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Operator Equalities Related to the Quantum $E(2)$ Group*

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Abstract. The paper deals with normal operators R and S satisfying simple commutation relations that we encounter investigating the quantum deformation of the $E(2)$ group. We show that $R+S$ admits a normal extension if and only if $R^{-1}S$ satisfies a certain spectral condition. A number of related formulae are derived. In particular, all the functions f satisfying the character equation $f(R+S)=f(R)f(S)$ are found.

0. Introduction

Investigating the quantum deformation (of the two-fold covering) of the group of motions of the Euclidean plane $E(2)$ we often deal with pairs of normal operators (R, S) satisfying in a strong sense the relations $SR=\mu^2 RS$ and $SR^*=R^*S$ (where μ is a real number, $0<\mu<1$). In the present paper we prove a number of results involving such operators. The results will be used in [3], where the quantum $E(2)$ and its Pontryagin dual are elaborated and in [4], where a quantum deformation of the Lorentz group is introduced and investigated.

Let μ be a real number such that $0<\mu<1$. We denote by D_μ the set of all pairs (R, S) of normal operators acting on a Hilbert space H satisfying the following five conditions:

$$\left.\begin{array}{l}
1.\ \operatorname{Ker} R = \operatorname{Ker} S = \{0\}\,,\\
2.\ (\operatorname{Phase} R)(\operatorname{Phase} S) = (\operatorname{Phase} S)(\operatorname{Phase} R)\,,\\
3.\ (\operatorname{Phase} R)^*|S|(\operatorname{Phase} R) = \mu|S|\,,\\
4.\ (\operatorname{Phase} S)|R|(\operatorname{Phase} S)^* = \mu|R|\,,\\
5.\ |S| \text{ and } |R| \text{ strongly commute}.
\end{array}\right\} \qquad (0.1)$$

One can easily show the following

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Proposition 0.1. *Let R and S be normal operators acting on H such that* $\operatorname{Ker} R = \{0\}$ *and* $\operatorname{Ker} S = \{0\}$. *Then* $(R, S) \in D_\mu$ *if and only if*

$$SR\varphi = \mu^2 RS\varphi\,, \qquad SR^*\varphi = R^*S\varphi$$

for all φ *belonging to a dense linear subset* $\mathscr{D} \subset H$ *consisting of vectors with compact R and S support and invariant under the action of R,* R^*, *S, and* S^*.

The paper is composed in the following way. In Sect. 2 we investigate the sum $R \dot{+} S$. It turns out that it need not be normal [despite the formal commutation $(R \dot{+} S)^*(R \dot{+} S) = (R \dot{+} S)(R \dot{+} S)^*$ the domain of $(R \dot{+} S)^*$ may be strictly larger than that of $R \dot{+} S$]. We show that $R \dot{+} S$ is normal if and only if $R^{-1}S$ satisfies a certain spectral condition. In this case $R \dot{+} S$ is unitarily equivalent to R. Sections 3 and 4 are devoted to the character equation $f(R \dot{+} S) = f(R)f(S)$. We find all solutions f of this equation satisfying a certain boundness condition. In Sect. 5 we show that for any C^*-algebra A, $R \dot{+} S \,\eta\, A$ whenever $R, S \,\eta\, A$. Here η denotes the C^*-affiliation relation introduced in [1] and elaborated in [2]. In Sect. 6 we collected a few formulae relating the fundamental solution of the character equation F_μ with the q-exponential function. Section 1 introduces the holomorphic continuation property which is the main technical tool used in this paper.

A few remarks about the notation. Let H be a Hilbert space and Q be a closed operator acting on H. $\mathscr{D}(Q)$ and $\operatorname{Sp}(Q)$ will denote the domain and the spectrum of Q. The partial unitary and the positive selfadjoint operator entering the polar decomposition of Q will be denoted by $\operatorname{Phase} Q$ and $|Q|$, respectively:

$$Q = (\operatorname{Phase} Q)|Q|\,.$$

We shall use this notation only for Q such that $\ker Q = \ker Q^* = \{0\}$. Then $\operatorname{Phase} Q$ is unitary. If Q is normal then $\operatorname{Phase} Q$ commutes with $|Q|$. In this case

$$Q = \int z dE_Q(z)\,,$$

where $dE_Q(z)$ is the spectral measure of Q. We say that a vector $\varphi \in H$ has a compact Q-support if the support of the measure $(\varphi | dE_Q(z)\varphi)$ is compact.

In all cases whenever we deal with products PQ of closed unbounded operators P and Q the operators $|P'| = (\operatorname{Phase} Q)^*|P|(\operatorname{Phase} Q)$ and $|Q|$ strongly commute and by definition

$$PQ = (\operatorname{Phase} P)(\operatorname{Phase} Q)m(|P'|, |Q|)\,,$$

where $m(x, y) = xy$ and consequently

$$m(|P'|, |Q|) = \int xy dE(x, y)\,,$$

where $E(\cdot\,, \cdot)$ is the spectral measure on $\mathbb{R}^2_+$ related to the strongly commuting pair of positive selfadjoint operators $|P'|$ and $|Q|$.

The closure of the sum of operators R and S is denoted by $R \dot{+} S$.

1. Holomorphic Continuation

Throughout the paper the following subgroup of $\mathbb{C} - \{0\}$ will play an essential role:

$$\mathbb{C}^\mu = \{t \in \mathbb{C} \colon |t| \in \mu^{\mathbb{Z}}\}\,. \tag{1.1}$$

We endow $\mathbb{C}^\mu$ with the Haar measure $dv(t)$ normalized in such a way that such $v(S^1)=1$. The closure of $\mathbb{C}^\mu$ will be denoted by $\bar{\mathbb{C}}^\mu : \bar{\mathbb{C}}^\mu = \mathbb{C}^\mu \cup \{0\}$. (In [2] this set was denoted by $\mathbb{C}_{(\mu)}$.) All the complex functions on $\mathbb{C}^\mu$ considered in this paper are square integrable on each circle contained in $\mathbb{C}^\mu$.

Let ϕ be such a function on $\mathbb{C}^\mu$. We say that ϕ has the holomorphic (meromorphic, respectively) continuation property if there exists a function $\tilde{\phi}$ holomorphic (meromorphic with a finite number of poles on each connected component) on $\mathbb{C} - \bar{\mathbb{C}}^\mu$ such that for all $z \in \mathbb{C}^\mu$ we have

$$\lim_{r \to 1-0} \tilde{\phi}(rz) = \phi(z), \qquad \lim_{r \to \mu+0} \tilde{\phi}(rz) = \phi'(\mu z),$$

where ϕ' is a function on $\mathbb{C}^\mu$ and the limits are understood in the sense of L^2-norm on each circle of $\mathbb{C}^\mu$. In this case we write $(\mathscr{H}\phi)(z)$ and $(\tilde{\mathscr{H}}\phi)(\zeta)$ instead of $\phi'(z)$ and $\tilde{\phi}(\zeta)$. The reader should notice that $\tilde{\phi}$ and ϕ' are uniquely determined by ϕ.

Let ϕ be a function on $\mathbb{C}^\mu$ having the meromorphic (holomorphic, respectively) continuation property. Then $\mathscr{H}\phi(z) = \phi(z)$ for all $z \in \mathbb{C}^\mu$ if and only if ϕ admits a meromorphic (holomorphic, respectively) extension on $\mathbb{C} - \{0\}$.

We shall use the following special function:

$$F_\mu(z) = \prod_{k=0}^{\infty} \frac{1+\mu^{2k}\bar{z}}{1+\mu^{2k}z}. \tag{1.2}$$

This formula defines $F_\mu(z)$ for all complex z except the values $z = -\mu^{-2k}$ $(k=0,1,\ldots)$. For these exceptional z we set $F_\mu(z) = -1$. Clearly $|F_\mu(z)| = 1$ for all $z \in \bar{\mathbb{C}}^\mu$. One can easily check that the function F_μ restricted to $\bar{\mathbb{C}}^\mu$ is continuous. Therefore, if Q is a normal operator affiliated with a C^*-algebra A and $\mathrm{Sp}(Q) \subset \bar{\mathbb{C}}^\mu$ then $F_\mu(Q)$ is a unitary element of the multiplier algebra $M(A)$. By straightforward computation one can verify that for $z \in \mathbb{C}^\mu$:

$$F_\mu(\mu^2 z) F_\mu(z^{-1}) = \chi(\mu z), \tag{1.3}$$

where $\chi(z) = (\mathrm{Phase}\, z)^{\log_\mu(|z|)}$.

Let us notice that F_μ has the holomorphic extension property and that

$$\mathscr{H} F_\mu(z) = (1+\mu^{-2}\bar{z}) F_\mu(z) \tag{1.4}$$

for all $z \in \bar{\mathbb{C}}^\mu$. In Sect. 4 we shall use

Proposition 1.1. *Let f be a function on $\bar{\mathbb{C}}^\mu$ such that f and f^{-1} are bounded and $\chi \in \mathbb{C}$. Assume that f has the holomorphic extension property and that*

$$\mathscr{H} f(z) = (1+\mu^{-2}\chi\bar{z}) f(z)$$

for all $z \in \bar{\mathbb{C}}^\mu$. Then $\chi \in \bar{\mathbb{C}}^\mu$ and $f(z) = \mathrm{const} \cdot F_\mu(\bar{\chi} z)$.

Proof. If $\chi = 0$ then $\mathscr{H} f(z) = f(z)$ and f admits a holomorphic extension $\tilde{f}$ on $\mathbb{C} - \{0\}$. By the maximum principle, $\tilde{f}$ is bounded and using the Liouville theorem we get $f = \mathrm{const}$.

If $\chi \neq 0$ then $\chi = \chi_0 \mu^s$, where $s \in \mathbb{Z}$ and $\mu < |\chi_0| \leqq 1$. For any $z \in \bar{\mathbb{C}}^\mu$ we set

$$F_{\mu\chi}(z) = \prod_{k=0}^{\infty} \frac{1+\mu^{2k+s}\chi_0 \bar{z}}{1+\mu^{2k+s}\chi_0^{-1} z}.$$

One can easily verify that

1. $$F_{\mu\chi}(z) = (\chi_0 \,\mathrm{Phase}\, \bar{z})^{-\log_\mu(|z|)-s+1} H(z),$$

where $\lim_{z\to\infty} H(z)=1$. In particular,

$$|F_{\mu\chi}(z)|=|\chi_0|^{-s+1}|H(z)|\,|z|^{\varrho}, \tag{1.5}$$

where $\varrho=-\log_\mu(|\chi_0|)\in]-1,0]$. If $\chi\notin\bar{\mathbb{C}}^\mu$ then $|\chi_0|<1$, $\varrho<0$ and $\lim_{z\to\infty} F_{\mu\chi}(z)=0$.
2. $F_{\mu\chi}$ has the holomorphic extension property, $(\tilde{\mathscr{H}}F_{\mu\chi})(\zeta)\neq 0$ for any $\zeta\in\mathbb{C}-\bar{\mathbb{C}}^\mu$ and $\mathscr{H}F_{\mu\chi}(z)=(1+\mu^{-2}\chi\bar{z})F_{\mu\chi}(z)$.

Therefore, the quotient $\phi(z)=f(z)/F_{\mu\chi}(z)$ has holomorphic extension property and $\mathscr{H}\phi(z)=\phi(z)$ for all $z\in\bar{\mathbb{C}}^\mu$. It shows that ϕ admits an extension $\tilde{\phi}$ holomorphic on $\mathbb{C}-\{0\}$. In a neighbourhood of 0, $F_{\mu\chi}(z)\cong 1$ and $\phi(z)$ is bounded. It means that there is no singularity at $z=0$ and $\tilde{\phi}(z)$ is an entire function. By virtue of (1.5) there exists a constant C such that for all sufficiently large $z\in\bar{\mathbb{C}}^\mu$ we have:

$$|\tilde{\phi}(z)|\leqq C|z|^{-\varrho},$$

where $-\varrho=\log_\mu(|\chi_0|)<1$. Repeating the standard proof of the Liouville theorem one can easily show that $\tilde{\phi}(z)=\text{const}$ is the only entire function satisfying the above estimate. Therefore, $f(z)=\text{const}\,F_{\mu\chi}(z)$. Now it is clear that $\chi\in\bar{\mathbb{C}}^\mu$ [otherwise $f(z)^{-1}$ would not be bounded]. In this case $|\chi_0|=1$, $\chi_0^{-1}=\bar{\chi}_0$ and $F_{\mu\chi}(z)=F_\mu(\bar{\chi}z)$. Q.E.D.

2. When is $R+S$ Normal?

The main results of this section are contained in the following theorems:

Theorem 2.1. *Let $(R,S)\in D_\mu$. Then the following conditions are equivalent:*

1. *$R+S$ admits a normal extension,*
2. *$R\dot{+}S$ is normal,*
3. $\mathrm{Sp}(R^{-1}S)\subset\bar{\mathbb{C}}^\mu$.

Theorem 2.2. *Let $(R,S)\in D_\mu$ and $\mathrm{Sp}(R^{-1}S)\subset\bar{\mathbb{C}}^\mu$. Then*

$$R\dot{+}S=F_\mu(R^{-1}S)RF_\mu(R^{-1}S)^*. \tag{2.1}$$

Proof. Let $(R,S)\in D_\mu$, $Q_0=R+S$, Q be the closure of Q_0 and $Q'=(Q^*|_{\mathscr{D}(Q)})^*$. Then Q' is an extension of Q. To show that in Theorem 2.1 Condition 1 implies Condition 2 (the converse is obvious) it is sufficient to prove that

$$\mathscr{D}(Q)=\mathscr{D}(Q^*)\cap\mathscr{D}(Q'). \tag{2.2}$$

Indeed, if $\tilde{Q}$ is a normal extension of Q then $\tilde{Q}^*\subset Q^*$, $Q^*|_{\mathscr{D}(Q)}\subset\tilde{Q}^*$, $Q'=(Q^*|_{\mathscr{D}(Q)})^*\supset\tilde{Q}$ and $\mathscr{D}(\tilde{Q})\subset\mathscr{D}(Q')$. On the other hand, $\mathscr{D}(\tilde{Q})=\mathscr{D}(\tilde{Q}^*)\subset\mathscr{D}(Q^*)$ and using (2.2) we see that $\mathscr{D}(\tilde{Q})\subset\mathscr{D}(Q)$, $\tilde{Q}=Q$ and Q is normal.

Proving the relation (2.2) and equivalence of Conditions 2 and 3 of Theorem 2.1 we may assume that (R,S) acts on H in the irreducible way (otherwise we use the direct integral decomposition into irreducible components). In this case Theorem 2.1 essentially coincides with Theorem 3.1 of [2]. We repeat the main steps of the proof, because the notation of [2] is not coherent with the present one and because the argument used in [2] to show that in certain cases $R+S$ has no normal extensions [on the contrary to the condition (2.2)] does not survive the direct integral synthesis.

Let (R, S) be a pair of normal operators acting on a Hilbert space in an irreducible way and satisfying all the five conditions (0.1). Then

$$\mathrm{Sp}(R)=t_R\bar{\mathbb{C}}^\mu, \quad \mathrm{Sp}(S)=t_S\bar{\mathbb{C}}^\mu,$$

where t_R, t_S are positive real numbers. The pair (R, S) is determined uniquely (up to a unitary equivalence) by (t_R, t_S): there exists an orthonormal basis $(e_{mn})_{m,n\text{-integers}}$ such that

$$\left.\begin{aligned} Re_{mn}&=t_R\mu^n e_{m+1;n},\\ Se_{mn}&=t_S\mu^m e_{m;n-1}\end{aligned}\right\} \tag{2.3}$$

for all $m, n\in\mathbb{Z}$. The set H_f of all finite linear combinations of basic vectors is a core for R and S.

We shall identify H with $L^2(\mathbb{C}^\mu)$: For any $\varphi\in H$ and any $z\in\mathbb{C}^\mu$ we set

$$\varphi(z)=\sum_{k=-\infty}^{+\infty}(e_{m+k,k}|\varphi)u^k, \tag{2.4}$$

where $m=\log_\mu(|z|)$ and $u=\mathrm{Phase}(z)$. One can easily verify that the series (2.4) is convergent in the sense of $L^2(\mathbb{C}^\mu)$-norm and that the correspondence $H\ni\varphi\to\varphi(\cdot)\in L^2(\mathbb{C}^\mu)$ is bijective and respects the Hilbert space structure of H and $L^2(\mathbb{C}^\mu)$.

Let $\varphi\in H$. One can easily verify that $\varphi\in\mathscr{D}(R)$ if and only if $\varphi(\cdot)$ has holomorphic continuation property and $(\mathscr{H}\varphi)(z)$ is square integrable on $\mathbb{C}^\mu$. In this case

$$(R\varphi)(z)=t_R(\mathscr{H}\varphi)(z). \tag{2.5}$$

Similarly, $\varphi\in\mathscr{D}(S)$ if and only if $\varphi(\cdot)$ has holomorphic continuation property and $|z|(\mathscr{H}\varphi)(z)$ is square integrable on $\mathbb{C}^\mu$. In this case

$$(S\varphi)(z)=t_S\mu^{-1}\bar{z}(\mathscr{H}\varphi)(z). \tag{2.6}$$

Consequently, $\varphi\in\mathscr{D}(Q_0)$ if and only if $\varphi(\cdot)$ has holomorphic continuation property and $(1+|z|)(\mathscr{H}\varphi)(z)$ is square integrable on $\mathbb{C}^\mu$. In this case

$$(Q_0\varphi)(z)=(t_R+t_S\mu^{-1}\bar{z})(\mathscr{H}\varphi)(z). \tag{2.7}$$

Comparing (2.5) and (2.7) we get

$$(Q_0\varphi)(z)=(1+[t_S/\mu t_R]\bar{z})(R\varphi)(z) \tag{2.8}$$

for any $\varphi\in\mathscr{D}(Q_0)$. The operator $R^{-1}S$ is diagonal in $L^2(\mathbb{C}^\mu)$. Comparing (2.5) and (2.6) we have

$$(R^{-1}S\varphi)(z)=[t_S\mu/t_R]\bar{z}\varphi(z). \tag{2.9}$$

Consequently,

$$\mathrm{Sp}(R^{-1}S)=[t_S/t_R]\bar{\mathbb{C}}^\mu. \tag{2.10}$$

Assume now that

$$\mathrm{Sp}(R^{-1}S)\not\subset\bar{\mathbb{C}}^\mu.$$

Then $t_S/t_R\notin\mathbb{C}^\mu$ and $t_R+t_S\mu^{-1}\bar{z}\neq 0$ for any $z\in\mathbb{C}^\mu$. In fact, there exists a positive constant c such that $t_R\leqq c(t_R+t_S\mu^{-1}\bar{z})$ for all $z\in\mathbb{C}^\mu$. Therefore, $\|R\varphi\|\leqq c\|Q_0\varphi\|$ for any $\varphi\in\mathscr{D}(Q_0)$. It shows that Q_0 is a closed operator (convergence of $Q_0\varphi_n$ implies that of $R\varphi_n$ and $S\varphi_n=Q_0\varphi_n-R\varphi_n$): $Q_0=Q$.

Briefly speaking [cf. (2.7)] the action of Q consists in the holomorphic continuation (which itself is a normal operator) followed by the multiplication by $t_R+t_S\mu^{-1}\bar{z}$. Therefore, $\varphi\in\mathscr{D}(Q^*)$ if and only if φ multiplied by $t_R+t_S\mu^{-1}z$ has the holomorphic continuation property (and the resulting function is square integrable). The crucial point is that $t_R+t_S\mu^{-1}\zeta$ vanishes at one point $\zeta\in\mathbb{C}-\bar{\mathbb{C}}^\mu$. Therefore, the function $\tilde{\mathscr{H}}\varphi$ [for $\varphi\in\mathscr{D}(Q^*)$] is allowed to have a first order pole in this point. More precisely we have (see the proof of Theorem 3.1 in [2] for the details):

A vector $\varphi\in\mathscr{D}(Q^*)$ if and only if $\varphi(\cdot)$ has meromorphic continuation property, the only possible singularity of $\tilde{\mathscr{H}}\varphi(\zeta)$ is a simple pole located at $\zeta=-\mu t_R/t_S$ and the function $(1+|z|)(\mathscr{H}\varphi)(z)$ is square integrable.

Moreover, using the similar analysis one can show that $\varphi\in\mathscr{D}(Q')$ [where $Q'=(Q^*|_{\mathscr{D}(Q)})^*$] if and only if $\varphi(\cdot)$ has meromorphic continuation property, the only possible singularity of $\tilde{\mathscr{H}}\varphi(\zeta)$ is a simple pole located at $\zeta=-t_R/\mu t_S$ and the function $(1+|z|)(\mathscr{H}\varphi)(z)$ is square integrable.

Comparing the above descriptions of $\mathscr{D}(Q)$, $\mathscr{D}(Q^*)$, and $\mathscr{D}(Q')$ we immediately obtain (2.1). Moreover, $\mathscr{D}(Q)$ is strictly smaller than $\mathscr{D}(Q^*)$ and Q is not normal.

To end this section we have to elaborate the case $\mathrm{Sp}(R^{-1}S)\subset\bar{\mathbb{C}}^\mu$. According to (2.9), $F_\mu(R^{-1}S)^*$ coincides with the multiplication by $f(z)=F_\mu([t_S\mu/t_R]z)$. If $\mathrm{Sp}(R^{-1}S)\subset\bar{\mathbb{C}}^\mu$ then [cf. (2.10)] t_S/t_R is an integer power of μ. In this case $f(\cdot)$ has holomorphic continuation property and [cf. (1.4)]

$$\mathscr{H}f(z)=(1+[t_S/\mu t_R]\bar{z})f(z) \tag{2.11}$$

for all $z\in\mathbb{C}^\mu$.

Let $\varphi\in\mathscr{D}(Q_0)$. Then using the descriptions of domains $\mathscr{D}(R)$ and $\mathscr{D}(Q_0)$ given in the introductory part of this proof we see that $F_\mu(R^{-1}S)^*\varphi\in\mathscr{D}(R)$ and [cf. (2.5), (2.11), and (2.7)],

$$\begin{aligned}(RF_\mu(R^{-1}S)^*\varphi)(z)&=t_R\mathscr{H}(f\varphi)(z)=t_R\mathscr{H}f(z)\mathscr{H}\varphi(z)\\&=(t_R+t_S\mu^{-1}\bar{z})f(z)\mathscr{H}(\varphi)(z)=(F_\mu(R^{-1}S)^*Q_0\varphi)(z).\end{aligned}$$

It shows that

$$Q_0=F_\mu(R^{-1}S)R|_{\mathscr{D}}F_\mu(R^{-1}S)^*, \tag{2.12}$$

where $\mathscr{D}=F_\mu(R^{-1}S)^*(Q_0)$. Let us notice that $\zeta=-\mu t_R|t_S$ is the only point in $\mathbb{C}$ where the holomorphic extension $\tilde{\mathscr{H}}f$ is approaching 0. Using this fact one can show that a vector $\varphi\in H_f$ belongs to $\mathscr{D}$ if and only if $\sum(-\mu)^k(e_{k+m,k}|\varphi)=0$ [where $m=\log_\mu(t_R/t_S)$]. Clearly, the functional $H_f\ni\varphi\to\sum(-\mu)^k(e_{k+m,k}|\varphi))\in\mathbb{C}$ is not continuous with respect to the graph norm of R. Therefore, $\mathscr{D}$ is a core of R and passing to the closures on both sides of (2.12) we obtain

$$Q=F_\mu(R^{-1}S)RF_\mu(R^{-1}S)^*.$$

This formula coincides with (2.1). It shows that Q is normal. In this case relation (2.2) is obvious.

3. The Character Property

In this section we prove the following remarkable

Theorem 3.1. *Let* $(R,S)\in D_\mu$ *and* $\mathrm{Sp}(R)$, $\mathrm{Sp}(S)\subset\bar{\mathbb{C}}^\mu$ *(in this case the condition* $\mathrm{Sp}(R^{-1}S)\subset\bar{\mathbb{C}}^\mu$ *is fulfilled automatically). Then*

$$F_\mu(R\dot{+}S)=F_\mu(R)F_\mu(S). \tag{3.1}$$

Proof. Let D'_μ be the set of all $(R,S)\in D_\mu$ such that the spectra of R and S are contained in $\bar{\mathbb{C}}^\mu$. Let us notice that the assumptions of Theorem 3.1 are very restrictive. Any $(R,S)\in D'_\mu$ is a direct sum of a number of copies of the unique (up to a unitary equivalence) irreducible $(R_0,S_0)\in D'_\mu$. Therefore, it is sufficient to prove (3.1) for a single pair.

Let H be a Hilbert space spanned by an orthonormal basis $(e_{kmn})_{k,m,n\text{-integers}}$ and P,Q,R be operators introduced by

$$Pe_{kmn}=\mu^n e_{k+1,m+1,n},$$
$$Re_{kmn}=\mu^m e_{k+1,m,n-1},$$
$$Se_{kmn}=\mu^k e_{k,m-1,n-1}$$

for all $k,m,n\in\mathbb{Z}$. The set H_f of all finite linear combinations of basic vectors is by definition the common core for P, Q, and S. One can check that (P,R), (P,S), (R,S), (P,PR), and (P,PS) belong to D'_μ.

Operator P strongly commutes with $R^{-1}S$. Using (2.1) we get

$$F_\mu(R^{-1}S)PF_\mu(R^{-1}S)^*=P,$$
$$F_\mu(R^{-1}S)PRF_\mu(R^{-1}S)^*=P(R\dot{+}S).$$

It shows that the pair (P,PR) and $(P,P(R\dot{+}S))$ are unitarily equivalent. Therefore, $(P,P(R\dot{+}S))\in D'_\mu$ and using (2.1) we get

$$F_\mu(R\dot{+}S)PF_\mu(R\dot{+}S)^*=P\dot{+}P(R\dot{+}S).$$

On the other hand, R strongly commutes with PS. Keeping this fact in mind and using twice (2.1) we get

$$\begin{aligned}F_\mu(R)F_\mu(S)PF_\mu(S)^*F_\mu(R)^*&=F_\mu(R)(P\dot{+}PS)F_\mu(R)^*\\&=F_\mu(R)PF_\mu(R)^*\dot{+}F_\mu(R)PSF_\mu(R)^*=(P\dot{+}PR)\dot{+}PS.\end{aligned}$$

Using twice Lemma 3.2 formulated below one can easily show that H_f is a core of $(P\dot{+}PR)\dot{+}PS$. Therefore, $(P\dot{+}PR)\dot{+}PS\subset P\dot{+}P(R\dot{+}S)$ (on H_f the two operators coincide) and $(P\dot{+}PR)\dot{+}PS=P\dot{+}P(R\dot{+}S)$ (normal operators have no normal extensions). Comparing the two formulae derived above we get

$$F_\mu(R\dot{+}S)^*F_\mu(R)F_\mu(S)P=PF_\mu(R\dot{+}S)^*F_\mu(R)F_\mu(S).$$

It shows that $F_\mu(R\dot{+}S)^*F_\mu(R)F_\mu(S)$ commutes with Phase P. On the other hand, Phase P scales R,S and $R\dot{+}S$ by the factor μ:

$$(\text{Phase }P)^*Q(\text{Phase }P)=\mu Q$$

for $Q=R, S$ and $R \dot{+} S$. Combining these two facts, for any integer n we obtain

$$F_\mu(R \dot{+} S)^* F_\mu(R) F_\mu(S) = F_\mu(\mu^n(R \dot{+} S))^* F_\mu(\mu^n R) F_\mu(\mu^n S). \tag{3.2}$$

Remembering that $\lim_{t\to 0} F_\mu(t) = F_\mu(0) = 1$ we get $\text{s-}\lim_{n\to\infty} F_\mu(\mu^n Q) = I$ for $Q=R, S$ and $R \dot{+} S$. Therefore, the right-hand side of (3.2) tends strongly to I for $n\to\infty$ and (3.1) follows.

To end this section we have to prove

Lemma 3.2. *Let $(R, S) \in D_\mu$ and $\mathscr{D}$ be a linear subset contained in $\mathscr{D}(R) \cap \mathscr{D}(S)$. Assume that $\mathscr{D}$ is a core for R and that $\mathscr{D}$ is invariant under spectral projections of $|S|$. Then $\mathscr{D}$ is a core for $R \dot{+} S$.*

Sketch of the Proof. The spectral projection of $|S|$ corresponding to the interval $[0, t]$ (where $t \in \mathbb{R}$) will be denoted by E_t. Let $\varphi \in \mathscr{D}(R+S)$. For any $\varepsilon > 0$ we choose $t > 0$ such that

$$\|E_t\varphi - \varphi\| \leqq \varepsilon/2,$$

$$\|E_t S\varphi - S\varphi\| \leqq \varepsilon/4,$$

$$\|E_{\mu t} R\varphi - R\varphi\| \leqq \varepsilon/4$$

and $\psi \in \mathscr{D}$ such that

$$\|\psi - E_t\varphi\| \leqq \min(\varepsilon/4t, \varepsilon/2),$$

$$\|R\psi - RE_t\varphi\| \leqq \varepsilon/4.$$

Then $E_t\psi \in \mathscr{D}$ and using the above inequalities, the relations $RE_t = E_{\mu t}R$ and $\|SE_t\| \leqq t$ we get

$$\|E_t\psi - \varphi\| \leqq \varepsilon,$$

$$\|(R+S)(E_t\psi - \varphi)\| \leqq \varepsilon. \quad \text{Q.E.D.}$$

4. The Character Equation

Throughout this section R and S are normal operators acting on a Hilbert space H. We assume that $(R, S) \in D_\mu$ and $\mathrm{Sp}(R), \mathrm{Sp}(S) \subset \bar{\mathbb{C}}^\mu$. It turns out that $F_\mu(\cdot)$ is essentially the only bounded function with bounded inverse satisfying the character equation (3.1).

Theorem 4.1. *Let f be a measurable function on $\bar{\mathbb{C}}^\mu$ such that f and f^{-1} are bounded. Assume that*

$$f(R \dot{+} S) = f(R) f(S). \tag{4.1}$$

Then there exists $\chi \in \bar{\mathbb{C}}^\mu$ such that

$$f(z) = F_\mu(\bar{\chi} z) \tag{4.2}$$

for almost all $z \in \bar{\mathbb{C}}^\mu$.

Remark. It follows immediately from Theorem 3.1 that all functions (4.2) satisfy the relation (4.1).

Proof. Keeping in mind the remarks at the beginning of the proof of Theorem 3.1 we may assume that the operators R and S are defined by (2.3) with $t_R=t_S=1$. According to (2.1), relation (4.1) means that

$$F_\mu(R^{-1}S)f(R)F_\mu(R^{-1}S)^*=f(R)f(S). \tag{4.3}$$

We shall use the Fourier decompositions

$$F_\mu(z)=\sum F_i(|z|)(\text{Phase}\,z)^i, \tag{4.4}$$

$$f(z)=\sum f_i(|z|)(\text{Phase}\,z)^i. \tag{4.5}$$

It follows immediately from (2.3) that Phase R acting on e_{mn} increases index m by one. Similarly, Phase S decreases index n by one, whereas Phase $R^{-1}S$ decreases both indices by one. Using this information one can easily compute the matrix elements of both sides of (4.3). We get

$$(e_{ab}|f(R)f(S)e_{cd})=f_{a-c}(\mu^b)f_{d-b}(\mu^c),$$

$$(e_{ab}|F_\mu(R^{-1}S)f(R)F_\mu(R^{-1}S)^*e_{cd})=\sum F_i(\mu^{a-b+1})\overline{F_j(\mu^{c-d+1})}f_{a+i-c-j}(\mu^{d+j}),$$

where the summation runs over all integers i,j such that $b+i=d+j$. Comparing the two expressions and replacing $d-b$ and $a-c$ by n and m, respectively, we obtain

$$f_m(\mu^b)f_n(\mu^c)=\sum F_i(\mu^{m+c-b+1})\overline{F_{i-n}(\mu^{c-n-b+1})}f_{m+n}(\mu^{b+i}), \tag{4.6}$$

where the summation runs over all integer i.

In what follows we need the infinitesimal version of this formula. We know that $F_\mu(z)$ is a real analytic function in a neighbourhood of 0. Using this fact one can prove that there exist positive constants A,ϱ such that for sufficiently small z

$$|F_k(|z|)|\leqq A(\varrho|z|)^{|k|}$$

for all $k\in\mathbb{Z}$. Moreover, it follows easily from (1.2) that

$$F_\mu(z)=1-(1-\mu^2)^{-1}z+(1-\mu^2)^{-1}\bar{z}+\text{higher order terms}.$$

Therefore,

$$|F_0(|z|)-1|\leqq A(\varrho|z|)^2,$$

$$|F_1(|z|)+(1-\mu^2)^{-1}|z||\leqq A(\varrho|z|)^3,$$

$$|F_{-1}(|z|)-(1-\mu^2)^{-1}|z||\leqq A(\varrho|z|)^3.$$

Inserting $n=-1$ in our main formula (4.6), dividing both sides by μ^c and sending $c\to\infty$ we see that on the right-hand side only the terms with $i=-1$ and $i=0$ survive. More precisely, we get

$$\lim_{c\to\infty} f_m(\mu^b)f_{-1}(\mu^c)/\mu^c=\frac{\mu^{m-b+1}}{1-\mu^2}f_{m-1}(\mu^{b-1})-\frac{\mu^{-b+2}}{1-\mu^2}f_{m-1}(\mu^b)$$

and

$$\mu^{m-1}f_{m-1}(\mu^{b-1})-f_{m-1}(\mu^b)=\chi\mu^{b-2}f_m(\mu^b),$$

where $\chi=(1-\mu^2)\lim\limits_{c\to\infty} f_{-1}(\mu^c)/\mu^c$.

Let $z=\mu^b u$, where $u=\text{Phase}\, z$. Multiplying both sides by u^{m-1} and summing over m we get [cf. (4.5)]

$$\sum f_m(|z|/\mu)(\mu \text{ Phase}\, z)^m=(1+\mu^{-2}\chi\bar{z})f(z).$$

It shows that f has the holomorphic continuation property and that $\mathscr{H}f(z)=(1+\mu^{-2}\chi\bar{z})f(z)$. Using Proposition 1.1 we get $\chi\in\bar{\mathbb{C}}^\mu$ and $f(z)=\text{const}\cdot F_\mu(\bar{\chi}z)$. Comparing (4.1) with (3.1) we get const = 1. Q.E.D.

Now let K be a Hilbert space and $f:\bar{\mathbb{C}}^\mu\ni z\to f(z)\in B(K)$ be a bounded measurable mapping. For any normal operator Q acting on H such that $\text{Sp}(Q)\subset\bar{\mathbb{C}}^\mu$ we set

$$f(Q)=\int f(z)\otimes dE_Q(z),$$

where $dE_Q(z)$ is the spectral measure of Q. We have the following unitary version of Theorem 4.1.

Theorem 4.2. *Let $f:\bar{\mathbb{C}}^\mu\ni z\to f(z)\in B(K)$ be a measurable mapping such that $f(z)$ is unitary for all $z\in\bar{\mathbb{C}}^\mu$. Assume that*

$$f(R\dot{+}S)=f(R)f(S).$$

Then there exists a normal operator X on K such that $\text{Sp}(X)\subset\bar{\mathbb{C}}^\mu$ and

$$f(z)=F_\mu(X^*z)$$

for almost all $z\in\bar{\mathbb{C}}^\mu$.

Proof. A moment of reflection shows that also in this non-scalar case relation (4.6) holds [now $f_i(|z|)$ introduced by (4.5) are operators on K and the order of factors on the left-hand side of (4.6) is relevant]. One can easily rewrite (1.3) in terms of Fourier coefficients introduced by (4.4). We have

$$\overline{F_r(\mu^k)}=F_{r-k+1}(\mu^{2-k})$$

for all integers r,k. Using this formula to eliminate the complex conjugate term in (4.6) we get

$$f_m(\mu^b)f_n(\mu^c)=\sum_{r\in\mathbb{Z}}F_{r-b}(\mu^{m+c-b+1})F_{r-c}(\mu^{n+b-c+1})f_{m+n}(\mu^r).$$

Let us notice that the right-hand side is manifestly invariant under exchange $(m,b)\leftrightarrow(n,c)$. Therefore, $[f_m(\mu^b),f_n(\mu^c)]=0$ and all unitaries $f(z)$ $(z\in\bar{\mathbb{C}}^\mu)$ mutually commute. By the spectral decomposition our problem is now reduced to the scalar one solved in the previous theorem. Q.E.D.

Let us notice the following continuity property:

Proposition 4.3. *Let $f:\bar{\mathbb{C}}^\mu\to B(K)$ be a bounded measurable mapping. Then*

$$\underset{n\to\infty}{\text{s-lim}}\, f(R\dot{+}\mu^n S)=f(R), \tag{4.7}$$

$$\underset{n\to\infty}{\text{s-lim}}\, f(\mu^n R\dot{+}S)=f(S). \tag{4.8}$$

Proof. Relation (4.7) follows immediately from the formula [cf. (2.1)]

$$f(R\dot{+}\mu^n S)=[I\otimes F_\mu(\mu^n R^{-1}S)]f(R)[I\otimes F_\mu(\mu^n R^{-1}S)]^*.$$

Inserting in (4.7) S^*, R^*, and f' [where $f'(z)=f(\bar{z})$] instead of R, S, and f we get (4.8). Q.E.D.

5. The Affiliation Relation

Let $A \subset B(H)$ be a C^*-algebra and Q be a closed operator acting on H. We always assume that A is nondegenerate. We recall (cf. [1, 2]) that Q is affiliated with A ($Q\,\eta\,A$) if and only if $Q(I+Q^*Q)^{-1/2} \in M(A)$ and $(I+Q^*Q)^{-1/2}A$ is dense in A. For normal operators $Q\,\eta\,A$ if and only if $\{f(Q)\colon f \in C_\infty(\mathrm{Sp}\,Q)\}$ is a subset of $M(A)$ and contains an approximate unity for A.

The main result of this section is the following:

Theorem 5.1. *Let $A \subset B(H)$ be a C^*-algebra and $R, S\,\eta\,A$. Assume that $(R,S) \in D_\mu$ and $\mathrm{Sp}(R^{-1}S) \subset \bar{\mathbb{C}}^\mu$. Then*

$$R \dot{+} S\,\eta\,A\,.$$

This result contains the essence of Theorem 3.2 of [2]. The proof presented here is completely new and much simpler than the one presented in [2]. We shall use

Proposition 5.2. *Let $A \subset B(H)$ be a C^*-algebra, Q be a normal operator acting on H and $\mathrm{Sp}(Q) \subset \bar{\mathbb{C}}^\mu$. Assume that for any $z \in \bar{\mathbb{C}}^\mu$, $F_\mu(zQ) \in M(A)$ and that the mapping*

$$\bar{\mathbb{C}}^\mu \ni z \to F_\mu(zQ) \in M(A) \tag{5.1}$$

(where $M(A)$ is equipped with the topology of almost uniform convergence) is continuous. Then $Q\,\eta\,A$.

Remark. If $Q\,\eta\,A$ then the mapping (5.1) is obviously continuous.

Proof. Let us notice [cf. (1.3)] that

$$F_\mu(z) \cong (\mathrm{Phase}\, z)^{\log_\mu(|z|)-1}$$

for large $z \in \bar{\mathbb{C}}^\mu$. Using this asymptotic behavior one can easily show that all the functions

$$f_{mn}(z) = \frac{1}{2\pi}\int_0^{2\pi} F_\mu(\mu^m e^{i\theta} z) e^{in\theta} d\theta$$

(where m, n runs over $\mathbb{Z}$) belong to $C_\infty(\bar{\mathbb{C}}^\mu)$ and separate points of $\bar{\mathbb{C}}^\mu$. Keeping in mind the assumptions of the lemma we conclude that $f_{mn}(Q) \in M(A)$ and (Stone-Weierstrass theorem) $f(Q) \in M(A)$ for all $f \in C_\infty(\bar{\mathbb{C}}^\mu)$. Moreover,

$$f_{m0}(z) = \frac{1}{2\pi}\int_0^{2\pi} F_\mu(\mu^m e^{i\theta} z) d\theta$$

converges almost uniformly to $F_\mu(0 \cdot Q) = I$ as $m \to \infty$. It shows that the set $\{f(Q)\colon f \in C_\infty(\bar{\mathbb{C}}^\mu)\}$ contains an approximate unity for A and the lemma follows.

Proof of Theorem 5.1. Rescaling if necessary we may assume that $\mathrm{Sp}(R) \subset \bar{\mathbb{C}}^\mu$ and $\mathrm{Sp}(S) \subset \bar{\mathbb{C}}^\mu$. By virtue of (3.1)

$$F_\mu(t(R \dot{+} S)) = F_\mu(tR) F_\mu(tS)$$

for any $t \in \bar{\mathbb{C}}^\mu$ and using Proposition 5.2 we obtain $R \dot{+} S\,\eta\,A$. Q.E.D.

6. Properties of $F_\mu(\cdot)$

In this section we collect some simple formulae that can be proved by straightforward computations. We have

$$F_\mu(z) = \frac{\exp_{1/\mu}\left(\frac{\bar{z}}{1-\mu^2}\right)}{\exp_{1/\mu}\left(\frac{z}{1-\mu^2}\right)},$$

where

$$\exp_{1/\mu}(\zeta) = \sum_{k=0}^{\infty} \frac{\zeta^k}{\mathrm{Fact}_{1/\mu}(k)},$$

where

$$\mathrm{Fact}_{1/\mu}(k) = \prod_{n=1}^{k} \frac{1-\mu^{-2n}}{1-\mu^{-2}} = \sum_{\mathrm{Perm}(k)} \mu^{-2(\text{number of inversions})},$$

where the summation runs over all permutations of k elements.

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