1 - \mathrm{e}^f = 0 \tag{3.6}$$

and Gauss' law

$$\nabla^2 \operatorname{Im}\eta - \mathrm{e}^f \operatorname{Im}\eta = 0\,. \tag{3.7}$$

Linearising (3.6), and noting $\dot{f} = 2\operatorname{Re}\eta$, we have

$$\nabla^2 \operatorname{Re}\eta - \mathrm{e}^f \operatorname{Re}\eta = 0\,, \tag{3.8}$$

and so, for $z \neq z_r$,

$$\nabla^2\eta - \mathrm{e}^f\eta = 0\,. \tag{3.9}$$

Turning now to the boundary conditions on f and η, we note that finiteness of the static energy requires

$$f \to 0 \quad \text{as} \quad |x| \to \infty\,. \tag{3.10}$$

Indeed, we know from (2.17) that f falls off exponentially fast. For $\dot{a}$ to be finite in the metric h, we require

$$\eta \to 0 \quad \text{as} \quad |x| \to \infty\,, \tag{3.11}$$

where, by (3.9, 3.10), the decay is again exponential.

Solutions of (3.9, 3.11) give the vectors $\dot{a}$ via (3.4, 3.5). Since $-\nabla^2 + e^f$ is a positive operator, non-trivial solutions must have singularities at one or more of the zeros z_r. Furthermore, finiteness of $\dot{a}$ implies that the singularities must be of the form

$$\eta \sim (z - z_r)^{-\varrho_r}\,, \qquad \varrho_r = 1, \ldots, n_r \tag{3.12}$$

where, as in (2.16), n_r is the multiplicity of the zero z_r. There are n zeros (counted with multiplicity), so we have $2n$ linearly independent solutions, as expected.

In the neighbourhood of z_r, the linear perturbation of ϕ corresponding to (3.12) is (recalling (2.16)),

$$\phi + \lambda\dot{\phi} = (z - z_r)^{n_r - \varrho_r}\,[(z - z_r)^{\varrho_r} + \lambda]h_r(x)\,. \tag{3.13}$$

If we now consider the extension to the whole plane, then ϕ has the form

$$\phi = P_n(z)h(x)\,, \tag{3.14}$$

where $h(x)$ is a smooth, non-vanishing function of x, and $P_n(z)$ is the polynomial (2.15). Since it is clear that each perturbation (3.13) corresponds to an independent

$O(\lambda)$ perturbation in the coefficients w_k of $P_n(z)$, the result follows. Note that while the perturbation of the w_k is smooth, that of the zeros z_r (if $\varrho_r > 1$) is not: (3.13) shows that the zero of multiplicity n_r at z_r splits into a zero of multiplicity $(n_r - \varrho_r)$, and ϱ_r simple zeros displaced from z_r by the ${\varrho_r}^{\text{th}}$ roots of $-\lambda$.

That the metric is well-defined everywhere on M_n determines the qualitative features of vortex scattering. For example consider the 2-vortex system, with the vortices placed symmetrically about the origin. Such a configuration is described by

$$P_2(z) = w - z^2 \,. \tag{3.15}$$

The zeros of the Higgs lie at $z = \pm\sqrt{w}$. Now consider, for instance, w real and decreasing through zero. This is a smooth motion on M_2 in which the zeros of the Higgs approach along the x_1-axis and separate along the x_2-axis. In essence, this is the remarkable 90° scattering behaviour found in [14]. It is straightforward to consider the local behaviour of higher-n collisions in the same way.

Finally, let us compare the vortices with a system of particles. One may imagine shrinking the vortices to zero size, to obtain identical point particles at the positions z_r. The resulting metric is flat. It is well-defined on the covering space $\mathbb{C}^n = \{z_r\}$, but unlike the vortex metric, is not well-defined everywhere on M_n, having conical singularities on the set of points Δ_n. Since the interactions between vortices are short-range, the two metrics will agree asymptotically, as shown explicitly later on. We shall often find it useful to regard the vortex metric as a smoothed version of the metric describing the particles.

4. Investigation of the Metric

We now turn to an investigation of the form of the vortex metric. We shall work on the subspace $M_n \backslash \Delta_n$, where the z_r are distinct and constitute good local coordinates. Our results will extend by continuity to all points of M_n. We aim to express the metric in terms of the z_r:

$$ds^2 = \sum_{r,s=1}^{n} (a_{rs} dz_r dz_s + b_{rs} dz_r d\bar{z}_s + \bar{a}_{rs} d\bar{z}_r d\bar{z}_s) \tag{4.1}$$

the metric coefficients depending on the z_r through gauge-invariant properties of the static solutions.

4.1. Coordinates and Fields

To begin, we make the dependence of the fields on the z_r as explicit as possible by extending the equations for f and η, derived above for $z \neq z_r$, to the whole plane.

We have assumed that the zeros z_r are distinct, so all have multiplicity one. In a neighbourhood of z_r, (2.16) implies that

$$f = \ln|\phi|^2 = \ln|z - z_r|^2 + \text{smooth} \tag{4.2}$$

and (together with the discussion in 2.2) that

$$\eta = \frac{-\dot{z}_r}{z - z_r} + \text{smooth}\,. \tag{4.3}$$

Noting that in two dimensions

$$\nabla^2 \ln |z - z_r|^2 = 4\pi\delta(x - x_r)\,, \tag{4.4}$$

we see that (3.6) and (3.9), extended to all points of $\mathbb{R}^2$, become

$$\nabla^2 f + 1 - \mathrm{e}^f = 4\pi \sum_{r=1}^{n} \delta(x - x_r) \tag{4.5}$$

and

$$\nabla^2 \eta - \mathrm{e}^f \eta = -\,4\pi \sum_{r=1}^{n} \dot{z}_r \partial_z \delta(x - x_r) \tag{4.6}$$

respectively. Note that (4.5) is the equation analysed by Taubes in his proof of the existence of vortex solutions.

The solutions of (4.5) and (4.6) may be related in a simple way. Differentiating (4.5) with respect to z_r gives

$$\nabla^2 \frac{\partial f}{\partial z_r} - \mathrm{e}^f \frac{\partial f}{\partial z_r} = -\,4\pi \partial_z \delta(x - x_r)\,. \tag{4.7}$$

Thus by the linearity of (4.6), and noting the boundary conditions (3.10, 3.11),

$$\eta = \sum_{r=1}^{n} \dot{z}_r \frac{\partial f}{\partial z_r}\,. \tag{4.8}$$

4.2. *Form of the Metric*

Now consider the kinetic energy (2.3), which in terms of A and ϕ is

$$T = \tfrac{1}{2} \int d^2x (4\dot{A}\dot{\bar{A}} + \dot{\phi}\dot{\bar{\phi}})\,. \tag{4.9}$$

Recalling the assumption that the zeros z_r are all distinct, let S_ε be a set of non-overlapping small discs of radius ε centred at the z_r, and divide the integral into two parts:

$$\int_{\mathbb{R}^2} = \int_{\mathbb{R}^2 \backslash S_\varepsilon} + \int_{S_\varepsilon}\,. \tag{4.10}$$

We evaluate each part and the let $\varepsilon \to 0$. Since the integrand is smooth the second part is $O(n\varepsilon^2)$, which vanishes as $\varepsilon \to 0$. The first part, using (3.4) and (3.5), becomes

$$\frac{1}{2} \int_{\mathbb{R}^2 \backslash S_\varepsilon} d^2x (4\partial_z \bar{\eta} \partial_{\bar{z}} \eta + \mathrm{e}^f \bar{\eta}\eta)\,. \tag{4.11}$$

Note that this expression is manifestly real. Since in $\mathbb{R}^2 \backslash S_\varepsilon$, η is smooth, we are free to rewrite it as

$$2 \int_{\mathbb{R}^2 \backslash S_\varepsilon} d^2x \partial_z (\bar{\eta} \partial_{\bar{z}} \eta) + \frac{1}{2} \int_{\mathbb{R}^2 \backslash S_\varepsilon} d^2x \bar{\eta} (-\nabla^2 \eta + \mathrm{e}^f \eta)\,. \tag{4.12}$$

The second integral vanishes by virtue of (4.6), leaving only the contributions from the neighbourhoods of the zeros S_ε:

$$T = -i \int\limits_{\partial S_\varepsilon} d\bar{z}\bar{\eta}\partial_{\bar{z}}\eta \,. \tag{4.13}$$

Using successively (4.3) and (4.8), and neglecting terms of $O(\varepsilon)$,

$$T = \sum_{s=1}^{n} \int\limits_{0}^{2\pi} d\theta_s \dot{\bar{z}}_s \partial_{\bar{z}}\eta = \sum_{r,s=1}^{n} \int\limits_{0}^{2\pi} d\theta_s \partial_{\bar{z}} \frac{\partial f}{\partial z_r} \dot{z}_r \dot{\bar{z}}_s \,, \tag{4.14}$$

where for each s the integration is around the circle $|z - z_s| = \varepsilon$. Now, near z_s, a Taylor expansion of the smooth part of f in (4.2) gives[3]

$$\begin{aligned} f = \ln|z - z_s|^2 + a_s + \tfrac{1}{2}\left\{b_s(z - z_s) + \bar{b}_s(\bar{z} - \bar{z}_s)\right\} \\ + c_s(z - z_s)^2 + d_s(z - z_s)(\bar{z} - \bar{z}_s) + \bar{c}_s(\bar{z} - \bar{z}_s)^2 + O(\varepsilon^3)\,, \end{aligned} \tag{4.15}$$

where to satisfy (4.5) we require

$$d_s = -\tfrac{1}{4} \,. \tag{4.16}$$

Hence for z near but not equal to z_s, it follows that

$$\partial_{\bar{z}} \frac{\partial f}{\partial z_r} = \frac{1}{2}\frac{\partial \bar{b}_s}{\partial z_r} + \frac{1}{4}\delta_{rs} + O(\varepsilon)\,. \tag{4.17}$$

Substituting into (4.14) and taking the limit $\varepsilon \to 0$ we obtain finally

$$T = \frac{1}{2}\pi \sum_{r,s=1}^{n} \left(\delta_{rs} + 2\frac{\partial \bar{b}_s}{\partial z_r}\right) \dot{z}_r \dot{\bar{z}}_s \,. \tag{4.18}$$

Since we began with a manifestly real expression for T, (4.18) must be real for arbitrary $\dot{z}_r$. Consequently

$$\frac{\partial \bar{b}_s}{\partial z_r} = \frac{\partial b_r}{\partial \bar{z}_s} \,. \tag{4.19}$$

It follows immediately that the metric

$$ds^2 = \frac{1}{2} \sum_{r,s=1}^{n} \left(\delta_{rs} + 2\frac{\partial \bar{b}_s}{\partial z_r}\right) dz_r d\bar{z}_s \tag{4.20}$$

is Hermitian. Physically, this means that the kinetic energy of a system of vortices is unchanged by a (fixed) rotation of all their velocity vectors. Note that we have chosen to normalise the metric relative to T by dividing by the vortex mass π.

The Kähler form associated with (4.20) is

$$\omega = \frac{i}{4} \sum_{r,s=1}^{n} \left(\delta_{rs} + 2\frac{\partial \bar{b}_s}{\partial z_r}\right) dz_r \wedge d\bar{z}_s \,. \tag{4.21}$$

[3] Were the zero of multiplicity n_s, the first term would be $n_s \ln|z - z_s|^2$

Taking the exterior derivative we find

$$d\omega = \frac{i}{4} \sum_{r,s,t=1}^{n} \left[\frac{\partial^2 \bar{b}_s}{\partial z_t \partial z_r} dz_t \wedge dz_r \wedge d\bar{z}_s + \frac{\partial^2 \bar{b}_s}{\partial \bar{z}_t \partial z_r} d\bar{z}_t \wedge dz_r \wedge d\bar{z}_s \right] = 0\,, \quad (4.22)$$

where the second term in the square brackets vanishes by virtue of (4.19). Thus ω is closed and the metric is Kähler. These results have been derived on $M_n \backslash \Delta_n$, but extend to all of M_n by continuity. The analysis is similar to that of Strachan in [13]; a different way of showing the Kähler property, due to Ruback, is described in Appendix B. We also remark that it is straightforward to generalize the analysis to vortices residing in a background 2-space with arbitrary metric. Provided the considerations of Sect. 2.2 still apply and the modified Bogomol'nyi equations admit a $2n$-dimensional manifold of solutions, the analysis is much the same and the metric on the moduli space is Kähler as before.

For a single vortex ($n = 1$), f is rotationally symmetric and the coefficient b_1 of the linear term in (4.15) vanishes. The metric then reduces to that describing a single free particle of mass π. When there is more than one vortex, the b_r are non-zero; they describe the leading local change in the fields at each vortex due to the presence of the rest. Since (see Appendix A) a system of well-separated vortices is approximated by the superposition of 1-vortex solutions (i.e. of the functions f) with an error exponentially small in the separation, the b_r will then be small of the same order, and the metric given approximately by

$$ds_0^2 = \frac{1}{2} \sum_{r=1}^{n} dz_r d\bar{z}_r\,. \quad (4.23)$$

This makes more precise the comments at the end of Sect. 3.2. The metric ds_0^2 is flat everywhere (except on Δ_n). It describes the motion of n non-interacting identical point-particles of mass π. The second term in the full metric (4.2) may be thought of as an "interaction" piece which has the effect of smoothing out the singularities of the particle metric on the set of coincident points Δ_n.

4.3. Centre of Mass Motion

The vortex metric inherits the translational and rotational symmetries of the parent field theory (2.1). This is manifest from (4.20) since the b_r depend only on the relative positions of the vortices and are unaffected by rigid motions of the complete system. The associated conserved quantities, corresponding to the total linear and angular momentum, may be obtained from (4.18) in the usual way. Noting

$$\frac{\partial}{\partial z_r} \sum_{s=1}^{n} \bar{b}_s = \sum_{s=1}^{n} \frac{\partial b_r}{\partial \bar{z}_s} = 0\,, \quad (4.24)$$

which follows from (4.19) and translational invariance, one obtains for the linear momentum P_i,

$$P_1 + iP_2 = \pi \sum_{r=1}^{n} \dot{z}_r\,. \quad (4.25)$$

Thus the total momentum is equal to that of n point particles of mass π located at the zeros of the Higgs field. An immediate consequence is that the centre of mass of the vortex system is

$$Z = \frac{1}{n} \sum_{r=1}^{n} z_r \, . \tag{4.26}$$

We may now define relative coordinates

$$\zeta_r = z_r - Z \, , \qquad r = 1, \ldots, n \, . \tag{4.27}$$

These satisfy

$$\sum_{r=1}^{n} \zeta_r = 0 \, . \tag{4.28}$$

Substituting into (4.20), and using (4.24), we find

$$ds^2 = \frac{1}{2} n dZ d\bar{Z} + \frac{1}{2} \sum_{r,s=1}^{n} \left(\delta_{rs} + 2 \frac{\partial \bar{b}_s}{\partial z_r} \right) d\zeta_r d\bar{\zeta}_s \, . \tag{4.29}$$

Thus M_n decomposes as an isometric product

$$M_n = \mathbb{C} \times M_n^0 \, , \tag{4.30}$$

where M_n^0 is the space of n-vortices with fixed centre.

4.4. A Computational Scheme

Equation (4.20) expresses the metric in terms of gauge invariant properties of the static solutions – or more precisely, in terms of the local behaviour of $|\phi|$ in the neighbourhood of its zeros. If one can compute all the static solutions then one can find the metric. Specifically, one must solve (4.5) for f for arbitrary configurations of vortices, extract the quantities b_r as functions of the z_s, and then substitute into (4.20).

Stated in this way, this procedure is not suitable for numerical work, since f has singularities. To remedy this, we define a smooth function Φ by moving all the singularities of f out to infinity:

$$\Phi = f - \sum_{r=1}^{n} \ln |z - z_r|^2 \, . \tag{4.31}$$

Φ satisfies

$$\left. \begin{aligned} & \nabla^2 \Phi + 1 - \prod_{r=1}^{n} |z - z_r|^2 \, e^{\Phi} = 0 \\ & \Phi \sim - \sum_{r=1}^{n} \ln |z - z_r|^2 \quad \text{as} \quad |z| \to \infty \, . \end{aligned} \right\} \tag{4.32}$$

To obtain the b_s in terms of Φ we differentiate (4.15) with respect to z and evaluate it at z_s. This gives

$$b_s = 2 \sum_{r \neq s} \frac{1}{z_s - z_r} + \tilde{b}_s \, , \tag{4.33}$$

where

$$\tilde{b}_s = 2\partial_z \Phi(z_s) \tag{4.34}$$

is smooth in the z_r. Finally, substituting (4.33) into (4.20) gives the result

$$ds^2 = \frac{1}{2} \sum_{r,s=1}^{n} \left(\delta_{rs} + 2 \frac{\partial \bar{\tilde{b}}_s}{\partial z_r} \right) dz_r d\bar{z}_s \,. \tag{4.35}$$

Equations (4.32), (4.34), and (4.35) provide a method of computing the n-vortex metric. For general n one must solve a $(2n-3)$-dimensional family of non-linear elliptic partial differential equations ($2n-3$ rather than $2n$ because of translational and rotational symmetry). It would be very interesting if all this work could be obviated in some way. In the case of monopoles the hyper-Kähler property is sufficiently strong to determine (together with the symmetries of the moduli space) the 2-monopole metric. Unfortunately the Kähler condition is much weaker; indeed, it is trivial in the 2-vortex case (though as we shall see, there is a non-trivial global residue).

It is possible that the ideas of Hitchin described in [15], and developed further by Ruback in [16], might allow further analytical progress to be made. There it is concluded that M_n^0 is the fixed point set of an isometric circle action in a $(4n-4)$-dimensional hyper-Kähler manifold, M. Unfortunately, the behaviour of M away from M_n^0 is not known (i.e. whether its metric is smooth), so it is not clear what constraints on the metric this information provides.

5. The 2-Vortex Metric

We now apply our general results to the simplest case, that of two vortices. The metric is found to depend on a single function of the vortex separation, and an integral constraint on this function obtained. We describe a numerical computation of the metric, using the method of Sect. 4.4. The geodesic prediction for the classical scattering is computed and compared with numerical simulations of the true scattering. Good agreement is found, even for quite large impact speeds.

5.1. Form of the Metric

When there are just two vortices the moduli space decomposes into two 2-dimensional spaces

$$M_2 = \mathbb{C} \times M_2^0 \,. \tag{5.1}$$

The centre of mass and relative coordinates are

$$Z = \tfrac{1}{2}(z_1 + z_2) \tag{5.2}$$

and

$$\zeta_1 = -\zeta_2 = \zeta \equiv \tfrac{1}{2}(z_1 - z_2) \tag{5.3}$$

respectively. For fixed Z, ζ and $-\zeta$ label the same point in moduli space and should be identified. ζ is only a good local coordinate on M_n^0 for $\zeta \neq 0$. As discussed in Sect. 2.2, a good global coordinate is w, where

$$w - \zeta^2 = 0 \,. \tag{5.4}$$

From the symmetry of f under $z - z \to -(z - Z)$ we have [see also the remark following Eq. (6.8) below]

$$b_1 = -b_2 \,. \tag{5.5}$$

The expression for the metric (4.29) then reduces to

$$ds^2 = dZd\bar{Z} + \left(1 + 2\,\frac{\partial \bar{b}_1}{\partial \zeta}\right) d\zeta d\bar{\zeta} \,. \tag{5.6}$$

It is convenient to introduce the polar coordinates (σ, ϑ) defined by

$$\zeta = \sigma\, \mathrm{e}^{i\vartheta} \tag{5.7}$$

where, by the remarks above, the range of ϑ is π. Rotation and parity symmetry then imply (see (4.15))

$$b_1 = b(\sigma)\,\mathrm{e}^{-i\vartheta} \quad \text{with} \quad b(\sigma) \quad \text{real} \tag{5.8}$$

and the metric describing the relative motion is

$$ds^2_{\text{rel}} = F^2(\sigma)\,(d\sigma^2 + \sigma^2 d\vartheta^2)\,, \tag{5.9}$$

where

$$F^2(\sigma) = 1 + \frac{1}{\sigma}\,\frac{d}{d\sigma}\,(\sigma b)\,. \tag{5.10}$$

We remark that symmetry under rotations and parity alone implies that

$$ds^2_{\text{rel}} = F^2(\sigma) d\sigma^2 + G^2(\sigma) d\vartheta^2 \tag{5.11}$$

as found in [14]. The reduction to just one unknown function $F(\sigma)$ is a consequence of the Hermiticity of the metric. The Kähler property, as we remarked before, is trivial for a 2-manifold.

For $\sigma \ll 1$, $b(\sigma)$ has the form

$$b(\sigma) = \frac{1}{\sigma} - \frac{1}{2}\,\sigma + O(\sigma^3)\,. \tag{5.12}$$

The singular term in (5.12) follows from (4.33), and does not contribute to the metric. The remaining terms are fixed by the requirement that the metric be non-singular when expressed in terms of the coordinate w. i.e.

$$\begin{aligned} ds^2_{\text{rel}} &= \frac{F^2(|w|^{1/2})}{|w|}\, dw d\bar{w} \\ &= O(1) dw d\bar{w} \quad \text{for} \quad |w| \ll 1 \end{aligned} \tag{5.13}$$

which is equivalent to

$$F^2(\sigma) = O(\sigma^2) \quad \text{for} \quad \sigma \ll 1\,. \tag{5.14}$$

An explicit calculation of the linear perturbation to the coincident 2-vortex configuration (given in Appendix C) confirms that the form (5.12) is indeed correct and our scheme consistent.

For $\sigma \gg 1$, the Higgs field at one vortex is perturbed by the exponential tail of the field produced by the other and one has (see Appendix A)

$$b(\sigma) = O(\mathrm{e}^{-2(1-\delta)\sigma}) \quad \text{any} \quad \delta > 0\,, \tag{5.15}$$

giving

$$F^2(\sigma) = 1 - O(\mathrm{e}^{-2(1-\delta)\sigma})\,. \tag{5.16}$$

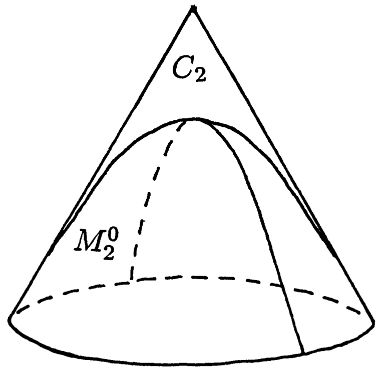


Fig. 2. A sketch of the smoothed cone representing M_2^0 as an embedding in $\mathbb{R}^3$, and the singular cone C_2 to which it is asymptotic. The difference in the areas of the cones is π. Also shown is a geodesic describing vortices in head-on collision

These results determine the qualitative form of ds^2_{rel}. A convenient way to represent this metric is by isometrically embedding M_2^0 as a surface of revolution in $\mathbb{R}^3$ [14]. The surface is asymptotic to the (singular) cone of deficit angle π, $C_2 = \mathbb{C}/\{\pm 1\}$ (see Fig. 2). Geodesics on C_2 describe the motion of a pair of identical non-interacting point particles. In accordance with our general picture, M_2^0 is a smoothed version of this cone in which the singularity at the vertex is removed. As pointed out in [14], the geodesics passing over the top ($\sigma = 0$) of the smoothed cone describe the 90° scattering of vortices in head-on collision.

Finally, we note that (5.10) and (5.12), together with the fast fall-off of b, imply the constraint

$$\int_0^\infty d\sigma\, \sigma[1 - F^2(\sigma)] = 1\,. \tag{5.17}$$

This expression, multiplied by π, has a simple geometrical meaning: it says that the difference between the areas of the cones C_2 and M_2^0 is π. In Sect. 6, we will try to understand this integral from a more general point of view. We shall also see that it appears naturally in the quantum mechanical scattering problem in the long-wavelength limit.

5.2. Numerical Computation of the Metric

Let the two zeros of the Higgs field lie on the x_1-axis at $(\pm\sigma, 0)$. The procedure described in Sect. 4.4 reduces to the solution of a one-parameter family of non-linear partial differential equations:

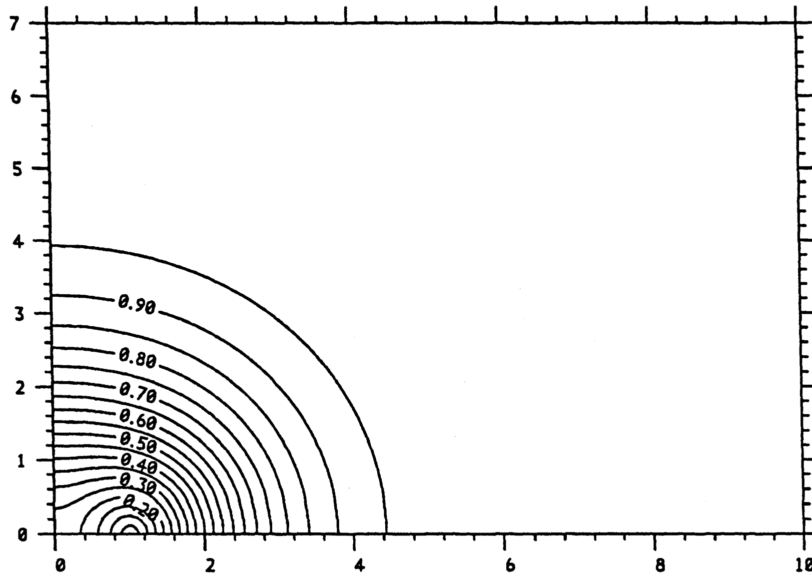
$$\left.\begin{aligned} &\nabla^2\Phi + 1 - R_+^2 R_-^2\, e^{\Phi} = 0 \\ &\Phi \sim -\ln R_+^2 R_-^2 \quad \text{as} \quad |x| \to \infty \end{aligned}\right\}, \tag{5.18}$$

where

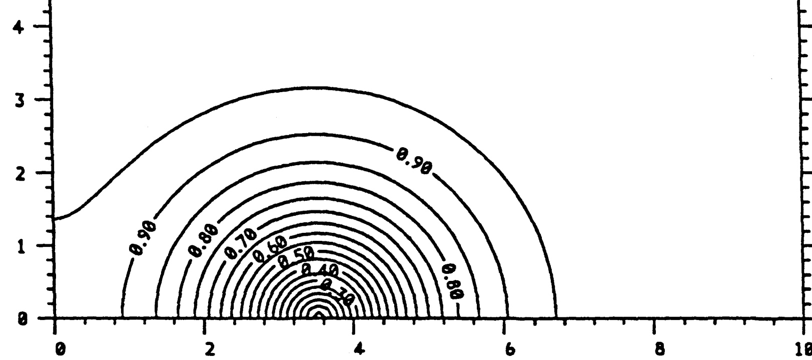
$$R_\pm(x) = \sqrt{(x_1 \pm \sigma)^2 + x_2^2}\,. \tag{5.19}$$

$b(\sigma)$ is then given by

$$b(\sigma) = \frac{1}{\sigma} + \tilde{b}(\sigma) \tag{5.20}$$



(a)



(b)

Fig. 3a and b. The magnitude of the Higgs field $|\phi|$ in the first quadrant, for two vortices at positions $(\pm\sigma, 0)$: **a** $\sigma = 1.0$; **b** $\sigma = 3.5$

with

$$\tilde{b}(\sigma) = \partial_1 \Phi(\sigma, 0)\,. \tag{5.21}$$

To solve (5.18) numerically a simultaneous over-relaxation method was used [30]. Since Φ is symmetric under reflection in the x_1 and x_2-axes one only need work in the first quadrant, requiring that the normal derivative of Φ vanish on the boundary. The domain taken was $0 \le x_1 \le 10$, $0 \le x_2 \le 7$, discretized with a square grid of spacing 0.1. The initial configuration Φ_0 to be relaxed was taken to be the superposition of 1-vortex solutions

$$\Phi_0 = \ln \varrho^2(R_+)\varrho^2(R_-) - \ln R_+^2 R_-^2 \tag{5.22}$$

with the 1-vortex profile $\varrho(r)$ approximated by $\tanh(0.6r)$. The values of Φ on $x_1 = 10$ and $x_2 = 7$ (which are very close to unity) were then left unchanged in the subsequent relaxation, which was repeated until the norm of the residual dropped below 10^{-4}. This required about 600 sweeps of the grid. Presumably more sophisticated techniques could speed up the rate of convergence, though since the equation to be solved is non-linear, obtaining convergence at all could be a delicate matter. The solution was obtained for σ ranging from 0 to 3.5 in steps of 0.05. Figure 3 shows a contour map of the magnitude of the Higgs field $|\phi|$ at two different separations. The domain appears to be sufficiently large to avoid significant boundary effects.

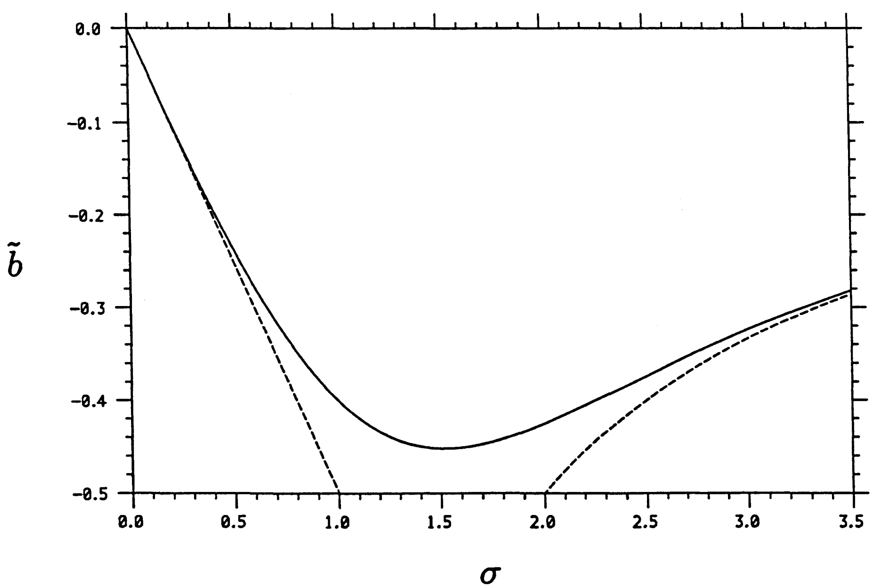


Fig. 4. The function $\tilde{b}(\sigma)$ (solid line), and the asymptotic forms $-\sigma/2(\sigma \to 0)$ and $-1/\sigma(\sigma \to \infty)$ (dashed lines)

The next step, that of calculating $\tilde{b}(\sigma)$ from (5.21), requires a differentiation of Φ at the point $(\sigma, 0)$. To ensure an accurate determination, the values of the solution on the 15×15 sub-grid centred at $(\sigma, 0)$ were fitted by a fourth order surface using a least-squares algorithm, and $\tilde{b}(\sigma)$ obtained algebraically from this fit. When the sub-grid overlapped the edge of the domain the values of Φ on the overlap were obtained by reflection symmetry. The result is shown in Fig. 4. For small and large σ, $\tilde{b}(\sigma)$ agrees with the analytical results (5.12), (5.15), as indicated by the broken lines in

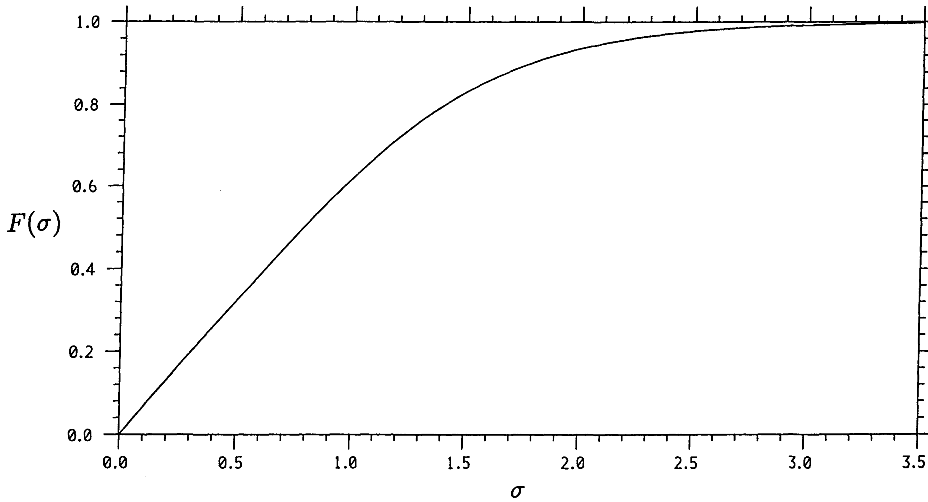


Fig. 5. The profile $F(\sigma)$ of the metric ds^2_{rel}. The slope at the origin is about 0.658

the graph. One may also check that there is good agreement with the size of the exponential decay in (5.15).

Finally, the profile $F(\sigma)$ was calculated from (5.10) (where we may replace b by $\tilde{b}$) using a standard 5-point differentiation formula. Note that since $\tilde{b}$ has the right asymptotic behaviour, the profile automatically satisfies the constraint (5.17). At separations σ of the same order as the grid spacing one would expect the results to be less reliable. Indeed, the slope of the profile obtained was not very smooth at these small separations ($\sigma \lesssim 0.2$). The profile was therefore smoothed by making a weighted polynomial fit for $0 \leq \sigma \leq 1.5$. The final result is shown in Fig. 5; the slope of $F(\sigma)$ at the origin is about 0.658. One may check numerically that this profile gives rise to a metric with positive curvature

$$\kappa = -\frac{1}{\sigma F^2}\frac{d}{d\sigma}\left(\frac{\sigma}{F}\frac{dF}{d\sigma}\right) > 0\,. \tag{5.23}$$

5.3. Classical 2-Vortex Scattering

It is straightforward to compute the geodesics of the metric ds^2_{rel}. The corresponding trajectories of the zeros of the Higgs in $\mathbb{R}^2$ are shown in Fig. 6 for various impact parameters (see also Fig. 7). In Fig. 8, the deflection angle Θ is plotted against the impact parameter a, here defined to be the perpendicular distance of each vortex from the x_1-axis at large separation. Since there is no orbiting, Θ is always equal to the observation angle ϑ. Note also that it is a monotonically decreasing function of a; this is a consequence of the positive curvature of the cone M_0^2 [4], and is important in the quantum scattering. The classical differential cross-section

$$\frac{d\sigma_{\mathrm{cs}}}{d\vartheta} = \left|\frac{d\Theta}{da}\right|^{-1} \tag{5.24}$$

is shown in Fig. 9.

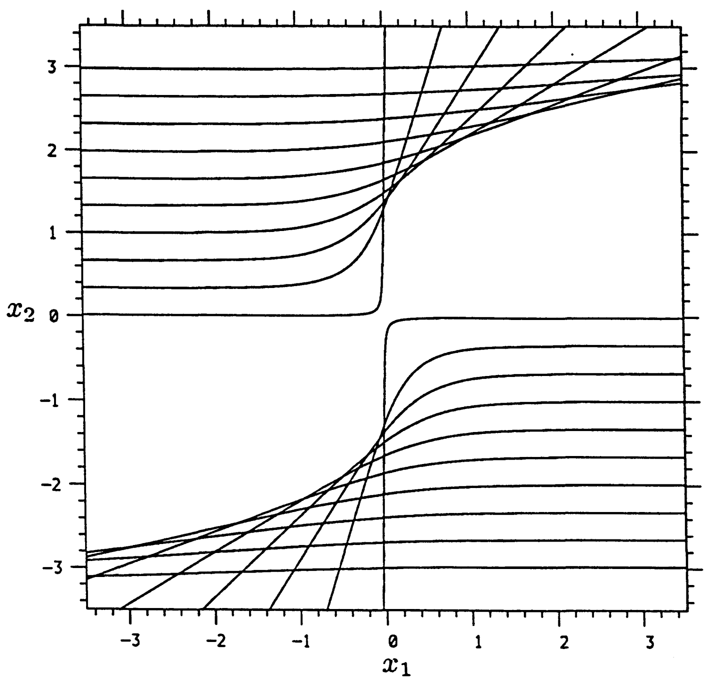


Fig. 6. Trajectories of the zeros of the Higgs field in the geodesic approximation

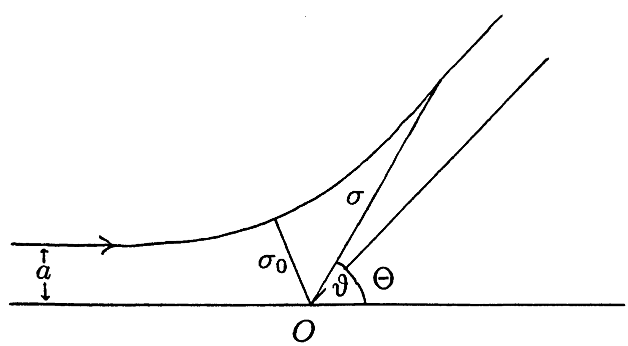


Fig. 7. The geometry of scattering. This diagram may be regarded as representing either the upper-half of $\mathbb{R}^2 = \{(x_1, x_2)\}$, or, identifying $\vartheta = 0$ and π, the smooth cone M_2^0

It is very interesting to compare these results with the true scattering of vortices, governed by the full equations of motion, investigated numerically in [17] and [18]. The scattering data obtained at various impact speeds are displayed together with the geodesic prediction in Fig. 8. We see that the geodesic description is remarkably robust. It holds to a good approximation up to speeds of at least 0.4 (at this speed the Lorentz factor is 0.92); only the numerical data for the very high impact speed 0.85 show a significant deviation[4]. This accords with the remarks made in Sect. 3.1.

[4] We assume that the anomalous point at impact parameter 2.5 for speed 0.16 results from inaccuracies in the simulation of [17]

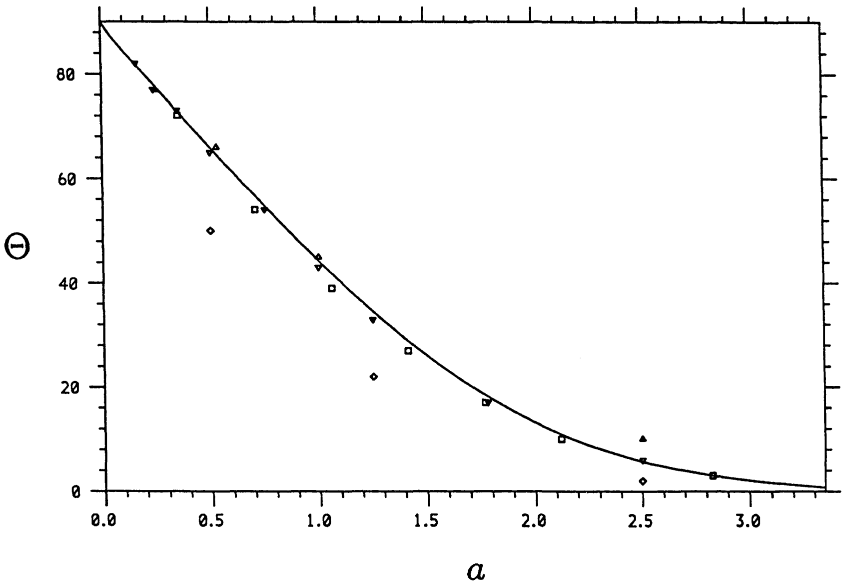


Fig. 8. The deflection angle as a function of impact parameter. The solid line is the geodesic prediction. The data points are from the numerical simulation of the full scattering problem at various impact speeds v: $v = 0.16$ ($\triangle$), $v = 0.4$ ($\triangledown$), $v = 0.85$ ($\Diamond$) (from [17]); $v = 0.5$ ($\Box$) (from [18]). For estimates of the errors in some of these data points see [17]

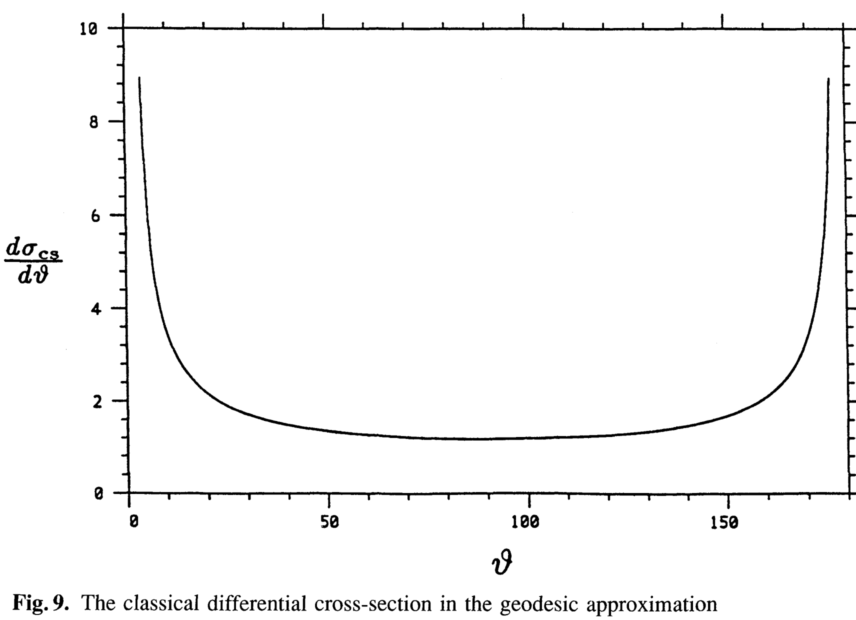


Fig. 9. The classical differential cross-section in the geodesic approximation

6. Integrals of the Kähler Form

In Sect. 5.1 we obtained an integral constraint (5.17) on the 2-vortex metric. We shall now try to understand this integral from a more general point of view.

Consider the decomposition of the Kähler form (4.21) on M_n into "free" and "interacting" parts

$$\omega = \omega_0 + \omega_1 \tag{6.1}$$

with

$$\omega_0 = \frac{i}{4} \sum_{r=1}^{n} dz_r \wedge d\bar{z}_r \,, \tag{6.2}$$

$$\omega_1 = \frac{i}{2} \partial \bar{b} \,, \tag{6.3}$$

where we have introduced the $(1,0)$-form

$$b = \sum_{s=1}^{n} b_s dz_s \,, \tag{6.4}$$

and where ∂ is the standard holomorphic exterior derivative. The 2-form ω_1 measures the difference between the geometry of the vortex space and that of the corresponding point particles. Now, let D be a 2-dimensional submanifold of M_n without boundary, and consider the integral

$$I(D) = -\frac{1}{\pi} \int_D \omega_1 \,. \tag{6.5}$$

ω_1 is closed, so this integral is invariant under local deformations of D. In fact, ω_1 is singular on Δ_n, so not closed there, but the singularities do not contribute to $I(D)$ (see the footnote below).

We shall see that (5.17) is an example of an integral of the type (6.5). To evaluate it in the general case we first show that we may replace the ∂ in (6.3) by the full exterior derivative $d = \partial + \bar{\partial}$. The condition (4.19) means that

$$\partial \bar{b} = -\bar{\partial} b \,. \tag{6.6}$$

It follows that the $(2,0)$-form ∂b is annihilated by both ∂ and $\bar{\partial}$, and hence $\partial b = \partial \alpha$ where α is a $(1,0)$-form satisfying $\bar{\partial}\alpha = 0$. The components of α are therefore holomorphic functions on $M_n \backslash \Delta_n$. Furthermore, since b decays exponentially fast at large vortex separation, so does $\partial\alpha$. Thus $\partial\alpha$ is in fact identically zero and we have

$$\partial b = 0 \,. \tag{6.7}$$

Locally on $M_n \backslash \Delta_n$ we may therefore write

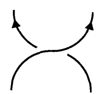
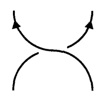
$$\omega_1 = \frac{i}{2} d\bar{b} \,. \tag{6.8}$$

We remark incidentally that (6.7) also implies $b = \partial K$, i.e. $b_r = \partial K / \partial z_r$, with K real by (6.6). Translation invariance then gives the interesting relation $\sum_{r=1}^{n} b_r = 0$.

If D is compact then $I(D)$ is zero, so we take it to be non-compact, and in particular, to be topologically a plane on which b is asymptotically zero. (This means that traversing the circle at infinity on D corresponds to the motion of infinitely separated vortices.) We assume also that D is generic, in the sense that it intersects Δ_n transversely in a finite number of points, and denote by D' the space obtained by removing from D small discs centred at these points[5]. Letting the radii of these discs tend to zero we have

$$I(D) = -\frac{i}{2\pi}\int_{D'} d\bar{b} = -\frac{i}{2\pi}\int_{\partial D'} \bar{b} = -\frac{i}{\pi}\sum_{\substack{r,s=1\\ r\neq s}}^{n}\int_{\partial D'}\frac{d\bar{z}_s}{\bar{z}_s - \bar{z}_r} = N \in \mathbb{Z}\,, \tag{6.9}$$

where in the third equality we have used (6.4) and (4.33). The integer N is determined by the intersection of D with Δ_n. It counts the number of times the zeros z_r wind round each other as one traverses a closed path Γ on D encircling $D \cap \Delta_n$. More precisely, since on Γ no two z_r coincide, it defines a braid. N is the oriented crossing number of this braid, with the assignment

$+1$ for  and -1 for  .

Let us consider two examples.

(i) D is the surface given by fixing all the z_r but one $-z_r$ say. Then

$$I(D) = -\frac{i}{\pi}\sum_{r\neq 1}\int_{\partial D'}\frac{d\bar{z}_1}{\bar{z}_1 - \bar{z}_r} = 2(n-1)\,. \tag{6.10}$$

(ii) D is the surface given by

$$P(z) = z^{n-k}(z^k - w)\,, \qquad w \in \mathbb{C}\,. \tag{6.11}$$

Note that this surface is not generic, but lies within Δ_n; we should really consider a perturbation of it, e.g.

$$P(z) = (z^{n-k} - \varepsilon)(z^k - w)\,, \qquad w \in \mathbb{C}\,. \tag{6.12}$$

Denote by ζ one of the k^{th} roots of w. Then the zeros (arbitrarily ordered) are

$$\begin{aligned} z_r &= \zeta \exp\frac{2\pi i r}{k}\,, \qquad & r &= 1, \ldots, k\,,\\ z_r &= 0\,, & r &= k+1, \ldots, n\,. \end{aligned} \tag{6.13}$$

The integral over D becomes

$$I(D) = \frac{i}{\pi}\sum_{s=1}^{k}\int\left[\sum_{r=k+1}^{n}\frac{d\bar{\zeta}}{\bar{\zeta}} + \sum_{\substack{r=1\\ r\neq s}}^{k}\frac{d\bar{\zeta}}{\bar{\zeta}\left(1 - \exp\dfrac{2\pi i(s-r)}{k}\right)}\right], \tag{6.14}$$

where the integration is around $\zeta = 0$ (in the positive sense). Since

$$\int\frac{d\bar{\zeta}}{\bar{\zeta}} = \frac{1}{k}\int\frac{d\bar{w}}{\bar{w}} = -\frac{2\pi i}{k} \tag{6.15}$$

---

[5] The contribution to $I(D)$ from such a disc of proper radius ε is $O(\varepsilon)$, and so vanishes in the limit $\varepsilon \to 0$

we obtain

$$I(D) = 2\left[n - k + \sum_{t=1}^{k-1} \frac{1}{1 - \exp\dfrac{2\pi i}{k}}\right] = 2n - k - 1\,. \tag{6.16}$$

If $n = 2$ and $k = 2$ then

$$I(D) = 1\,. \tag{6.17}$$

Noting that for $n = 2$

$$\omega_1 = -(1 - F^2(\sigma))\sigma d\sigma \wedge d\vartheta\,, \tag{6.18}$$

we see that (6.17) is precisely the constraint (5.17) on the 2-vortex metric obtained in Sect. 5.1.

7. Quantum Scattering

We have seen that at low (and not so low) energies vortex scattering is well approximated by geodesic motion on the vortex moduli space. We now consider the quantum scattering problem. As mentioned in the Introduction, an approximate quantization is obtained by considering a wave function Ψ on M_n, obeying the Schrödinger equation

$$i\hbar\,\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{\pi}\,\nabla_n^2\Psi\,, \tag{7.1}$$

where ∇_n^2 is the covariant Laplacian on M_n. The factor π results from our choice of normalization of the metric, and corresponds to a reduced mass of $\pi/2$ for the 2-vortex system. The centre of mass motion may be split off via

$$\Psi = \mathrm{e}^{-iP\cdot X/\hbar}\psi \tag{7.2}$$

and in a stationary state we obtain

$$\left(\frac{\hbar^2}{\pi}\,\nabla^2 + E\right)\psi = 0\,, \tag{7.3}$$

where ∇^2 is the covariant Laplacian on M_n^0 and E is the energy of the relative motion.

As in the description of the classical motion we neglect excitations of field modes orthogonal to M_n. Since the fields are now subject to quantum fluctuations, it is not clear that this will still give such a good approximation to the true dynamics. Indeed, it is probable that quantizing these modes leads to an effective potential on M_n. Nevertheless, we shall ignore this possibility and just consider the simple problem of free motion (7.3), confining the discussion to the 2-vortex case. This prescription was employed to quantize the 2-monopole system in [5].

7.1. Quantum 2-Vortex Scattering

When $n = 2$ the problem reduces to quantum scattering on the smoothed cone M_2^0[6]. Recalling the metric (5.9), the wave function on M_2^0, $\psi = \psi(\sigma, \vartheta)$, satisfies

$$\sigma \frac{\partial}{\partial \sigma}\left(\sigma \frac{\partial \psi}{\partial \sigma}\right) + \frac{\partial^2 \psi}{\partial \vartheta^2} + k^2 \sigma^2 F^2(\sigma)\psi = 0\,, \tag{7.4}$$

where we have set

$$E = \frac{\hbar^2 k^2}{\pi}\,. \tag{7.5}$$

The angular coordinate ϑ has the range π. We therefore impose the periodic boundary condition

$$\psi(\sigma, 0) = \psi(\sigma, \pi)\,. \tag{7.6}$$

For large σ, M_2^0 is asymptotic to C_2, i.e. locally flat. The appropriate boundary condition for the scattering problem is thus

$$\psi(\sigma, \vartheta) \sim \mathrm{e}^{ik\sigma \cos\vartheta} + \mathrm{e}^{-ik\sigma\cos\vartheta} + \frac{\mathrm{e}^{ik\sigma}}{\sqrt{\sigma}}\, f(\vartheta) \qquad \sigma \to \infty\,, \tag{7.7}$$

and the differential cross-section is given by

$$\frac{d\sigma_{\mathrm{cs}}}{d\vartheta} = |f(\vartheta)|^2\,. \tag{7.8}$$

It is straightforward to adapt the usual partial wave analysis to the present case. The partial wave decomposition of ψ is

$$\psi(\sigma, \vartheta) = \sum_n u_n(\sigma)\,\mathrm{e}^{in\vartheta}\,, \tag{7.9}$$

where, owing to (7.6), only the even waves will contribute. Denoting differentiation by σ by a prime, the equation for u_n is

$$\sigma(\sigma u_n')' + (k^2\sigma^2 F^2 - n^2)u_n = 0\,. \tag{7.10}$$

For large σ this reduces to Bessel's equation

$$\sigma(\sigma u_n')' + (k^2\sigma^2 - n^2)u_n)0\,, \tag{7.11}$$

hence the asymptotic behaviour

$$\begin{aligned} u_n(\sigma) &\sim a_n J_n(k\sigma) + b_n N_n(k\sigma) \\ &\sim \sqrt{\frac{2}{\pi k\sigma}}\, c_n \cos\left(k\sigma - \left(n + \frac{1}{2}\right)\frac{\pi}{2} + \delta_n\right), \end{aligned} \tag{7.12}$$

where $a_n = c_n \cos\delta_n$, $b_n = -\,c_n \sin\delta_n$. Using

$$\mathrm{e}^{ik\sigma\cos\vartheta} = \sum_n i^n J_n(k\sigma)\,\mathrm{e}^{in\vartheta} \tag{7.13}$$

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[6] We note that the scattering problem on a singular cone of arbitrary deficit angle has recently been studied in connection with cosmic strings, and also 2-dimensional quantum gravity [31]

and comparing with (7.7) with (7.12) we obtain

$$f(\vartheta) = \sum_{n \text{ even}} f_n(k)\, e^{in\vartheta}\,, \tag{7.14}$$

where

$$f_n(k) = \sqrt{\frac{8i}{\pi k}}\, e^{i\delta_n} \sin\delta_n\,. \tag{7.15}$$

It is useful to relate the phase shifts to the metric in the following way [32]. We compare (7.10) with the free equation (7.11) whose regular solution (normalized for later convenience) is

$$\bar{u}_n(\sigma) = a_n J_n(k\sigma)\,. \tag{7.16}$$

One obtains a standard form by writing $u_n = v_n/\sqrt{\sigma}$ and $\bar{u}_n = \bar{v}_n/\sqrt{\sigma}$. Then (7.10) becomes

$$v_n'' + \left\{ k^2F^2 - \frac{1}{\sigma^2}\left(n^2 - \frac{1}{4}\right)\right\} v_n = 0 \tag{7.17}$$

and similarly for (7.11). The Wronskian then satisfies

$$(v_n\bar{v}_n' - v_n'\bar{v}_n)' = k^2(1-F^2)v_n\bar{v}_n\,. \tag{7.18}$$

Integrating from $\sigma = 0$ to ∞ and employing (7.12) and (7.16), we obtain

$$\tan\delta_n = -\frac{1}{2}\pi k^2 \int_0^\infty d\sigma\, \sigma(1-F^2)\, \frac{u_n}{a_n}\, J_n(k\sigma)\,. \tag{7.19}$$

To describe the quantum scattering at a general k, one must sum the contributions of partial waves up to at least order $n \sim k$. We consider just the large and small wavelength limits, where it is possible to obtain some results without lengthy computation.

7.2. Large and Small Wavelength Limits

In our units the size of a vortex, and thus the length scale in the scattering problem, is $O(1)$. We consider the scattering in the two limits $k \ll 1$ and $k \gg 1$. In terms of the classical velocity $v \sim \hbar k$ we require $v \ll \hbar$ and $\hbar \ll v \ll 1$ respectively.

First consider $k \ll 1$. When $\sigma \gg 1$, Eqs. (7.10) and (7.11) agree; when $\sigma \sim 1$ they differ by $O(k^2)$. Thus with corrections of the same order, we have $u_n \cong \bar{u}_n$. Substituting into (7.19) we obtain (the first Born approximation)

$$\tan\delta_n = -\frac{1}{2}k^2\pi \int_0^\infty d\sigma\, \sigma(1-F^2)J_n(k\sigma)^2\,. \tag{7.20}$$

Further, since for $m > 0$

$$J_{\pm m}(x) \sim \frac{(\pm 1)^m x^m}{2^m m!} \quad \text{as} \quad x \to 0 \tag{7.21}$$

it follows that

$$\delta_n = O(k^{2|n|+2})\,. \tag{7.22}$$

Thus, when $k \ll 1$, the lowest partial wave dominates the scattering and, noting (7.14) and (7.15), the scattering amplitude is

$$f(\vartheta) = \sqrt{\frac{8i}{\pi k}}\,\delta_0 \tag{7.23}$$

with

$$\delta_0 = -\frac{1}{2}\,k^2\pi \int_0^\infty d\sigma\,\sigma(1-F^2)\,. \tag{7.24}$$

The quantity appearing here is precisely the area deficit between M_2^0 and C_2 discussed in Sect. 5.1. Thus, using (5.17), the differential cross section for vortex scattering at small k is

$$\frac{d\sigma_{\mathrm{cs}}}{d\vartheta} = 2\pi k^3\,, \tag{7.25}$$

and the total cross-section

$$\sigma_{\mathrm{cs}} = 2\pi^2 k^3\,. \tag{7.26}$$

Now consider $k \gg 1$; this is the semiclassical limit of the quantum scattering. Using (7.17), the semiclassical phase shift, as modified by Langer, is found to be

$$\delta_n = \int_{\sigma_0}^\infty d\sigma \left[\left(k^2F^2 - \frac{n^2}{\sigma^2}\right)^{1/2} - k\right] - k\sigma_0 + \frac{1}{2}\,n\pi\,, \tag{7.27}$$

where σ_0 is the classical distance of closest approach – the zero of the function in round brackets [33]. In the case of the lowest partial wave, we have $\sigma_0 = 0$ and

$$\delta_0 = -k\int_0^\infty d\sigma(1-F)\,. \tag{7.28}$$

The quantity appearing here is now the length deficit between the two cones M_2^0 and C_2, i.e. the difference in the geodesic distances to the apex in each case.

Semiclassically, of course, the $n = 0$ phase shift has little significance. At a given angle ϑ, the dominant contribution to the scattering amplitude is from waves with $n \sim l_i(\vartheta)/\hbar$, where the $l_i(\vartheta)$ are the angular momenta of classical paths i scattering to ϑ. The sum over waves may be replaced by an integral, and at generic ϑ, evaluated by the method of stationary phase, though in certain special regions (in the "forward" scattering region $\vartheta \sim 0$, π, and also where $d\Theta/da = 0$) one has to be more careful [33]. Away from these regions one obtains

$$f(\vartheta) = \sum_i \left(\frac{d\Theta(a_i)}{da}\right)^{1/2} \exp 2iS(a_i)/\hbar\,, \tag{7.29}$$

where $S(a)$ is the classical action associated with the path with impact parameter a. In our problem, the deflection angle depends monotonically on the impact parameter, so for every scattering angle there is just one contributing classical path. This means that the only semiclassical effects in the quantum scattering amplitude are in the region $\vartheta \sim 0$, π, where there is a "forward diffraction peak." Outside this region the semiclassical differential cross-section is just the classical expression (5.24).

Appendix A: Well-Separated Vortices

We show without rigour that a solution corresponding to well-separated vortices is the superposition of single vortex solutions, up to corrections exponentially small in the separation.

Recall that the quantity $f = \ln|\phi|^2$ satisfies

$$\nabla^2 f + 1 - \mathrm{e}^f = 4\pi \sum_{r=1}^{n} \delta(x - x_r) \tag{A.1}$$

with

$$f = O(\mathrm{e}^{-(1-\delta)|x|}) \quad \text{as} \quad |x| \to \infty\,. \tag{A.2}$$

Denote the solution for a single vortex at the origin by $f_0(x)$ and write $f_r(x) = f_0(x - x_r)$. For n well-separated vortices we expect a solution of the form

$$f = \sum_{r=1}^{n} f_r + g\,, \tag{A.3}$$

where the smooth function g is small. Substituting into (A.1) gives

$$\nabla^2 g + 1 - n + \sum_{r=1}^{n} \mathrm{e}^{f_r} - \mathrm{e}^g \exp \sum_{r=1}^{n} f_r = 0 \tag{A.4}$$

with

$$g \to 0 \quad \text{as} \quad |x| \to \infty\,. \tag{A.5}$$

Let the minimum separation between any two x_r be $2R$, with $R \gg 1$, and consider the domain $D = \{x : |x - x_1| \le R\}$. In the limit $R \to \infty$ the 1-vortex problem is recovered, and $g \equiv 0$. We treat the problem with $R \gg 1$ as a perturbation of this case. Thus, retaining only first order terms in g, and noting that in D, $f_r = O(\mathrm{e}^{-(1-\delta)R})$ for $r \neq 1$, (A.4) gives

$$\nabla^2 g + (1 - \mathrm{e}^{f_1}) \sum_{r \neq 1} f_r - g \exp \sum_{r=1}^{n} f_r = O(\mathrm{e}^{-2(1-\delta)R})\,. \tag{A.6}$$

Now, the second, inhomogeneous term in this equation is also $O(\mathrm{e}^{-2(1-\delta)R})$. Thus

$$\left(-\nabla^2 + \exp \sum_{r=1}^{n} f_r \right) g = O(\mathrm{e}^{-2(1-\delta)R})\,, \tag{A.7}$$

and since this equation is symmetrical in the index r, it holds *throughout* $\mathbb{R}^2$. Noting the boundary condition (A.5) we conclude that

$$g = O(\mathrm{e}^{-2(1-\delta)R})\,. \tag{A.8}$$

Appendix B: A Different Way of Showing Kähler

We briefly sketch the formal steps of an argument due to Ruback [34] which shows in a different way that the metric on M_n is Kähler. The reasoning is similar to that of Atiyah and Hitchin in [4], in their proof that the monopole metric is hyper-Kähler.

The linearized Bogomol'nyi equations, and Gauss' law may be written

$$\dot{A} = i\partial_z(\dot{\bar{\phi}}/\bar{\phi}), \tag{B.1}$$

$$4\partial_{\bar{z}}\dot{A} = i\dot{\bar{\phi}}\phi. \tag{B.2}$$

(The imaginary part of (B.2) is the linearized second Bogomol'nyi equation, and the real part is Gauss' law.) The map I given by

$$I:(\dot{A},\dot{\phi}) \to (-i\dot{A}, i\dot{\phi}) \tag{B.3}$$

leaves (B.1) and (B.2) invariant and satisfies $I^2 = -1$, so defines an almost complex structure on M_n. Further, recalling (4.8)

$$\frac{\dot{\phi}}{\phi} = \sum_{r=1}^{n} \dot{z}_r \frac{\partial}{\partial z_r} \ln|\phi|^2, \tag{B.4}$$

we see that under I, $\dot{z}_r \to i\dot{z}_r$, so I coincides with the complex structure defined by the coordinates z_r on M_n. Thus I is truly a complex structure on M_n, and not just an almost complex one. The metric on the space of all fields $\mathscr{A}$,

$$h(\dot{a},\dot{b}) = \tfrac{1}{2}\int d^2x(4\dot{A}\dot{\bar{B}} + 4\dot{\bar{A}}\dot{B} + \dot{\phi}\dot{\bar{\psi}} + \dot{\bar{\phi}}\dot{\psi}) \tag{B.5}$$

is invariant under I, and provided Gauss' law is satisfied, well-defined on M_n. Thus the induced metric on M_n, g say, is also invariant under I, i.e. it is Hermitian.

To show Kähler one must show that the Kähler form ω defined by

$$\omega(\dot{a},\dot{b}) = g(I\dot{a},\dot{b}) \tag{B.6}$$

is closed. We define a Kähler form on $\mathscr{A}$ by

$$\tilde{\omega}(\dot{a},\dot{b}) = \frac{i}{4}\int d^2x(4\dot{A}\dot{\bar{B}} - 4\dot{\bar{A}}\dot{B} - \dot{\phi}\dot{\bar{\psi}} + \dot{\bar{\phi}}\dot{\psi}). \tag{B7}$$

If $\dot{\lambda}$ is an infinitesimal gauge transformation $(\partial_z\Lambda, i\Lambda\phi)$ then after an integration by parts, and using (the imaginary part of) (B.2), we find $\tilde{\omega}(\dot{a},\dot{\lambda}) = 0$, so $\tilde{\omega}$ is well-defined on M_n and reduces to ω there. $\tilde{\omega}$ is constant on $\mathscr{A}$, so closed. Pulling back to any set of solutions of the Bogomol'nyi equations we obtain a closed form there. Hence the result.

Appendix C: Perturbing the Coincident 2-Vortex

To check the expression (5.12) for $b(\sigma)$ at small σ we consider the linear perturbation which splits the coincident 2-vortex. We may suppose that the zeros of the Higgs are perturbed to $(\pm\sigma, 0)$ and restrict our attention to the x_1-axis.

In the neighbourhood of the unperturbed 2-vortex, the function $f = \ln|\phi|^2$ is

$$f = 2\ln x_1^2 + \text{const} - \tfrac{1}{4}x_1^2 + O(x_1^4). \tag{C.1}$$

The relevant perturbation is given by $(1 + \lambda\eta)\phi$ with

$$\eta = -\frac{1}{x_1^2} + O(x_1^2)\,. \tag{C.2}$$

f is then perturbed to

$$\ln(x_1^2 - \lambda + O(\lambda x_1^2))^2 + \text{const} - \tfrac{1}{4}\, x_1^2 + O(x_1^4)\,. \tag{C.3}$$

Thus

$$\lambda = \sigma^2 + O(\sigma^4)\,, \tag{C.4}$$

and the higher, $O(\lambda^2)$, corrections to f are only $O(\sigma^4)$.

$b(\sigma)$ is the coefficient of the linear term in the expansion of f about the zero at $x_1 = \sigma$. Writing $x_1 = \sigma + s$ with $s \ll \sigma$ and retaining only terms up to $O(s)$ we find

$$f = \ln s^2 + \frac{s}{\sigma} + \text{const} - \frac{1}{2}\sigma s + O(\sigma^3 s) \tag{C.5}$$

and thus

$$b(\sigma) = \frac{1}{\sigma} - \frac{1}{2}\sigma + O(\sigma^3)\,. \tag{C.6}$$

Acknowledgements. I would like to thank N. Manton for many valuable suggestions and criticisms, P. Ruback for stimulating correspondence and discussions, B. Schroers and G. Watts for useful conversations, and N. Hitchin for pointing out an error in an earlier draft. I acknowledge a SERC research studentship.

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Communicated by S.-T. Yau

Note added in proof. E. Myers, C. Rebbi, and R. Strilka have now performed further simulations of the full equations of motion for the two vortex system [Phys. Rev. D, 45 (4) to appear]. They also carry out a computation of the 2-vortex metric, using a different method from that employed here.