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Bicovariant Differential Calculus on Quantum Groups $SU_q(N)$ and $SO_q(N)$

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Abstract. Following Woronowicz's proposal the bicovariant differential calculus on the quantum groups $SU_q(N)$ and $SO_q(N)$ is constructed. A systematic construction of bicovariant bimodules by using the $\hat{R}_q$ matrix is presented. The relation between the Hopf algebras generated by the linear functionals relating the left and right multiplication of these bicovariant bimodules, and the q-deformed universal enveloping algebras is given. Imposing the conditions of bicovariance and consistency with the quantum group structure the differential algebras and exterior derivatives are defined. As an application the Maurer-Cartan equations and the q-analogue of the structure constants are formulated.

1. Introduction

Recently a class of non-commutative non-cocommutative Hopf algebra has been found in the investigations of the integrable systems. These Hopf algebras are q-deformed function algebras of classical groups. This structure is called "quantum group" [Dri].

The structure of the quantum groups suggests the possibility of investigating a geometry where we can even consider discarding the commutativity of the algebra of coordinate functions. It is interesting to ask whether one can find applications of this new class of symmetry to some physical systems other than the integrable models.

Following this idea, the first step one has to make is to provide the appropriate tools for this investigation. To this end we take the usual application of the group theory as a guiding principle for the generalization of the q-deformed quantities. This is also useful since in the limit $q \to 1$ we wish to reproduce the results obtained in the ordinary classical group case.

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In this paper we formulate the differential calculus on quantum groups to investigate their geometrical aspects. This also gives an example of the non-commutative geometry. The framework of such a non-commutative differential calculus has been developed by Woronowicz following the general ideas of Connes [Connes]. In a series of papers, introducing the bimodule over the quantum group various theorems concerning the differential forms and exterior derivatives were presented. Generalizations of the construction of bicovariant bimodules to other quantum groups are also investigated in [Rosso]. However, the concrete constructions of the differential calculus must still be developed. For the $SU_q(2)$ case, two types of differential calculi called $3D$ [Wor1] which is not bicovariant and the bicovariant $4D_{\pm}$ calculus [Wor3, Stach, PW, Weich] have been given. [The 3D calculus has been extended to the case of $GL_q(1|1)$ [SVZ] and $GL_{p,q}(2)$ [SWZ].]

In [Wor3] the author presented the bicovariant differential calculus for quantum groups where the differential forms transform under both left and right transformations covariantly. This is a natural q-deformation of the differential calculus on classical groups.

The aim of this paper is to develop the concrete bicovariant differential calculus for various other known quantum groups following Woronowicz's programme in [Wor3] applying the formulation of the quantum groups proposed by Faddeev et al. [FRT, Takh].

The paper is organized as follows. In Sect. 2 we introduce briefly the quantum groups and the concept of bicovariant bimodules following Woronowicz to establish our notations. In Sect. 3 we construct the fundamental bicovariant bimodules which provide the building block for constructing any bicovariant bimodules. We also show that the algebras of the functionals defining the relation between the left and right multiplication are equivalent to the universal enveloping algebras. The bicovariant bimodule including the adjoint representation is constructed in Sect. 4. In Sect. 5 using the result of Sect. 4, we construct the first order differential calculus using the idea of the extended module of [Wor1, Wor3]. In Sect. 6 the higher order differential calculus is constructed. As an application in Sect. 7 we write down the q-analogue of the Maurer-Cartan equation and the structure constants.

2. Quantum Groups and Bicovariant Bimodule

In this section we introduce briefly some necessary concepts in order to formulate the differential calculus on the quantum groups.

2.1. Quantum Groups

The quantum group $\mathscr{A}$ is a non-commutative non-cocommutative Hopf algebra generated by N^2 elements $M^i{}_j$ $(i,j=1,\dots,N)$. The algebra $\mathscr{A}$ has a unit element which we denote by $\mathbf{1}$. The coalgebra of the quantum group is defined by the following maps.

The coproduct Δ of the quantum groups, a multiplicative algebra homomorphism $\Delta:\mathscr{A}\to\mathscr{A}\otimes\mathscr{A}$, is defined for the generators M^i_j as

$$\Delta(M^i{}_j)=M^i{}_k\otimes M^k{}_j\,, \tag{2.1}$$

where the summation over repeated indices k runs from 1 to N.

The antipode (coinverse) κ of $\mathscr{A}$ is a linear antimultiplicative map $\kappa:\mathscr{A}\to\mathscr{A}$ which is defined for the generators by

$$\kappa(M^i{}_k)M^k{}_j=M^i{}_k\kappa(M^k{}_j)=\delta^i_j\mathbf{1}\,. \tag{2.2}$$

The counit ε, an algebra homomorphism $\varepsilon:\mathscr{A}\to\mathbf{C}$, is defined for the generators as

$$\varepsilon(M^i{}_j)=\delta^i{}_j\,. \tag{2.3}$$

These maps defining the coalgebra on $\mathscr{A}$ satisfy the standard axioms of a Hopf algebra [Abe].

The quantum groups are a class of Hopf algebras obtained by the q-deformation of the algebra Fun(G), the algebra of the functions on the group G. [Strictly speaking it is developed on the polynomials of the generators $M^i{}_j$, a subset of $C^\infty(G)$.]

In these algebras the non-commutativity is controlled by the $\hat{R}_q$ matrix which is a solution of the Yang-Baxter equation

$$\hat{R}_q^{ij}{}_{j'i'}\hat{R}_q^{i'k}{}_{k'i''}\hat{R}_q^{j'k'}{}_{k''j''}=\hat{R}_q^{jk}{}_{k'j'}\hat{R}_q^{ik'}{}_{k''i'}\hat{R}_q^{i'j'}{}_{j''i''}\,. \tag{2.4}$$

The commutation relation between the generators $M^i{}_j$ is then given by

$$\hat{R}_q^{ij}{}_{j'i'}M^{j'}{}_{j''}M^{i'}{}_{i''}=M^i{}_{i'}M^j{}_{j'}\hat{R}_q^{i'j'}{}_{j''i''}\,. \tag{2.5}$$

In this paper we consider mainly the quantum groups $SU_q(N)$ and $SO_q(N)$ which were introduced with the help of $\hat{R}_q$ matrices [FRT, Takh, Rosso], and the parameter of the deformation, q is in general a positive real number.

The $SU_q(N)$ is a q-deformed Fun($SU(N)$) and its generators satisfy the unimodularity condition

$$\det M=1\,. \tag{2.6}$$

We also consider the quantum group $SO_q(N)$. The $SO_q(N)$ is a q-deformed Fun($SO(N)$) and instead of the unimodularity condition (2.6) the generators $M^i{}_j$ satisfy the orthogonality condition

$$C_{ij}M^i{}_{i'}M^j{}_{j'}=C_{i'j'}\,, \tag{2.7}$$

where C_{ij} is an $N\times N$ matrix corresponding to the metric.

Furthermore in order to define these quantum groups we have to consider the $*$-structure [Wor2]. The $*$-structure is defined by an antilinear $*$-operation:

$$*:\mathscr{A}\ni a\to a^*\in\mathscr{A}\,, \tag{2.8}$$

such that $\forall a,\,b\in\mathscr{A}$ and $\forall\lambda\in\mathbf{C}$:

$$(\lambda ab)^*=\lambda^*b^*a^*\,, \tag{2.9}$$

$$\kappa(\kappa(a^*)^*)=a\,, \tag{2.10}$$

where λ^* is the complex conjugate of λ. The coproduct and counit are $*$-homomorphisms. We call the Hopf algebra with $*$-structure a $*$-Hopf algebra.

The conjugated element is denoted as

$$(M^i{}_j)^*=M^{*i}{}_j\equiv M^{\dagger j}{}_i\,. \tag{2.11}$$

Note that we introduce the $M^\dagger$ in order to keep the manifest covariance as in the commuting case, i.e. a lower suffix transforms as a covariant and an upper index as a contravariant quantity.

Then the unitarity condition is represented as

$$M^{\dagger i}{}_j = \kappa(M^i{}_j). \tag{2.12}$$

The $*$-operation is a generalization of the complex conjugation. Therefore it is convenient to consider the bigger algebra generated by $M^i{}_j$ and $M^{\dagger i}{}_j$ with the commutation relations (2.5) and

$$\hat{R}_q^{ij}{}_{k'l'} M^{\dagger l'}{}_l M^{\dagger k'}{}_k = M^{\dagger j}{}_{j'} M^{\dagger i}{}_{i'} \hat{R}_q^{i'j'}{}_{kl}, \tag{2.13}$$

and

$$\hat{R}_q^{i'j}{}_{k'l} M^{\dagger i}{}_{i'} M^{k'}{}_k = M^j{}_{j'} M^{\dagger l'}{}_l \hat{R}_q^{ij'}{}_{kl'}, \tag{2.14}$$

or

$$\hat{R}_q^{-1 i'j}{}_{k'l} M^{\dagger i}{}_{i'} M^{k'}{}_k = M^j{}_{j'} M^{\dagger l'}{}_l \hat{R}_q^{-1 ij'}{}_{kl'}. \tag{2.15}$$

Then the unitarity condition reduces the bigger algebra to the original $\mathscr{A}$. The commutation relations (2.13)–(2.15) are simply defined so that they are equivalent to (2.5) when one substitutes the unitarity condition (2.12).

2.2. Bicovariant Bimodule

The differential 1-forms on a Lie group manifold are sections of the cotangent bundle. The space of all sections on the cotangent bundle $C^\infty(T^*(G))$ is a bimodule over $C^\infty(G)$. On this space there is a natural action of the group G which is expressed by the coaction of $C^\infty(G)$ in the Hopf algebra terminology. In order to construct the differential calculus on the quantum groups we employ these algebraic structures. Therefore we introduce the bimodule-bicomodule over $\mathscr{A}$. We consider here especially the case that those bimodule-bicomodules are bicovariant, i.e. bicovariant bimodules over $\mathscr{A}$ [Wor3].

On the bicovariant bimodule Γ there exist left coaction Δ_L and right coation Δ_R of $\mathscr{A}$

$$\Delta_L : \Gamma \to \mathscr{A} \otimes \Gamma, \tag{2.16}$$

$$\Delta_R : \Gamma \to \Gamma \otimes \mathscr{A}. \tag{2.17}$$

Following the general definition of the bicovariant bimodule [Definitions 2.1–2.3 in [Wor3]] we require that the coactions have the following properties:

After identifying coactions and coproduct on $\mathscr{A}$ the coactions are bimodule homomorphisms

$$\Delta_L(a\varrho b) = \Delta(a)\Delta_L(\varrho)\Delta(b), \tag{2.18}$$

$$\Delta_R(a\varrho b) = \Delta(a)\Delta_R(\varrho)\Delta(b), \tag{2.19}$$

and they satisfy

$$(\varepsilon \otimes \mathrm{id})\Delta_L(\varrho) = \varrho, \tag{2.20}$$

$$(\mathrm{id} \otimes \varepsilon)\Delta_R(\varrho) = \varrho. \tag{2.21}$$

Furthermore we require that the left coaction and the right coaction commute:

$$(\mathrm{id} \otimes \Delta_R)\Delta_L = (\Delta_L \otimes \mathrm{id})\Delta_R. \tag{2.22}$$

We call an element $\varrho \in \Gamma$ left invariant if

$$\Delta_L(\varrho) = \mathbf{1} \otimes \varrho, \tag{2.23}$$

and right invariant if

$$\Delta_R(\varrho) = \varrho \otimes \mathbf{1}, \tag{2.24}$$

where $\mathbf{1}$ is the unit of the algebra $\mathscr{A}$.

The $*$-structure in the algebra $\mathscr{A}$ can be extended to the bicovariant bimodule Γ in a natural way. There exists a unique antilinear antimultiplicative map [Wor3]:

$$*: \Gamma \ni \varrho \to \varrho^* \in \Gamma^*, \tag{2.25}$$

such that $\forall a, b \in \mathscr{A}$:

$$(a\varrho b)^* = b^* \varrho^* a^*. \tag{2.26}$$

The $*$-operation commutes with the coactions

$$\Delta_L(\varrho)^* = \Delta_L(\varrho^*), \tag{2.27}$$

$$\Delta_R(\varrho)^* = \Delta_R(\varrho^*). \tag{2.28}$$

In any bicovariant bimodule, one can find a linear right invariant subspace Γ_{inv}. Let the basis of this subspace be $\eta^J \in \Gamma_{\mathrm{inv}}$. Then any element $\varrho \in \Gamma$ can be represented in the form

$$\varrho = \sum_J a_J \eta^J, \tag{2.29}$$

where the index J runs over all elements of the right invariant basis and the elements $a_J \in \mathscr{A}$ are determined uniquely (Theorem 2.3 in [Wor3]).

The left coaction on right invariant elements is defined by

$$\Delta_L(\eta^I) = \mathbf{T}^I{}_J \otimes \eta^J, \tag{2.30}$$

where $\mathbf{T}^I{}_J \in \mathscr{A}$ is uniquely determined when we fix the basis η^J. Note that throughout this paper we use the upper case indices such as I, J, K to distinguish the right invariant basis and we also abbreviate the summation symbol over upper case indices if it is apparent.

The left invariant basis $\omega^J \in \Gamma$ can be introduced as

$$\omega^J = \kappa(\mathbf{T}^J{}_K)\eta^K, \tag{2.31}$$

with the $\mathbf{T}^I{}_J$ being the matrix defined in Eq. (2.30). It is easy to confirm that ω^J is left invariant using the above definitions. The ω^J form the left invariant basis of Γ since any element of Γ can be represented as in Eq. (2.29). The right coaction on ω^J is given by

$$\Delta_R(\omega^J) = \omega^K \otimes \kappa(\mathbf{T}^J{}_K). \tag{2.32}$$

The bicovariant bimodule is characterized by the following functionals which relate the right multiplication to the left multiplication of $a \in \mathscr{A}$ on $\varrho \in \Gamma$.

Let η^J be the basis of the right invariant subspace Γ_{inv}, there exist linear functionals $f^I{}_J$,

$$f^I{}_J : \mathscr{A} \ni a \to f^I{}_J(a) \in \mathbf{C}, \tag{2.33}$$

such that $\forall a, b \in \mathscr{A}$:

$$\eta^I b = \sum_J (b * f^I{}_J)\eta^J, \tag{2.34}$$

$$a\eta^I = \sum_J \eta^J (a * f^I{}_J \circ \kappa), \tag{2.35}$$

where the indices I, J run over the full basis of right invariant elements. The convolution product of an element $a \in \mathscr{A}$ and a functional f is defined as [Wor3]:

$$a * f \equiv (f \otimes \mathrm{id})\Delta(a) = \sum_S f(a'_S)a''_S, \tag{2.36}$$

where $\Delta(a) = \sum_S a'_S \otimes a''_S$.

The functionals introduced above satisfy $\forall a, b \in \mathscr{A}$:

$$f^I{}_J(ab) = \sum_K f^I{}_K(a) f^K{}_J(b), \tag{2.37}$$

and

$$f^I{}_J(\mathbf{1}) = \delta^I{}_J. \tag{2.38}$$

The main problem of the explicit construction of the differential calculus is therefore to find the explicit form of the $f^I{}_J$. The functionals $f^I{}_J$ introduced above are elements of the linear functionals on $\mathscr{A}$, $\mathrm{Hom}(\mathscr{A}, \mathbf{C})$. In $\mathrm{Hom}(\mathscr{A}, \mathbf{C})$, one can define a product, the convolution product [Wor3]: for two functionals f_1, $f_2 \in \mathrm{Hom}(\mathscr{A}, \mathbf{C})$ and $a \in \mathscr{A}$

$$f_1 * f_2(a) \equiv (f_1 \otimes f_2)\Delta(a). \tag{2.39}$$

Definition. *$\mathscr{A}'$ is the unital $\mathbf{C}$-algebra generated by the functionals $f^I{}_J$ with the convolution product (2.39).*

The Hopf algebra structure of $\mathscr{A}$ induces a Hopf algebra structure on $\mathscr{A}'$. From Eqs. (2.37) and (2.38) we can read off how the coproduct Δ' and the counit ε' of $\mathscr{A}'$ are acting on the functionals $f^I{}_J$:

$$\Delta'(f^I{}_J) = f^I{}_K \otimes f^K{}_J, \tag{2.40}$$

$$\varepsilon'(f^I{}_J) = \delta^I{}_J. \tag{2.41}$$

We can also prove that they satisfy $\forall a \in \mathscr{A}$:

$$\sum_J f^I{}_J * (f^J{}_K \circ \kappa)(a) = \delta^I_K \varepsilon(a), \tag{2.42}$$

and therefore the antipode κ' of the algebra $\mathscr{A}'$ is

$$\kappa'(f^I{}_J) = f^I{}_J \circ \kappa. \tag{2.43}$$

This means that the functionals $f^I{}_J$ are a special set of elements of $\mathscr{A}'$ such that their coproducts are represented by matrix multiplication and the antipode is given by the inverse of the matrix.

Since $\mathscr{A}$ is a $*$-Hopf algebra, one of the important properties of the algebra $\mathscr{A}'$ is *that one can find an induced $*$-structure:*

Proposition 1. *Let $\chi \in \mathscr{A}'$. Define the $*$-operation of $\mathscr{A}'$ as $\forall a \in \mathscr{A}$:*

$$\chi^*(a) := \{\chi(\kappa(a^*))\}^*. \tag{2.44}$$

Then this $*$-operation in $\mathscr{A}'$ is an antimultiplicative involution and satisfies

$$\kappa'(\kappa'(\chi^*)^*)=\chi\,. \tag{2.45}$$

Proof. 1) The operation is an involution of $\mathscr{A}'$ since for all elements $a\in\mathscr{A}$ it satisfies:

$$\begin{aligned}(\chi^*)^*(a)&=\{\chi^*(\kappa(a^*))\}^*\\&=\chi(\kappa(\kappa(a^*)^*))\\&=\chi(a)\,.\end{aligned} \tag{2.46}$$

2) The operation is antimultiplicative: Let ξ, $\chi\in\mathscr{A}'$, then $\forall a\in\mathscr{A}$,

$$\begin{aligned}(\chi*\xi)^*(a)&=\{(\chi*\xi)(\kappa(a^*))\}^*\\&=\{(\chi\otimes\xi)\Delta(\kappa(a^*))\}^*\\&=\sum_S\{(\chi\otimes\xi)(\kappa(a_S''^*)\otimes\kappa(a_S'^*))\}^*\\&=\sum_S\chi^*(a_S'')\xi^*(a_S')\\&=(\xi^**\chi^*)(a)\,,\end{aligned} \tag{2.47}$$

where $\sum_S a_S'\otimes a_S''=\Delta(a)$.

3) The coproduct Δ' commutes with the $*$-operation: For any element a, $b\in\mathscr{A}$,

$$\begin{aligned}\Delta'(\chi^*)(a\otimes b)&=\chi^*(ab)\\&=\{\chi(\kappa(a^*)\kappa(b^*))\}^*\\&=\sum_S\{\chi_S'(\kappa(a^*))\chi_S''(\kappa(b^*))\}^*\\&=\sum_S(\chi_S'^*\otimes\chi_S''^*)(a\otimes b)=\Delta'(\chi)^*(a\otimes b)\,,\end{aligned} \tag{2.48}$$

where $\Delta'(\chi)=\sum_S(\chi_S'\otimes\chi_S'')$.

4) Equation (2.45) can be shown using Eq. (2.10):

$$\begin{aligned}\kappa'(\kappa'(\chi^*)^*)(a)&=\{\kappa'(\chi^*)(\kappa(\kappa(a)^*))\}^*\\&=\{\kappa'(\chi^*)(a^*)\}^*\\&=\{\chi^*(\kappa(a^*))\}^*\\&=\chi(\kappa(\kappa(a^*)^*))=\chi(a)\,.\quad \text{Q.E.D.}\end{aligned} \tag{2.49}$$

From this it follows that

$$(\chi\circ\kappa^{-1})^*=\chi^*\circ\kappa\,. \tag{2.50}$$

In the following section we explicitly construct the functionals $f^I{}_J$ for various bicovariant bimodules. Given the definitions above we write down the defining conditions for $f^I{}_J$.

Condition 1. Bicovariance: The bicovariance of Eq. (2.34) requires that the functionals $f^I{}_J$ must satisfy

$$\Delta_R(\eta^I)\Delta(b)=\sum_J\Delta(b*f^I{}_J)\Delta_R(\eta^J)\,, \tag{2.51}$$

$$\Delta_L(\eta^I)\Delta(b)=\sum_J\Delta(b*f^I{}_J)\Delta_L(\eta^J)\,. \tag{2.52}$$

By the definition of the convolution product $b * f^I{}_J$ the right covariance (2.51) is trivially satisfied. The condition for $f^I{}_J$ required from the left covariance is obtained by the following considerations:

In order to define the functionals $f^I{}_J$ it is sufficient to define the values of $f^I{}_J(M^i{}_j)$, i.e. when its argument is a generator. Therefore in Eq. (2.34) we take the element $b = M^i{}_j$. Using the definition of the convolution product we get

$$\eta^I M^k{}_l = f^I{}_J(M^k{}_n) M^n{}_l \eta^J . \tag{2.53}$$

The left coaction on the left-hand side of Eq. (2.53) gives

$$\begin{aligned} \Delta_L(\eta^I M^k{}_l) &= \mathbf{T}^I{}_J M^k{}_n \otimes \eta^J M^n{}_l \\ &= \mathbf{T}^I{}_J M^k{}_n \otimes f^J{}_K(M^n{}_m) M^m{}_l \eta^K . \end{aligned} \tag{2.54}$$

The left coaction of the right-hand side gives

$$\begin{aligned} \Delta_L(f^I{}_J(M^k{}_n) M^n{}_l \eta^J) &= f^I{}_J(M^k{}_n) \Delta_L(M^n{}_l \eta^J) \\ &= f^I{}_J(M^k{}_n)(M^n{}_m \mathbf{T}^J{}_K \otimes M^m{}_l \eta^K) . \end{aligned} \tag{2.55}$$

The condition (2.52) requires that (2.54) and (2.55) are equivalent. Therefore comparing these two equations, we get an equation for $f^I{}_J(M^i{}_j)$:

$$\mathbf{T}^I{}_J M^k{}_n f^J{}_K(M^n{}_m) = f^I{}_J(M^k{}_n) M^n{}_m \mathbf{T}^J{}_K . \tag{2.56}$$

(This is the analogue to the Eq. (2.39) in [Wor3].)

Condition 2. Consistency with the quantum group relations:

a) The consistency with the commutation relations of the generators $M^i{}_j$ given in Eq. (2.5) leads to the following condition:

$$\begin{aligned} &\eta^I(\hat{R}_q{}^{ij}{}_{j'i'} M^{j'}{}_{j''} M^{i'}{}_{i''} - M^i{}_{i'} M^j{}_{j'} \hat{R}_q{}^{i'j'}{}_{j''i''}) \\ &= ((\hat{R}_q{}^{ij}{}_{j'i'} M^{j'}{}_{j''} M^{i'}{}_{i''} - M^i{}_{i'} M^j{}_{j'} \hat{R}_q{}^{i'j'}{}_{j''i''}) * f^I{}_K)\eta^K . \end{aligned} \tag{2.57}$$

The left-hand side is zero due to Eq. (2.5). The right-hand side is obtained by using the commutation relations (2.34). Because of the uniqueness of the expansion of Eq. (2.29) the coefficients on the right-hand side have to vanish. This leads to

$$\hat{R}_q{}^{ij}{}_{j'i'} f^I{}_J(M^{j'}{}_{j''}) f^J{}_K(M^{i'}{}_{i''}) = f^I{}_J(M^i{}_{i'}) f^J{}_K(M^j{}_{j'}) \hat{R}_q{}^{i'j'}{}_{j''i''} . \tag{2.58}$$

b) Other quantum group relations such as the unimodularity (2.6) or orthogonality (2.7) must also be compatible with the bimodule structure.

3. Fundamental Bimodule of $SU_q(N)$ and $SO_q(N)$

The construction of the bicovariant bimodule can be performed in an analogous way to the construction of the representations of classical groups. In this section we construct the fundamental bicovariant bimodules of $SU_q(N)$ and $SO_q(N)$. They are the analogues to the sections in the bundles of the fundamental representations over the groups $SU(N)$ or $SO(N)$. Other bicovariant bimodules can be constructed using them as building blocks.

3.1. Fundamental Bimodule of $SU_q(N)$

We use the $\hat{R}_q$ matrix for the $SU_q(N)$ given in [FRT]. As for our convention see [CSSW]. The commutation relations of the generators $M^i{}_j$ are given in Eq. (2.5). Another condition for the $SU_q(N)$ is the unitarity which is formulated by using the ε tensor, the q-deformed antisymmetric tensor. Since we need some properties of this antisymmetric tensor for the construction of the bimodule we first give its definition and its relation to the $\hat{R}_q$ matrix. For later discussion we also introduce the graphical representation which clarifies the relation to the braid group.

The definition of the N^{th} rank antisymmetric tensor ε is

$$\varepsilon_{i_1\ldots i_N} = q^{\frac{N(N-1)}{4}}(-q)^{\ell(\sigma)}, \tag{3.1}$$

where σ denotes the permutation of the suffices $(i_1,\ldots,i_N)=\sigma(1,2,\ldots,N)$ and $\ell(\sigma)$ is the minimal number of inversions in the permutation σ [Dri, Wor4]. The overall constant is chosen such that the formulas below become simple. To keep the manifest covariance we also introduce the ε tensor with upper indices as:

$$\varepsilon^{i_1\ldots i_N} = (-1)^{N-1} q^{\frac{N(N-1)}{4}} \times (-q)^{\ell(\sigma)}. \tag{3.2}$$

In Fig. 1 we gave the graphical representation of these fundamental quantities [Res, Wor4].

a $\quad 1^{ij}{}_{kl} = \delta^i{}_k \delta^j{}_l =$

b $\quad \hat{R}^{ij}{}_{kl} =$

c $\quad \hat{R}^{-1ij}{}_{kl} =$

d $\quad \epsilon_{i_1 i_2 \ldots i_N} =$

e $\quad \epsilon^{i_1 i_2 \ldots i_N} =$

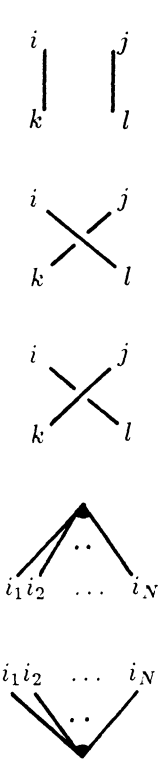


Fig. 1a–e. The graphical representation of the basic quantities of $SU_q(N)$. **a** The unit operator, **b, c** the matrices $\hat{R}^{ij}_{q\,kl}$ and $\hat{R}_q^{-1ij}{}_{kl}$, **d, e** the ε-tensors $\varepsilon^{i_1\ldots i_N}$ and $\varepsilon_{i_1\ldots i_N}$ defined in Eqs. (3.1) and (3.2)

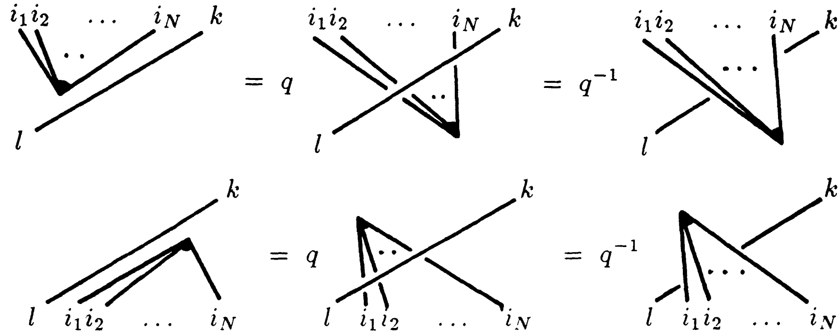


Fig. 2. The graphical representation of the relations among the $\hat{R}_q$ matrix and the ε-tensor. The multiplications of the $\hat{R}_q$ matrices and the ε-tensors are represented by drawing the diagrams one below the other and connecting the lines corresponding to the indices which are summed

Some important properties for the construction of the modules are

$$\varepsilon_{i_1\dots i_N}\hat{R}_q^{ki_1}{}_{i'_1k_1}\hat{R}_q^{k_1i_2}{}_{i'_2k_2}\cdots\hat{R}_q^{k_{s-1}i_s}{}_{i'_sk_s}\cdots\hat{R}_q^{k_{N-1}i_N}{}_{i'_Nl}=q\varepsilon_{i'_1\dots i'_N}\delta^k_l\,, \tag{3.3}$$

$$\varepsilon^{i_1\dots i_N}\hat{R}_q^{ki'_1}{}_{i_1k_1}\hat{R}_q^{k_1i'_2}{}_{i_2k_2}\cdots\hat{R}_q^{k_{s-1}i'_s}{}_{i_sk_s}\cdots\hat{R}_q^{k_{N-1}i'_N}{}_{i_Nl}=q\varepsilon^{i'_1\dots i'_N}\delta^k_l\,, \tag{3.4}$$

(see Fig. 2) and

$$\begin{aligned}\varepsilon_{j_1\dots j_lk_{l+1}\dots k_N}\varepsilon^{k_{l+1}\dots k_Ni_1\dots i_l}&=\varepsilon^{i_1\dots i_lk_{l+1}\dots k_N}\varepsilon_{k_{l+1}\dots k_Nj_1\dots j_l}\\&=(-1)^{(l-1)(N-1)}[\![N-l]\!]!\,[\![l]\!]!\,\mathcal{P}_l^{i_1\dots i_l}{}_{j_1\dots j_l}\,.\end{aligned} \tag{3.5}$$

$\mathcal{P}_l$ is the projector to the l^{th} order antisymmetric tensor representation (see Fig. 3) and

$$[\![N]\!]!=[\![N]\!][\![N-1]\!]\dots[\![1]\!]\,, \tag{3.6}$$

with the definition of the q-number $[\![x]\!]$

$$[\![x]\!]=\frac{q^x-q^{-x}}{q-q^{-1}}\,. \tag{3.7}$$

Especially the projection operator to the second rank antisymmetric tensor $\mathcal{P}_2$ and the one to the second rank symmetric tensor play an important role in the

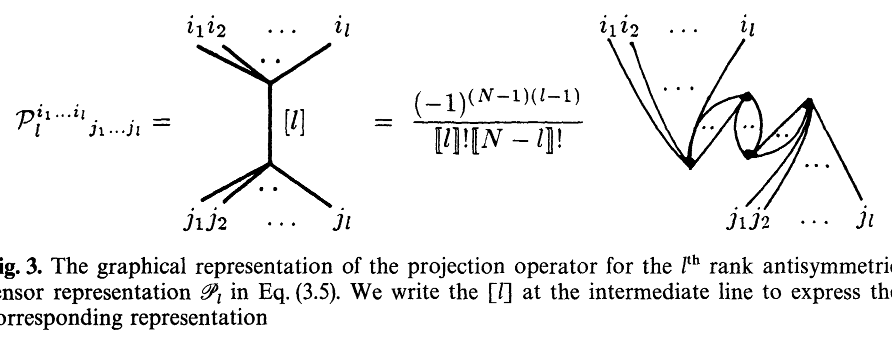


Fig. 3. The graphical representation of the projection operator for the l^{th} rank antisymmetric tensor representation $\mathcal{P}_l$ in Eq. (3.5). We write the $[l]$ at the intermediate line to express the corresponding representation

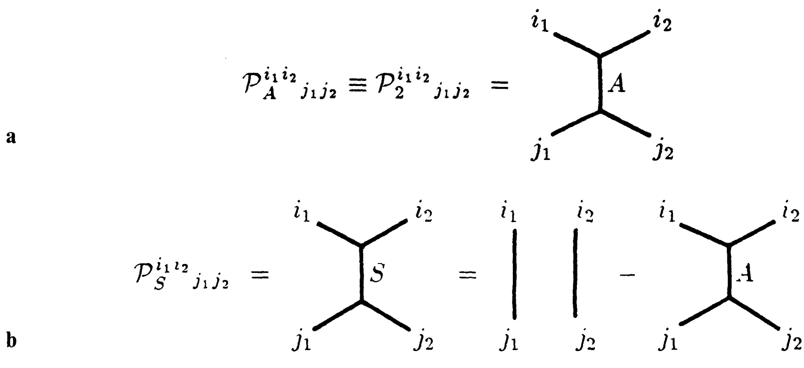


Fig. 4a, b. The projection operators to the second rank tensor in Eqs. (3.8) and (3.9): the antisymmetrizer $\mathscr{P}_A$ in **a**, and the symmetrizer $\mathscr{P}_S$ in **b**

following. We denote them as

$$\mathscr{P}_A \equiv \mathscr{P}_2 , \tag{3.8}$$

and

$$\mathscr{P}_S \equiv \mathbf{1} - \mathscr{P}_A , \tag{3.9}$$

where $\mathbf{1}^{ij}{}_{kl} = \delta^i_k \delta^j_l$. The graphical representation of these projectors is given in Fig. 4.

Using the projectors, we can represent the $\hat{R}_q$ matrix as [Res, Wor4, FRT]

$$\hat{R}^{ij}_{q\,kl} = (q\mathscr{P}_S - q^{-1}\mathscr{P}_A)^{ij}{}_{kl} . \tag{3.10}$$

The unimodularity condition is represented by using the ε-tensor:

$$\det M = \frac{(-1)^{N-1}}{[\![N]\!]!}\, \varepsilon^{k_1 \dots k_N} M^{l_1}{}_{k_1} M^{l_2}{}_{k_2} \dots M^{l_N}{}_{k_N} \varepsilon_{l_1 \dots l_N} = 1 . \tag{3.11}$$

The antipode is given by

$$\kappa(M^i{}_j) = \frac{1}{[\![N-1]\!]!}\, \varepsilon^{ik_1 \dots k_{N-1}} M^{l_1}{}_{k_1} M^{l_2}{}_{k_2} \dots M^{l_{N-1}}{}_{k_{N-1}} \varepsilon_{l_1 \dots l_{N-1} j} , \tag{3.12}$$

and the inverse of the antipode is

$$\kappa^{-1}(M^i{}_j) = \frac{1}{[\![N-1]\!]!}\, \varepsilon_{jl_1 \dots l_{N-1}} M^{l_1}{}_{k_1} M^{l_2}{}_{k_2} \dots M^{l_{N-1}}{}_{k_{N-1}} \varepsilon^{k_1 \dots k_{N-1} i} . \tag{3.13}$$

Now we can introduce the fundamental bicovariant bimodule of $SU_q(N)$ as follows:

Definition of the Fundamental Bicovariant Bimodule. The fundamental bicovariant bimodule is the bimodule where the right invariant subspace is an N dimensional linear space with the basis η^i $(i = 1, \dots, N)$. The left coaction on η^i is defined by

$$\Delta_L(\eta^i) = M^i{}_j \otimes \eta^j . \tag{3.14}$$

Therefore we can identify the matrix $\mathbf{T}^I{}_J$ in Eq. (2.30) with $M^i{}_j$. Then Eq. (2.34) implies the existence of functionals $f^i{}_j$ such that

$$\eta^i b = (b * f^i{}_j)\eta^j \tag{3.15}$$

holds for any element $b \in \mathscr{A}$, and $i, j = 1, \ldots, N$.

As we described in the previous section, to define the functionals $f^i{}_j$, it is sufficient to find their values on the generators $M^k{}_l$, i.e. to find the tensor $f^i{}_j(M^k{}_l)$.

From Eq. (2.53) we get as the defining equation of the tensor $f^i{}_j(M^k{}_l)$:

$$\eta^i M^k{}_l = f^i{}_j(M^k{}_n) M^n{}_l \eta^j . \tag{3.16}$$

We impose Conditions 1–2 given in the previous section.

Condition 1.

$$M^i{}_j M^k{}_n f^j{}_l(M^n{}_m) = f^i{}_j(M^k{}_n) M^n{}_m M^j{}_l . \tag{3.17}$$

This means that the tensor $f^i{}_j(M^k{}_l)$ must be represented by a linear combination of $\hat{R}_q$ and $\hat{R}_q^{-1}$.

Condition 2.

a) The consistency with the commutation relation of the generators is

$$\hat{R}_q^{ij}{}_{j'i'} f^l{}_n(M^{j'}{}_{j''}) f^n{}_k(M^{i'}{}_{i''}) = f^l{}_n(M^i{}_{i'}) f^n{}_k(M^j{}_{j'}) \hat{R}_q^{i'j'}{}_{j''i''} . \tag{3.18}$$

b) Another requirement for $SU_q(N)$ is that the determinant of $M^i{}_j$ defined in Eq. (3.11) commutes with any element $\varrho \in \Gamma$.

This implies that

$$f^i{}_j(\det M) = \frac{(-1)^{N-1}}{[\![N-1]\!]!} \varepsilon_{k_1 \ldots k_N} f^i{}_{i_1}(M^{k_1}{}_{l_1}) \ldots f^{i_{N-1}}{}_j(M^{k_N}{}_{l_N}) \varepsilon^{l_1 \ldots l_N} = \delta^i{}_j . \tag{3.19}$$

From these requirements we get two solutions. We call them $f^i_{\pm j}$:

$$f^i_{+j}(M^k{}_l) = q^{-1/N} \hat{R}_q^{ik}{}_{lj} , \tag{3.20}$$

and

$$f^i_{-j}(M^k{}_l) = q^{1/N} \hat{R}_q^{-1ik}{}_{lj} . \tag{3.21}$$

**-Conjugation of the Fundamental Bimodule.* For a complex representation of the ordinary $SU(N)$ there exists the complex conjugate representation with the same dimension. In the case of the quantum group $SU_q(N)$ there are also the modules conjugate to the above fundamental bimodules.

Applying the *-operation on both sides of Eq. (3.14) the coaction on the conjugate representation is obtained

$$\Delta_L(\eta^i)^* = (M^i{}_j)^* \otimes (\eta^j)^* . \tag{3.22}$$

Since the coaction commutes with the *-operation this defines the coaction on the conjugate module. Denoting the conjugate module as

$$(\eta^i)^* = \eta^{*i} \equiv \bar{\eta}_i , \tag{3.23}$$

we can rewrite Eq. (3.22) as

$$\Delta_L(\bar{\eta}_i) = M^{\dagger j}{}_i \otimes \bar{\eta}_j = \kappa(M^j{}_i) \otimes \bar{\eta}_j . \tag{3.24}$$

We denote the functional which defines the commutation relation between any element $b \in \mathscr{A}$ and $\bar{\eta}$ as $\bar{f}^i{}_j$

$$\bar{\eta}_i b = (b * \bar{f}^j{}_i)\bar{\eta}_j . \tag{3.25}$$

As we shall show below, there is a one to one correspondence between the functionals $\bar{f}^i{}_j$ and $f^i{}_j$. And therefore, we have two functionals corresponding to $f^i_{\pm j}$ in Eqs. (3.20) and (3.21). We denote these two functionals as $\bar{f}^i_{\pm j}$. Their relation is as follows:

Proposition 2. *The functionals $f^i_{\pm j}$ corresponding to the right invariant bases $\eta^i_\pm$ defined in Eq.* (3.15) *and the functionals $\bar{f}^i_{\pm j}$ corresponding to the bases $\bar{\eta}_{\pm i}$ defined in Eq.* (3.25) *are related by*

$$\bar{f}^i_{\pm j} = f^{*j}_{\pm\ i} \equiv f^{\dagger i}_{\pm\ j} \tag{3.26}$$

with the $*$-operation defined in Eq. (2.44).

Proof. From Eq. (3.15) and using the general properties of the functionals f we get $\forall a \in \mathscr{A}$:

$$a^* \eta^i = \sum_S \eta^j f^i{}_j(\kappa(a'^*_S))a''^*_S , \tag{3.27}$$

where $\Delta(a) = \sum_S (a'_S \otimes a''_S)$.

Applying the $*$-operation on both sides of (3.27) and using the definition of the $*$-operation on $\mathscr{A}'$ given in Eq. (2.44), we get

$$\begin{aligned} \eta^{*i} a &= \sum_S \{f^i{}_j(\kappa(a'^*_S))\}^* a''_S \eta^{*j} \\ &= \sum_S f^{*i}{}_j(a'_S) a''_S \eta^{*j} \\ &= (f^{*i}{}_j \otimes \mathrm{id})\Delta(a)\eta^{*j} . \end{aligned} \tag{3.28}$$

Therefore, using the definitions (3.23) and (2.44) we obtain

$$\bar{\eta}_i a = (a * f^{*i}{}_j)\bar{\eta}_j . \tag{3.29}$$

Comparing (3.25) and (3.29) we get the relation (3.26). Q.E.D.

Using the unitarity condition (2.12), we can find a further identity between $f^i_{\pm j}$ and $\bar{f}^i_{\pm j}$ for $SU_q(N)$, which allows us to relate them by using κ'^{-1}.

Proposition 3.

$$f^{*j}_{\pm i} = f^{\dagger i}_{\pm\ j} = \kappa'^{-1}(f^i_{\mp j}) . \tag{3.30}$$

Proof. We apply the $*$-operation on both sides of Eq. (3.16). Since the $\hat{R}_q$ matrix is real we can write the result as

$$M^{*k}{}_l \eta^{*i} = f^i_{\pm j}(M^k{}_t)\eta^{*j} M^{*t}{}_l . \tag{3.31}$$

Substituting the unitarity condition (2.12) into $M^{*i}{}_j$, we get

$$\begin{aligned} \kappa(M^l{}_k)\bar{\eta}_i &= f^i_{\pm j}(M^k{}_t)\bar{\eta}_j \kappa(M^l{}_t) \\ &= f^i_{\pm j}(M^k{}_s)\bar{f}^n_{\pm j}(\kappa(M^t{}_s))\kappa(M^l{}_t)\bar{\eta}_n , \end{aligned} \tag{3.32}$$

where we have used the definition of $\bar{f}$, (3.25). This equation gives a condition for the value of the functionals f and $\bar{f}$:

$$\sum_{s,j} f^{i}_{\pm j}(M^{k}{}_{s})\bar{f}^{n}_{\pm j}(\kappa(M^{t}{}_{s}))=\delta^{n}_{i}\delta^{t}_{k}. \tag{3.33}$$

The symmetry of the $\hat{R}_q$ matrix $\hat{R}^{ij}_{q\,kl}=\hat{R}^{kl}_{q\,ij}$, implies that

$$\bar{f}^{i}_{\pm j}(\kappa(M^{k}{}_{l}))=f^{i}_{\mp j}(M^{k}{}_{l}). \tag{3.34}$$

Since this relation can be generalized to any element $a\in\mathscr{A}$ using the properties of the functionals f we get the relation between the f and f^* given in Eq. (3.30). Q.E.D.

Since we have two independent functionals which define the relation between the right and left multiplication, we obtain two types of fundamental bimodules and their conjugates. Therefore we distinguish their bases η^i and $\bar{\eta}^i$ by the suffix $\pm$ corresponding to these functionals $f^{i}_{\pm j}$ respectively. We specify the bicovariant bimodule by a pair consisting of a basis and the corresponding functionals $(\eta^{i}{}_{+}, f^{i}_{+j})$. Then the relations under the $*$-operation are

$$*:(\eta^{i}{}_{+}, f^{i}_{+j})\rightarrow(\bar{\eta}_{+i}, \bar{f}^{i}_{+j})=(\bar{\eta}_{+i}, f^{\dagger i}_{+j}), \tag{3.35}$$

$$*:(\eta^{i}{}_{-}, f^{i}_{-j})\rightarrow(\bar{\eta}_{-i}, \bar{f}^{i}_{-j})=(\bar{\eta}_{-i}, f^{\dagger i}_{-j}). \tag{3.36}$$

Due to Eq. (3.30), the fundamental bicovariant bimodules and their conjugates are completely defined by Eqs. (3.20) and (3.21).

In order to analyze the structure of the product representation it is convenient to use the quantities with upper indices instead of the ones with lower indices such as $\bar{\eta}_{\pm i}$. This also enables us to use the graphical representations.

The representation with upper index of $\bar{\eta}$ is defined by using the ε tensor given in Eq. (3.2).

$$\bar{\eta}^{[j_1\cdots j_{N-1}]}_{\pm}\equiv\bar{\eta}_{\pm i}\varepsilon^{ij_1\cdots j_{N-1}}, \tag{3.37}$$

where the symbol [...] is introduced to remind that the indices are antisymmetrized. To simplify the notation we denote $N-1$ antisymmetrized indices collectively as $[j]$:

$$[j]\equiv[j_1\cdots j_{N-1}]. \tag{3.38}$$

Using this notation Eq. (3.37) becomes

$$\bar{\eta}^{[j]}_{\pm}\equiv\bar{\eta}_{\pm i}\varepsilon^{i[j]}. \tag{3.37'}$$

Then the coaction is

$$\begin{aligned}\Delta_L(\bar{\eta}^{[j]}_{\pm})&=\Delta_L(\bar{\eta}^{[j_1\cdots j_{N-1}]}_{\pm})\\&=M^{j_1}{}_{i_1}\ldots M^{j_{N-1}}{}_{i_{N-1}}\otimes\bar{\eta}^{[i_1\cdots i_{N-1}]}_{\pm}.\end{aligned} \tag{3.39}$$

The corresponding functional $\bar{f}$ is defined by $\forall a\in\mathscr{A}$:

$$\bar{\eta}^{[i]}_{\pm}a=(a*\bar{f}^{[i]}_{\pm\,[j]})\bar{\eta}^{[j]}, \tag{3.40}$$

where the relation to the $f^{\dagger}_{\pm}$ is given by

$$\bar{f}^{[i]}_{\pm\,[j]}=\frac{1}{[\![N-1]\!]!}\varepsilon_{[j]j'}f^{\dagger j'}_{\pm\ i'}\varepsilon^{i'[i]}. \tag{3.41}$$

Using Eq. (3.30),

$$\bar{f}_{\pm}^{[i]}{}_{[k]} = f_{\mp}^{i_{N-1}}{}_{j_{N-1}} * \dots * f_{\mp}^{i_2}{}_{j_2} * f_{\mp}^{i_1}{}_{j_1} \mathscr{P}_{N-1}^{j_1 \dots j_{N-1}}{}_{k_1 \dots k_{N-1}} . \tag{3.42}$$

This shows the following equivalence of the two right invariant bases:

$$\bar{\eta}_{\pm j} \varepsilon^{j i_1 \dots i_{N-1}} \sim \eta_{\mp}^{[i_1} \eta_{\mp}^{i_2} \dots \eta_{\mp}^{i_{N-1}]} = \mathscr{P}_{N-1}^{i_1 \dots i_{N-1}}{}_{k_1 \dots k_{N-1}} \eta_{\mp}^{k_1} \dots \eta_{\mp}^{k_{N-1}} . \tag{3.43}$$

Clearly, this relation among the bimodules is the analogue of the well-known relation between the fundamental representation and its complex conjugate in the ordinary $SU(N)$. [For the definition of the product of modules in the right-hand side of (3.43) see Eq. (4.4).]

3.2. Fundamental Bicovariant Bimodule of $SO_q(N)$

We take the $\hat{R}_q$ matrix and the metric given in [FRT]. [As for our convention see also [CSW].] The commutation relations of the generators $M^i{}_j$ are the same as in Eq. (2.5). The extra condition for the $SO_q(N)$ is the orthogonality condition which is given by the metric C_{ij} as in Eq. (2.7).

The antipode is

$$\kappa(M^i{}_j) = C^{ii'} M^{j'}{}_{i'} C_{j'j} , \tag{3.44}$$

and the inverse of the antipode is

$$\kappa^{-1}(M^i{}_j) = C_{jj'} M^{j'}{}_{i'} C^{i'i} , \tag{3.45}$$

where C^{ij} is the inverse matrix of C_{ij}.

We also need to consider the following projection operators: The symmetrizer $\mathscr{P}_S$, the antisymmetrizer $\mathscr{P}_A$ and the projection operator to the singlet $\mathscr{P}_1$. Using them the $\hat{R}_q$ matrix is represented as

$$\hat{R}_q = q\mathscr{P}_S - q^{-1}\mathscr{P}_A + q^{1-N}\mathscr{P}_1 . \tag{3.46}$$

For the graphical representation see Fig. 5.

The $\hat{R}_q$ matrix and the metric satisfy the following relations (see Fig. 6):

$$C_{i'k'} \hat{R}_q^{i'j'}{}_{ji} \hat{R}_q^{k'l}{}_{j'k} = \delta^l_j C_{ik} , \tag{3.47}$$

$$C^{i'k'} \hat{R}_q^{ij'}{}_{ji'} \hat{R}_q^{kl}{}_{j'k'} = \delta^l_j C^{ik} . \tag{3.48}$$

As in the case of $SU_q(N)$ we construct the fundamental bicovariant bimodule.

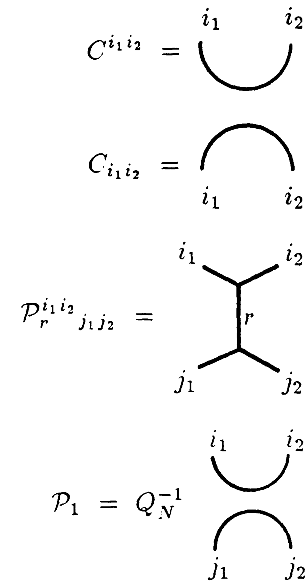
Definition of the Fundamental Bicovariant Bimodule. The fundamental bicovariant bimodule of $SO_q(N)$ is the bimodule where the right invariant subspace is an N dimensional linear space with the basis η^i $(i = 1, \dots, N)$. The left coaction on η^i is defined by

$$\Delta_L(\eta^i) = M^i{}_j \otimes \eta^j . \tag{3.49}$$

Therefore we identify the matrix $\mathbf{T}^I{}_J$ in Eq. (2.30) with $M^i{}_j$. Equation (2.34) implies that there exist functionals $f^i{}_j$ such that $\forall b \in \mathscr{A}$:

$$\eta^i b = (b * f^i{}_j) \eta^j , \tag{3.50}$$

where the indices i, j run $1, \dots, N$.



a

b

c

Fig. 5. **a** The graphical representation of $C^{i_1 i_2}$ and $C_{i_1 i_2}$, the analogue of the metric in the usual $SO(N)$ group. The graphical representation of the $\hat{R}_q$ matrix for $SO_q(N)$ is the same as for the $SU_q(N)$ case. **b** The projection operators $\mathcal{P}_S$ and $\mathcal{P}_A$ corresponding to the antisymmetric and the symmetric traceless product of two N dimensional representations, respectively. The index r represents either S or A. **c** The projection operator to the singlet representation corresponding to the trace part of the product of two N dimensional representations. Q_N^{-1} is a normalization constant

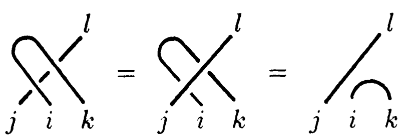


Fig. 6. Some graphical relations including the metric and the $\hat{R}_q$-matrix

As in the case of $SU_q(N)$ to define the functionals $f^i{}_j$ it is sufficient to find the tensor $f^i{}_j(M^k{}_l)$ appearing in the relation:

$$\eta^i M^k{}_l = f^i{}_j(M^k{}_n) M^n{}_l \eta^j . \tag{3.51}$$

The defining condition for the $f^i{}_j(M^k{}_l)$ are Eqs. (3.17) and (3.18) replacing the generators $M^i{}_j$ by those of $SO_q(N)$:

Condition 1.

$$M^i{}_j M^k{}_n f^j{}_k(M^n{}_m) = f^i{}_j(M^k{}_n) M^n{}_m M^j{}_k . \tag{3.52}$$

This means that the tensor $f^i{}_j(M^k{}_l)$ must be represented by a linear combination of $\hat{R}_q$, $\hat{R}_q^{-1}$, and $\mathcal{P}_1$.

Condition 2.

a) The requirement of consistency with the commutation relations of the generators leads to

$$\hat{R}_q^{ij}{}_{j'i'} f^l{}_n(M^{j'}{}_{j''}) f^n{}_k(M^{i'}{}_{i''}) = f^l{}_n(M^i{}_{i'}) f^n{}_k(M^j{}_{j'}) \hat{R}_q^{i'j'}{}_{j''i''} . \tag{3.53}$$

The difference between the quantum groups $SU_q(N)$ and $SO_q(N)$ appears in Condition 2b). Instead of the condition on $\det M$ we require.

Condition 2b′). The orthogonality condition (2.7) must be consistent with the bimodule structure. This implies that

$$f^i{}_j(C_{k'l'}M^{k'}{}_kM^{l'}{}_l)=C_{k'l'}f^i{}_{i'}(M^{k'}{}_k)f^{i'}{}_j(M^{l'}{}_l)=\delta^i{}_jC_{kl}\,. \tag{3.54}$$

From these requirements we again get two solutions. We call them $f_\pm$:

$$f^i_{+j}(M^k{}_l)=\hat{R}^{ik}_{q\;lj}\,, \tag{3.55}$$

and

$$f^i_{-j}(M^k{}_l)=\hat{R}_q^{-1ik}{}_{lj}\,. \tag{3.56}$$

We also distinguish the two right invariant bases corresponding to the two solutions $f_\pm$ as $\eta^i_\pm$ like in the case of the $SU_q(N)$. This result as well as (3.20) and (3.21) agrees with the one obtained in [Rosso] which has been constructed from the quasitriangular structure of q-deformed universal enveloping algebra.

Reality Condition. For the quantum group $SO_q(N)$ we also consider the $*$-operation since by construction we have to distinguish $M^i{}_j$ and M^{i*}_j like in $SU_q(N)$. This means that in order to get the q-deformed $SO_q(N)$ we have to divide by a $\mathbf{Z}_2$ symmetry which identifies an element with its conjugate to reduce the number of degrees of freedom. This identification corresponds to the reality condition which restricts $SO(N,\mathbf{C})$ to $SO(N,\mathbf{R})$ in the limit $q\to 1$.

The $*$-operation in the $SO_q(N)$ can be defined in the same way like in the $SU_q(N)$ and we can also prove

$$f^{*j}_{\pm i}=f^{\dagger i}_{\pm j}=\kappa'^{-1}(f^i_{\mp j})\,. \tag{3.57}$$

The reality condition for the generators $M^i{}_j$ of $SO_q(N)$ is given by the unitarity condition (2.12):

$$M^i{}_j=\kappa^{-1}(M^{*j}{}_i)=\kappa(M^j{}_i)^*\,. \tag{3.58}$$

The operation $*$ on the element of the bimodule is defined by the action on the right invariant basis $\eta^i_\pm$ and the condition (2.26).

Then the reality condition for the fundamental bimodule is

$$\bar{\eta}_{\pm j}=\eta^i_\mp C_{ij} \tag{3.59}$$

with the definition

$$\bar{\eta}_{\pm i}\equiv(\eta^i_\mp)^*\,. \tag{3.60}$$

Note that Eq. (3.59) means that the fundamental bicovariant bimodule of $SO_q(N)$ is not real since the $*$-conjugation maps η_+ into η_- and vice versa. However, we shall see that the bicovariant bimodule corresponding to the adjoint representation constructed by using these fundamental bimodules becomes real.

3.3. Relation to q-Deformed Universal Enveloping Algebra

Before we finish this section, let us establish the relations between the functionals $f^i_{\pm j}$ and the generators of the q-deformed universal enveloping algebras given in [FRT].

As a consequence of the above results, we can show the following relations among the functionals:

$$\hat{R}_q{}^{ij}{}_{j'i'} f_{+}{}^{i'}{}_{k} * f_{+}{}^{j'}{}_{l} = f_{+}{}^{j}{}_{l'} * f_{+}{}^{i}{}_{k'} \hat{R}_q{}^{k'l'}{}_{lk}, \tag{3.61}$$

$$\hat{R}_q{}^{ij}{}_{j'i'} f_{-}{}^{i'}{}_{k} * f_{-}{}^{j'}{}_{l} = f_{-}{}^{j}{}_{l'} * f_{-}{}^{i}{}_{k'} \hat{R}_q{}^{k'l'}{}_{lk}, \tag{3.62}$$

$$\hat{R}_q{}^{ij}{}_{j'i'} f_{+}{}^{i'}{}_{k} * f_{-}{}^{j'}{}_{l} = f_{-}{}^{j}{}_{l'} * f_{+}{}^{i}{}_{k'} \hat{R}_q{}^{k'l'}{}_{lk}. \tag{3.63}$$

To prove this it is sufficient to prove the equivalence of the value of both sides when we apply those functionals on the generators $M^i{}_j$ and $\mathbf{1}$. For $\mathbf{1}$ it is trivially satisfied. To show the equivalence on the $M^i{}_j$ we apply the left-hand side of (3.61) on $M^s{}_t$ and get

$$\begin{aligned}\hat{R}_q{}^{ij}{}_{j'i'}(f_{+}{}^{i'}{}_{k} * f_{+}{}^{j'}{}_{l})(M^s{}_t) &= \hat{R}_q{}^{ij}{}_{j'i'} f_{+}{}^{i'}{}_{k}(M^s{}_{s'}) f_{+}{}^{j'}{}_{l}(M^{s'}{}_t)\\ &= \alpha^2 \hat{R}_q{}^{ij}{}_{j'i'} \hat{R}_q{}^{i's}{}_{s'k} \hat{R}_q{}^{j's'}{}_{tl}.\end{aligned} \tag{3.64}$$

On the other hand the right-hand side of (3.61) gives

$$\begin{aligned}(f_{+}{}^{j}{}_{l'} * f_{+}{}^{i}{}_{k'})(M^s{}_t) \hat{R}_q{}^{k'l'}{}_{lk} &= f_{+}{}^{j}{}_{l'}(M^s{}_{s'}) f_{+}{}^{i}{}_{k'}(M^{s'}{}_t) \hat{R}_q{}^{k'l'}{}_{lk}\\ &= \alpha^2 \hat{R}_q{}^{js}{}_{s'l'} \hat{R}_q{}^{is'}{}_{tk'} \hat{R}_q{}^{k'l'}{}_{lk},\end{aligned} \tag{3.65}$$

where for $SU_q(N)$ we have substituted the value of the $f_{+}{}^{k}{}_{l}(M^i{}_j)$ given in Eq. (3.20) and $\alpha = q^{-\frac{1}{N}}$. For $SO_q(N)$ we use Eq. (3.55) and $\alpha = 1$. Therefore both sides of Eq. (3.61) are equivalent due to the Yang-Baxter equation for the $\hat{R}_q$ matrix. The proof of the other relations (3.62) and (3.63) can be performed analogously.

Using the same method as above we can also prove for the case of $SU_q(N)$

$$\varepsilon_{i_1 \dots i_N} f_{\pm}{}^{i_N}{}_{j_N} * \dots * f_{\pm}{}^{i_1}{}_{j_1} = \varepsilon_{j_1 \dots j_N} \boldsymbol{\varepsilon}. \tag{3.66}$$

For $SO_q(N)$ the functionals satisfy

$$C_{i_1 i_2} f_{\pm}{}^{i_2}{}_{j_2} * f_{\pm}{}^{i_1}{}_{j_1} = C_{j_1 j_2} \boldsymbol{\varepsilon}. \tag{3.67}$$

The relations (3.61–63), (3.66) or (3.67) and the $*$-operations on the functionals (3.30) or (3.57) are equivalent to the relations which were imposed on the subalgebra of $\mathrm{Hom}(\mathscr{A}, \mathbf{C})$ by Faddeev et al. [FRT]. Therefore, the algebra $\mathscr{A}'$ generated by the functionals $f_{\pm}{}^{i}{}_{j}$ with induced $*$-operation and imposing the relations (3.61–63), (3.66) or (3.67) is equivalent to the one introduced by Faddeev et al. with the identification:

$$f_{\pm}{}^{i}{}_{j} = L_{\pm}{}^{i}{}_{j}, \tag{3.68}$$

where $L_{\pm}{}^{i}{}_{j}$ is the one in [FRT]. Therefore, in this way the algebra generated by the functionals which relate left and right multiplication of the bimodule coincides with the q-deformed universal enveloping algebra in [Dri, Jim].

4. Bimodule with Adjoint Representation

The right or left invariant one forms in the ordinary differential calculus on the group manifold belong to the adjoint representation. Therefore to construct the differential calculus on the quantum group which coincides with the commuting case in the limit $q \to 1$, we need the bicovariant bimodule which contains the right invariant basis of the adjoint representation.

To generalize the transformation of the adjoint representation to the quantum group one can use the left coaction ad_L and right coaction ad_R [Wor3]. Their actions on the generators $M^i{}_j$ is

$$\mathrm{ad}_L(M^i{}_j) \equiv M^i{}_{i'}\kappa(M^{j'}{}_j) \otimes M^{i'}{}_{j'}, \tag{4.1}$$

$$\mathrm{ad}_R(M^i{}_j) \equiv M^{i'}{}_{j'} \otimes \kappa(M^i{}_{i'}) M^{j'}{}_j. \tag{4.2}$$

In our construction therefore we need consider the right invariant basis on which the left coaction acts like ad_L in Eq. (4.1). We denote such a right invariant basis as $\theta^i{}_j$ and the corresponding bicovariant bimodule as Γ_{Ad}.

Then the left coaction on the basis $\theta^i{}_j$ must be

$$\Delta_L(\theta^i{}_j) = M^i{}_{i'}\kappa(M^{j'}{}_j) \otimes \theta^{i'}{}_{j'}. \tag{4.3}$$

One easily sees that such a right invariant left covariant basis $\theta^i{}_j$ can be obtained by simply multiplying the two fundamental modules defined in the previous section.

The product of the two bimodules Γ_1 and Γ_2 can be defined by the tensor product over $\mathscr{A}$: For $\varrho_1 \in \Gamma_1$ and $\varrho_2 \in \Gamma_2$ we have

$$\Gamma_1 \otimes_{\mathscr{A}} \Gamma_2 \ni \varrho_1 \varrho_2 \equiv \varrho_1 \otimes_{\mathscr{A}} \varrho_2, \tag{4.4}$$

where $\otimes_{\mathscr{A}}$ means that $\forall a \in \mathscr{A}$:

$$\varrho_1 a \otimes_{\mathscr{A}} \varrho_2 = \varrho_1 \otimes_{\mathscr{A}} a \varrho_2. \tag{4.5}$$

In this way the product $\Gamma_1 \otimes_{\mathscr{A}} \Gamma_2$ becomes a bimodule as well. The coactions on this bimodule are defined by

$$\Delta_R(\varrho_1 \varrho_2) \equiv \Delta_R(\varrho_1) \Delta_R(\varrho_2), \tag{4.6}$$

$$\Delta_L(\varrho_1 \varrho_2) \equiv \Delta_L(\varrho_1) \Delta_L(\varrho_2), \tag{4.7}$$

where the product on the right-hand side is defined as $\forall a, b \in \mathscr{A}$:

$$(a \otimes \varrho_1)(b \otimes \varrho_2) = (ab \otimes \varrho_1 \varrho_2), \tag{4.8}$$

$$(\varrho_1 \otimes a)(\varrho_2 \otimes b) = (\varrho_1 \varrho_2 \otimes ab). \tag{4.9}$$

The $*$-operation is generalized on the bimodule $\Gamma_1 \otimes_{\mathscr{A}} \Gamma_2$ as

$$(\varrho_1 \varrho_2)^* = \varrho_2^* \varrho_1^*. \tag{4.10}$$

In order to define the bicovariant differential calculus with the $*$-structure we have to require that the $*$-operation is a bimodule antiautomorphism:

$$(\Gamma_{\mathrm{Ad}})^* = \Gamma_{\mathrm{Ad}}. \tag{4.11}$$

With this requirement we can find two different types of right invariant bases containing the adjoint representation. They are given by $\eta^i{}_+ \bar{\eta}_{+j}$ and $\eta^i{}_- \bar{\eta}_{-j}$. For example we know for the first choice

$$(\eta^i{}_+ \bar{\eta}_{+j})^* = \eta^j{}_+ \bar{\eta}_{+i}. \tag{4.12}$$

Consequently the bimodule generated by this basis is closed under the $*$-operation and thus (4.11) holds.

According to (4.6) the left coaction is

$$\begin{aligned}\Delta_L(\eta^i_+\bar{\eta}_{+j}) &= \Delta_L(\eta^i_+)\Delta_L(\bar{\eta}_{+j}) \\ &= (M^i{}_{i'}\otimes\eta^{i'}_+)(\kappa(M^{j'}{}_j)\otimes\bar{\eta}_{+j'}) \\ &= (M^i{}_{i'}\kappa(M^{j'}{}_j)\otimes\eta^{i'}_+\bar{\eta}_{+j'}) .\end{aligned} \tag{4.13}$$

Comparing (4.13) with (4.3) we can identify

$$\theta^i{}_j = \eta^i_+\bar{\eta}_{+j}, \tag{4.14}$$

to get the bimodule Γ_{Ad} as the product of two fundamental bimodules.

The other choice, i.e. $\eta^i_-\bar{\eta}_{-j}$ can be also taken as the right invariant basis. It has a different bimodule structure; however the following constructions are performed in a completely parallel way. In the following we choose the first possibility (4.14) to construct the right invariant basis of Γ_{Ad}.

The basis $\theta^i{}_j$ given in Eq. (4.14) corresponds to the basis of the tensor product of the two bundles of the fundamental and its conjugate representation in the limit $q\to 1$. Therefore this basis of Γ_{Ad} is reducible. We impose no constraints to make them irreducible as a representation of the quantum group and thus the basis $\theta^i{}_j$ has N^2 components. To extract the irreducible components belonging to the adjoint representation we may multiply the corresponding projection operators (see below). However considering the bimodule generated by the basis projected to the adjoint representation one sees that such a projection does not close with respect to the left and right bimodule structure. In other words Γ_{Ad} is the smallest bicovariant bimodule containing the adjoint representation. Consequently to construct the bicovariant bimodule which contains the adjoint representation it is necessary to keep the basis with N^2 components.

The induced bimodule structure of Γ_{Ad} is given as follows:

For any element $a\in\mathscr{A}$

$$\theta^i{}_j a = (a * f_{\mathrm{Ad}}{}^i{}_{jk}{}^l)\theta^k{}_l, \tag{4.15}$$

where

$$f_{\mathrm{Ad}}{}^i{}_{jk}{}^l = f^{\dagger l}_{+j} * f^i_{+k}, \tag{4.16}$$

and where the convolution product of the two functionals χ and ξ is defined as $\forall a\in\mathscr{A}$:

$$\chi * \xi(a) = (\chi\otimes\xi)\Delta(a). \tag{4.17}$$

Proof of Eq. (4.15). Using the representation of the basis $\theta^i{}_j$ by fundamental modules in Eq. (4.14) the left-hand side of Eq. (4.15) is

$$\begin{aligned}(\eta^i_+\otimes_{\mathscr{A}}\bar{\eta}_{+j})a &= \eta^i_+\otimes_{\mathscr{A}}(a * f^{\dagger l}_{+j})\bar{\eta}_{+l} \\ &= ((a * f^{\dagger l}_{+j}) * f^i_{+k})(\eta^k_+\otimes_{\mathscr{A}}\bar{\eta}_{+l}) .\end{aligned} \tag{4.18}$$

Using the property of the convolution product we get Eq. (4.15). Q.E.D.

In order to analyze the structure of the functional f_{Ad} it is convenient to use only upper indices.

1) $SU_q(N)$ case. Using the tensor ε we introduce the following basis:

$$\theta^{j_0[j]} = \theta^{j_0}{}_{j'}\varepsilon^{j'[j]}, \tag{4.19}$$

where the notation $[\cdot]$ is the one introduced in (3.38).

In this basis the left coaction is now

$$\Delta_L(\theta^{j_0[j]}) = M^{j_0}{}_{k_0} M^{j_1}{}_{k_1} \dots M^{j_{N-1}}{}_{k_{N-1}} \otimes \theta^{k_0[k]}. \tag{4.20}$$

The relation between the left and right multiplication is

$$\theta^{i_0[i]} a = (a * f_{\mathrm{Ad}}{}^{i_0[i]}{}_{j_0[j]}) \theta^{j_0[j]}, \tag{4.21}$$

with

$$\begin{aligned} f_{\mathrm{Ad}}{}^{i_0[i]}{}_{j_0[j]} &= \frac{1}{[\![N-1]\!]!} \varepsilon_{[j]j'} f_{\mathrm{Ad}}{}^{i_0}{}_{i' j_0}{}^{j'} \varepsilon^{i'[i]} \\ &= \bar{f}_{+}^{[i]}{}_{[j]} * f_{+}^{i_0}{}_{j_0}, \end{aligned} \tag{4.22}$$

where we have used the definition of $\bar{f}^{[i]}{}_{[j]}$ given in Eq. (3.42).

The value of this functional acting on the generators is

$$\begin{aligned} & f_{\mathrm{Ad}}{}^{i_0[i]}{}_{j_0[j]}(M^k{}_l) \\ & = q^{\frac{N-2}{N}} \hat{R}_q^{i_0 l_0}{}_{l j_0} \hat{R}_q^{-1\, i_1 l_1}{}_{l_0 j_1'} \hat{R}_q^{-1\, i_2 l_2}{}_{l_1 j_2'} \dots \hat{R}_q^{-1\, i_{N-1} k}{}_{l_{N-2} j_{N-1}'} \mathscr{P}_{N-1}^{j_1' \dots j_{N-1}'}{}_{j_1 \dots j_{N-1}}. \end{aligned} \tag{4.23}$$

The structure of this equation can be seen more easily by the graphical representation. See Fig. 7a.

2) $SO_q(N)$ case.

For $SO_q(N)$ we consider the basis

$$\theta^{i_1 i_2} = \theta^{i_1}{}_j C^{j i_2}. \tag{4.24}$$

Then the coaction is given by

$$\Delta_L(\theta^{i_1 i_2}) = M^{i_1}{}_{i_1'} M^{i_2}{}_{i_2'} \otimes \theta^{i_1' i_2'}. \tag{4.25}$$

The relation between the left and right multiplication is

$$\theta^{i_1 i_2} a = (a * f_{\mathrm{Ad}}{}^{i_1 i_2}{}_{j_1 j_2}) \theta^{j_1 j_2}, \tag{4.26}$$

where

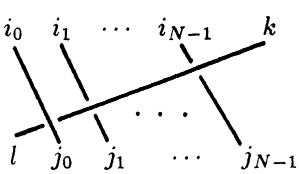
$$\begin{aligned} f_{\mathrm{Ad}}{}^{i_1 i_2}{}_{j_1 j_2} &= \kappa'(\bar{f}_{+}^{i_2}{}_{j_2}) * f_{+}^{i_1}{}_{j_1} \\ &= f_{-}^{i_2}{}_{j_2} * f_{+}^{i_1}{}_{j_1}. \end{aligned} \tag{4.27}$$

Using Eqs. (2.39), (3.55), and (3.56) we obtain

$$f_{\mathrm{Ad}}{}^{i_1 i_2}{}_{j_1 j_2}(M^k{}_l) = \hat{R}_q^{i_1 k'}{}_{l j_1} \hat{R}_q^{-1\, i_2 k}{}_{k' j_2}. \tag{4.28}$$

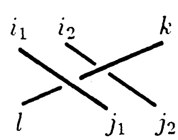
See also Fig. 7b.

$$f_{Ad}{}^{i_0[i]}{}_{j_0[j]}(M^k{}_l) \;=\; q^{\frac{N-2}{N}}$$



a

$$f_{Ad}{}^{i_1 i_2}{}_{j_1 j_2}(M^k{}_l) \;=$$



d

Fig. 7. The graphical representation of the value of the functionals f_{Ad}: **a** $SU_q(N)$ case corresponding to Eq. (4.23). **b** $SO_q(N)$ case corresponding to Eq. (4.28)

Note that for $SO_q(N)$ the two bases $\eta^i_+\bar{\eta}_{+j}$ and $\eta^i_-\bar{\eta}_{-j}$ yield the same bicovariant bimodule. This is due to the following fact.

Consider the basis $\theta'^{i_1i_2}$ which is a linear combination of the second choice of basis $\eta^i_-\bar{\eta}_{-j}$:

$$\theta'^{i_1i_2}=\hat{R}_q^{i_1i_2}{}_{k_1k_2}\eta^{k_1}_-\bar{\eta}_{-j}C^{jk_2}. \tag{4.29}$$

Then we can construct the bicovariant bimodule Γ_{Ad}' using the basis θ'. For this bimodule Γ_{Ad}' the relation between the left and right multiplication is given by the same functional f as for the bimodule Γ_{Ad}, i.e.

$$\theta'^{i_1i_2}a=(a*f_{\mathrm{Ad}}{}^{i_1i_2}{}_{j_1j_2})\theta'^{j_1j_2}, \tag{4.30}$$

due to the relation (3.63). Therefore the two bimodules Γ_{Ad} and Γ_{Ad}' are equivalent.

5. Differential Calculus

In the differential calculus on the ordinary classical group G we can consider the exterior derivative d as a map from the space of smooth functions over G onto the space of the sections of the cotangent bundle $C^\infty(T^*(G))$:

$$\mathrm{d}:C^\infty(G)\to C^\infty(T^*(G)). \tag{5.1}$$

In order to generalize the differential calculus to the quantum group $\mathscr{A}$ we adopt this picture. As the algebra $\mathscr{A}$ corresponds to the algebra $C^\infty(G)$ the bicovariant bimodule over $\mathscr{A}$ corresponds to $C^\infty(T^*(G))$. Since we want to formulate the differential calculus which coincides with the one on the group manifold in the limit $q\to 1$, we take the Γ_{Ad} constructed in the previous section as the bimodule of 1-forms. Thus we introduce the exterior derivative d on the quantum group as a map from the algebra $\mathscr{A}$ to the bicovariant bimodule Γ_{Ad} following [Wor3]:

$$\mathrm{d}:\mathscr{A}\to\Gamma_{\mathrm{Ad}}. \tag{5.2}$$

We also require that the derivative d satisfies the Leibniz rule

$$\forall a,b\in\mathscr{A}:\mathrm{d}(ab)=(\mathrm{d}a)b+a(\mathrm{d}b). \tag{5.3}$$

Therefore, once the bicovariant bimodule is defined it is rather straightforward to develop the first order differential calculus on the quantum group. Since the first order differential calculus has the same structure for $SU_q(N)$ and $SO_q(N)$ we consider both cases simultaneously.

In this paper we are constructing the bicovariant differential calculus and therefore the left and right coaction and the derivative d have to satisfy the relations [Wor3]

$$\Delta_L(\mathrm{d}a)=(\mathrm{id}\otimes\mathrm{d})\Delta(a), \tag{5.4}$$

$$\Delta_R(\mathrm{d}a)=(\mathrm{d}\otimes\mathrm{id})\Delta(a). \tag{5.5}$$

Using the right invariant basis introduced in the previous section we can find an explicit form of the exterior derivative which satisfies the above requirements, i.e. Leibniz rule (5.3) and bicovariance (5.4) and (5.5) as follows:

As already remarked the right invariant basis $\theta^i{}_j$ constructed in the previous section is not an irreducible representation. Among its representations there is a singlet representation which is both left and right invariant. We denote this element by $\mathbf{X}$ which is defined as

$$\mathbf{X}=\varepsilon_{i_0[i]}\theta^{i_0[i]}, \tag{5.6}$$

for $SU_q(N)$ and

$$\mathbf{X}=C_{i_1i_2}\theta^{i_1i_2}, \tag{5.7}$$

for $SO_q(N)$. The left invariance of $\mathbf{X}$ is apparent. This left right invariant element $\mathbf{X}$ plays the role of the additional scalar element introduced in Woronowicz's extended module.

We define the exterior derivative as $\forall a\in\mathscr{A}$:

$$\mathrm{d}a\equiv\frac{1}{\mathscr{N}}[\mathbf{X},a]_-=\frac{1}{\mathscr{N}}(\mathbf{X}a-a\mathbf{X}), \tag{5.8}$$

where $\mathscr{N}\in\mathbf{C}$ is the normalization constant which will be defined later.

By this definition of the exterior derivative, the Leibniz rule is trivially satisfied. It is also easy to show that the left and right coaction on $\mathrm{d}a$ satisfies the properties required in Eqs. (5.4) and (5.5) due to Eqs. (2.18) and (2.19).

As discussed by Woronowicz we also preserve the $*$-structure so that the resulting calculus becomes a $*$-differential calculus. For example, for $SU_q(2)$

$$\mathbf{X}^*=\mathbf{X}, \tag{5.9}$$

we take the normalization constant as pure imaginary:

$$\mathscr{N}^*=-\mathscr{N}. \tag{5.10}$$

In this way, with the appropriate choice of the normalization constant, we can always achieve that the following relation holds

$$(\mathrm{d}a)^*=\mathrm{d}(a^*). \tag{5.11}$$

Since the difference between the multiplication from the left and right is defined by the bimodule structure the commutator on the right-hand side of Eq. (5.8) can be evaluated in terms of the functional f_{Ad}.

Right Invariant Vector Field. In order to obtain the concrete relation between the derivative and the functional f_{Ad} we introduce functionals χ_I

$$\chi_I:\mathscr{A}\to\mathbf{C}, \tag{5.12}$$

where the suffix I denotes $I=(i_0,[i])$ for $SU_q(N)$ and $I=(i_1,i_2)$ for $SO_q(N)$. They are defined as

$$\chi_I\equiv\chi_{i_0[i]}=-\frac{1}{\mathscr{N}}(\varepsilon_{j_0[j]}f_{\mathrm{Ad}}{}^{j_0[j]}{}_{i_0[i]}\circ\kappa-\varepsilon_{i_0[i]}\boldsymbol{\varepsilon}), \tag{5.13}$$

for $SU_q(N)$ and

$$\chi_I\equiv\chi_{i_1i_2}=-\frac{1}{\mathscr{N}}(C_{j_1j_2}f_{\mathrm{Ad}}{}^{j_1j_2}{}_{i_1i_2}\circ\kappa-C_{i_1i_2}\boldsymbol{\varepsilon}), \tag{5.14}$$

for $SO_q(N)$.

Since the exterior derivative of $a\in\mathscr{A}$, $\mathrm{d}a$ defined in Eq. (5.8) is an element of Γ_{Ad}, it can be represented by using the right invariant basis θ^I. It is rather straightforward to show that such a representation is given by

$$\mathrm{d}a = \theta^I(a * \chi_I), \tag{5.15}$$

with the functional χ_I defined above.

Proof. We give the proof for the $SU_q(N)$ case. Using Eqs. (5.8) and (4.21) we can show:

$$\begin{aligned}
\mathrm{d}a &= \frac{1}{\mathcal{N}}\, \varepsilon_{i\mathrm{o}[i]}(\theta^{i\mathrm{o}[i]}a - a\theta^{i\mathrm{o}[i]}) \\
&= \frac{1}{\mathcal{N}}\, \varepsilon_{i\mathrm{o}[i]}(\theta^{i\mathrm{o}[i]}a - \theta^{j\mathrm{o}[j]}(a * f_{\mathrm{Ad}}{}^{i\mathrm{o}[i]}{}_{j\mathrm{o}[j]} \circ \kappa)) \\
&= \frac{1}{\mathcal{N}}\, \theta^{j\mathrm{o}[j]}(a * (\varepsilon_{j\mathrm{o}[j]}\varepsilon - \varepsilon_{i\mathrm{o}[i]} f_{\mathrm{Ad}}{}^{i\mathrm{o}[i]}{}_{j\mathrm{o}[j]} \circ \kappa)) .
\end{aligned} \tag{5.16}$$

Comparing with Eq. (5.15) we obtain Eq. (5.13). Q.E.D.

For $SO_q(N)$ the proof is performed analogously.

χ_I corresponds to a generalization of the right invariant vector field and defines the derivative on the quantum group. Therefore denoting

$$\nabla_I a \equiv (a * \chi_I), \tag{5.17}$$

we can consider ∇_I as differential operators on the quantum group. These differential operators satisfy the following generalized Leibniz rule:

$$\nabla_I\{ab\} = (\nabla_I a)b + (a * f_{\mathrm{Ad}}{}^J{}_I \circ \kappa)(\nabla_J b). \tag{5.18}$$

In order to define the constant $\mathcal{N}$ let us compute the values of $\chi_I(M^k{}_l)$. These are the derivatives of the matrix elements evaluated at unity:

$$\chi_I(a) = \varepsilon(a * \chi_I) = \nabla_I a|_{\mathrm{at}\, M^i{}_j = \delta^i_j} . \tag{5.19}$$

Using the definition of the derivative we get for $SU_q(N)$

$$\chi_{j\mathrm{o}[j]}(M^k{}_l) = -q^{1/N}\frac{(q-q^{-1})}{\mathcal{N}}\left\{\left[\!\left[\frac{1}{N}\right]\!\right] \varepsilon_{j\mathrm{o}[j]}\delta^k{}_l + (-1)^N q^{1/N-N}\delta^k_{j\mathrm{o}}\varepsilon_{[j]l}\right\}, \tag{5.20}$$

and for $SO_q(N)$

$$\chi_{j_1j_2}(M^k{}_l) = \frac{q-q^{-1}}{\mathcal{N}}(q^{1-N}\delta^k_{j_1}C_{j_2l} - \hat{R}_q^{-1\,kk'}{}_{j_1j_2}C_{k'l}) . \tag{5.21}$$

We give the graphical representation of Eq. (5.21) in Fig. 8. In Fig. 9 we give also the graphical representation of the Leibniz rule (5.18) for the $SO_q(N)$ case.

If we want to get nonzero values in the limit $q\to 1$ the normalization constant $\mathcal{N}$ must be proportional to $(q-q^{-1})$. Consequently we define

$$\mathcal{N} = (q-q^{-1})\mathcal{N}_0, \tag{5.22}$$

where the constant $\mathcal{N}_0$ has a nonzero value in the limit $q\to 1$.

In the commutative differential calculus on the ordinary group manifold a basis of the right invariant 1-forms can be constructed with the entries of $\mathrm{d}UU^{-1}$ with the matrix representation U. Thus to see further relations to the usual differential

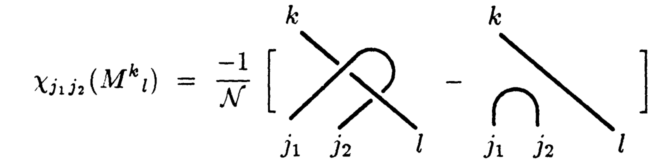


Fig. 8. The graphical representation of the differential operator when acting on the generator $M^i{}_j$ for the $SO_q(N)$ case corresponding to Eq. (5.21)

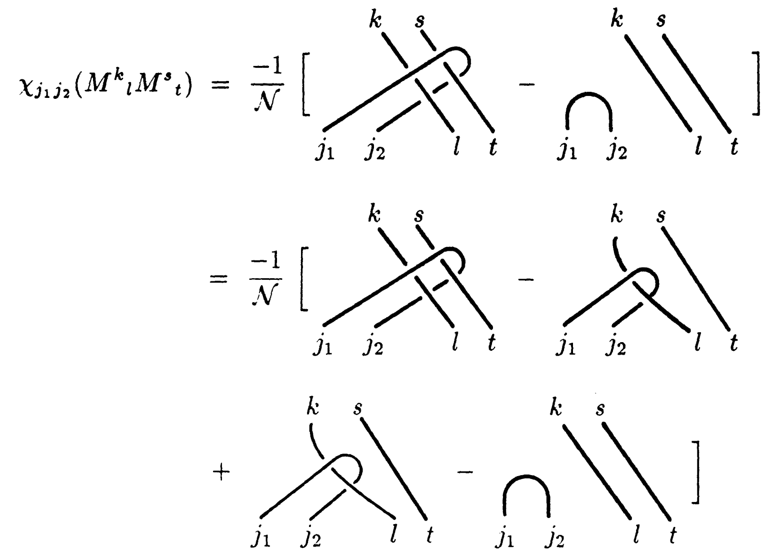


Fig. 9. The graphical representation of the Leibniz rule (5.18). We only give here the derivative of the product of the generators for the $SO_q(N)$ case

calculus on the group manifold it is instructive to consider the following new right invariant basis $\tilde{\theta}$:

$$\tilde{\theta}^i{}_j = \mathrm{d}M^i{}_s\kappa(M^s{}_j)\,. \tag{5.23}$$

Using the definition of the derivative d the relation between the basis $\theta^i{}_j$ and $\tilde{\theta}^i{}_j$ can be easily found

$$\tilde{\theta}^k{}_l = \theta^J(M^k{}_{l'} * \chi_J)\kappa(M^{l'}{}_l) = \theta^J\chi_J(M^k{}_l)\,. \tag{5.24}$$

Therefore the basis $\tilde{\theta}$ is simply the linear transform over **C** of the basis θ.

Substituting the value of $\chi_I(M^i{}_j)$ in Eqs. (5.20) and (5.21) into Eq. (5.24) we get the explicit formulas for $SU_q(N)$ and $SO_q(N)$, respectively.

1) $SU_q(N)$.

$$\begin{aligned}\tilde{\theta}^{k_o[k]} &\equiv \mathrm{d}M^{k_o}{}_s\kappa(M^s{}_l)\varepsilon^{l[k]} \\ &= \frac{-q^{1/N}}{\mathcal{N}_0}\left\{\left[\!\left[\frac{1}{N}\right]\!\right]\mathbf{X}\varepsilon^{k_o[k]} + (-1)^N [\![N-1]\!]!\,q^{1/N-N}\theta^{k_o[k]}\right\},\end{aligned} \tag{5.25}$$

$$\tilde{\theta}^{k_o[k]}\varepsilon_{k_o[k]} = \frac{(-1)^N q^{1/N}}{\mathcal{N}_0}\mathbf{X}[\![N]\!]!\left\{\left[\!\left[\frac{1}{N}\right]\!\right] - \frac{1}{[\![N]\!]}q^{1/N-N}\right\}. \tag{5.26}$$

2) $SO_q(N)$

$$\begin{aligned}\tilde{\theta}^{k_1k_2} &= \mathrm{d}M^{k_1}{}_s\kappa(M^s{}_l)C^{lk_2}\\ &= \frac{1}{\mathcal{N}_0}\theta^{j_1j_2}(q^{1-N}\delta^{k_1}_{j_1}\delta^{k_2}_{j_2}-\hat{R}_q^{-1\,k_1k_2}{}_{j_1j_2})\\ &= \frac{q}{\mathcal{N}_0}\theta^{j_1j_2}((1+q^{-N})\mathcal{P}_A^{k_1k_2}{}_{j_1j_2}+(q^{-N}-q^{-2})\mathcal{P}_S^{k_1k_2}{}_{j_1j_2}+(q^{-N}-q^{N-2})\mathcal{P}_1^{k_1k_2}{}_{j_1j_2}).\end{aligned} \tag{5.27}$$

Equation (5.26) shows that there exists the singlet component in the basis $\tilde{\theta}$ of $SU_q(N)$ therefore in the differential calculus on $SU_q(N)$ the 1-form basis $\tilde{\theta}$ has also N^2 components. The projector expansion given in Eq. (5.27) shows that the basis $\tilde{\theta}$ also has N^2 components for $SO_q(N)$.

In the limit $q\to 1$ the components belonging to the adjoint representation remain nonzero. On the other hand the additional components drop. Therefore $\tilde{\theta}$ coincides with the usual right invariant 1-form in the limit $q\to 1$.

6. Higher Order Differential Form

6.1. Exterior Product

Automorphism σ. In this section, we define the higher order differential calculus introducing a q-deformed $\wedge$-product and p-forms. For this aim we consider the freely generated algebra with the $\otimes_{\mathcal{A}}$-product of the bimodule Γ_{Ad}:

$$\Gamma_{\mathrm{Ad}}^{\otimes p} = \underbrace{\Gamma_{\mathrm{Ad}}\otimes_{\mathcal{A}}\Gamma_{\mathrm{Ad}}\otimes_{\mathcal{A}}\ldots\otimes_{\mathcal{A}}\Gamma_{\mathrm{Ad}}}_{p}. \tag{6.1}$$

Then we divide it by the ideals corresponding to the symmetric product in the limit $q\to 1$ keeping the bicovariance. Therefore the basic operation to define the higher order differential calculus is the bicovariant bimodule automorphism

$$\sigma : \Gamma_{\mathrm{Ad}}^{\otimes 2}\to\Gamma_{\mathrm{Ad}}^{\otimes 2}, \tag{6.2}$$

such that $\forall a, b\in\mathcal{A}$ and $\forall\tau\in\Gamma_{\mathrm{Ad}}^{\otimes 2}$:

$$\sigma(a\tau b) = a\sigma(\tau)b. \tag{6.3}$$

This map σ generalizes the permutation operation to the case of the tensor product of two bicovariant bimodules [Wor3].

We can find the bimodule automorphism σ which is bicovariant by using the basis θ^I as

$$\sigma(\omega^I\otimes_{\mathcal{A}}\theta^J) = \theta^J\otimes_{\mathcal{A}}\omega^I. \tag{6.4}$$

ω^I is the left invariant basis defined by

$$\omega^J = \kappa(\mathbf{T}^J{}_I)\theta^I, \tag{6.5}$$

where the indices I, J represent a set of indices $I=(i_0,[i])$ for $SU_q(N)$, and $I=(i_1,i_2)$ for $SO_q(N)$. The matrix $\mathbf{T}^I{}_J$ is defined from the left coaction on θ^I as in Eq. (2.30):

For $SU_q(N)$ we get from Eq. (4.20),

$$\mathbf{T}^I{}_J = \mathbf{T}^{i_0[i]}{}_{j_0[j]} = M^{i_0}{}_{j_0}M^{i_1}{}_{j_1'}\ldots M^{i_{N-1}}{}_{j_{N-1}'}\mathcal{P}_{N-1}^{j_1'\ldots j_{N-1}'}{}_{j_1\ldots j_{N-1}}, \tag{6.6}$$

and for $SO_q(N)$ from Eq. (4.25),

$$\mathbf{T}^I{}_J=\mathbf{T}^{i_1i_2}{}_{j_1j_2}=M^{i_1}{}_{j_1}M^{i_2}{}_{j_2}\,. \tag{6.7}$$

Due to the property (6.3) the map σ is then defined completely by the action on the basis of $\Gamma_{\mathrm{Ad}}^{\otimes 2}$ given in Eq. (6.4).

The bicovariant symmetric and antisymmetric $\otimes_{\mathscr{A}}$-product of two bimodules are determined by this automorphism σ. The wedge product is defined by using the antisymmetric $\otimes_{\mathscr{A}}$-product of bimodules. Therefore it is necessary to analyze the structure of this operator σ in detail.

Using Eqs. (6.3) and (6.5) the definition of σ (6.4) is

$$\kappa(\mathbf{T}^I{}_{I'})\sigma(\theta^{I'}\otimes_{\mathscr{A}}\theta^J)=\theta^J\otimes_{\mathscr{A}}\kappa(\mathbf{T}^I{}_{J'})\theta^{J'}\,. \tag{6.8}$$

From this we get

$$\begin{aligned}\sigma(\theta^I\otimes_{\mathscr{A}}\theta^J)&=\mathbf{T}^I{}_{I'}(\kappa(\mathbf{T}^{I'}{}_{J'})*f_{\mathrm{Ad}}{}^J{}_{J''})(\theta^{J''}\otimes_{\mathscr{A}}\theta^{J'})\\&=f_{\mathrm{Ad}}{}^J{}_{I'}(\kappa(\mathbf{T}^I{}_{J'}))(\theta^{I'}\otimes_{\mathscr{A}}\theta^{J'})\,.\end{aligned} \tag{6.9}$$

This equation provides the matrix representation of σ on the basis $\theta^I\otimes_{\mathscr{A}}\theta^J$. Since this matrix representation is given in terms of a combination of the $\hat{R}_q$ matrix it is easy to show that σ satisfies the Yang-Baxter equation

$$(\mathrm{id}\otimes\sigma_{23})(\sigma_{12}\otimes\mathrm{id})(\mathrm{id}\otimes\sigma_{23})=(\sigma_{12}\otimes\mathrm{id})(\mathrm{id}\otimes\sigma_{23})(\sigma_{12}\otimes\mathrm{id})\,, \tag{6.10}$$

where $(\mathrm{id}\otimes_{\mathscr{A}}\sigma_{23})$ and $(\sigma_{12}\otimes\mathrm{id})$ act on $\Gamma_{\mathrm{Ad}}^{\otimes 3}$ and σ_{23} (σ_{12}) acts on the second and third (first and second) elements of the $\Gamma_{\mathrm{Ad}}^{\otimes 3}$. (The relation of σ with the $\hat{R}_q$-matrix has also been pointed out in [Rosso].)

As we expect, the property of the σ discussed above shows that it is a generalized $\hat{R}_q$ matrix of the tensor representation corresponding to the right invariant basis θ^I. Therefore to define the antisymmetric product defined by the operation σ we must find the expansion of the matrix $f_{\mathrm{Ad}}{}^I{}_{I'}(\kappa(\mathbf{T}^J{}_{J'}))$ in terms of the projection operators to the irreducible representations.

Using the matrix representation of the operator σ derived above we first consider the characteristic equation satisfied by σ.

1) For $SU_q(N)$ the $\hat{R}_q$ matrix representation of σ in Eq. (6.9) gives

$$\sigma=f_{\mathrm{Ad}}{}^{j_0[j]}{}_{i_0'[i']}(\kappa(M^{i_0}{}_{j_0'}M^{i_1}{}_{l_1}\ldots M^{i_{N-1}}{}_{l_{N-1}}))\mathscr{P}_{N-1}^{l_1\ldots l_{N-1}}{}_{j_1'\ldots j_{N-1}'}\,. \tag{6.11}$$

Using the Hecke relation

$$\hat{R}_q{}^{ij}{}_{k'l'}\hat{R}_q{}^{k'l'}{}_{kl}=(q-q^{-1})\hat{R}_q{}^{ij}{}_{kl}+\delta^i_k\delta^j_l\,, \tag{6.12}$$

we obtain

$$(\sigma-\mathrm{id})(\sigma+q^2\,\mathrm{id})(\sigma+q^{-2}\,\mathrm{id})=0\,. \tag{6.13}$$

In order to derive this equation we had to know the $\hat{R}_q$ matrix corresponding to the conjugate of the fundamental representation, i.e. $\bar{\eta}_i$. Since to analyze the structure of the matrix σ in Eq. (6.11) we use the $N-1$ rank antisymmetric tensor $\bar{\eta}^{[i]}$. We have to introduce the $\hat{R}_q$ matrix of the $N-1$ rank antisymmetric tensor representation $\hat{\tilde{R}}_q{}^{[i][j]}{}_{[k][l]}$ which is defined by

$$\hat{\tilde{R}}_q{}^{[i][j]}{}_{[k][l]}=\left(\frac{1}{[\![N-1]\!]!}\right)^2\varepsilon^{[i]i'}\varepsilon^{[j]j'}\hat{R}_q{}^{l'k'}{}_{j'i'}\varepsilon_{k'[k]}\varepsilon_{l'[l]}\,. \tag{6.14}$$

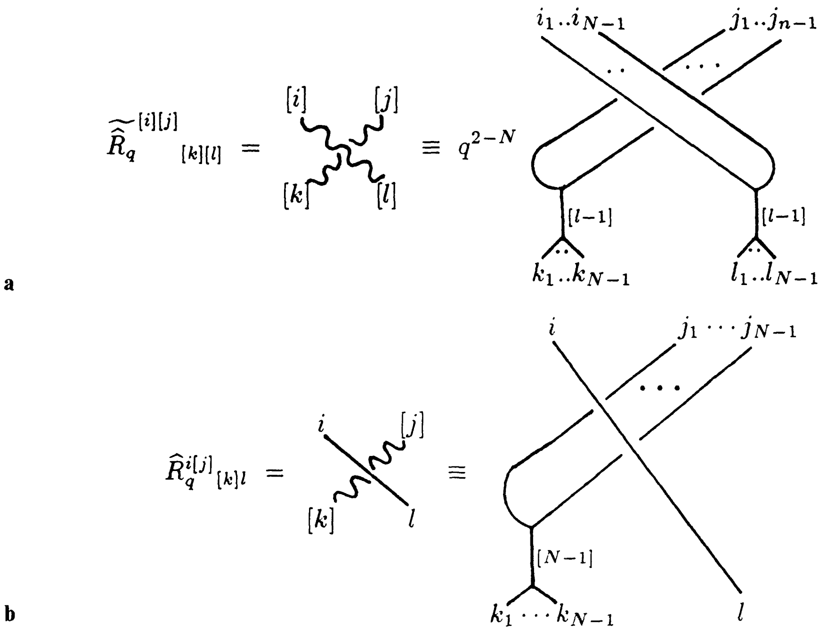


Fig. 10. **a** The graphical representation of the $\tilde{\hat{R}}_q$ matrix corresponding to the commutation of the two $(N-1)^{\text{th}}$ rank antisymmetric tensor representations. **b** The $\hat{R}_q$ matrix corresponding to the commutation of a fundamental representation and a $(N-1)^{\text{th}}$ rank antisymmetric tensor representation. We use the wavy line to represent the $N-1$ antisymmetrized lines

This can be represented by a product of $(N-1)^2$ of $\hat{R}_q^{ij}{}_{kl}$ which is given graphically in Fig. 10a.

The factor in the definition of the matrix $\tilde{\hat{R}}_q$ is chosen such that it also satisfies the Hecke relation:

$$\tilde{\hat{R}}_q^{[i][j]}{}_{[k'][l']}\tilde{\hat{R}}_q^{[k'][l']}{}_{[k][l]} = (q-q^{-1})\tilde{\hat{R}}_q^{[i][j]}{}_{[k][l]} + \delta^{[i]}_{[k]}\delta^{[j]}_{[l]} . \tag{6.15}$$

The proof can be easily performed by using the Hecke relation (6.12) for $\hat{R}_q^{ij}{}_{kl}$.

We also have to introduce the $\hat{R}_q$ matrix for commuting the fundamental representation and the $(N-1)$-rank antisymmetric representation:

$$\hat{R}_q^{i[j]}{}_{[k]l} = \hat{R}_q^{ij_1}{}_{k'_1 i_1}\hat{R}_q^{i_1 j_2}{}_{k'_2 i_2}\cdots\hat{R}_q^{i_{N-2} j_{N-1}}{}_{k'_{N-1} l}\mathscr{P}_{N-1}^{k'_1\ldots k'_{N-1}}{}_{k_1\ldots k_{N-1}} , \tag{6.16}$$

and its inverse $\hat{R}_q^{-1[i]j}{}_{k[l]}$. The graphical representation of $\hat{R}_q^{[i]j}{}_{k[l]}$ is also given in Fig. 10b.

Using these quantities the matrix representation of the σ operation on the basis $\theta^I \otimes_{\mathscr{A}} \theta^J$ in Eq. (6.11) can be represented as

$$f_{\mathrm{Ad}}{}^{i_0[i]}{}_{j_0[j]}(\kappa(\mathbf{T}^{k_0[k]}{}_{l_0[l]})) = \hat{R}_q^{-1 k_0 i'_0}{}_{j_0 k'_0}\hat{R}_q^{-1[k]i_0}{}_{i'_0[k']}\hat{R}_q^{k'_0[i']}{}_{[j]l_0}\tilde{\hat{R}}_q^{[k'][i]}{}_{[i'][l]} . \tag{6.17}$$

With this representation of σ in terms of $\hat{R}_q$ and $\tilde{\hat{R}}_q$ and the graphical representation given in Fig. 11a we can easily derive the characteristic equation (6.13).

2) For $SO_q(N)$ the $\hat{R}_q$ matrix representation of σ in Eq. (6.9) is (see Fig. 11b)

$$\sigma = f_{\mathrm{Ad}}{}^{j_1 j_2}{}_{i'_1 i'_2}(\kappa(M^{i_1}{}_{j'_1}M^{i_2}{}_{j'_2})) = \hat{R}_q^{-1 i_1 k_1}{}_{i'_1 l_1}\hat{R}_q^{-1 i_2 j_1}{}_{k_1 l_2}\hat{R}_q^{l_1 k_2}{}_{i'_2 j'_1}\hat{R}_q^{l_2 j_2}{}_{k_2 j'_2} . \tag{6.18}$$

$$\sigma = f_{Ad}{}^{i_0[i]}{}_{j_0[j]}(\kappa(\mathbf{T}^{k_0[k]}{}_{l_0[l]})) =$$

a

$$\sigma = f_{Ad}{}^{j_1 j_2}{}_{k_1 k_2}\left(\kappa(M^{i_1}{}_{l_1} M^{i_2}{}_{l_2})\right) =$$

b

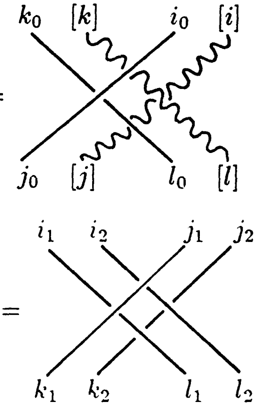


Fig. 11a, b. The graphical representation of the σ: **a** The $SU_q(N)$ case given in Eq. (6.17). **b** The $SO_q(N)$ case given in Eq. (6.18)

In this case the $\hat{R}_q$ matrix satisfies instead of (6.12)

$$((\hat{R}_q - q\mathbf{1})(\hat{R}_q + q^{-1}\mathbf{1})(\hat{R}_q - q^{1-N}\mathbf{1}))^{ij}{}_{kl} = 0\,, \tag{6.19}$$

where $\mathbf{1}^{ij}{}_{kl} = \delta^i_k \delta^j_l$. Therefore we get the characteristic equation for σ of $SO_q(N)$

$$(\sigma - \mathrm{id})(\sigma - q^N \mathrm{id})(\sigma - q^{-N}\mathrm{id})(\sigma + q^2 \mathrm{id})(\sigma + q^{-2}\mathrm{id})(\sigma + q^{2-N}\mathrm{id})(\sigma + q^{N-2}\mathrm{id}) = 0\,. \tag{6.20}$$

Definition of p-Forms. The 2-form for the $SU_q(N)$ case has been defined by Woronowicz as

$$\Gamma_{\mathrm{Ad}}^{\wedge 2} \equiv \Gamma_{\mathrm{Ad}} \otimes_{\mathscr{A}} \Gamma_{\mathrm{Ad}} / [\ker(\sigma - \mathrm{id})]\,. \tag{6.21}$$

This means that the basis of the 2-forms satisfies the following equation:

$$(\sigma + q^2 \mathrm{id})(\sigma + q^{-2}\mathrm{id})(\theta^{i_0[i]} \wedge \theta^{j_0[j]}) = 0\,. \tag{6.22}$$

On the other hand for $SO_q(N)$ from the structure of the characteristic Eq. (6.20) we can read off that the definition of the symmetric $\otimes_{\mathscr{A}}$-product is not simply given by $\ker(\sigma - \mathrm{id})$. Since acting with the operators $(\sigma - q^{\pm N})$ on $\theta \otimes_{\mathscr{A}} \theta$ also reproduces a symmetric product in the limit $q \to 1$. For the q-deformed antisymmetric product we have to impose the additional conditions that $\ker(\sigma - q^N \mathrm{id})$ and $\ker(\sigma - q^{-N}\mathrm{id})$ vanish. Consequently we define the $\wedge$ product in $SO_q(N)$:

$$\Gamma_{\mathrm{Ad}}^{\wedge 2} \equiv \Gamma_{\mathrm{Ad}} \otimes_{\mathscr{A}} \Gamma_{\mathrm{Ad}} / [\ker(\sigma - \mathrm{id}), \ker(\sigma - q^N \mathrm{id}), \ker(\sigma - q^{-N}\mathrm{id})]\,. \tag{6.23}$$

To define the space of p-forms $\Gamma_{\mathrm{Ad}}^{\wedge p}$, we generalize the action of σ on the i^{th} and the $(i+1)^{\text{th}}$ component of $\Gamma_{\mathrm{Ad}}^{\otimes p}$ for $i = 1, \ldots, p-1$ as

$$\begin{aligned}&\sigma_{ii+1}(\theta^{I_1} \otimes_{\mathscr{A}} \ldots \otimes_{\mathscr{A}} \theta^{I_i} \otimes_{\mathscr{A}} \theta^{I_{i+1}} \otimes_{\mathscr{A}} \ldots \otimes_{\mathscr{A}} \theta^{I_p}) \\ &= f_{\mathrm{Ad}}{}^{I_{i+1}}{}_{I'}(\kappa(\mathbf{T}^{I_i}{}_{J'}))\theta^{I_1} \otimes_{\mathscr{A}} \ldots \otimes_{\mathscr{A}} \theta^{I'} \otimes_{\mathscr{A}} \theta^{J'} \otimes_{\mathscr{A}} \ldots \otimes_{\mathscr{A}} \theta^{I_p}\,.\end{aligned} \tag{6.24}$$

Then we can define the space of p-forms by

$$\Gamma_{\mathrm{Ad}}^{\wedge p} \equiv \Gamma_{\mathrm{Ad}}^{\otimes p} \Big/ \prod_{i=1}^{p-1} [\ker(\sigma_{ii+1} - \mathrm{id})]\,, \tag{6.25}$$

for $SU_q(N)$ and by

$$\Gamma_{\mathrm{Ad}}^{\wedge p} \equiv \Gamma_{\mathrm{Ad}}^{\otimes p} \Big/ \prod_{i=1}^{p-1} [\ker(\sigma_{ii+1} - \mathrm{id}), \ker(\sigma_{ii+1} - q^N \mathrm{id}), \ker(\sigma_{ii+1} - q^{-N} \mathrm{id})] \,, \tag{6.26}$$

for $SO_q(N)$.

6.2. *Exterior Derivative of p-Forms*

The action of the exterior derivative d on $\mathscr{A}$ can be generalized on p-forms as in the usual differential calculus. Similar to the definition of the first order derivative in Sect. 5 we define the exterior derivative acting on the p-form as the map

$$\mathrm{d}: \Gamma_{\mathrm{Ad}}^{\wedge p} \to \Gamma_{\mathrm{Ad}}^{\wedge p+1} \,, \tag{6.27}$$

which is defined by $\forall \Omega \in \Gamma_{\mathrm{Ad}}^{\wedge p}$:

$$\mathrm{d}\Omega \equiv \frac{1}{\mathscr{N}} [\mathbf{X}, \Omega]_{\pm} = \frac{1}{\mathscr{N}} (\mathbf{X} \wedge \Omega - (-1)^p \Omega \wedge \mathbf{X}) \,, \tag{6.28}$$

where we introduced the graded commutator $[\cdot,\cdot]_{\pm}$. Apparently the map d defined above respects the bicovariance. It is also easy to show that the map d satisfies for any elements $\Omega_1 \in \Gamma_{\mathrm{Ad}}^{\wedge p}$ and $\Omega_2 \in \Gamma_{\mathrm{Ad}}^{\wedge p'}$

$$\mathrm{d}(\Omega_1 \wedge \Omega_2) = (\mathrm{d}\Omega_1) \wedge \Omega_2 + (-1)^p \Omega_1 \wedge (\mathrm{d}\Omega_2) \,, \tag{6.29}$$

which is the Leibniz rule of the exterior derivative acting on the $(p+p')$-form.

Furthermore as we shall prove in the next section, the map d satisfies the nilpotency:

$$\mathrm{d}^2 = 0 \,. \tag{6.30}$$

This completes the definition of the exterior derivative of the p-forms, since we can calculate the graded commutator on the right-hand side of Eq. (6.28) using the definition of the $\wedge$-product given in Eqs. (6.25) and (6.26). However, practically it is not so easy to perform these computations. In the next section we present the explicit expression using the graphical representation. Then the Maurer-Cartan equation and the structure constants are derived.

7. Maurer-Cartan Equation

7.1. *$SU_q(N)$ Case*

In order to analyze the structure of the $\wedge$-product we need the projector expansion of σ. Due to the Hecke relation (6.12) the matrix $\tilde{\hat{R}}_q$ can be represented by the projectors to the antisymmetric and symmetric representations.

$$\tilde{\hat{R}}_q = q\tilde{\mathscr{P}}_S - q^{-1}\tilde{\mathscr{P}}_A \,, \tag{7.1}$$

with

$$\tilde{\mathscr{P}}_r{}^{[i][j]}{}_{[k][l]} = \left(\frac{1}{[\![N-1]\!]!}\right)^2 \varepsilon^{[i]i'} \varepsilon^{[j]j'} (\mathscr{P}_r{}^{l'k'}{}_{j'i'}) \varepsilon_{k'[k]} \varepsilon_{l'[l]} \,, \tag{7.2}$$

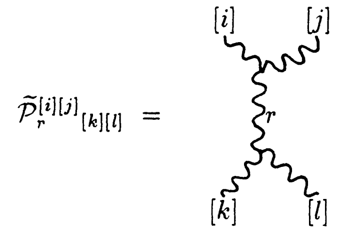


Fig. 12. The graphical representation of the projectors in Eq. (7.2)

where $r = A$, S. We also use the graphical representation of them (see Fig. 12).

Using the projector expansion of $\hat{R}_q$ and $\tilde{\hat{R}}_q$ we get

$$\sigma = [(\mathscr{P}_S, \tilde{\mathscr{P}}_S) + (\mathscr{P}_A, \tilde{\mathscr{P}}_A) - q^{-2}(\mathscr{P}_S, \tilde{\mathscr{P}}_A) - q^2(\mathscr{P}_A, \tilde{\mathscr{P}}_S)]^{i_0[i]k_0[k]}{}_{j_0[j]l_0[l]}, \tag{7.3}$$

where

$$(\mathscr{P}_r, \tilde{\mathscr{P}}_{r'})^{i_0[i]j_0[j]}{}_{k_0[k]l_0[l]} = \hat{R}_q^{-1\,[i]j_0}{}_{j_0'[i']}\mathscr{P}_r^{i_0j_0'}{}_{k_0l_0'}\tilde{\mathscr{P}}^{[i'][j]}{}_{[k'][l]}\hat{R}_q^{l_0'[k']}{}_{[k]l_0}, \tag{7.4}$$

and $\hat{R}_q^{i[j]}{}_{[k]l}$ is the one introduced in Eq. (6.16).

Here r, r' are either S or A. One can understand the structure of this operator $(\mathscr{P}_r, \mathscr{P}_{r'})$ better through the graphical representation given in Fig. 13a. Especially it is easy to see that each operator is a projector and they are orthogonal to each other, i.e.

$$(\mathscr{P}_{r_1}, \tilde{\mathscr{P}}_{r_1'})^{i_0[i]j_0[j]}{}_{i_0'[i']j_0'[j']}(\mathscr{P}_{r_2}, \tilde{\mathscr{P}}_{r_2'})^{i_0'[i']j_0'[j']}{}_{k_0[k]l_0[l]} = \delta_{r_1,r_2}\delta_{r_1'r_2'}(\mathscr{P}_{r_1}, \tilde{\mathscr{P}}_{r_1'})^{i_0[i]j_0[j]}{}_{k_0[k]l_0[l]} \tag{7.5}$$

Then the conditions given in Eq. (6.25) are equivalent to set the following $\dfrac{N^2(N^2+1)}{2}$ relations to zero:

$$(\mathscr{P}_S, \tilde{\mathscr{P}}_S)^{i_0[i]j_0[j]}{}_{k_0[k]l_0[l]}(\theta^{k_0[k]} \otimes_{\mathscr{A}} \theta^{l_0[l]}) = 0, \tag{7.6}$$

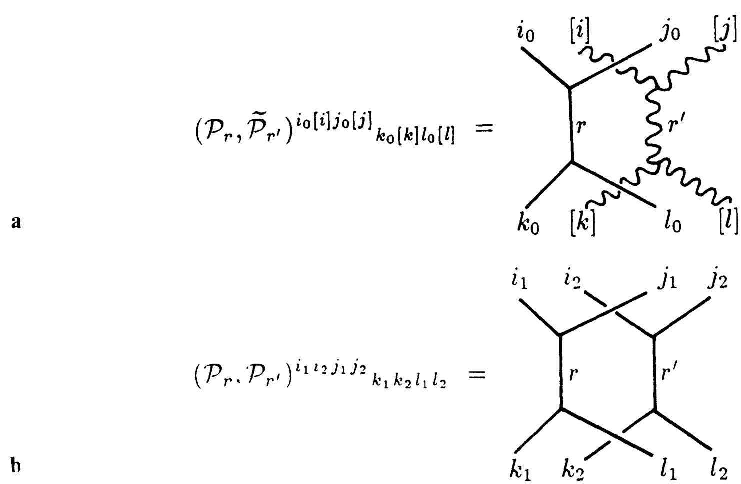


Fig. 13a, b. The graphical representation of the projectors of the tensor representation: **a** $SU_q(N)$ case corresponding to Eq. (7.4). **b** $SO_q(N)$ case corresponding to Eq. (7.23) and

$$(\mathscr{P}_A, \tilde{\mathscr{P}}_A)^{i_0[i] j_0[j]}{}_{k_0[k] l_0[l]} (\theta^{k_0[k]} \otimes_{\mathscr{A}} \theta^{l_0[l]}) = 0 . \tag{7.7}$$

To evaluate the exterior derivative using the definition (5.8) and (6.28) we rewrite these relations in terms of the irreducible components.

For this purpose we introduce the following projectors

$$\mathscr{P}_X^{i_0[i]}{}_{j_0[j]} = \frac{(-1)^{N-1}}{[\![N]\!]!} \varepsilon^{i_0[i]} \varepsilon_{j_0[j]} , \tag{7.8}$$

and

$$\mathscr{P}_{\mathrm{Ad}}^{i_0[i]}{}_{j_0[j]} = (\mathbf{1} - \mathscr{P}_X)^{i_0[i]}{}_{j_0[j]} , \tag{7.9}$$

where $\mathbf{1}^{i_0[i]}{}_{j_0[j]}$ is the identity matrix.

With these projectors we can decompose the right invariant basis $\theta^{i_0[i]}$ into the irreducible components corresponding to a singlet

$$\theta_X = \mathscr{P}_X^{i_0[i]}{}_{j_0[j]} \theta^{j_0[j]} = \frac{(-1)^{N-1}}{[\![N]\!]!} \varepsilon^{i_0[i]} \mathbf{X} , \tag{7.10}$$

and the adjoint representation

$$\theta_{\mathrm{Ad}}^{i_0[i]} = \mathscr{P}_{\mathrm{Ad}}^{i_0[i]}{}_{j_0[j]} \theta^{j_0[j]} . \tag{7.11}$$

Then the relations (7.6) and (7.7) can be rewritten for these irreducible components as

$$(\mathscr{P}_r, \mathscr{P}_r)^{i_0[i] j_0[j]}{}_{k_0[k] l_0[l]} (\theta_{\mathrm{Ad}}^{k_0[k]} + \theta_X^{k_0[k]}) \wedge (\theta_{\mathrm{Ad}}^{l_0[l]} + \theta_X^{l_0[l]}) = 0 , \tag{7.12}$$

where r is either S or A. The relations which contain the element $\mathbf{X}$ among the $\frac{N^2(N^2+1)}{2}$ relations can be written in a convenient form:

1) Applying $\mathscr{P}_X$ to the indices $i_0[i]$ and $j_0[j]$ on the left-hand side of Eq. (7.12) we get

$$\mathbf{X} \wedge \mathbf{X} = 0 . \tag{7.13}$$

2) Applying $\mathscr{P}_X$ on the indices $i_0[i]$ and $\mathscr{P}_{\mathrm{Ad}}$ on $j_0[j]$ on the left-hand side of Eq. (7.12) we get after some tedious calculation:

$$\mathbf{X} \wedge \theta_{\mathrm{Ad}}^{i_0[i]} + \theta_{\mathrm{Ad}}^{i_0[i]} \wedge \mathbf{X} = (q - q^{-1}) \mathscr{K} F^{i_0[i]}_{j_0[j] k_0[k]} \theta_{\mathrm{Ad}}^{j_0[j]} \wedge \theta_{\mathrm{Ad}}^{k_0[k]} , \tag{7.14}$$

where

$$\mathscr{K} = \frac{(-1)^{N-1}}{\left(q^N - \frac{(q-q^{-1})}{[\![N]\!]} \right)} , \tag{7.15}$$

and

$$F^{i_0[i]}_{j_0[j] k_0[k]} = \mathscr{P}_{\mathrm{Ad}}^{i_0[i]}{}_{i_0'[i']} \mathscr{P}_{\mathrm{Ad}}^{i_0[j']}{}_{j_0[j]} \mathscr{P}_{\mathrm{Ad}}^{k_0'[i']}{}_{k_0[k]} \varepsilon_{[j']k_0} . \tag{7.16}$$

The rest of the conditions gives the relation among the $\theta_{\mathrm{Ad}}^{i_0[i]} \wedge \theta_{\mathrm{Ad}}^{j_0[j]}$ such as

$$\varepsilon_{k_0[l]} \varepsilon_{[k] l_0} (\theta_{\mathrm{Ad}}^{k_0[k]} \wedge \theta_{\mathrm{Ad}}^{l_0[l]}) = 0 . \tag{7.17}$$

By using the above results we prove the nilpotency of the exterior derivative defined in Eq. (6.28):

The first Eq. (7.13) means that

$$\mathrm{d}\mathbf{X}=0, \tag{7.18}$$

and consequently

$$\begin{aligned} \mathrm{dd}\Omega &= \frac{1}{\mathcal{N}}\,\mathrm{d}(\mathbf{X}\wedge\Omega-(-1)^{p}\Omega\wedge\mathbf{X}) \\ &= \frac{1}{\mathcal{N}^{2}}((-1)^{p}\mathbf{X}\wedge\Omega\wedge\mathbf{X}-(-1)^{p}\mathbf{X}\wedge\Omega\wedge\mathbf{X}) \\ &= 0\,. \end{aligned} \tag{7.19}$$

The Maurer-Cartan equation can be formulated by using Eq. (7.14):

$$\mathrm{d}\theta^{i_0[i]}_{\mathrm{Ad}}=\frac{\mathcal{K}}{\mathcal{N}_0}\,F^{i_0[i]}_{j_0[j]k_0[k]}\theta^{j_0[j]}_{\mathrm{Ad}}\wedge\theta^{k_0[k]}_{\mathrm{Ad}}\,. \tag{7.20}$$

Therefore the quantities $F^{i_0[i]}_{j_0[j]k_0[k]}$ are the q-analogues of the structure constants. The graphical representation of these structure constants is given in Fig. 14.

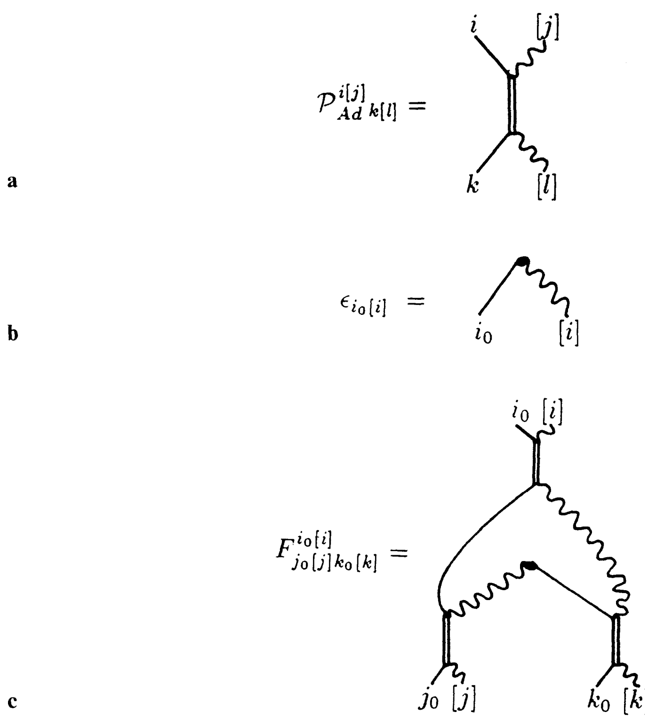


Fig. 14. a The graphical representation of the projection operator to the adjoint representation given in Eq. (7.9). **b** The graphical representation of $\varepsilon_{i_0\ldots i_{N-1}}$ using the wavy line for $N-1$ lines corresponding to the indices $[i]=\{i_1\ldots i_{N-1}\}$. **c** We give the graphical representation of the structure constants of $SU_q(N)$ using the graphical representations in **a, b**

Applying the exterior derivative on both sides of Eq. (7.20) we get the q-deformed Jacobi identity of the structure constants:

$$(F^{j'_0[j']}{}_{j_0[j]k_0[k]}F^{i_0[i]}{}_{j'_0[j']l_0[l]}-F^{i_0[i]}{}_{j_0[j]k'_0[k']}F^{k'_0[k']}{}_{k_0[k]l_0[l]})\theta^{j_0[j]}_{\mathrm{Ad}}\wedge\theta^{k_0[k]}_{\mathrm{Ad}}\wedge\theta^{l_0[l]}_{\mathrm{Ad}}=0\,. \tag{7.21}$$

7.2. $SO_q(N)$ Case

For the case of $SO_q(N)$ we use the projection operators introduced in Eq. (3.46). Then we get the projector expansion of the σ matrix in Eq. (6.18):

$$\begin{aligned}\sigma=&(\mathscr{P}_S,\mathscr{P}_S)+(\mathscr{P}_A,\mathscr{P}_A)+(\mathscr{P}_1,\mathscr{P}_1)+q^{-N}(\mathscr{P}_S,\mathscr{P}_1)+q^{N}(\mathscr{P}_1,\mathscr{P}_S)\\&-q^{-2}(\mathscr{P}_S,\mathscr{P}_A)-q^{2}(\mathscr{P}_A,\mathscr{P}_S)-q^{N-2}(\mathscr{P}_1,\mathscr{P}_A)-q^{2-N}(\mathscr{P}_A,\mathscr{P}_1)\end{aligned} \tag{7.22}$$

with

$$(\mathscr{P}_r,\mathscr{P}_{r'})^{i_1i_2j_1j_2}{}_{k_1k_2l_1l_2}=\hat{R}_q^{-1\,i_2j_1}{}_{j'_1i'_2}\mathscr{P}_r^{i_1j'_1}{}_{k_1l'_1}\mathscr{P}_{r'}^{i'_2j_2}{}_{k_2l'_2}\hat{R}_q^{l'_1k'_2}{}_{k_2l_1}\,, \tag{7.23}$$

where $r,r'=S$, A or 1, respectively. Note that as in the $SU_q(N)$ case, all the terms of Eq. (7.22) are projectors orthogonal to each other:

$$(\mathscr{P}_{r_1},\mathscr{P}_{r'_1})^{i_1i_2j_1j_2}{}_{i'_1i'_2j'_1j'_2}(\mathscr{P}_{r_2},\mathscr{P}_{r'_2})^{i'_1i'_2j'_1j'_2}{}_{k_1k_2l_1l_2}=\delta_{r_1,r_2}\delta_{r'_1r'_2}(\mathscr{P}_{r_1},\mathscr{P}_{r'_1})^{i_1i_2j_1j_2}{}_{k_1k_2l_1l_2}\,. \tag{7.24}$$

This can be proven easily using the graphical representation of these operators given in Fig. 13b.

Then the conditions given in Eq. (6.25) are equivalent to the following $\frac{N^2(N^2+1)}{2}$ relations:

$$(\mathscr{P}_r,\mathscr{P}_r)^{i_1i_2j_1j_2}{}_{k_1k_2l_1l_2}(\theta^{k_1k_2}\otimes_{\mathscr{A}}\theta^{l_1l_2})=0\,, \tag{7.25}$$

with $r=S$, A, 1 and

$$(\mathscr{P}_S,\mathscr{P}_1)^{i_1i_2j_1j_2}{}_{k_1k_2l_1l_2}(\theta^{k_1k_2}\otimes_{\mathscr{A}}\theta^{l_1l_2})=0\,, \tag{7.26}$$

$$(\mathscr{P}_1,\mathscr{P}_S)^{i_1i_2j_1j_2}{}_{k_1k_2l_1l_2}(\theta^{k_1k_2}\otimes_{\mathscr{A}}\theta^{l_1l_2})=0\,. \tag{7.27}$$

We also decompose the right invariant basis into the irreducible components using the projection operator $\mathscr{P}_r$ with $r=A$, S, 1 as

$$\theta^{i_1i_2}=\theta_A^{i_1i_2}+\theta_S^{i_1i_2}+\theta_1^{i_1i_2}\,, \tag{7.28}$$

where

$$\theta_A^{i_1i_2}=\mathscr{P}_A^{i_1i_2}{}_{j_1j_2}\theta^{j_1j_2}\,, \tag{7.29}$$

$$\theta_S^{i_1i_2}=\mathscr{P}_S^{i_1i_2}{}_{j_1j_2}\theta^{j_1j_2}\,, \tag{7.30}$$

and

$$\begin{aligned}\theta_1^{i_1i_2}&=\mathscr{P}_1^{i_1i_2}{}_{j_1j_2}\theta^{j_1j_2}\\&=Q_N^{-1}C^{i_1i_2}\mathbf{X}\,,\end{aligned} \tag{7.31}$$

where $Q_N=\dfrac{(1-q^N)(1+q^{2-N})}{1-q^2}$.

To write down the Maurer-Cartan equation we again substitute the decomposition (7.28) into Eqs. (7.25)–(7.27). We list here some of the essential relations in these bases.

1) Applying $\mathscr{P}_1$ on the indices $i_1 i_2$ and $j_1 j_2$ on the left-hand side of Eq. (7.25) we get

$$\mathbf{X} \wedge \mathbf{X} = 0, \tag{7.32}$$

$$C_{k_1 l_2} C_{k_2 l_1} (\theta_A^{k_1 k_2} \wedge \theta_A^{l_1 l_2}) = 0, \tag{7.33}$$

$$C_{k_1 l_2} C_{k_2 l_1} (\theta_S^{k_1 k_2} \wedge \theta_S^{l_1 l_2}) = 0. \tag{7.34}$$

2) Applying the antisymmetrizer $\mathscr{P}_A$ on the indices $i_1 i_2$ and $\mathscr{P}_1$ on $j_1 j_2$ of Eq. (7.25) we get

$$\theta_A^{i_1 i_2} \wedge \mathbf{X} + \mathbf{X} \wedge \theta_A^{i_1 i_2} = (q - q^{-1}) \mathscr{K}_1 F^{i_1 i_2}_{j_1 j_2 k_1 k_2} (\theta_A^{j_1 j_2} \wedge \theta_A^{k_1 k_2}). \tag{7.35}$$

where $\mathscr{K}_1$ is a q dependent constant

$$\mathscr{K}_1 = \frac{(1 + q^{2-N})(q + q^{-1})^2}{(q^{N-3} + q^3 + q^{-3} + q^{3-N})}. \tag{7.36}$$

The structure constants $F^{i_1 i_2}_{j_1 j_2 k_1 k_2}$ are given by

$$F^{i_1 i_2}_{j_1 j_2 k_1 k_2} = \mathscr{P}_A{}^{i_1 i_2}{}_{i'_1 i'_2} \mathscr{P}_A{}^{i'_1 j'_2}{}_{j_1 j_2} \mathscr{P}_A{}^{k'_1 i'_2}{}_{k_1 k_2} C_{j'_2 k'_1}. \tag{7.37}$$

3) Applying the symmetrizer $\mathscr{P}_S$ on the indices $i_1 i_2$ and $\mathscr{P}_1$ on $j_1 j_2$ of Eq. (7.25) we get

$$\theta_S^{i_1 i_2} \wedge \mathbf{X} + \mathbf{X} \wedge \theta_S^{i_1 i_2} = (q - q^{-1}) \mathscr{K}_2 \left(G \begin{bmatrix} S \\ SA \end{bmatrix}^{i_1 i_2}_{j_1 j_2 k_1 k_2} (\theta_S^{j_1 j_2} \wedge \theta_A^{k_1 k_2}) \right.$$

$$\left. + G \begin{bmatrix} S \\ AS \end{bmatrix}^{i_1 i_2}_{j_1 j_2 k_1 k_2} (\theta_A^{j_1 j_2} \wedge \theta_S^{k_1 k_2}) \right), \tag{7.38}$$

where the constant $\mathscr{K}_2$ is

$$\mathscr{K}_2 = \frac{(1 + q^{2-N})(q + q^{-1})^2}{(2q^{N-1} + q^3 + q + q^{-1} + q^{-3} + 2q^{1-N})}. \tag{7.39}$$

The tensors $G \begin{bmatrix} r_1 \\ r_2 r_3 \end{bmatrix}^{i_1 i_2}_{j_1 j_2 k_1 k_2}$ are given by

$$G \begin{bmatrix} r_1 \\ r_2 r_3 \end{bmatrix}^{i_1 i_2}_{j_1 j_2 k_1 k_2} = \mathscr{P}_{r_1}{}^{i_1 i_2}{}_{i'_1 i'_2} \mathscr{P}_{r_2}{}^{i'_1 j'_2}{}_{j_1 j_2} \mathscr{P}_{r_3}{}^{k'_1 i'_2}{}_{k_1 k_2} C_{j'_2 k'_1}, \tag{7.40}$$

where r_1, r_2, and r_3 stand for the representations A or S. The tensors $G \begin{bmatrix} r_1 \\ r_2 r_3 \end{bmatrix}$ are the q-analogue of the Clebsch-Gordon coefficient of the fusion of two representations: $r_2 \otimes r_3 \to r_1$. The structure constants F in Eq. (7.37) are equivalent to $G \begin{bmatrix} A \\ AA \end{bmatrix}$. [See also Fig. 15.]

We also get the following relations

$$\theta_S^{i_1 i_2} \wedge \mathbf{X} + \mathbf{X} \wedge \theta_S^{i_1 i_2} = \frac{q(q - q^{-1})(q + q^{-1})^2}{(q^{N-2} - 1)^2} G \begin{bmatrix} S \\ AA \end{bmatrix}^{i_1 i_2}_{j_1 j_2 k_1 k_2} (\theta_A^{j_1 j_2} \wedge \theta_A^{k_1 k_2}), \tag{7.41}$$

$$G\begin{bmatrix} r_1 \\ r_2 r_3 \end{bmatrix}^{i_1 i_2}_{j_1 j_2 k_1 k_2} =$$

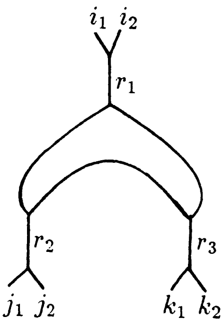


Fig. 15. The graphical representation of the constants $G\begin{bmatrix} r_1 \\ r_2 r_3 \end{bmatrix}$ in Eq. (7.40), where r_1, r_2, and r_3 are either A or S. The structure constants of $SO_q(N)$ are given by $G\begin{bmatrix} A \\ AA \end{bmatrix}$

and

$$\theta_S^{i_1 i_2} \wedge \mathbf{X} + \mathbf{X} \wedge \theta_S^{i_1 i_2} = \frac{q(q-q^{-1})(q+q^{-1})^2}{\left(\frac{q^N}{Q_N}(q^2-q^{-2})(1+q^{2-N})+1-q^{N+2}\right)} \times G\begin{bmatrix} S \\ SS \end{bmatrix}^{i_1 i_2}_{j_1 j_2 k_1 k_2} (\theta_S^{j_1 j_2} \wedge \theta_S^{k_1 k_2}) . \tag{7.42}$$

The first relation (7.32) proves the nilpotency of the exterior derivative defined in Eq. (6.28) analogously to (7.19). The Maurer-Cartan equation can be read off from Eq. (7.35). Using the definition (6.28) we get

$$d\theta_A^{i_1 i_2} = \frac{\mathcal{K}_1}{\mathcal{N}_0} F^{i_1 i_2}_{j_1 j_2 k_1 k_2} (\theta_A^{j_1 j_2} \wedge \theta_A^{k_1 k_2}) . \tag{7.43}$$

From Eq. (7.38) we also get the exterior derivative of the bases θ_S,

$$d\theta_S^{i_1 i_2} = \frac{\mathcal{K}_2}{\mathcal{N}_0} \left(G\begin{bmatrix} S \\ SA \end{bmatrix}^{i_1 i_2}_{j_1 j_2 k_1 k_2} (\theta_S^{j_1 j_2} \wedge \theta_A^{k_1 k_2}) + G\begin{bmatrix} S \\ AS \end{bmatrix}^{i_1 i_2}_{j_1 j_2 k_1 k_2} (\theta_A^{j_1 j_2} \wedge \theta_S^{k_1 k_2}) \right) . \tag{7.44}$$

Knowing the explicit form of the Maurer-Cartan equations we can also investigate the structure of the algebra of the differential operators χ_I defined in Eqs. (5.13) and (5.14). Then we get the definitions of the q-analogue of Lie brackets for $SU_q(N)$ and $SO_q(N)$. Due to the result of Sect. 3.3, these algebras of χ_I give different formulations of the q-analogue of the universal enveloping algebras which is now under investigation.

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