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Quantization of $SL(2, \mathbf{R})$ Chern–Simons Theory

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Abstract. We discuss Chern–Simons gauge theory with an $SL(2, \mathbf{R})$ gauge group on an arbitrary 3-manifold M. The $SL(2, \mathbf{R})$ Chern–Simons action is defined for gauge bundles over M of arbitrary topological type. The geometric quantization of $SL(2, \mathbf{R})$ Chern–Simons theory is discussed and related to the quantization of Teichmüller space. The generalization to Chern–Simons theory with an $SL(n, \mathbf{R})$ gauge group is also considered.

1. Introduction

One of the most interesting recent developments in theoretical physics is the realization that there are non-trivial quantum field theories defined on smooth manifolds which are independent of any choice of metric on the manifold. The observables of such a topological quantum field theory are then automatically topological invariants of the situation. The Donaldson invariants of smooth 4-manifolds and the Floer homology groups of 3-manifolds are related to a certain topological gauge theory in $3 + 1$ dimensions [1]. Similarly the Gromov invariants of symplectic manifolds may be interpreted in terms of the quantum field theory of a topological sigma model in $1 + 1$ dimensions [2]. In $2 + 1$ dimensions an interesting topological quantum field theory is defined by the Chern–Simons action [3]. If M is a compact oriented 3-manifold and G is a compact simple simply-connected Lie group, then the partition function of Chern–Simons theory defines a topological invariant of M. If the topology of M is such that the flat connections on M are isolated, then in the semi-classical limit this invariant is related to the Ray–Singer torsion of M. If the manifold M contains an embedded link then the expectation value of the corresponding Wilson lines in M yields an invariant of the link, which in the simplest case is just the Jones polynomial of the link [3].

An important observation made in [3] is that there is a direct connection between Chern–Simons theory in $2 + 1$ dimensions and conformal field theory in $1 + 1$ dimensions. The canonical quantization of Chern–Simons theory associates a Hilbert space to a 2-dimensional surface Σ. This Hilbert space may be interpreted as the space of conformal blocks of a rational conformal field theory in $1+1$ dimensions. The $1+1$-dimensional conformal field theory is just the current algebra of G [3–6]. In this paper we consider Chern–Simons theory for the non-compact real group $SL(2,\mathbf{R})$, and more generally for the groups $SL(n,\mathbf{R})$, $n>2$. Chern–Simons theory for $SL(2,\mathbf{R})$ is a natural generalization of Chern–Simons theory for a compact group G, and is in addition related to two-dimensional quantum gravity and the representation theory of the loop group of $SL(2,\mathbf{R})$. It should also be noted that three-dimensional quantum gravity with a negative cosmological constant may be viewed as a Chern–Simons gauge theory with gauge group $SL(2,\mathbf{R})\times SL(2,\mathbf{R})$ [7]. Chern–Simons theory of the complex groups $SL(n,\mathbf{C})$ has been discussed recently in [8]. This paper is organized as follows. In Sect. 2, we define the Chern–Simons action for an $SL(2,\mathbf{R})$ gauge bundle over M of arbitrary topological type. The geometric quantization of the $SL(2,\mathbf{R})$ theory is discussed in Sect. 3 and this is related to the quantization of Teichmüller space in Sect. 4. Finally, in Sect. 5, we discuss the generalization to $SL(n,\mathbf{R})$ Chern–Simons theory, and very briefly the relation between $SL(2,\mathbf{R})$ Chern–Simons theory and representations of the loop group of $SL(2,\mathbf{R})$.

2. The Chern–Simons Action for an $SL(2,\mathbf{R})$ Gauge Group

We wish to define the Chern–Simons action for an $SL(2,\mathbf{R})$ gauge theory over a compact 3-manifold M. Let us recall first the well-known case of the Chern–Simons action for a compact simple Lie group G. Let $P\to M$ be a fixed principal G-bundle over M and let $\mathrm{ad}(P)=P\times_G\mathbf{g}$ be the bundle associated to P by the adjoint action of G on its Lie algebra $\mathbf{g}$. The space of $\mathrm{ad}(P)$-valued p-forms on M is denoted by $\Omega^p(M;\mathrm{ad}(P))$. The space $\mathscr{A}$ of all connections on P is an affine space with associated vector space $\Omega^1(M;\mathrm{ad}(P))$. If the G-bundle $P\to M$ is trivial, i.e., if $P=M\times G$, then a gauge connection $A\in\mathscr{A}$ is globally well defined on M and the Chern–Simons action may be written explicitly as

$$\check{S}[A]=\frac{k}{4\pi}\int_M \mathrm{Tr}\left(A\wedge dA+\frac{2}{3}A\wedge A\wedge A\right), \tag{1}$$

where Tr represents an adjoint invariant inner product on $\mathbf{g}$ and k is a constant. If the group G is simply connected then any G-bundle $P\to M$ is trivial and the Chern–Simons action may be written in the form (1).

Let $\mathscr{G}$ denote the group of gauge transformations of P, i.e., the group of fibre-reeserving automorphisms of P. If $P\to M$ is trivial, then $\mathscr{G}$ is isomorphic to the group of smooth maps from M to G: $\mathscr{G}=\mathrm{Map}(M;G)$. The group of components of $\mathscr{G}$ is then given by $\pi_0(\mathscr{G})\cong[M;G]$, where $[M;G]$ denotes the set of homotopy classes of maps from M to G. If G is simply connected, then it may be shown [9] that $[M;G]\cong\pi_3(G)$. Thus, as G is simple, $\pi_0(\mathscr{G})\cong\mathbf{Z}$. It is well known that under a gauge transformation $g\in\mathscr{G}$, the action $\check{S}$ changes by $\deg(g)$, the degree of g corresponds to the isomorphism $\pi_0(\mathscr{G})\cong\mathbf{Z}$. The requirement that $e^{i\check{S}}$ should be single-valued implies that the constant k in (1) must be quantized; if $G=SU(n)$ then k is quantized as an integer.

If the gauge group G is not simply connected then there may exist non-trivial G-bundles P over the 3-manifold M. If $P\to M$ is non-trivial then the connection cannot be defined globally on M and the Chern–Simons action cannot be written in the form (1). However, there is an alternative way of defining the Chern–Simons action in this case, which has been discussed for simple groups in [10] and for the non-semisimple group $U(n)$ in [11]. As we are interested here in the group $SL(2,\mathbf{R})$ which is not simply connected ($\pi_1(SL(2,\mathbf{R})\cong\mathbf{Z})$, it is necessary to use such a method to define the Chern–Simons action.

Before considering how to define the $SL(2,\mathbf{R})$ Chern–Simons action let us consider how $SL(2,\mathbf{R})$-bundles over a 3-manifold M are classified. Associated to any Lie group G there is a space BG, the universal classifying space of G, which is determined up to homotopy type by being the base space of a fibration

$$G\to EG\to BG \tag{2}$$

in which the total space is contractible [12]. It is a standard result in topology [12] that any G-bundle P over a manifold M may be obtained by pulling-back the universal bundle $EG\to BG$ by a suitable map $g:M\to BG$. If two maps $g_1, g_2: M\to BG$ are homotopic, then the bundles $g_1^*(EG)$ and $g_2^*(EG)$ are isomorphic. Thus the set of isomorphism classes of G-bundles over M is equal to the set $[M;BG]$ of homotopy classes of maps from M to BG. We are interested here in the case $G=SL(2,\mathbf{R})$. Recall that $SL(2,\mathbf{R})$ is diffeomorphic to $\mathbf{R}^2\times S^1$ and hence $SL(2,\mathbf{R})$ has the homotopy type of S^1. It follows from the exact homotopy sequence corresponding to (2) that $\pi_i(BG)\cong\pi_{i-1}(G)$. The first three homotopy groups of $BSL(2,\mathbf{R})$ are therefore: $\pi_1(BSL(2,\mathbf{R}))=0, \pi_2(BSL(2,\mathbf{R}))=\mathbf{Z}, \pi_3(BSL(2,\mathbf{R}))=0$. It follows, therefore, that up to dimension 3, $BSL(2,\mathbf{R})$ is approximated homotopically by the Eilenberg–MacLane space $K(\mathbf{Z},2)$. We recall that for a positive integer n and group π (with π Ableian for $n>1$) a space X is defined to be an Eilenberg–MacLane space $K(\pi,n)$ if

$$\pi_i(X)=\begin{cases}\pi, & i=n\\ 0, & \text{otherwise.}\end{cases} \tag{3}$$

For any n and π, such a space exists and is unique up to homotopy type [13]. The set of isomorphism classes of $SL(2,\mathbf{R})$-bundles over a 3-manifold M is therefore given by $[M;BSL(2,\mathbf{R})]\cong[M;K(\mathbf{Z},2)]$. The Eilenberg–MacLane space $K(\pi,n)$ is classifying for cohomology, i.e.,

$$[M;K(\pi,n)]\cong H^n(M;\pi). \tag{4}$$

Hence, $[M;BSL(2,\mathbf{R})]\cong H^2(M;\mathbf{Z})$ and consequently an $SL(2,\mathbf{R})$-bundle over a 3-manifold M is classified by a class in $H^2(M;\mathbf{Z})$. For certain 3-manifolds the only $SL(2,\mathbf{R})$-bundle over M is the trivial bundle (i.g., when M is a homology 3-sphere $H^1(M;\mathbf{Z})=H^2(M;\mathbf{Z})=0$), however, for a general 3-manifold M, there will exist non-trivial $SL(2,\mathbf{R})$-bundles over M.

We return now to the problem of defining the Chern–Simons action for the group $SL(2,\mathbf{R})$. It is important to note at this point that there is a fundamental difference between Chern–Simons gauge theories and gauge theories of Yang–Mills type, which has been discussed for the case of complex gauge groups in [8]. The standard Yang–Mills action, for a theory with gauge group G, is

$$I[A]=\frac{1}{4g^2}\int_M \mathrm{Tr}(F\wedge *F),$$

where F is the curvature of the connection A and $*$ is the Hodge duality operator associated to a Riemannian metric on M. If the gauge group G is compact then the bilinear form Tr on $\mathbf{g}$ is positive definite and the action $I[A]$ is bounded below. If, however, G is non-compact (e.g., $G = SL(2, \mathbf{R})$) then Tr is not positive definite and $I[A]$ is unbounded below. Thus, in order to ensure that Yang–Mills theory has positive energy it is necessary to restrict Yang–Mills gauge groups to be compact groups. The situation for Chern–Simons gauge theory is, however, quite different. The Chern–Simons action (1) is independent of any choice of metric on M (i.e., Chern–Simons gauge theory is generally covariant), which implies that the Hamiltonian in Chern–Simons theory vanishes identically. Hence, for any choice of gauge group G, Chern–Simons gauge theory always has a Hamiltonian which is bounded below. Furthermore, for any gauge group G, Chern–Simons theory, as a bosonic theory with a real Lagrangian, is always unitary. It makes sense, therefore, to consider Chern–Simons gauge theory for the non-compact gauge group $G = SL(2, \mathbf{R})$.

If the $SL(2, \mathbf{R})$ gauge bundle $P \to M$ is trivial, then the Chern–Simons action may be defined directly as

$$S[A] = \frac{k}{2\pi} \int_M \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right), \tag{5}$$

where Tr is an invariant bilinear form on the Lie algebra $sl(2, \mathbf{R})$ of $SL(2, \mathbf{R})$. Under a gauge transformation $g: M \to SL(2, \mathbf{R})$ the action (5) changes in exactly the same way as in the case of a compact group, i.e., S changes by the degree of g. However, in the case of $SL(2, \mathbf{R})$ we can no longer conclude that k is quantized as $\pi_3(SL(2, \mathbf{R}) = 0$. The quantization of k is discussed in greater detail below.

If $P \to M$ is a non-trivial bundle then the Chern–Simons action cannot be defined globally on M and we must adopt the following procedure. Assume that there exists a compact oriented 4-manifold Y such that M is the boundary of Y, i.e., $M = \partial Y$. Assume further that the $SL(2, \mathbf{R})$-bundle $P \to M$ extends to an $SL(2, \mathbf{R})$-bundle $\hat{P} \to Y$. In this case any connection A on $P \to M$ extends to a connection $\hat{A}$ on $\hat{P} \to Y$, and we will denote the curvature of $\hat{A}$ by F. We can now define the functional

$$S[A] = \frac{k}{2\pi} \int_Y \mathrm{Tr}(F \wedge F). \tag{6}$$

In fact, as will be discussed below, it may be shown that every compact oriented 3-manifold M bounds a compact oriented 4-manifold Y. If $P \to M$ is the trivial $SL(2, \mathbf{R})$-bundle then P clearly extends to the trivial bundle $\hat{P}$ over Y. It is well known that $\mathrm{Tr}(F \wedge F) = d\,\mathrm{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$ and hence, for the trivial bundle, $S[A]$ defined by (6) is identical to the standard Chern–Simons action defined by (5). In fact we shall now show that (6) is well defined for an arbitrary $SL(2, \mathbf{R})$-bundle over M and thus gives a general definition of the Chern–Simons action. To write the Chern–Simons action, for an arbitrary $SL(2, \mathbf{R})$-bundle $P \to M$, in the form (6) it is necessary that the gauge bundle P extends over some 4-dimensional coboundary Y. To show that this is always possible it is necessary to enter into a short mathematical digression concerned with cobordism theory (see [14]).

Let us first recall that two oriented n-manifolds M_1 and M_2 are oriented cobordant if there exists an $(n+1)$-dimensional oriented manifold Y such that the disjoint union of M_1 and M_2 is the boundary of Y. The set of equivalence classes of oriented cobordant n-manifolds forms a group (see [14]) which is denoted by Ω_n^{SO}. The identity element of Ω_n^{SO} consists of those oriented n-manifolds which bound an $(n+1)$-manifold. Thus if $\Omega_n^{SO}=0$, then every oriented n-manifold bounds some oriented $(n+1)$-dimensional manifold. It may be shown [14] that $\Omega_3^{SO}=0$; hence, any 3-manifold M bounds a 4-manifold Y, as was stated above. We now wish to consider the situation in which we have maps from the manifolds to some additional space. Let M_1 and M_2 be two oriented n-manifolds and let $f_1: M_1 \to X$ and $f_2: M_2 \to X$ be maps to some additional space X. We define (M_1, f_1) and (M_2, f_2) to be oriented cobordant if there exists an $(n+1)$-dimensional oriented coboundary Y of M_1 and M_2 together with a map $\hat{f}: Y \to X$ such that $\hat{f}|_{M_1}=f_1$ and $\hat{f}|_{M_2}=f_2$. The corresponding n-dimensional oriented cobordism group is denoted by $\Omega_n^{SO}(X)$ (see [14]). If $\Omega_n^{SO}(X)=0$, then for any oriented n-manifold M and map $f: M \to X$, there exists an oriented coboundary Y such that f extends to a map $\hat{f}: Y \to X$.

To apply cobordism theory to the case of an $SL(2,\mathbf{R})$-bundle V over a 3-manifold M we simply take the space X to be the classifying space $BSL(2,\mathbf{R})$ of $SL(2,\mathbf{R})$. If $\Omega_3^{SO}(BSL(2,\mathbf{R}))=0$, then for any oriented 3-manifold M and any map $f: M \to BSL(2,\mathbf{R})$, there exists an oriented coboundary Y such that f extends to a map $\hat{f}: Y \to BSL(2,\mathbf{R})$. This is exactly equivalent to saying that the $SL(2,\mathbf{R})$-bundle $V \to M$ classified by the map $f: M \to BSL(2,\mathbf{R})$ extends to an $SL(2,\mathbf{R})$-bundle $\hat{V} \to Y$ classified by the map $\hat{f}: Y \to BSL(2,\mathbf{R})$.

If follows from general results in cobordism theory [15] that for any space Y (which satisfies certain mild topological restrictions) the cobordism groups $\Omega_n^{SO}(X)$ are given by

$$\Omega_n^{SO}(X) = \Omega_n^{SO} \bigotimes_{\mathbf{Z}} H_n(X), \tag{7}$$

where $H_n(X)$ is the n^{th} homology group of X. Thus, for $X = BSL(2,\mathbf{R})$ we obtain

$$\begin{aligned} \Omega_3^{SO}(BSL(2,\mathbf{R})) &= \Omega_3^{SO} \bigotimes_{\mathbf{Z}} H_3(BSL(2,\mathbf{R})) \\ &= 0 \end{aligned} \tag{8}$$

as $H_3(BSL(2,\mathbf{R})) \cong H_3(BS^1) = 0$. It follows, therefore, that for any $SL(2,\mathbf{R})$-bundle V over a 3-manifold M there exists a coboundary Y over which V extends. Thus, we can define the Chern–Simons action for an arbitrary $SL(2,\mathbf{R})$-bundle over M by (6).

The only issue that remains to be addressed in order to show that (6) is completely well defined is the way in which S, as defined by (6), depends upon the choice of coboundary. Let Y_1 and Y_2 be two oriented coboundaries of a given 3-manifold M, and let $S_1 = \frac{k}{2\pi} \int_{Y_1} \mathrm{Tr}(F \wedge F)$ and $S_2 = -\frac{k}{2\pi} \int_{Y_2} \mathrm{Tr}(F \wedge F)$ be the actions defined using Y_1 and Y_2 respectively. For the quantum field theory to be well-defined we require that $e^{iS_1} = e^{iS_2}$, i.e., that $S_1 = S_2 + 2\pi N$, where N is an integer. It follows from the form of S_1 and S_2 that $N = kn$, where n is given by

$$n = \frac{1}{4\pi^2} \int_{\bar{Y}} \mathrm{Tr}(F \wedge F), \tag{9}$$

where $\bar{Y}$ is the closed four-manifold $Y_1 + Y_2$. Topologically the characteristic class $\frac{1}{4\pi^2}\mathrm{Tr}(F \wedge F)$ represents the obstruction to extending an $SL(2,\mathbf{R})$-bundle V defined over the 3-skeleton of the manifold $\bar{Y}$ to the 4-skeleton. The obstruction to such an extension comes from $\pi_3(SL(2,\mathbf{R}))$. However, $SL(2,\mathbf{R})$ is diffeomorphic to $\mathbf{R}^2 \times S^1$ and hence $\pi_3(SL(2,\mathbf{R})) \cong \pi_3(S^1) = 0$. It follows, therefore, that the class $\frac{1}{4\pi}\mathrm{Tr}(F \wedge F)$ is trivial for an $SL(2,\mathbf{R})$-bundle over the 4-manifold $\bar{Y}$. It is a consequence of this that the constant k is not required to satisfy a quantization condition. In this respect, $SL(2,\mathbf{R})$ Chern–Simons theory is similar to $U(1)$ Chern–Simons theory where there is also no quantization condition implied directly from the definition of the action. In both cases this follows from the fact that $\pi_3(SL(2,\mathbf{R})) = \pi_3(U(1)) = 0$. It should be noted however that there is strong evidence that it is necessary to impose a quantization condition on k in $U(1)$ Chern–Simons theory if one wishes to define general operators on an arbitrary three-manifold [16]. It may as well be, therefore, that a deeper understanding of $SL(2,\mathbf{R})$ Chern–Simons theory will require k to be quantized, even though such a quantization condition does not follow directly from the action.

Given that the Chern–Simons action (6) is well defined for an arbitrary $SL(2,\mathbf{R})$-bundle V over a 3-manifold M we may formally define the path integral

$$Z = \int_{\mathscr{A}} \mathscr{D}A e^{iS[A]}, \tag{10}$$

where $\mathscr{A}$ is the space of connections on $V \to M$. More generally, if R is an irrducible representation of $SL(2,\mathbf{R})$ and $C: S^1 \to M$ is an embedded curve, the Wilson line, or holonomy operator, $W_R(A;C) = \mathrm{Tr}_R P \exp \int_C A_i dx^i$ is the natural gauge invariant and generally covariant operator in the theory and has the expectation value

$$\langle W_R(A;C) \rangle = \int_{\mathscr{A}} \mathscr{D}A e^{iS[A]} W_R(A;C). \tag{11}$$

In Chern–Simons theory with a compact group the exact analogue of (11) defines an invariant of the knot C. It should be noted that in the case of Chern–Simons theory with a compact group in order to define the analogues of (10) and (11) it is necessary to give a framing of the three-manifold M and C. Pressumably in the correct definition of the path integrals (10) and (11) such framings play an important role, however, we will have nothing further to say on this issue here. In the next section we will consider the canonical quantization of $SL(2,\mathbf{R})$ Chern–Simons theory.

3. Canonical Quantization of $SL(2,\mathrm{R})$ Chern–Simons Theory

We have seen in the last section that for an arbitrary $SL(2,\mathbf{R})$ gauge bundle over a 3-manifold M the Chern–Simons action may be defined by (6). Let us now consider the canonical quantization of the theory defined by this action. To canonically quantize $SL(2,\mathbf{R})$ Chern–Simons theory we take the 3-manifold M to be of the form $M = \mathbf{R} \times \Sigma$, where Σ is a compact oriented surface. The $SL(2,\mathbf{R})$

gauge bundle over M defines an $SL(2,\mathbf{R})$-bundle V over Σ. Such a bundle $V\to\Sigma$ is classified by a map $\Sigma\to BSL(2,\mathbf{R})$. Up to dimension two we have the homotopy equivalence $BSL(2,\mathbf{R})\sim K(\mathbf{Z},2)$ and thus an $SL(2,\mathbf{R})$-bundle over Σ is determined by a class in $[\Sigma;BSL(2,\mathbf{R})]\cong[\Sigma;K(\mathbf{Z},2)]\cong H^2(\Sigma;\mathbf{Z})\cong\mathbf{Z}$. This class may be interpreted in the following way. Let P be a principal $SL(2,\mathbf{R})$-bundle over Σ; $SL(2,\mathbf{R})$ acts naturally on $\mathbf{R}^2$ and we let $E_P=P\times_{SL(2,\mathbf{R})}\mathbf{R}^2$ denote the associated real 2-plane bundle over Σ. The bundle $E_P\to\Sigma$ has an Euler class $e(E_P)\in H^2(\Sigma;\mathbf{Z})\cong\mathbf{Z}$ and this class determines P. Thus an $SL(2,\mathbf{R})$-bundle $P\to\Sigma$ is classified topologically by the Euler class $e(E_P)\in H^2(\Sigma;\mathbf{Z})\cong\mathbf{Z}$. We will let r denote the Euler class of an $SL(2,\mathbf{R})$-bundle over Σ.

To canonically quantize the $SL(2,\mathbf{R})$ Chern–Simons theory on $M=\mathbf{R}\times\Sigma$, with gauge bundle $V\to\Sigma$, we introduce an oriented coboundary B of Σ over which V extends. We then have $M=\partial Y$, where $Y=\mathbf{R}\times B$. Choosing the gauge $A_0=0$ (where A_0 is the component of the connection in the $\mathbf{R}$ direction) the action (6) takes the form

$$S=\frac{k}{\pi}\int dt\int_B d^3x\varepsilon^{\alpha\beta\gamma}\mathrm{Tr}(\dot{A}_\alpha F_{\beta\gamma}). \tag{12}$$

The Gauss' law constraint $\delta S/\delta A_0=0$ is given by

$$F_\Sigma=0, \tag{13}$$

where F_Σ is the curvature of the connection on Σ. The Chern–Simons action is first order in time derivatives and thus the physical phase space of the theory is the subspace of the space of connections on the $SL(2,\mathbf{R})$-bundle $V\to\Sigma$ on which the constraint (13) holds. The phase space is, therefore, the space of flat connections on $V\to\Sigma$ modulo gauge transformations. We will denote this space by $\mathscr{M}$. The phase space $\mathscr{M}$ has a natural symplectic structure which determines the Poisson bracket on $\mathscr{M}$. As the action (6) is independent of any metric on M, Chern–Simons theory is generally covariant and the Hamiltonian of the theory vanishes identically; therefore the dynamics of Chern–Simons theory is determined completely by the Poisson bracket (i.e., symplectic structure) on the phase space $\mathscr{M}$.

(a) *The Symplectic Structure on* $\mathscr{M}$. A symplectic structure on $\mathscr{M}$ is a closed non-degenerate 2-form on $\mathscr{M}$. If we let $\mathscr{F}$ denote the space of flat corrections on an $SL(2,\mathbf{R})$-bundle $V\to\Sigma$ and let $\mathscr{G}$ denote the group of gauge transformations on V, then $\mathscr{M}=\mathscr{F}/\mathscr{G}$. It is well known that the space of flat connections on a bundle over Σ may be described in terms of the representations of $\pi_1(\Sigma)$. Let P_ϕ be a principal $SL(2,\mathbf{R})$-bundle over Σ with a flat connection. The flat bundle P_ϕ is equivalent to a representation $\phi:\pi_1(\Sigma)\to SL(2,\mathbf{R})$. Hence the space $\mathscr{F}$ may be identified with $\mathrm{Hom}(\pi_1(\Sigma),SL(2,\mathbf{R}))$ and $\mathscr{F}/\mathscr{G}=\mathrm{Hom}(\pi_1(\Sigma),SL(2,\mathbf{R}))/SL(2,\mathbf{R})$, where the quotient is with respect to the natural action of $SL(2,\mathbf{R})$ on $\mathrm{Hom}(\pi_1(\Sigma),SL(2,\mathbf{R}))$.

Now let us define a natural symplectic structure on $\mathscr{M}=\mathrm{Hom}(\pi_1(\Sigma),SL(2,\mathbf{R}))/SL(2,\mathbf{R})$ (see [17]). To define a 2-form on $\mathscr{M}$ we must first determine the tangent space $T_\phi\mathscr{M}$ to $\mathscr{M}$ at $\phi\in\mathrm{Hom}(\pi_1(\Sigma),SL(2,\mathbf{R}))$. Let $P_\phi\to\Sigma$ be the principal $SL(2,\mathbf{R})$-bundle associated to ϕ as above; P_ϕ has a canonical flat connection D_ϕ. Let $\mathrm{ad}(P_\phi)$ be the $SL(2,\mathbf{R})$-bundle associated to P_ϕ by the adjoint representation $\rho:SL(2,\mathbf{R})\to\mathrm{Aut}(sl(2,\mathbf{R}))$, i.e., $\mathrm{ad}(P_\phi)=P_\phi\times_\rho sl(2,\mathbf{R})$. Then $\mathrm{ad}(P_\phi)$ has a canonical flat connection associated to D_ϕ, which we denote by $\mathcal{D}_\phi$. Now let us consider the de Rham cohomology of Σ with coefficients in the flat vector bundle $\mathrm{ad}(P_\phi) \to \Sigma$. The cohomology of the complex $\Omega^*(\Sigma; \mathrm{ad}(P_\phi))$ of $\mathrm{ad}(P_\phi)$-valued forms on Σ is defined as follows. Locally a p-form $\omega \in \Omega^p(\Sigma; \mathrm{ad}((P_\phi))$ is $\omega = \alpha \otimes \theta$, where $\alpha \in \Omega^p(\rho)$ and θ is a section of $\mathrm{ad}(P_\phi)$. The differential is defined to be

$$d_\phi(\alpha \otimes \theta) = d\alpha \otimes \theta + (-1)^p \alpha \otimes \nabla_\phi \theta, \tag{14}$$

where ∇_ϕ is the covariant derivative associated to the canonical flat connection $\mathcal{D}_\phi$ on $\mathrm{ad}(P_\phi)$. The flatness of ∇_ϕ implies that $d_\phi^2 = 0$ and thus the cohomology of the complex $(\Omega^*(\Sigma; \mathrm{ad}(P_\phi)), d_\phi)$ may be defined. We will denote this cohomology ring by $H^*(\Sigma; \mathrm{ad}(P_\phi))$.

It may be shown (see [17]) that the tangent space $T_\phi \mathcal{M}$ to $\mathcal{M}$ at $\phi \in \mathrm{Hom}(\pi_1(\Sigma), SL(2, \mathbf{R}))/SL(2, \mathbf{R})$ is given by

$$T_\phi \mathcal{M} = H^1(\Sigma; \mathrm{ad}(P_\phi)). \tag{15}$$

Given two tangent vectors $\alpha, \beta \in T_\phi \mathcal{M}$, their product $\alpha \wedge \beta \in \Omega^2(\Sigma; \mathrm{ad}(P_\phi) \otimes \mathrm{ad}(P_\phi))$. The bilinear form $\mathrm{Tr}: sl(2, \mathbf{R}) \times sl(2, \mathbf{R}) \to \mathbf{R}$ defines a bundle map, also denoted by Tr, from $\mathrm{ad}(P_\phi) \otimes \mathrm{ad}(P_\phi)$ to the trivial $\mathbf{R}$-bundle over Σ. Hence

$$\omega_\phi(\alpha, \beta) = \int_\Sigma \mathrm{Tr}(\alpha \wedge \beta) \tag{16}$$

defines a 2-form on $T_\phi \mathcal{M}$. The skew form ω_ϕ may be viewed as coming from the cup product on cohomology

$$\omega_\phi : H^1(\Sigma; \mathrm{ad}(P_\phi)) \times H^1(\Sigma; \mathrm{ad}(P_\phi)) \to H^2(\Sigma; \mathbf{R}) \cong \mathbf{R}. \tag{17}$$

The two-form ω defined on $\mathcal{M}$ may be shown to be closed [17] and thus defines a symplectic structure on $\mathcal{M}$.

The space $\mathcal{M} = \mathrm{Hom}(\pi_1(\Sigma), SL(2, \mathbf{R}))/SL(2, \mathbf{R})$ consists of different connected components corresponding to the topological type of the $SL(2, \mathbf{R})$-bundle over Σ. Recall from the above discussion that a principal $SL(2, \mathbf{R})$-bundle $P \to \Sigma$ is classified by the Euler class $e(E_P) \in H^2(\Sigma; \mathbf{Z}) \cong \mathbf{Z}$, where $E_P = P \times_{SL(2,\mathbf{R})} \mathbf{R}^2$. For any integer r, there exists an $SL(2, \mathbf{R})$-bundle $P \to \Sigma$ such that $e(E_P) = r$. The situation for a flat $SL(2, \mathbf{R})$-bundle over Σ is, however, more subtle. Let $\phi \in \mathrm{Hom}(\pi_1(\Sigma), SL(2, \mathbf{R}))/SL(2, \mathbf{R})$ and let P_ϕ be the associated flat principal $SL(2, \mathbf{R})$-bundle introduced above. Let $E_\phi = P_\phi \times_{SL(2,\mathbf{R})} \mathbf{R}^2$ be the flat 2-plane bundle over Σ associated to P_ϕ by the natural action of $SL(2, \mathbf{R})$ on $\mathbf{R}^2$. The 2-plane bundle $E_\phi \to \Sigma$ has an Euler class $e(E_\phi) \in H^2(\Sigma; \mathbf{Z}) \cong \mathbf{Z}$, which will be denoted by $e(\phi)$. It is a basic theorem [18] that $e(\phi)$ satisfies the bound $|e(\phi)| \leqq |\chi(\Sigma)| = 2g - 2$ (where g is the genus of Σ). Thus we have a map $e: \mathrm{Hom}(\pi_1(\Sigma), SL(2, \mathbf{R}))/SL(2, \mathbf{R}) \to \{2 - 2g, 3 - 2g, \ldots, 2g - 3, 2g - 2\}$ and the inverse images $e^{-1}(r)$, for $r \in \{2 - 2g, 3 - 2g, \ldots, 2g - 3, 2g - 2\}$, are precisely the connected components of $\mathcal{M} = \mathrm{Hom}(\pi_1(\Sigma), SL(2, \mathbf{R}))/SL(2, \mathbf{R})$. We shall denote the connected component $e^{-1}(r)$ by $\mathcal{M}_r$. The physical phase space of $SL(2, \mathbf{R})$ Chern–Simons theory, for a gauge bundle $V \to \Sigma$ of Euler class $e(V) \in H^2(\Sigma; \mathbf{Z}) \cong \mathbf{Z}$ such that $|e(V)| \leqq 2g - 2$, is $\mathcal{M}_r$ where $r = e(V)$. If the gauge bundle V does not satisfy the bound $|e(V)| \leqq 2g - 2$ then V is incompatible with the constraint (13) $F = 0$.

The physical phase space $\mathcal{M}_r$ has a natural symplectic structure defined by the symplectic form ω. However, to quantize the phase space $\mathcal{M}_r$, using the standard principles of geometric quantization (see [19] for example) it is necessary that $\mathcal{M}_r$ should admit a natural polarization. In practice, there are two natural polarizations that a symplectic manifold can have. The first is a real polarization, which is given, for example, by the structure of a cotangent bundle and the second is a complex polarization, which is given by a Kähler structure. If a phase space W is a cotangent bundle T^*Z then the quantum Hilbert space $\mathcal{H}$ may be taken to be the space $L^2(Z)$ of square integrable functions on the configuration space Z. If W has a Kähler structure, i.e., a complex structure compatible with the symplectic structure on W, then the quantum Hilbert space may be taken to be the space of holomorphic functions on W, or more generally, the space of holomorphic sections of a holomorphic line bundle over W. We shall now see that the phase space $\mathcal{M}_r$ of $SL(2,\mathbf{R})$ Chern–Simons theory has a Kähler structure and can, therefore, be quantized using the complex polarization.

(b) *The Kähler Structure on* $\mathcal{M}$. One obtains a natural Kähler structure on $\mathcal{M}_r$ via a connection with 2-dimensional self-duality equations that has been studied in [20]. Here we shall simply outline the main ideas; further details may be found in [20].

Consider a principal G-bundle P over a compact Riemann surface Σ (G a compact Lie group). Let A be a connection on P (with curvature F_A) and let Φ be a complex Higgs field $\Phi\in\Omega^{1,0}(\Sigma;\mathrm{ad}(P)\otimes\mathbf{C})$. The pair (A,Φ) are solutions of the self-duality equations if

$$F_A+[\Phi,\Phi^*]=0,\quad d''_A\Phi=0. \tag{18}$$

From now on we restrict our attention to the case $G=SO(3)$. A principal $SO(3)$-bundle $P\to\Sigma$ is classified by the second Stiefel–Whitney class $w_2(P)\in H^2(\Sigma;\mathbf{Z}_2)\cong\mathbf{Z}_2$. If $w_2(P)=0$, then P is covered by a principal $SU(2)$-bundle, associated to which is a rank-two vector bundle V, with $c_1(V)=0$. If $w_2(P)\neq 0$, then there is a principal $U(2)$-bundle $\hat{P}$ to which P is associated via the homomorphism $U(2)/\text{centre}\cong SO(3)$. Associated to $\hat{P}$ is a rank-two vector bundle V, with $c_1(V)$ odd. In both cases the Higgs field Φ may be viewed as a section of $\mathrm{End}(V)\otimes K$, where K is the canonical bundle of Σ. Let $L\subset V$ be a line subbundle. L is said to be Φ-invariant if $\Phi(L)\subset L\otimes K$. There is an important connection between solutions of the self-duality equations (18) and a certain notion of stability of vector bundles. Let V be a holomorphic vector bundle of rank-two over Σ and let Φ be a holomorphic section of $\mathrm{End}(V)\otimes K$, the pair (V,Φ) is defined to be *stable* if, for every Φ-invariant line subbundle $L\subset V$, $\deg(L)<\frac{1}{2}\deg(\Lambda^2V)$. When $\Phi=0$, this definition reduces to the standard definition of stability for a rank two vector bundle. It may be shown [20] that if (A,Φ) is a solution of (18), then the associated pair (V,Φ) is stable. Conversely, to each stable pair (V,Φ) there corresponds a solution of self-duality equations (18) which is unique up to gauge equivalence. If we let $\mathcal{N}$ denote the moduli space of solutions of (18) (or equivalently the moduli space of stable pairs (V,Φ)), then [20] $\mathcal{N}$ is a smooth manifold of dimension $12(g-1)$, where g is the genus of Σ.

The space $\mathcal{N}$ of solutions of (18) has a subtle geometrical structure: $\mathcal{N}$ is a hyperkähler manifold [20]. To see this, consider the infinite-dimensional space $N=\mathcal{A}\times\Omega^{1,0}(\Sigma;\mathrm{ad}(P)\otimes\mathbf{C})$, where $\mathcal{A}$ is the space of connections on P. The tangent space to $\mathcal{A}$ at A is $T_A\mathcal{A}=\Omega^1(\Sigma;\mathrm{ad}(P))$, which may be identified with the complex space $\Omega^{0,1}(\Sigma; \mathrm{ad}(P) \otimes \mathbf{C})$. The space N has a natural Kähler metric

$$g_1(\alpha, \Phi) = 2i \int_\Sigma \mathrm{Tr}(\alpha^* \alpha + \Phi \Phi^*). \tag{19}$$

The gauge group $\mathscr{G}$ acts on N preserving the Kähler form ω_1 of the metric (19). The moment map associated to this symplectic action is [20]

$$\mu(A, \Phi) = F_A + [\Phi, \Phi^*]. \tag{20}$$

Thus, $\mu(A, \Phi) = 0$ corresponds to the first equation of (18). The second equation of (18), $d''_A \Phi = 0$, may be obtained as follows. The tangent space to N at (A, Φ) is $\Omega^{0,1}(\Sigma; \mathrm{ad}(P) \otimes \mathbf{C}) \oplus \Omega^{1,0}(\Sigma; \mathrm{ad}(P) \otimes \mathbf{C})$. Define a complex symplectic form ω on N by

$$\omega((\alpha_1, \Phi_1), (\alpha_2, \Phi_2)) = \int_\Sigma \mathrm{Tr}(\Phi_2 \alpha_1 - \Phi_1 \alpha_2). \tag{21}$$

The real and imaginary parts of ω give two Kähler forms on N, denoted by ω_2 and ω_3, and $d''_A \Phi = 0$ is the zero set of the moment maps of ω_2 and ω_3 with respect to the action of $\mathscr{G}$. The forms ω_1, ω_2 and ω_3 are the Kähler forms for a flat hyperkähler metric on N. If μ_1, μ_2 and μ_3 denote the moment maps corresponding to ω_1, ω_2 and ω_3, with respect to $\mathscr{G}$, then the self-duality equations (18) are given by

$$\mu_i(A, \Phi) = 0, \tag{22}$$

where $i = 1, 2, 3$. The moduli space $\mathscr{N}$ of solutions to the self-duality equations (18), modulo gauge euivalence is

$$\mathscr{N} = \bigcap_{i=1}^{3} \mu_i^{-1}(0)/\mathscr{G}. \tag{23}$$

It is a standard theorem that the quotient of a hyperkähler manifold, defined in this way, has a natural hyperkähler structure [21]. Hence, the moduli space $\mathscr{N}$ is naturally hyperkähler. The hyperkähler metric g on $\mathscr{N}$ is Kähler with respect to three complex structures I, J and K which satisfy the algebraic relations of the quaternions. The Kähler forms corresponding to each complex structure are $\omega_1(X, Y) = g(IX, Y), \omega_2(X, Y) = g(JX, Y)$ and $\omega_3(X, Y) = g(KX, Y)$.

The connection between the moduli space $\mathscr{N}$ and the space of flat $SL(2, \mathbf{R})$ connection $\mathscr{M}_r$ comes from considering $\mathscr{N}$ with the different complex structures. I, J and K. Recall that with respect to I the tangent space of N is $\Omega^{0,1}(\Sigma; \mathrm{ad}(P) \otimes \mathbf{C}) \oplus \Omega^{1,0}(\Sigma; \mathrm{ad}(P) \otimes \mathbf{C})$. The complex structure J may be defined by $J(A, B) = (iB^*, -iA^*)$ and the complex structure K by $K(A, B) = (-B^*, A^*)$. We can define an isomorphism $\beta: N \to \mathscr{A} \times \bar{\mathscr{A}}$ by $\beta(A, \Phi) = (d''_A + \Phi^*, d'_A + \Phi)$, where $d_A = d'_A + d''_A$ is the covariant derivative of the unitary connection A. The map β identifies N with the complex structure J with the space $\mathscr{A} \times \bar{\mathscr{A}}$ with its natural complex structure. An element of $\mathscr{A} \times \bar{\mathscr{A}}$ is a pair of operators (d''_1, d'_2) or, equivalently, $d = d'_1 + d''_2$ is an $SL(2, \mathbf{C})$ connection. The self-duality equations (18) imply under the isomorphism β that the $SL(2, \mathbf{C})$ connection $d'_A + d''_A + \Phi + \Phi^*$ is *flat*. Conversely, it is proved in [22] that every flat $SL(2, \mathbf{C})$ connection arises from a solution of (18). Thus the moduli space $\mathscr{N}$ with the complex structure J may be identified with $\mathrm{Hom}(\pi_1(\Sigma), SL(2, \mathbf{C}))/SL(2, \mathbf{C})$.

There is an involution θ on $\mathscr{N}$ induced by $(A, \Phi) \to (A, -\Phi)$ which is holomorphic with respect to the complex structure I, but antiholomorphic with respect to the complex structure J. Thus θ defines a real structure on $(\mathcal{N}, J)$. The fixed points of θ consist of complex submanifolds $\mathcal{N}_0, \mathcal{N}_{2d-1}$ $(1 \leqq d \leqq g-1)$ each of dimension $3g-3$. Consider now the map $\beta: \mathcal{N} \to \mathcal{R}$ which takes (A, Φ) to the equivalence class of the flat connection $d'_A + d''_A + \Phi + \Phi^*$. As θ is an anti-holomorphic involution on $(\mathcal{N}, J)$ and β is holomorphic, the connection $\beta(A, \Phi)$ corresponding to a pair (A, Φ) which is a fixed point of θ must satisfy a reality condition. Now consider the fixed points of θ. If (A, Φ) is fixed by θ then $\Phi = -\Phi = 0$ and $d'_A + d''_A$ is a flat $SO(3)$-connection over Σ. Thus this component $\mathcal{N}_0$ of the fixed point set of θ may be identified with the moduli space of stable rank 2 bundles over Σ. If, however, the gauge equivalence class only of (A, Φ) (which corresponds to a stable pair (V, Φ)) is fixed by θ then A is reducible to a $U(1)$ connection on the decomposition

$$V = L \oplus L^* \Lambda^2 V \tag{24}$$

and Φ has the form

$$\Phi = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}. \tag{25}$$

An antilinear homomorphism

$$T: V \to V \oplus \Lambda^2 V^* \tag{26}$$

may be defined by

$$T(u_1, u_2) = (\bar{u}_2, \bar{u}_1). \tag{27}$$

It follows that $\Phi T = T\Phi^*$. Hence the connection $d'_A + d''_A + \Phi + \Phi^*$ commutes with T, and therefore defines an $\mathbf{R}^2$-bundle over Σ associated to the flat $SL(2,\mathbf{R})$ connection $d'_A + d''_A + \Phi + \Phi^*$. Under β the fixed point sets of θ correspond to the spaces $\mathcal{M}_r$ of equivalence classes of flat $SL(2,\mathbf{R})$ connections whose associated $\mathbf{R}^2$-bundle has Euler class r. It follows, therefore, that the spaces $\mathcal{M}_r$ inherit a Kähler structure from the hyperkähler structure on $\mathcal{N}$. The symplectic structure corresponding to this Kähler structure on $\mathcal{M}_r$ is that defined by (16).

We can now quantize the phase space $\mathcal{M}_r$ using the Kähler polarization. In the above discussion of the Kähler structure on $\mathcal{M}_r$, we were considering a fixed Riemann surface Σ. Thus, for a fixed smooth surface Σ, a choice of complex structure σ on Σ induces a Kähler structure on the space of flat $SL((2,\mathbf{R})$ connections $\mathcal{M}_r = \mathrm{Hom}(\pi_1(\Sigma), SL(2,\mathbf{R}))_r / SL(2,\mathbf{R})$, where $\mathrm{Hom}(\pi_1(\Sigma), SL(2,\mathbf{R}))_r$ denotes the space of representations $\phi: \pi_1(\Sigma) \to SL(2,\mathbf{R})$ with $e(\phi) = r$. Note that while the Kähler structure on $\mathcal{M}_r$ depends on σ, the symplectic structure on $\mathcal{M}_r$ determined by the Kähler form (i.e., ω of (16)) depends only on the smooth surface Σ. Let Σ_σ denote the Riemann surface determined by the complex structure σ on Σ. There exists a Hermitian holomorphic line bundle $\mathcal{L} \to \mathcal{M}_r$ with a connection whose curvature is equal to the symplectic form ω on $\mathcal{M}_r$ defined by (16). The quantum Hilbert space $\mathcal{H}$ is then given by the space $\Gamma(\mathcal{L})$ of holomorphic sections of $\mathcal{L}$. Although we will refer to $\mathcal{H}$ as the quantum "Hilbert" space, as is conventional, all we actually require is that $\mathcal{H}$ is a vector space canonically associated to Σ, which is exchanged with its dual space when the orientation of Σ is reversed. Since $\mathcal{M}_r$ is non-compact, the Hilbert space $\mathcal{H} = \Gamma(\mathcal{L})$ is infinite dimensional. Therefore, for any choice of complex structure σ on Σ we obtain an infinite dimensional Hilbert space $\mathcal{H}_\sigma$. Let $\mathcal{T}_g$ denote the Teichmüller space of the smooth surface Σ of genus g. As σ varies over $\mathcal{T}_g$ the spaces $\mathcal{H}_\sigma$ vary holomorphically to form an infinite rank holomorphic vector bundle $\mathcal{V} \to \mathcal{T}_g$. In direct analogy with the case of a compact group one might expect that this bundle should be canonically projectively flat (see [3–6] and [23–24]). However, in view of the supposed connection between $SL(2,\mathbf{R})$ Chern–Simons theory and two-dimensional quantum gravity it may be that the geometrical structure of $\mathscr{V}$ is more subtle. Understanding the geometry of the bundle of Hilbert spaces $\mathscr{V}$ defined by the quantization of $SL(2,\mathbf{R})$ Chern–Simons theory is an important open problem, the solution of which may well involve further ideas from [20].

4. Quantization of Teichmüller Space

We saw in the last section that the space $\mathscr{M}$ of flat $SL(2,\mathbf{R})$ connections on a surface Σ (of genus g) has connected components $\mathscr{M}_r = e^{-1}(r)$, where $e:\mathrm{Hom}(\pi_1(\Sigma), SL(2,\mathbf{R}))/SL(2,\mathbf{R}) \to \mathbf{Z}$ is the map defined by the Euler class of the 2-plane bundle associated to a flat $SL(2,\mathbf{R})$ connection, and $r \in \{2-2g, 3-2g, \ldots, 2g-3, 2g-2\}$. These components have a natural Kähler structure and can be quantized geometrically. In this section we will consider the maximal component $\mathscr{M}_{g-2}$, which is related to the geometry of the surface Σ.

The relation between $\mathscr{M}_{2g-2}$ and the geometry of Σ is as follows. A flat $SL(2,\mathbf{R})$ connection on the smooth surface Σ is given by a representation $\phi:\pi_1(\Sigma)\to SL(2,\mathbf{R})$. The uniformation theorem asserts that every Riemann surface S (S should be regarded as the surface Σ together with a choice of complex structure) may be represented as a quotient of the upper half-space $\mathbf{H}^2$ by a discrete group of isometries Γ. Such a uniformation determines an isomorphism of $\pi_1(S)$ with a discrete subgroup Γ of the group $SL(2,\mathbf{R})$ of isometries of $\mathbf{H}^2$. If $\phi:\pi_1(\Sigma)\to SL(2,\mathbf{R})$ is an isomorphism onto a discrete subgroup of $SL(2,\mathbf{R})$ then $S \cong \mathbf{H}^2/\phi(\pi_1(\Sigma))$. Thus the Teichmüller space $\mathscr{T}_g$ of the surface Σ of genus g is the connected component of $\mathscr{M} = \mathrm{Hom}(\pi_1(\Sigma), SL(2,\mathbf{R}))/SL(2,\mathbf{R}))$, consisting of equivalence classes $[\phi]\in\mathscr{M}$, where ϕ is an isomorphism of $\pi_1(\Sigma)$ onto a discrete subgroup of $SL(2,\mathbf{R})$. It is a theorem [25] that $\phi\in\mathrm{Hom}(\pi_1(\Sigma), SL(2,\mathbf{R}))/SL(2,\mathbf{R})$ is an isomorphism onto a discrete subgroup $SL(2,\mathbf{R})$ if and only if $e(\phi) = 2g-2$. Hence the Teichmüller space $\mathscr{T}_g$ is isomorphic to the maximal component $\mathscr{M}_{2g-2}$ of $\mathscr{M}$. Quantizing the component $\mathscr{M}_{2g-2}$ is thus equivalent to quantizing Teichmüller space.

The Teichmüller space $\mathscr{T}_g$ has a well-known Kähler structure given by the Weil–Petersson Kähler form. The symplectic structure defined by the Weil–Petersson form on $\mathscr{T}_g$ is precisely the symplectic form ω on $\mathscr{M}$ (defined by (16)) restricted to the component $\mathscr{T}_g \cong \mathscr{M}_{2g-2}$ [17]. Teichmüller space may be quantized geometrically using the Weil–Petersson Kähler structure in a straightforward way. Over $\mathscr{T}_g$ there is a natural holomorphic line bundle given by the determinant bundle of the $\bar\partial$ operator. A point $y\in\mathscr{T}_g$ corresponds to a complex structure on Σ. On the Riemann surface (Σ, y), there is a $\bar\partial$ operator. The determinant line $\mathscr{L}_y$ over $y\in\mathscr{T}_g$ is defined by

$$\mathscr{L}_y = \overset{\max}{\wedge} (\mathrm{Ker}\,\bar\partial)^{-1} \otimes \overset{\max}{\wedge} (\mathrm{Ker}\,\bar\partial^*). \tag{28}$$

As y varies over $\mathscr{T}_g$ the lines $\mathscr{L}_y$ vary holomorphically to form a holomorphic line bundle $\mathscr{L}\to\mathscr{T}_g$; the determinant line bundle $\mathrm{Det}\,\bar\partial$ [26]. The determinant bundle $\operatorname{Det}\bar{\partial}\to\mathscr{T}_g$ has a natural connection, the curvature of which is equal to the Weil–Petersson Kähler form on $\mathscr{T}_g$ [26]. The quantum Hilbert space $\hat{\mathscr{H}}$ is then given by the space of holomorphic sections of $\mathscr{L}=\operatorname{Det}\bar{\partial}:\hat{\mathscr{H}}=\Gamma(\mathscr{L})$. A discussion of the quantization of Teichmüller space related to this approach is given in [27] and also [6].

In Sect. 3, we studied the quantization of the spaces of $\mathscr{M}_r$ of flat $SL(2,\mathbf{R})$ connections on Σ. In particular, the space $\mathscr{M}_{2g-2}$ (which as we have seen may be identified with $\mathscr{T}_g$) has a Kähler structure induced from the hyperkähler structure on the moduli space $\mathscr{N}$. The Kähler structure on $\mathscr{M}_{2g-2}$ obtained in this way depends upon a choice of complex structure σ on Σ. The symplectic 2-form on $\mathscr{M}_{2g-2}$ defined by the Kähler structure is ω (defined by (16)), which is independent of the complex structure σ and is equal to the Weil–Petersson symplectic form on $\mathscr{T}_g$. Hence the spaces $\mathscr{T}_g$ and $\mathscr{M}_{2g-2}$ coincide symplectically, but are distinct as Kähler manifolds; the Kähler structure on $\mathscr{M}_{2g-2}$ depends upon a choice of complex structure on Σ whereas the Weil–Petersson Kähler structure on $\mathscr{T}_g$ requires no such choice. For a choice of complex structure σ on Σ (i.e., for a choice of a point $\sigma\in\mathscr{T}_g$) we obtain a Kähler structure on $\mathscr{M}_{2g-2}$. Let $\mathscr{H}_\sigma$ denote the Hilbert space obtained by quantizing $\mathscr{M}_{2g-2}$ with the Kähler polarization, as in Sect. 3. It would seem to be worthwhile to study further the relation between the two Hilbert spaces $\mathscr{H}_\sigma$ and $\hat{\mathscr{H}}$ obtained by the quantizations of Teichmüller space considered here.

It is interesting to note that there is a different way of quantizing Teichmüller space which has been considered in [28]. In this approach one notes that in terms of the length and twist co-ordinates τ_i $(0\leqq\tau_i<\infty)$ and θ_j $(-\infty<\theta_j<\infty)$ of Teichmüller space the symplectic form (defined by (16)) is given by [29]

$$\omega=\sum_j d\theta_j\wedge d\tau_j.$$

The canonical commutation relations are

$$[\theta_i,\tau_j]=-i\delta_{ij},\quad [\theta_i,\theta_j]=[\tau_i,\tau_i]=0.$$

If we let Z denote the product of the real half-lines $\tau_i\in[0,\infty)$, then the Hilbert space obtained by quantizing Teichmüller space with this polarization is given by $L^2(Z)$.

5. Chern–Simons Theory with $SL(n,\mathbf{R})$ Gauge Group

So far in this paper we have discussed Chern–Simons theory for the gauge group $SL(2,\mathbf{R})$. Let us now consider the generalization to the gauge group $SL(n,\mathbf{R})$, for $n\geqq 3$. Let M be a compact 3-manifold and let $V\to M$ be an $SL(n,\mathbf{R})$-bundle. Recall that $SL(n,\mathbf{R})$ is diffeomorphic to $\mathbf{R}^{(1/2)n(n+1)-1}\times SO(n)$, so there is a homotopy equivalence of classifying spaces $BSL(n,\mathbf{R})\sim BSO(n)$. The first three homotopy groups of $BSL(n,\mathbf{R})$ $(n\geqq 3)$ are given by $\pi_1(BSL(n,\mathbf{R}))=0,\pi_2(BSL(n,\mathbf{R}))=\mathbf{Z}_2$ and $\pi_3(BSL(n,\mathbf{R}))=0$. Hence, up to dimension three $BSL(n,\mathbf{R})$ is approximated homotopically by a $K(\mathbf{Z}_2,2)$ space. The set of isomorphism classes of $SL(n,\mathbf{R})$-bundles over M is given by $[M;BSL(n,\mathbf{R})]\cong[M;K(\mathbf{Z}_2,2)]\cong H^2(M;\mathbf{Z}_2)$. In general, therefore, there are non-trivial $SL(n,\mathbf{R})$-bundles over a 3-manifold M.

We wish to define the Chern–Simons action for an arbitrary $SL(n,\mathbf{R})$-bundle over M by (6). To do this we need to know that for any $SL(2,\mathbf{R})$-bundle $V \to M$, there exists a coboundary Y of M over which V extends, i.e., that $\Omega_3^{SO}(BSL(n,\mathbf{R})) = 0$. It follows from (7) that

$$\begin{aligned}\Omega_3^{SO}(BSL(n,\mathbf{R})) &= \Omega_3^{SO} \otimes H_3(BSL(n,\mathbf{R})) \\ &= 0\end{aligned} \tag{29}$$

as $H_3(BSL(n,\mathbf{R})) \cong H_3((BSO(n)) = 0$. The $SL(n,\mathbf{R})$ Chern–Simons action may now be defined (as in (6)) to be

$$S = \frac{k}{2\pi} \int_Y \operatorname{Tr}(F \wedge F). \tag{30}$$

Requiring that e^{iS} be independent of the coboundary Y is equivalent to demanding that for two coboundaries Y_1 and Y_2 of M the corresponding actions S_1 and S_2 should satisfy $S_1 = S_2 + 2\pi N$, where N is an integer. $N = kn$, where N is given by (cf., (9))

$$n = \frac{1}{4\pi^2} \int_{\bar{Y}} \operatorname{Tr}(F \wedge F). \tag{31}$$

$\bar{Y}$ is the closed 4-manifold $Y_1 + Y_2$. The expression (31) represents a 4-dimensional characteristic class of an $SL(n,\mathbf{R})$-bundle $\hat{V} \to \bar{Y}$. If we let $\lambda(\hat{V}) = \frac{1}{4\pi^2}\operatorname{Tr}(F \wedge F)$ denote this class then $\lambda(\hat{V}) \in H^4(\bar{Y};\mathbf{R})$. Viewing $\lambda(\hat{V})$ as an integral class in $H^4(\bar{Y};\mathbf{Z})$ it pulls back from a universal class in $H^4(BSL(n,\mathbf{R});\mathbf{Z})$. We have that $H^4(BSL(n,\mathbf{R});\mathbf{Z}) \cong H^4(BSO(n);\mathbf{Z}) \cong \mathbf{Z}$ and $\lambda(\hat{V}) \in H^4(\bar{Y};\mathbf{Z}) \cong \mathbf{Z}$. Thus for any $SL(n,\mathbf{R})$-bundle $\hat{V} \to \bar{Y}, n$ defied by (31) is an integer. Geometrically, the class $\lambda(\hat{V})$ represents the obstruction to trivially extending $\hat{V}$ from the 3-skeleton of $\bar{Y}$ to the 4-skeleton. The obstructions to a trivial extension come from $\pi_3(SL(n,\mathbf{R})) \cong \pi_3(SO(n)) \cong \mathbf{Z}$. This contrasts with the case of $SL(2,\mathbf{R})$ discussed in Sect. 2. For $SL(2,\mathbf{R})$, the class $\frac{1}{4\pi^2}\operatorname{Tr}(F \wedge F)$ is trivial as $H^4(BSL(2,\mathbf{R});\mathbf{Z}) \cong H^4(BS^1;\mathbf{Z}) = 0$; or geometrically because $\pi_3(SL(2,\mathbf{R})) \cong \pi_3(S^1) = 0$. The requirement that $N = kn$ be an integer then imposes a quantization condition on $SL(n,\mathbf{R})$ Chern–Simons theory: the constant k in (30) must be an integer. This is, of course, the situation for Chern–Simons theory for a compact non-Abelian gauge group.

Now let us move on to discuss the quantization of $SL(n,\mathbf{R})$ Chern–Simons theory. Canonically quantizing the theory on $M = \mathbf{R} \times \Sigma$, as in Sect. 3, results in the physical phase space $\hat{\mathcal{M}}$ being the space of flat $SL(n,\mathbf{R})$ connections over Σ. That is $\hat{\mathcal{M}} = \operatorname{Hom}(\pi_1(\Sigma), SL(n,\mathbf{R}))/SL(n,\mathbf{R})$. Consider a representation $\phi : \pi_1(\Sigma) \to SL(n,\mathbf{R})$ and let P_ϕ be the associated flat principal $SL(n,\mathbf{R})$-bundle over Σ. Let $E_\phi = P_\phi \times_{SL(n,\mathbf{R})} \mathbf{R}^n$ be the real n-plane bundle associated to P_ϕ by the natural action of $SL(n,\mathbf{R})$ on $\mathbf{R}^n$. The bundle $E_\phi \to \Sigma$ is canonically flat and has a second Stiefel–Whitney class $w_2(E_\phi) \in H^2(\Sigma;\mathbf{Z}_2) \cong \mathbf{Z}_2$. The moduli space $\hat{\mathcal{M}}$ has two connected components $\hat{\mathcal{M}}_0$ and $\hat{\mathcal{M}}_1$, consisting of representations $\phi : \pi_1(\Sigma) \to SL(n,\mathbf{R})$ for which $w_2(E_\phi)$ is zero or non-zero, respectively.

The procedure used to define a symplectic structure on $\mathcal{M}$ in Sect. 3 generalizes to define a symplectic structure on $\hat{\mathcal{M}}$. The sumplectic pairing is given by the cup product on cohomology (cf., (17))

$$H^1(\Sigma;\mathrm{ad}(P_\phi)) \times H^1(\Sigma;\mathrm{ad}(P_\phi)) \to H^2(\Sigma;\mathbf{R}) \cong \mathbf{R}, \tag{32}$$

where $\mathrm{ad}(P_\phi)$ is the bundle associated to P_ϕ by the adjoint action of $SL(n,\mathbf{R})$ on $sl(n,\mathbf{R})$. The procedure used to define a Kähler structure on $\mathscr{M}_r$ also generalizes to the case of $SL(n,\mathbf{R})$. The self-duality equations considered in Sect. 3 may be defined for an arbitrary compact Lie group G. The case $G=SU(2)$ or $SO(3)$ is related to the moduli space of flat $SL(2,\mathbf{R})$ connections over Σ. For $G=SU(n)$ one obtains a relation with the moduli space of flat $SL(n,\mathbf{R})$ connections over Σ. The essential features of the discussion in Sect. 3 remain true for $G=SU(n)$. The moduli space $\hat{\mathscr{N}}$ of solutions of the self-duality equations has a natural hyperkähler structure and this induces a Kähler structure on $\hat{\mathscr{M}}_i$ $(i=0,1)$, which is the fixed-point set of an involution on $\hat{\mathscr{N}}$. The Kähler structure on $\hat{\mathscr{M}}_i$ depends on a choice of complex structure σ on Σ. For a given complex structure $\sigma\in\mathscr{T}_g$, quantizing $\hat{\mathscr{M}}_i$ with the Kähler polarization gives a Hilbert space $\hat{\mathscr{H}}_\sigma$. As σ varies over $\mathscr{T}_g$ the spaces $\hat{\mathscr{H}}_\sigma$ fit together to form a canonically projectively flat vector bundle $\hat{\mathscr{V}} \to \mathscr{T}_g$, just as for the case of $SL(2,\mathbf{R})$. In general, then the quantization of $SL(n,\mathbf{R})$ Chern–Simons theory proceeds in complete analogy with the $SL(2,\mathbf{R})$ case.

Finally, we will give a very brief and somewhat speculative, discussion of the relation between $SL(2,\mathbf{R})$ Chern–Simons theory and the representations of the loop group of $SL(2,\mathbf{R})$. The connection between $SL(2,\mathbf{R})$ Chern–Simons theory and representations of the loop group $LSL(2,\mathbf{R})$ of $SL(2,\mathbf{R})$ is formally the same as for a compact group (see [3]), and also holds for a complex group (see [8]). Consider quantizing $SL(2,\mathbf{R})$ Chern–Simons theory on $M=\mathbf{R}\times D$, where D is a two-dimensional disc. Let $\mathscr{G}_*$ denote the group of gauge transformations that are the identity on the boundary of $\mathbf{R}\times D$, i.e., on $\partial(\mathbf{R}\times D)=\mathbf{R}\times S^1$. The classical phase space of flat $SL(2,\mathbf{R})$ connections on D modulo $\mathscr{G}_*$ is $LSL(2,\mathbf{R})/SL(2,\mathbf{R})$. We then quantize the phase space $LSL(2,\mathbf{R})/SL(2,\mathbf{R})$ using a Kähler polarization as described above. The quantum Hilbert space $\mathscr{H}$ then furnishes a projective unitary representation of $LSL(2,\mathbf{R})$. Representations of $LSL(2,\mathbf{R})$ obtained in this way may be related to the representations of $LSL(2,\mathbf{R})$ described in [30].

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