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Complements to Various Stone-Weierstrass Theorems for C^*-algebras and a Theorem of Shultz

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Abstract. J. Glimm's Stone-Weierstrass theorem states that if A is a C^*-algebra, $P(A)$ is the set of pure states of A, and B is a C^*-subalgebra which separates $\overline{P(A)}\cup\{0\}$, then $B=A$. We show that if B is a C^*-subalgebra of A and x an element of A such that any two elements of $\overline{P(A)}\cup\{0\}$ which agree on B agree also on x, then $x\in B$. Similar complements are given to other Stone-Weierstrass theorems. A theorem of F. Shultz states that if $x\in A^{**}$, the enveloping von Neumann algebra of A, and if x, x^*x, and xx^* are uniformly continuous on $P(A)\cup\{0\}$, then there is an element of A which agrees with x on $P(A)$. We show that the hypotheses on x^*x and xx^* can be dropped.

The Stone-Weierstrass conjecture is that if B is a C^*-subalgebra of A and if B separates $P(A)\cup\{0\}$, then $B=A$. It was shown essentially by Kaplansky (see p. 16 of [2] for the history of this result) that this is true if A or B is GCR. It was shown by Sakai [18] that the conjecture is true if B is separable and nuclear. In Theorem 5(a) below we show that if $x\in A$, if any two elements of $P(A)\cup\{0\}$ which agree on B agree also on x, and if one of the above extra hypotheses is satisfied, then $x\in B$. Theorem 5(a) also implies single-element versions of some other Stone-Weierstrass theorems that have been proved, and possibly others that will be proved.

The factorial Stone-Weierstrass conjecture is that if B is a C^*-subalgebra of A which separates $F(A)\cup\{0\}$, where $F(A)$ is the set of factorial states of A, then $B=A$. This was proved in the separable case by Longo [14] and Popa [16], independently. In Theorem 5(b) below we show that if B is separable and if any two elements of $F(A)\cup\{0\}$ which agree on B agree also on x, then $x\in B$. A major part of the proofs of the factorial conjecture was the solution of the factorial state extension problem. Theorem 6.1 of [14] states that if B is separable, then any factorial state of B extends to a factorial state of A. (Theorem 4 of [16] states the same result for A separable.) This result as well as the factorial Stone-Weierstrass theorem itself is used in our proof.

Glimm's Stone-Weierstrass theorem appears in [12], and our complement to it, stated in the abstract, is Theorem 5(c) below.

Since the author is not an expert, the above should not be considered as definitive with regard to either the history or current status of the Stone-Weierstrass problem. Also, it should be emphasized that this paper makes no contribution to the solution of the Stone-Weierstrass problem. A goal of the paper, instead, is to make it easier to apply the Stone-Weierstrass theorems proved by others.

The theorem of Shultz stated in the abstract appears in [19], and we need one more piece of notation to discuss it. Let z be the maximal atomic projection in A^{**}. Thus $A^{**}=zA^{**}\oplus(1-z)A^{**}$, zA^{**} is the direct sum of type I factors, and $(1-z)A^{**}$ has no type I factor direct summands. Any element of $P(A)$ can be regarded as a normal state on A^{**}, supported by z, and thus it can also be regarded as a normal state on zA^{**}. In the actual notation of [19], Shultz's theorem states that if $x\in zA^{**}$ and if x, x^*x, and xx^* are uniformly continuous on $P(A)\cup\{0\}$, then $x\in zA$. Our proof that the hypotheses on x^*x and xx^* can be dropped is in Theorem 6 below, which follows from Theorem 5(c).

This entire paper was inspired by [19], although the full extent of Shultz's influence may be difficult to detect from the present form of the paper. We are also grateful for helpful comments from R. Archbold.

We are going to prove an abstract lemma which can be used to deduce single-element Stone-Weierstrass theorems from "ordinary" Stone-Weierstrass theorems. In the applications of what follows Y will be $P(A)$, $F(A)$, or $\overline{P(A)}\cap S(A)$, where $S(A)$ is the state space of A.

If A is a C^*-algebra and Y is a subset of $S(A)$, then Y will be called *r-closed* if whenever π is the GNS representation induced from a state in Y and v is a unit vector in H_π, then the state $(\pi(\cdot)v, v)$ is in Y. We remark that the concept of r-closed set is just a formalization of ideas which have been known for a long time. Kadison [13] contains abstract results on r-closed sets, expressed in a different terminology, and related results on $P(A)$ and $F(A)$.

Lemma 1. *Suppose Y is a norm closed subset of $S(A)$ such that whenever $f\in Y$, $a\in A$, and $f(a^*a)=1$, then $f(a^*\cdot a)\in Y$. Then Y is r-closed.*

Proof. Let $\pi: A\to B(H)$ be the GNS representation induced from an element f of Y. Let v be a unit vector in H, and let w be a cyclic vector such that $f=(\pi(\cdot)w, w)$. Then there is a sequence (a_n), $a_n\in A$, such that $\pi(a_n)w\to v$. Since $\|v\|=1$, $f(a_n^*a_n)\to 1$, and there is no loss of generality in assuming $f(a_n^*a_n)=1$, $\forall n$. Then if $f_n=f(a_n^*\cdot a_n)$, $f_n\in Y$ and $f_n\to(\pi(\cdot)v, v)$ in norm.

Lemma 2. *If Y is an r-closed subset of $S(A)$, and if F_Y is the smallest norm closed face of $Q(A)$, the quasi-state space of A, containing $Y\cup\{0\}$, then F_Y is a split face of $Q(A)$.*

Proof. If p is the smallest projection in A^{**} such that $Y\subset pA^*p$, then $F_Y=\{f\in A^*: f\geqq 0, \|f\|\leqq 1, \text{and } f\in pA^*p\}$. (See [11 or 17].) We need to show that p is central in A^{**}. Let V be the norm closed linear subspace of A^* generated by Y. If $f\in Y$ and $a\in A$, then $f(a^*\cdot a)$ is a multiple of an element of Y. Therefore $a^*Va\subset V$. By polarization $a^*Vb\subset V$ for all a, b in A. If (e_i) is an approximate unit of A, then $e_if\to f$ and $fe_i\to f$ in norm for all f in A^*. Thus V is closed under left and right multiplication by elements of A, and it follows that $V=wA^*$ for some central projection w in A^{**}. It is clear that $w=p$.

Lemma 3. *Let A be a C^*-algebra, B a C^*-subalgebra, x an element of A, and Y an r-closed subset of $S(A)$. If A is the C^*-algebra generated by B and x, and if any two elements of $Y\cup\{0\}$ which agree on B agree also on x, then B separates F_Y, where F_Y is the smallest norm closed face of $Q(A)$ containing $Y\cup\{0\}$.*

Proof. Let π be the GNS representation induced from an element of Y, and let M be an invariant (closed) subspace for $\pi|_B$. We will show that M is invariant for π. Choose vectors v_0 in M and v_1 in $M^\perp$ such that $\|v_0\|=\|v_1\|=2^{-1/2}$. For real θ, let $v_\theta=v_0+e^{i\theta}v_1$ and let $f_\theta=(\pi(\cdot)v_\theta,v_\theta)$. Then $f_\theta\in Y$ and all f_θ's agree on B. Therefore all f_θ's agree on x. Since

$$f_\theta(x)=(\pi(x)v_0,v_0)+(\pi(x)v_1,v_1)+(\pi(x)v_1,v_0)e^{i\theta}+(\pi(x)v_0,v_1)e^{-i\theta},$$

it is clear that $(\pi(x)v_1,v_0)=(\pi(x)v_0,v_1)=0$. This shows that M reduces $\pi(x)$. Since $\{a\in A: M \text{ reduces } \pi(a)\}$ is a C^*-algebra, M is invariant for π.

Next we show that $\pi|_B$ is non-degenerate. If not, let M be the degeneracy subspace for $\pi|_B$. If v is a unit vector in M and if $f=(\pi(\cdot)v,v)$, then $f\in Y$ and $f|_B=0$. It follows that $f(x)=0$. Since v is arbitrary, this shows that the compression of $\pi(x)$ to M is 0. But M is reducing for $\pi(x)$ by the previous paragraph. Thus $\pi(x)|_M=\pi(x)^*|_M=0$. Since $\{a\in A:\pi(a)|_M=\pi(a)^*|_M=0\}$ is a C^*-algebra, this shows that $\pi(A)|_M=0$, a contradiction.

Now since $\pi(B)'$, the commutant, is generated by projections, the first paragraph shows that $\pi(A)\subset\pi(B)''$. Since $\pi|_B$ is non-degenerate, the von Neumann density theorem implies that $\pi(A)$ and $\pi(B)$ generate the same von Neumann algebra.

Next let $\pi_1:A\to B(H_1)$ and $\pi_2:A\to B(H_2)$ be two GNS representations induced from elements of Y, and assume that π_1 and π_2 are disjoint. We will show that $\pi_1|_B$ and $\pi_2|_B$ are disjoint. If this is not so, there are non-trivial subrepresentations of $\pi_1|_B$ and $\pi_2|_B$ which are equivalent. Since every B-invariant subspace for π_i is also A-invariant, we may change notation and assume $\pi_1|_B$ is equivalent to $\pi_2|_B$. (Note that a subrepresentation of a cyclic representation is cyclic.) Now there is a unitary $U:H_1\to H_2$ such that $U^{-1}\pi_2(b)U=\pi_1(b)$ for all b in B. Let v_1 be a unit vector in H_1, let $v_2=Uv_1$, and let $f_i=(\pi_i(\cdot)v_i,v_i)$. Then f_1 and f_2 are elements of Y which agree on B. Hence $f_1(x)=f_2(x)$; or, in other words,

$$(U^{-1}\pi_2(x)Uv_1,v_1)=(\pi_1(x)v_1,v_1).$$

Since v_1 is arbitrary, we conclude that $\pi_1(x)=U^{-1}\pi_2(x)U$. Since

$$\{a\in A:\pi_1(a)=U^{-1}\pi_2(a)U\}$$

is a C^*-algebra, it now follows that $\pi_1(a)=U^{-1}\pi_2(a)U$ for all a in A, a contradiction.

If π is any representation of A, we will say that a quasi-state f is associated with π if $f=\mathrm{tr}(\pi(\cdot)T)$ for some positive operator T on H_π with $\mathrm{tr}(T)\leq 1$. The set of quasi-states associated to π is a split face of $Q(A)$ and depends only on the central support of π in A^{**}. (In other words we may replace π with a quasi-equivalent representation.) Let π_Y be the direct sum of all the GNS representations induced from elements of Y. Then F_Y is just the set of quasi-states associated with π_Y.

Suppose f_1 and f_2 are elements of F_Y which agree on B. Since every trace class operator is supported on a separable Hilbert space, we can find representations $\pi_1,\pi_2,\ldots$ such that each π_n is the GNS representation induced from an element of Y

and both f_1 and f_2 are associated with $\oplus_1^\infty \pi_n$. Then there are subrepresentations π_n' of π_n which are mutually disjoint such that $\oplus_1^\infty \pi_n$ and $\oplus_1^\infty \pi_n'$ have the same central support. Let $\pi = \oplus_1^\infty \pi_n'$, so that f_1 and f_2 are associated with π. By what has been proved above, $\pi(A)$ and $\pi(B)$ generate the same von Neumann algebra. [Since the $\pi_n'|_B$ are mutually disjoint, this von Neumann algebra is just $\oplus_1^\infty \pi_n'(B)''$, the ℓ^∞ direct sum.] From this and the fact that $f_1|_B = f_2|_B$, we conclude that $f_1 = f_2$. QED

Remark. Let w be the central projection in A^{**} constructed in the proof of Lemma 2. Then the conclusion of the lemma could be restated: $wB^{**} = wA^{**}$. (The countability used in the last paragraph of the proof is not really needed. It was used only to avoid Zorn's lemma.) This way of stating the conclusion would be more pleasing if we could also assert that w is in B^{**}. When $Y = P(A)$, this is true by [1, III.2], since we know that B separates $P(A) \cup \{0\}$ [$w = z$ when $Y = P(A)$]. If $Y = \overline{P(A)} \cap S(A)$ or if B is separable and $Y = F(A)$, this is again true, since we can invoke the theorems of Glimm or Longo and Popa to conclude that $B = A$.

Corollary 4. *Let A be a C^*-algebra, B a C^*-subalgebra, and x an element of A. If any two elements of $P(A) \cup \{0\}$ which agree on B agree also on x, then B separates $P(C) \cup \{0\}$, where C is the C^*-algebra generated by B and x.*

Proof. Since every pure state of C can be extended to a pure state of A, we may change notation and assume $A = C$. Now $P(A)$ is obviously r-closed. Thus the conclusion follows immediately from Lemma 3.

Remark. For fixed A and B let D be the set of x satisfying the hypothesis of Corollary 4. Corollary 4 does not imply that D is an algebra, or even that $C \subset D$, since the restriction to C of a pure state need not be pure. Nevertheless, it can be proved that D is a C^*-algebra without introducing any new ideas.

More generally it can be proved that if Y is any r-closed subset of $S(A)$ and if D is the set of x in A such that any two elements of $Y \cup \{0\}$ which agree on B agree also on x, then $D = \{x \in A : wx \in wB^{**}\}$, where w is as in Lemma 2. After reading an earlier version of this paper, Archbold told us of a single-element Stone-Weierstrass theorem for $\overline{P(A)} \cap F(A)$ (in the separable case), and the above assertion can be used to simplify Archbold's proof. The point is that if Y_1 and Y_2 are r-closed subsets of $S(A)$ such that $F_{Y_1} = F_{Y_2}$, then a single-element Stone-Weierstrass theorem for Y_1 is equivalent to one for Y_2. Therefore we sketch the proof of the assertion about D.

First note that it is obvious that $\{x \in A : wx \in wB^{**}\}$ is contained in D, since every element of Y is supported by w and any two elements of A^* which agree on B agree also on B^{**}. Thus assume $x \in D$. Let π_Y be the representation of A defined in the proof of Lemma 3. Then it is enough to show $\pi_Y(x) \in \pi_Y(B)''$, a property which depends only on the restriction, π', of π_Y to C, the C^*-algebra generated by B and x. Now assume, as we may, that A is unital and $1 \in B$ and let $Y' = \{f|_C : f \in Y\}$. Then Y' is an r-closed subset of $S(C)$. Also, π' is the direct sum of some GNS representations of C induced by elements of Y'. To see this last fact, note that although the restriction to C of a GNS representation from Y need not be cyclic, it is a direct sum of cyclic representations. Because Y is r-closed each of these cyclic representations comes from an element of Y'. Now the proof of Lemma 3 and the remark after it (applied to C and Y' instead of A and Y) show that $\pi'(B)$ and $\pi'(C)$ generate the same von Neumann algebras (note that this statement depends only on the quasi-equivalence class of π'). Therefore $\pi'(x) \in \pi'(B)''$, as desired.

Theorem 5. (a) *Assume that B is a C^*-subalgebra of A, $x \in A$, and any two elements of $P(A) \cup \{0\}$ which agree on B agree also on x. If A or B is* GCR *or if B is separable and nuclear, then $x \in B$. Also, if for every C^*-algebra C such that $B \subset C \subset A$, the pair (B, C) fails to be a counterexample for the Stone-Weierstrass conjecture, then $x \in B$.*
(b) *Assume that B is a separable C^*-subalgebra of A, $x \in A$, and any two elements of $F(A) \cup \{0\}$ which agree on B agree also on x. Then $x \in B$.*
(c) *Assume that B is a C^*-subalgebra of A, $x \in A$, and any two elements of $\overline{P(A)} \cup \{0\}$ which agree on B agree also on x. Then $x \in B$.*

Proof. (a) follows immediately from Corollary 4 and the Stone-Weierstrass theorems of Kaplansky and Sakai quoted above.
(b) Let C be the C^*-algebra generated by B and x. Then C is separable and 6.1 of [14] implies that every factorial state of C extends to a factorial state of A. Therefore we may change notation and assume $A = C$. Now the result follows immediately from Lemma 3 [with $Y = F(A)$] and the Stone-Weierstrass theorem of Longo and Popa.
(c) Let C be as above. Since every element of $P(C)$ extends to an element of $P(A)$, then every element of $\overline{P(C)}$ extends to an element of $\overline{P(A)}$. (The closures are with respect to the weak* topologies of C^* and A^*, and the restriction map is continuous.) Thus again we may change notation and assume $A = C$.

Let $Y = \overline{P(A)} \cap S(A)$. We prove that Y is r-closed by verifying the hypotheses of Lemma 1. Obviously Y is norm closed. Assume $f \in Y$, $a \in A$, and $f(a^*a) = 1$, and choose f_i in $P(A)$ such that $f_i \to f$. Then $f_i(a^*a) \to 1$. If $g_i = f_i(a^*a)^{-1} f_i(a^* \cdot a)$, then $g_i \in P(A)$ and $g_i \to f(a^* \cdot a)$. Therefore $f(a^* \cdot a)$ is in Y and Y is r-closed.

It now follows from Lemma 3 that B separates F_Y, and it is not hard to see that F_Y contains $\overline{P(A)} \cup \{0\}$. The quickest way is to quote Lemma 11 of [19], which implies that $\overline{P(A)}$ is a union of weak* closed faces of $Q(A)$ (cf. [5], bottom of page 136). But it is less technical (and routine for this type of problem) to avoid this issue by adjoining an identity to A and B. Once we know that B separates $\overline{P(A)} \cup \{0\}$, it follows from Glimm's Stone-Weierstrass theorem that $B = A$ and hence $x \in B$.

Remark. Lemma 11 of [19] can be generalized (see Theorem 3.8 of [4]). In the terminology of this paper one has: If Y is r-closed, then the norm closure of Y is a union of norm closed faces of $S(A)$ and the weak* closure of Y is a union of weak* closed faces of $Q(A)$.

Theorem 6. *If A is a C^*-algebra, $x \in zA^{**}$, and x is uniformly continuous on $P(A) \cup \{0\}$, then $x \in zA$.*

Proof. We have noted above that each pure state of A determines a normal state of zA^{**}. Let X be the set of states of zA^{**} obtained in this way. Then X determines the order of zA^{**}. Therefore $\bar{X} \supset \overline{P(zA^{**})}$, where the closure is with respect to the weak* topology of $(zA^{**})^*$ (cf. [10, Lemma 3.4.1]).

Suppose f_1 and f_2 are in $\bar{X} \cup \{0\}$ and $f_1|_{zA} = f_2|_{zA}$. Then there are nets (g_i) and (h_j) in $P(A) \cup \{0\}$ such that $g_i \to f_1$ and $h_j \to f_2$, pointwise on zA^{**}. It follows that $g_i - h_j \to 0$, pointwise on zA. In other words $g_i - h_j \to 0$ in the weak* topology of A^*. Since x is uniformly continuous on $P(A) \cup \{0\}$, this implies $g_i(x) - h_j(x) \to 0$, which in turn implies $f_1(x) - f_2(x) = 0$. Now the conclusion $x \in zA$ follows from Theorem 5(c) with zA^{**} playing the role of A and zA the role of B.

Remark. Glimm's Stone-Weierstrass theorem is in some sense the weakest of the ones discussed here (though it is the only one with no extra hypotheses on A or B). Therefore it might be thought that by assuming A is GCR (say), we could replace uniform continuity with continuity. The memoir [3] of Akemann and Shultz makes it clear that this is wrong. In fact the basic results of [3], including in particular Proposition 2.16 on p. 28, show that there is a separable GCR algebra A and an element x of zA^{**} such that every element of $C^*(x)$ is continuous on $Q_{\mathrm{at}}(A)$ and $x \notin zA$. Here $C^*(x)$ is the C^*-algebra generated by x and $Q_{\mathrm{at}}(A)$ is the set of atomic quasi-states of A.

Corollary 7. *Let w be the smallest central projection in A^{**} which supports every element of $P(A)$. If $x \in A^{**}$ and x is continuous on $\overline{P(A)}$, then there is a in A such that $wa = wx$.*

Proof. If A is non-unital, then $\overline{P(A)}$ contains 0; and if A is unital, then 0 is isolated in $\overline{P(A)} \cup \{0\}$. Thus the hypothesis implies that x is uniformly continuous on $P(A) \cup \{0\}$. By the theorem, there is a in A such that $za = zx$. Since a and x are both continuous on $\overline{P(A)}$, this implies that a and x agree on $\overline{P(A)}$. Hence a and x agree on V, the norm closed subspace of A^* generated by $\overline{P(A)}$. Now it follows from Lemma 2 and its proof that V is wA^*. Hence $wa = wx$.

Remark. It might be interesting to study w if this has not already been done. When A is NGCR w is the projection tacitly considered by Shultz in the proofs of Lemma 16 and Theorem 17 of [19].

Corollary 8. *If $x \in zA^{**}$ and x is uniformly continuous on $P(A)$, then $x \in z\tilde{A}$, where $\tilde{A}$ is the C^*-algebra generated by A and the identity of A^{**}.*

Proof. There is nothing to prove if A is unital. Therefore assume A non-unital. If (f_i) is a net in $P(A)$ converging to 0, then the hypothesis implies that $(f_i(x))$ is Cauchy. Thus there is λ in $\mathbb{C}$ such that $(f_i(x))$ converges to λ for all (f_i) as above. The result now follows from Theorem 6 applied to $x - \lambda z$.

If I is a (closed, two-sided) ideal of A, then every positive functional on I has a unique norm-preserving extension to a positive functional on A, and in this way $Q(I)$ is identified with a (split, not weak* closed in general) face of $Q(A)$. This identification is homeomorphic on $S(I)$, but not generally on $Q(I)$, for the two weak* topologies. A remark at the end of Sect. 1 of [5] suggests the question: When is the map indicated above uniformly continuous from $P(I)$ to $P(A)$?

Corollary 9. *Let I be an ideal of a C^*-algebra A and let θ be the natural map from A to $M(I)$, where $M(I)$ is the multiplier algebra of I.*
(a) *The following are equivalent:*
 (i) *The natural map from $S(I)$ to $S(A)$ is uniformly continuous.*
 (ii) *The natural map from $P(I)$ to $P(A)$ is uniformly continuous.*
 (iii) *$\theta(A) \subset \tilde{I}$.*
(b) *The natural map from $P(I) \cup \{0\}$ to $P(A) \cup \{0\}$ is uniformly continuous if and only if $A = I \oplus I^{\perp}$, where $I^{\perp}$ is the annihilator of I.*
(c) (i)–(iii) *of* (a) *are also equivalent to:*
(iv) *$\dim(A/I \oplus I^{\perp}) \leqq 1$, and if the dimension is 1, then $A/I^{\perp}$ is unital.*

Proof. (a) (i)$\Rightarrow$(ii) is obvious.

For (ii)$\Rightarrow$(iii) let z_I be the maximal atomic projection in I^{**}. Then for a in A the hypothesis implies that $z_I a$ is uniformly continuous on $P(I)$. Then by Corollary 8 applied to I there is b in $\tilde{I}$ such that $z_I b = z_I a$. Then $\theta(a) - b$ is an element of $M(I)$ whose atomic part is 0, and hence $\theta(a) = b$.

For (iii)⇒(i) we need only prove that every element of A is uniformly continuous on $S(I)$. But a and $\theta(a)$ have the same restriction to $S(I)$, and it is obvious that any element of $\tilde{I}$ is uniformly continuous on $S(I)$.

(b) A proof similar to the above, using Theorem 6 instead of Corollary 8, shows that uniform continuity on $P(I)\cup\{0\}$ is equivalent to $\theta(A)\subset I$. But clearly $\theta(A)\supset\theta(I)=I$, and the kernel of θ is $I^\perp$. Thus the condition is equivalent to $A=I\oplus I^\perp$.

(c) (iii)⇒(iv). In view of (b) we may assume $\theta(A)=\tilde{I}$ (and I non-unital). Then $I\oplus I^\perp$ has codimension one. If $\theta(e)=1$, then the image of e is an identity for $A/I^\perp$. (iv)⇒(iii) is also easy since (iv) allows us to compute θ explicitly.

Remarks. Corollary 9(b) can be deduced from Shultz [19] instead of from Theorem 6, but we do not know whether Corollary 9(a) can be deduced from [19].

We now consider a hereditary C^*-subalgebra, B, of A instead of an ideal. Most of the remarks made before Corollary 9 still apply to the relationship of $Q(B)$ and $Q(A)$. The only exception is that $Q(B)$ is no longer a split face of $Q(A)$. It is only a norm closed face.

Corollary 9′. *Let B be a hereditary C^*-subalgebra of A and let θ be the natural map from A to $QM(B)$, where $QM(B)$ is the space of quasi-multipliers of B.*

(a) *The following are equivalent:*

(i) *The natural map from $S(B)$ to $S(A)$ is uniformly continuous.*

(ii) *The natural map from $P(B)$ to $P(A)$ is uniformly continuous.*

(iii) $\theta(A)\subset\tilde{B}$.

(b) *The following are equivalent:*

(i′) *The natural map from $Q(B)$ to $Q(A)$ is continuous.*

(ii′) *The natural map from $P(B)\cup\{0\}$ to $P(A)\cup\{0\}$ is uniformly continuous.*

(iii′) $\theta(A)\subset B$.

(iv′) *B is a corner of A.*

(c) (i)–(iii) *imply that B is a corner of an ideal of A.*

Proof. The proof of (a) is the same as for Corollary 9(a). One minor change is that the symbol "$z_I a$" has to be replaced by "$z_B\theta(a)$" or "$z_B qaq$," where q is the open projection corresponding to B. (Thus q is the identity of B^{**}.)

(b) (i′)⇔(ii′)⇔(iii′). A proof similar to the above shows that (ii′) and (iii′) are equivalent to uniform continuity on $Q(B)$. But since $Q(B)$ is compact, this is equivalent to (i′).

(iii′)⇔(iv′). Since B is a corner if and only if $q\in M(A)$, $\theta(a)=qaq$, and $(qAq)\cap A=B$, this is equivalent to: $qAq\subset A\Leftrightarrow q\in M(A)$. Perhaps the least technical way to see this is to quote [7], 2.23 (ii), p. 880.

(c) This follows from [7], 2.23 (i).

Remark. It would be desirable to have a structural criterion for the conditions in 9′(a), analogous to condition (iv) in Corollary 9. A reasonably satisfactory analysis can be carried out, but it does not produce a simply stated conclusion. In the case where A is unital the following is a simply stated sufficient condition for 9′(a), but it is far from necessary, as the analysis sketched below shows: B is a corner of an ideal of A and the hereditary C^*-subalgebra generated by B and $B^\perp$ is a maximal hereditary C^*-subalgebra of A. (The two parts of this condition are independent.)

Let I be the ideal generated by B, $A_1 = A/I$, $B^{\perp}$ the two-sided annihilator of B, which is again a hereditary C^*-subalgebra, and B' the hereditary C^*-subalgebra generated by B and $B^{\perp}$. B' can also be described as the largest hereditary C^*-subalgebra such that B is a corner of it. Since 9′(c) implies that B is a corner of I, $I \subset B'$. Also, B' and $B^{\perp}$ have the same image in A_1. We regard I as an algebra of 2×2 matrices whose upper left corner is in B, lower right corner is in $B^{\perp}\cap I$, and upper right corner is in $(BAB^{\perp})^{-}$. According to Busby [9], the rest of the structure of A is determined by a $*$-homomorphism $\tau: A_1 \to M(I)/I$. If $M(I)/I$ is also regarded as an algebra of 2×2 matrices, then (iii) is equivalent to the statement that the upper left corner of $\tau(x)$ is a scalar, $\forall x \in A_1$. Denote this scalar by $f(x)$. [Equivalently, $f(x)$ is the scalar component of $\theta(a)$ for a in the pre-image of x.] Then $f \in Q(A_1)$, $f \neq 0$ unless the conditions of 9′(b) are satisfied (an uninteresting case), and $\|f\|$ can be less than 1 if A is non-unital. The image of $B^{\perp}$ in A_1 is $\{x \in A_1 : f(x^*x) = f(xx^*) = 0\}$. Thus B' is maximal if and only if f is a multiple of a pure state. (The justification of the sufficient condition of the first paragraph of this remark uses 3.13.6 of [15] to prove that $\theta(a)$ is in B whenever $a+I$ is in the kernel of f. In this context we are given the relationship between the image of $B^{\perp}$ and f stated above, but we are not given that $\theta(a) - f(a+I)$ is in B.)

If we are given B (non-unital), A_1, and f, it is easy to construct an example for this data. Let $\pi: A_1 \to B(H)$ be the GNS representation constructed from f (if $\|f\| < 1$ we add a degeneracy subspace to H), and let v be a unit vector in H such that $f = (\pi(\cdot)v, v)$. Let $I = B \otimes \mathscr{K}(H)$ and identify B with $B \otimes p$, where $\mathscr{K}(H)$ is the algebra of compact operators on H and p is the rank one projection on $\mathbb{C}v$. Let $\tau(x)$ be the image in $M(I)/I$ of $1 \otimes \pi(x)$. (We note in passing that I must be strongly Morita equivalent to B [8], and in the separable case I is stably isomorphic to B [6].)

References

1. Akemann, C.A.: The general Stone-Weierstrass problem. J. Funct. Anal. **4**, 277–294 (1969)
2. Akemann, C.A., Anderson, J.: The Stone-Weierstrass problem for C^*-algebras. In: Invariant subspaces and other topics, Operator Theory: Adv. Appl. **6**, 15–32 (1982)
3. Akemann, C.A., Shultz, F.W.: Perfect C^*-algebras. Mem. Am. Math. Soc. **326** (1985)
4. Archbold, R.J., Batty, C.J.K.: On factorial states of operator algebras. III. J. Op. Theory **15**, 53–81 (1986)
5. Batty, C.J.K., Archbold, R.J.: On factorial states of operator algebras. II. J. Op. Theory **13**, 131–142 (1985)
6. Brown, L.G.: Stable isomorphism of hereditary subalgebras of C^*-algebras. Pac. J. Math. **71**, 335–348 (1977)
7. Brown, L.G.: Semicontinuity and multipliers of C^*-algebras. Canad. J. Math. **40**, 865–988 (1988)
8. Brown, L.G., Green, P., Rieffel, M.A.: Stable isomorphism and strong Morita equivalence of C^*-algebras. Pac. J. Math. **71**, 349–363 (1977)
9. Busby, R.C.: Double centralizers and extensions of C^*-algebras. Trans. Am. Math. Soc. **132**, 79–99 (1968)
10. Dixmier, J.: *C^*-algebras*. Amsterdam: North-Holland 1977
11. Effros, E.G.: Order ideals in a C^*-algebra and its dual. Duke Math. J. **30**, 391–412 (1963)
12. Glimm, J.: A Stone-Weierstrass theorem for C^*-algebras. Ann. Math. **72**, 216–244 (1960)
13. Kadison, R.V.: States and representations. Trans. Am. Math. Soc. **103**, 304–319 (1962)
14. Longo, R.: Solution of the factorial Stone-Weierstrass conjecture. An application of the theory of standard split W^*-inclusions. Invent. Math. **76**, 145–155 (1984)

15. Pedersen, G.K.: C^*-algebras and their automorphism groups. London: Academic Press 1979
16. Popa, S.: Semi-regular maximal abelian $*$-subalgebras and the solution to the factor state Stone-Weierstrass problem. Invent. Math. **76**, 157–161 (1984)
17. Prosser, R.T.: On the ideal structure of operator algebras. Mem. Am. Math. Soc. **45** (1963)
18. Sakai, S.: On the Stone-Weierstrass theorem of C^*-algebras. Tohoku Math. J. **22**, 191–199 (1970)
19. Shultz, F.W.: Pure states as a dual object for C^*-algebras. Commun. Math. Phys. **82**, 497–509 (1982)

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