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On The Algebraic Structure of $N=2$ String Theory

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Abstract. In $N = 2$ string theory the chiral algebra expresses the generations and anti-generations of the theory and the Yukawa couplings among them and is thus crucial to the phenomenological properties of the theory. Also the connection with complex geometry is largely through the algebras. These algebras are systematically investigated in this paper. A solution for the algebras is found in the context of rational conformal field theory based on Lie algebras. A statistical mechanics interpretation for the chiral algebra is given for a large family of theories and is used to derive a rich structure of equivalences among the theories (dihedralities). The Poincaré polynomials are shown to obey a resolution series which cast these in a form which is a sum of complete intersection Poincaré polynomials. It is suggested that the resolution series is the proper tool for studying all $N = 2$ string theories and, in particular, exposing their geometrical nature.

1. Introduction

In many respects, a viable physical theory is not unlike a basic mathematical one. Apart from being experimentally correct, all good physical theories are marked by a set of simple concepts and the depth and elegance of their results. The study of nature, from this viewpoint, may be considered as the uncovering of the principles from which it stems, with experimental data supplying the lead. String theory, although as yet only partially understood and thus hard to confront directly with the realm of particle physics and their interactions, passes the criteria above with flying colors. So far, it is experimentally correct, in the sense that string theories which closely imitate nature can be constructed. Equal in significance, it has been offering depth and elegance that are perhaps unprecedented, fusing into its set of notions many mathematical fields in a nontrivial fashion and giving rise to almost miraculous interrelations among these.

The subject of this paper is $N = 2$ string theory, first constructed in ref. [1] which in its structure appears to embody the aforementioned properties (for a review, see, ref. [2]). Its origin is in the pursual of semi-realism in the framework of non-trivial string theory. The requirements of supersymmetry and sufficiently large gauge group were shown to have a canonical solution. Namely, for any two dimensional quantum field theory with the so-called $N=2$ superconformal invariance, there is a unique construction of a fully consistent string theory in four dimensions which is space-time supersymmetric. It is further shown that the requirement of modular invariance can be satisfied, in general, only if the underlying gauge group is $E_8 \times E_6$ or $SO(26) \times U(1)$. This class of theories has a very good phenomenological potential. For example, a three generation string theory has been studied in detail [3] and appears to be quite promising.

Owing to the severely constrained consistency requirements of string theory, the fact that string theory can be constructed in such a generality is quite surprising. Even more amazing is the fact that to each such string theory one can attach a geometry, and that the entire massless spectrum of the string theory and many of the interactions are completely geometrical in nature. This interplay between geometrical and algebraic structures in $N=2$ string theory is as yet only partially understood. This paper is a step in the exploration of this.

The $N=2$ superconformal algebra, N_2 is a graded super-Lie algebra. The commutation relations are [4],

$$\begin{aligned}
&[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^2 - m)\delta_{n+m,0},\\
&[L_m, J_n] = -nJ_{m+n},\\
&[L_m, G_r^{\pm}] = \left(\frac{m}{2} - r\right)G_{m+r}^{\pm},\\
&[J_m, J_n] = \frac{nc}{3}\delta_{n+m},\\
&[J_m, G_r^{\pm}] = \pm G_{m+r}^{\pm},\\
&\{G_r^+, G_s^-\} = 2L_{r+s} + (r-s)J_{r+s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}.
\end{aligned} \tag{1.1}$$

The first equation is the usual Virasoro algebra for the moments of the energy-momentum tensor. The J_n generate to the usual Heisenberg algebra ($U(1)$ current algebra), while the G_n are fermionic generators, obeying anti-commutation relations among each other. c is the central charge of the Virasoro algebra. The indices r and s take their values either in Z (the integers), in which case this is called the Ramond sector, or in $Z+1/2$ which is called the Neveu–Schwarz sector. We shall assume that all the representations are highest weight modules. Then the Cartan subalgebra consists of J_0 and L_0. Accordingly, each such highest weight representation is labeled by the value of c (the central charge), the value of L_0 (denoted by Δ and called the conformal dimension) and the value of J_0 (the $U(1)$ charge, denoted by Q), when acting on the highest weight vector.

By an $N=2$ superconformal field theory we shall refer to a fully reducible highest weight representation, $\mathscr{H}$, of the product of two such algebras, $N_2 \times N_2$. We shall refer to each as the left and right $N=2$ superconformal algebras. The values of Δ and $\bar{\Delta}$ as the left and right $U(1)$ charges and the values of Q and $\bar{Q}$ as the left and right $U(1)$ charges. The representation is such that c has a certain value (it acts as the identity). We shall call the decomposition of an $N=2$ superconformal field theory into its irreducible representations, the spectrum of the theory. In addition we shall assume that this representation is modular invariant, unitary, and that the representation $Q=\Delta=\bar{Q}=\bar{\Delta}=0$ appears exactly once in the spectrum of the theory.

The property of modular invariance in this context is the following. Consider the quantity,

$$Z(\tau)=\operatorname*{Tr}_{\mathscr{H}} e^{2\pi i\tau(L_0-c/24)}e^{-2\pi i\tau^*(\bar{L}_0-c/24)}, \tag{1.2}$$

which is a certain specialized character of this representation. Here τ is an arbitrary complex number in the upper half plane. Modular invariance is the condition that $Z(\tau)=Z(\tau+1)=Z(-1/\tau)$. $Z(\tau)$ is called the partition function of the theory, corresponding to its Feynman path integral on the torus, and hence the modular invariance property.

A conformal field theory is marked with an additional structure. Primarily, one can consider the space of fields in the theory, each denoted by $\phi_i(z,\bar{z})$, where z and $\bar{z}$ are complex numbers, which forms an isomorphic representation of the $N_2\times N_2$ algebra, where the isomorphism sends the field $\phi(z,\bar{z})$ to the state $|\phi\rangle=\phi(0)|0\rangle$, where $|0\rangle$ is the vacuum, i.e., the unique highest weight state with $\Delta=\bar{\Delta}=Q=\bar{Q}=0$. The action of $N_2\times N_2$ on the fields is related to superconformal maps on the variables $z,\bar{z}$ and four Grassmann variables $\theta_\pm$ and $\bar{\theta}_\pm$ (superspace).

The fields in the theory which correspond to highest vectors of the superconformal algebra are called primary fields. The primary fields obey the inequalities

$$\Delta\geqq\frac{|Q|}{2} \quad\text{and}\quad \bar{\Delta}\geqq\frac{|\bar{Q}|}{2}, \tag{1.3}$$

which is simplest to see by computing $\langle\phi|[G^+_{1/2},G^-_{-1/2}]|\phi\rangle\geqq 0$, implied by unitarity. The fields for which the equality holds in Eq. (1.3) are special and are called chiral primary fields. One can define a product structure on the set of chiral fields which obey $\Delta=Q/2$, $\bar{\Delta}=\bar{Q}/2$ via the operator products of the fields. Denote by C_i and C_j two such chiral fields. It follows from the conformal properties of the theory that

$$C_i(z_1)C_j(z_2)=f^k_{ij}C_k(z_1)+O(z_1-z_2), \tag{1.4}$$

where the product is regular as $z_1\to z_2$, and the constant term is itself a chiral field with $Q=Q_i+Q_j$ and $\bar{Q}=\bar{Q}_i+\bar{Q}_j$. Thus, endowed with this product structure, the set of chiral fields becomes a bigraded associative commutative algebra (see, e.g., ref. [5, 6, 7]), called the chiral algebra. Similarly, one can define a product structure for the fields which obey $\Delta=Q/2$, $\bar{\Delta}=-\bar{Q}/2$, called the anti-chiral algebra. The other two algebras one can define are related to these two through complex conjugation of the structure constants.

Since any unitary N_2 theory is a tensor product of a $U(1)$ current algebra (generated by the Heisenberg algebra J_n) and a unitary conformal field theory, it follows that all the fields in such a theory obey the inequalities,

$$\Delta\geqq\frac{3Q^2}{2c} \quad\text{and}\quad \bar{\Delta}\geqq\frac{3\bar{Q}^2}{2c}. \tag{1.5}$$

For the chiral fields this implies that there is a unique chiral field with maximal $U(1)$ charge, with $Q = c/3$. Bosonizing the $U(1)$ current, $J = i\sqrt{c/3}\partial_z\phi$, where ϕ is a canonical free boson, this field assumes the form,

$$C_{\max} = e^{i\sqrt{c/3}\phi}, \quad \bar{C}_{\max} = e^{i\sqrt{c/3}\bar{\phi}}, \tag{1.6}$$

where we have similarly defined the right moving field $\bar{C}_{\max}$. The product of these two fields, $C_{\max}\bar{C}_{\max}$ is a field with the $U(1)$ charges $(c/3, c/3)$. This field might or might not appear in the spectrum of the actual conformal field theory. From here on we shall limit ourself to N_2 theories for which this field is indeed in the spectrum of the theory. In case that ambiguity cannot arise we shall denote this field simply by $C_{\max}$.

In all the known N_2 conformal field theories, the $U(1)$ charges have a rational value. Moreover, the set of charges appearing in a theory is of the form s/D, where D is some fixed integer and s is an integer which depends on the field. It is natural to conjecture that in all N_2 theories this is indeed the form of the charges, and we shall assume it from here on. The existence of a chiral field of a maximal $U(1)$ charge, $C_{\max}$, then implies that the chiral algebra $\mathscr{C}$ is finite dimensional. Hence also all the fields other than the identity are nilpotent.

Owing to the presence of the field $C_{\max}$ in the theory, one can define a transposition operation [8]. Let C be any chiral field. The transpose of this field is then defined by

$$C^t = C_{\max} C^{\dagger}, \tag{1.7}$$

where the field $C^{\dagger}$ is the complex conjugate of the field C, and the product above is defined as the most singular term in the operator product of the two fields. It can be seen that the field C^t is a chiral field with the $U(1)$ charges $\left(\frac{c}{3} - Q, \frac{c}{3} - \bar{Q}\right)$. The transposition operation is a 1–1 map form the chiral algebra, $\mathscr{C}$, onto itself which is of order 2, $(C^t)^t = C$.

It is strongly believed, through the connection with complex manifolds described below, that this pair of algebras completely determines an $N = 2$ superconformal field theory.

To every superconformal field theory with central charge $c = 12$, one can associate a superstring theory. More precisely, it is a string theory in $D + 2 =$ even dimensions if the conformal field theory is a tensor product of a $c = 12 - 3D/2$ superconformal field theory with D free bosons X^{μ} (the string coordinates) and D free fermions (their supersymmetry partners). The physically interesting case is $D = 2$ which corresponds to a four dimensional string theory. By applying a certain projection, the resulting string theory can be made space-time supersymmetric. As a result of this projection all the $U(1)$ charges become odd integers.

It is convenient to actually split the representations appearing in the theory to smaller representations, which are representations of the algebra (which is not a Lie algebra) generated by the products of any even number of the G_i, which will be denoted by $\hat{N}_2$. We can then define the full character of the algebra $\hat{N}_2$, of some irreducible highest weight representation $\mathscr{H}_p$,

$$\chi_p(\tau, z, u) = e^{-2\pi i u} \operatorname*{Tr}_{\mathscr{H}_p} e^{2\pi i z J_0} e^{2\pi i \tau (L_0 - c/24)}, \tag{1.8}$$

which is the generating function of the number of fields in the representation $\mathscr{H}_p$ with a given dimension and a $U(1)$ charge. There is an action of the modular group on the full character described by,

$$\chi_p\left(\frac{a\tau+b}{m\tau+n}, \frac{z}{m\tau+n}, u+\frac{mcz^2}{6(m\tau+n)}\right)=\sum_q S_{p,q}\chi_q(\tau,z,u), \tag{1.9}$$

where $S_{p,q}$ is some (unitary) matrix, and the sum ranges over some set of representations, all with the same central charge c. Here a, b, m and n are arbitrary integers such that $an-bm=1$, i.e., the elements of the modular group $SL_2(Z)$. In a string theory the central charge is $c=12$, as explained above, and we shall assume this value from here on. Written in terms of the characters, the partition function of the theory becomes,

$$Z(\tau)=\sum_{p,q} N_{p,q}\chi_p(\tau,0,0)\chi_q(\tau,0,0)^* \tag{1.10}$$

which corresponds to the decomposition of the Hilbert space into irreducible representations,

$$\mathscr{H}=\bigoplus_{p,q} N_{p,q}\mathscr{H}_p\times\mathscr{H}_q, \tag{1.11}$$

and $N_{p,q}$ are some non-negative integers.

Now, given an $N=2$ theory with $c=12$ with a partition function as above, we can project and make the theory space-time supersymmetric, while retaining its modular invariance, by summing over the action of the operator $(-1)^n Q^n = (-1)^n e^{in\phi}$. The supersymmetrized partition function would then be

$$Z^{\text{susy}}(\tau)=\sum_{p,q} N_{p,q}\delta(p)\delta(q)\chi_p^{\text{sum}}(\tau,0,0)\chi_q^{\text{sum}}(\tau,0,0)^*, \tag{1.12}$$

where $\delta(q)$ is equal to one if the $U(1)$ charge Q_q is an odd integer, and is zero otherwise. χ_q^{sum} is defined by

$$\chi_q^{\text{sum}}(\tau,z,u)=\sum_{n\in Z}(-1)^n\chi_p\left(\tau, z+\frac{n\tau}{2}, u-\frac{n^2c\tau}{24}-\frac{znc}{6}\right). \tag{1.13}$$

Supersymmetry implies that the partition function so defined is equal identically to zero, $Z^{\text{susy}}(\tau)=0$, since there is an equal number of bosons and fermions at each mass level and these appear with opposite signs in the partition function. Thus, Eq. (1.12) is understood as a generating function for the spectrum when the full dependence on z and u is retained. Modular invariance then follows from Eq. (1.9).

A four dimensional string theory includes the space-time fermions and space-time bosons X^μ and ψ^μ, where $\mu=1,2$, as explained above. The inclusion is both for the left and for the right movers. The theory can be made heterotic-like via the replacement of $\bar{\psi}^\mu$ (i.e., the right moving fermions with space-time index), with internal fermions, representing either the level one $SO(26)$ or the $E_8\times SO(10)$ current algebras. In the latter case which is the one of physical interest since it admits chiral fermions, the gauge group becomes $E_8\times E_6\times G$, where G is some enhanced symmetry group (which is generically empty). The massless spectrum can then be read from Eq. (1.12) and contains in the matter sector only particles in the 27, $\bar{27}$ and the singlet representations of E_6. All the particles are E_8 singlets.

The particles in the 27 representation of E_6 (generations) correspond to all the chiral fields in the theory with $Q=\bar{Q}=1$. Similarly, the particles in the $\bar{27}$ (anti-generations) are described by the anti-chiral fields, $Q=-\bar{Q}=1$. In addition, the Yukawa couplings among the generations are determined by the graded chiral algebras introduced above, as follows [8]. Consider the three point Yukawa coupling of some three particles in the 27 representation of E_6. Then, each of these correspond to a chiral field C_i with $Q_i=\bar{Q}_i=1$. Multiply the three chiral fields, $C=C_iC_jC_k$ with the product structure introduced above. The result is then a chiral field with $Q=\bar{Q}=3$. However, this chiral field is a chiral field also of the $c=9$ theory with $Q=\bar{Q}=c/3=3$. Thus, $C=\alpha_{ijk}C_{\max}$, where $C_{\max}$ is the unique maximal chiral field of the $c=9$ theory and α_{ijk} is some constant of proportionality. The constant α_{ijk} is the Yukawa coupling of the three fields. Similarly, the Yukawa couplings among the anti-generations are determined by the anti-chiral algebra.

This description of the chiral algebra is very close in spirit to the geometry of manifolds with a vanishing first Chern class, although absolutely no geometrical input was entered in the construction of this class of string theories. In particular, in field theory compactifications on such manifolds ref. [9] the 27 correspond to (2, 1) forms and their Yukawa couplings to the wedging of three such forms, where one extracts the coefficient of the unique (3, 0) form. We thus have roughly the identification of the chiral algebra with the Delbeult cohomology complex of the manifold in question and the Yukawa couplings as the graded algebra structure of this complex given by the wedging of forms.

Via a case study, this correspondence can be made very precise and detailed, in particular, singling out the manifold which corresponds to each $N=2$ string theory. For example, the discrete unitary series of the N_2 algebra has the central charge $c=3k/(k+2)$ for any positive integer k [10]. The chiral algebra of the theories is given by the cyclical polynomial algebra, $P(x)/(x^{k+1})$, where (x^{k+1}) denotes the ideal generated by x^{k+1}. Taking $k+2$ such theories results in the central charge $c=3k$, which can be used to construct a string theory in $10-2k$ dimensions. It can then be seen that the corresponding manifold is [11],

$$\sum_{i=1}^{k+2} Z_i^{k+2}=0, \tag{1.14}$$

where the Z_i are complex variables in CP^{k+1}, $\{Z_i\}=\{\lambda Z_i\}$, for any complex λ. The Fermat surface, Eq. (1.14), is a complex manifold with dimension k and a vanishing first Chern class. The cohomology algebra of the manifold is then seen to be identical to that of the corresponding $N=2$ string theory.

A theorem was conjectured in ref. [11] and proved in some cases (see [2] for an account), that, indeed, all $N=2$ string theories are geometrical in nature, and that their chiral algebras correspond to cohomologies of complex manifolds with vanishing first Chern class. The purpose of this paper is to further investigate this question in the context of a large class of N_2 theories.

To simplify our considerations we will assume that the N_2 theory before the supersymmetry projection is left-right symmetric. Namely, we shall assume that there are no anti-chiral fields and that all the chiral fields obey $Q=\bar{Q}$. In this case the bigrading of the chiral algebra becomes a unique grading according to $Q=\bar{Q}$.

Let $C_1, C_2, \ldots, C_r$, be a set of generating chiral fields in the sense that all the fields in the chiral algebra, $\mathscr{C}$, can be obtained by products of these. We can then define the "normal order" map, τ, as follows [8]. Consider the free algebra of polynomials $P = P(x_1, x_2, \ldots, x_r)$, in which the x_i are graded in the same way as the chiral fields $C_1, C_2, \ldots, C_r$. Given a polynomial $p = x_1^{a_1} x_2^{a_2} \cdots x_r^{a_r} \in P$ we define $\tau(p)$ to be the chiral field $C_1^{a_1} C_2^{a_2} \cdots C_r^{a_r}$, where the product is in the operator product sense, Eq. (1.4). The fact the operator product algebra among the chiral fields forms an associative algebra is crucial for the definition of the normal ordered map. This map is a graded algebras homomorphism. The kernel of this map $K = \mathrm{Ker}(\tau)$ consists of all the polynomials $p(x_i)$ for which $\tau(p) = 0$ and is a graded ideal of the algebra P. The ideal K is generated by a set of a finite number of relations that the generating chiral fields obey, syzigies in Hilbert's terminology. The chiral algebra itself is given as the quotient algebra $\mathscr{C} = P/K$. The chiral algebra is completely determined by this set of syzigies. Thus, if one knows which Yukawa couplings among the chiral fields vanish, one can determine the precise values of the non-vanishing ones, as well.

An important tool in the study of graded algebras (referred to as G-algebras) is the Poincaré series [6, 7, 12] which is a character of the algebra. Let M be any graded module of the G-algebra A. We then define the Poincaré series of M as

$$P_M(t) = \sum_{m \in M} t^{Q_m}, \tag{1.15}$$

where Q_m is the grade of $m \in M$. Since A itself is an A-module, one can similarly define the Poincaré series of A, denoted by $P(t)$.

Since the chiral algebras are finite dimensional, the Poincaré series becomes a polynomial. The transposition map, Eq. (1.7), implies that $\dim(Q) = \dim(c/3 - Q)$ where $\dim(Q)$ is the dimension of the vector subspace of charge Q. Thus, the Poincaré polynomial of chiral algebras obey the duality property,

$$P(t) = t^{c/3} P\left(\frac{1}{t}\right). \tag{1.16}$$

The Poincaré series of G-algebras obey a number of properties [6, 7, 12]. Let M_1 and M_2 be two graded A-modules. We can then define the direct sum module, $M = M_1 \oplus M_2$ and the tensor product module $N = M_1 \otimes M_2$. The Poincaré series of M and N are then

$$P_M(t) = P_{M_1}(t) + P_{M_2}(t), \quad P_N(t) = P_{M_1}(t) P_{M_2}(t). \tag{1.17}$$

The Poincaré series of a free polynomial algebra, $P = P(x_1, x_2, \ldots, x_r)$, is easily seen to be given by

$$P(t) = \prod_{i=1}^{r} \frac{1}{1 - t^{d_i}}, \tag{1.18}$$

where the d_i are the grades assigned to the variables x_i. A theorem of Hilbert and Serre asserts that every Poincaré series of a module of a G-algebra is a product of the series above $P(t)$, where the d_i are the degrees of the generators, times a finite polynomial. The proof, which can be used to compute the polynomial, is based on the identities above, Eq. (1.17), via the writing of an exact sequence of free modules which "resolves" the given module. This sequence is called a finite free resolution.

The Poincaré series encodes much of the structure of the algebra and many results have been established concerning the characterization of G-algebras through their Poincaré series (see, e.g., ref. [13]).

An important class of G-algebras is the complete intersection algebras. The polynomial algebra L,

$$L = P(x_1, x_2, \ldots, x_m)/(\theta_1, \theta_2, \ldots, \theta_n), \tag{1.19}$$

where $\theta_1, \theta_2, \ldots, \theta_n$ is a set of syzigies among the x_i and $(\theta_1, \theta_2, \ldots, \theta_n)$ denotes the ideal generated by these syzigies, is a complete intersection algebra, if the relations form a A-sequence, i.e.,

$$(\theta_1, \theta_2, \ldots, \theta_n)P \neq P \tag{1.20}$$

and θ_i is not a zero divisor modulo $\theta_1, \theta_2, \ldots, \theta_{i-1}$ for all $1 \leqq i \leqq n$.

If the degrees of the variables are d_i and those of the syzigies are f_i, then the Poincaré polynomial of this algebra is

$$P(t) = \frac{\prod_{i=1}^{n} (1 - t^{f_i})}{\prod_{i=1}^{m} (1 - t^{d_i})}. \tag{1.21}$$

This is proved using the properties of the Poincaré series Eq. (1.17) [13, 12]. Denote the ideal $(\theta_1, \theta_2, \ldots, \theta_n)$ by K. Consider P as a K module. The complete intersection condition implies that P is a free K module, and the basis elements are given by the basis of the algebra $L = P/K$. It follows that P is isomorphic to the tensor product of L and K as modules over the base field. The property Eq. (1.17) implies that $P_P(t) = P_L(t)P_K(t)$. Since both P and K are free polynomial algebras (using the algebraic independence of the θ_i which follows from the complete intersection condition) their Poincaré polynomials are of the form Eq. (1.18) and the result follows.

There are many examples of complete intersection algebras. If A is a coinvariant polynomial algebra of some complex reflection group, then it is a complete intersection algebra with $n = m$, $d_i = 1$ and the f_i are some exponents which depend on the group. A well known theorem of Todd and Chevalay [14] asserts that G is a complex reflection group if and only if its coinvariant algebra has this form.

Another example of complete intersection algebra is the local ring of a quasi-homogeneous singularity [15]. In this case, the syzigies, ϕ_i are given by the derivatives of the singularity. If an N_2 theory is described by a two dimensional Wess–Zumino model (scalar field theory) [16, 17, 8] then the graded chiral algebra of the theory is given by this local ring [8]. Namely, the algebra is of the form $P/(\phi_i)$, where $V(x_1, x_2, \ldots, x_r)$ is some quasi-homogeneous potential, of degree d and $\phi_i = \dfrac{\partial V}{\partial x_i}$. It follows that in this case $f_i = d - d_i$.

In all these examples, the Poincaré polynomials have a very specific form: the zeros and poles of the polynomial are all primitive roots of unity. The degrees f_i and d_i can then be read from the polynomial uniquely, thus obtaining the algebraic information about the degrees of the generators and the syzigies and the fact that they are algebraically independent. Such information can be obtained also in case that the algebra is not a complete intersection algebra, however, some guesswork might be involved. The advantage of doing so is that it is generally much easier to compute the Poincaré polynomial than to study the algebra directly.

Our path in studying $N = 2$ string theory is clear then. First the Poincaré polynomial is computed and is cast in a form which reveals its algebraic structure. This is the combinatorial part of the problem. Then this combinatorial information is used to decode the algebraic structure of the chiral algebra. Finally, this algebraic structure is used to find the associated geometry, by exposing a complex manifold whose cohomology ring has this structure. This has been done in many instances [11, 17, 8].

We shall use as our testing ground for these ideas a large class of N_2 superconformal field theories, the $N = 2$ coset models, which are based on affine current algebra. These may be considered as the rational conformal field theories of the N_2 algebra. The problem of classifying these theories, is, to a large extent, equivalent to the classification of all rational conformal field theories. This classification, in turn, proceeds by linking them with complex manifolds.

We find here the chiral algebra of these theories in complete generality. It is then seen that the chiral fields assume a remarkably simple form, being in correspondence with certain subsets of the Weyl group. We then proceed to calculate the Poincaré polynomials.

It turns out that the generic Poincaré polynomial are never of the complete intersection form Eq. (1.21), except for a few isolated cases. However, quite generally, we show that the Poincaré polynomials of the theories can be written as a sum of polynomials of the form Eq. (1.21), with some of the exponents being negative. The exponents that appear are then seen to obey many remarkable properties, which almost determine them completely.

It is thus suggested that the proper tool for classifying these theories, and perhaps all N_2 conformal field theories, is to consider sums of polynomials of the complete intersection type. We call such sums the resolution series of a theory. Examples of resolution series abound in all known N_2 theories, but are not always easy to calculate. This essentially solves the combinatorial part of the problem. Unfortunately, less is known at the present concerning the algebraic and geometrical part of it and some directions are outlined in the discussion.

2. $N = 2$ Quotient Theories

Conformal field theories associated with quotients of Lie algebras, G/H, were introduced in ref. [18, 19] for abelian H, and generalized to other H in [20]. For a special choices of G and H the quotient theory has and $N = 2$ superconformal invariance [21]. The $N = 2$ quotient conformal field theories are constructed as follows. One takes a Lie algebra G and a subgroup of it H. We shall assume that $\mathrm{rank}(G) = \mathrm{rank}(H)$ and that H is a diagram subgroup. Namely, $H = H_1 \times H_2 \times \cdots H_m$, where each of the H_i is either a semi-simple Lie subalgebra of G or a $U(1)$ subalgebra, and H is obtained by the removal of some nodes from the

Dynkin diagram of G. Such a pair of (G,H) is called a reductive pair.

One then forms the quotient theory $\dfrac{G_k \times SO(n)_1}{H^1_{k+g-h_1} \times H^2_{k+g-h_2} \times \cdots \times H^m_{k+g-h_m}}$, where G_k stands for a $\hat{G}$ current algebra at level k, whose currents are denoted by J^a, and $n = \dim(G) - \dim(H)$. Similarly the SO and H_i factors correspond to the current algebras at the appropriate levels. g and h_i are the dual Coxeter numbers of G and H^1_i. H is the diagonal subalgebra of this product (the H currents are the sum of the G and $SO(n)$ ones). The $SO(n)_1$ current algebra may be represented in terms of n free Majorana fermions, denoted by ρ^α, where α is a root of G.

The fields in the theory are obtained by decomposing a $G \times SO(n)_1$ into the H current algebra times the quotient theory. Explicitly, let Λ and λ stand for integrable highest weights of the G algebra at level k and the H algebra at level $k+g-h$. In addition, let s stand for the four integrable representations of $SO(n)_1$, the singlet, the vector and the two spinor representations. The hallmark of a quotient conformal field theory is the decomposition of fields $g = \phi h$, where g is any field in the G current algebra, h a field in the H current algebra, and ϕ is a field in the quotient theory [18, 19, 20, 22]. In this context, a general field of the quotient theory, denoted by $\Phi^{\Lambda,a}_\lambda$ is obtained from,

$$G^\Lambda V^a = \Phi^{\Lambda,a}_\lambda H^\lambda. \tag{2.1}$$

Here G^Λ, V^s and H^λ stand for any three fields in the corresponding representations of $G, SO(n)_1$ and H. Equation (2.1) enables the calculation of the dimensions and operator products of the fields in the quotient theory from those of the fields in the current algebra.

The roots of G may be divided into positive and negative roots. It is convenient to use the order relation on roots $\alpha > \beta$ if $\alpha - \beta$ is a positive root. The positive (negative) roots are a sum of the simple roots with positive (negative) coefficients. Consider the following field in the $G \times SO(n)$ current algebra

$$G^+ = \frac{1}{\sqrt{k+g}}\left[\sum_{\alpha>0} \rho^{-\alpha} J_\alpha + i \sum_{\substack{\beta,\gamma>0\\ \alpha<0}} f^{\alpha\beta\gamma} \rho^\alpha \rho^\beta \rho^\gamma\right], \tag{2.2}$$

where α, β and γ range over the roots of G, Δ_G which are not roots of H, $\alpha \in \Delta_G - \Delta_H = \Delta_{G/H}$, and $G^- = G^{+\dagger}$, i.e., replacing i with $-i$ in Eq. (2.2) and exchanging "$<$" with "$>$". The fields $G^\pm$ are singlets under H and thus are also fields in the quotient theory. It can be verified that the fields $G^\pm$ are the super-energy momentum tensors of an $N = 2$ superconformal algebra and their moments obey the commutation relations Eq. (1.1). The energy-momentum tensor of the quotient theory is the difference, $T = T_G + T_{SO} - T_H$, where T_G is the energy momentum of the G current algebra, etc. The $U(1)$ current is

$$J = -\sum_{\alpha>0} \rho^\alpha \rho^{-\alpha} + \frac{1}{k+g} h_\alpha, \tag{2.3}$$

[1] If H_i is a subalgebra of short roots, the level is actually $2(k+g)/\theta_i^2 - h_i$, where θ_i is the longest root of H in the normalization given by the embedding, which is the length of the short roots. This is easy to see by writing down the current algebra commutation relations

where h_α is the element of the Cartan subalgebra of h dual to the root α. All the sums over roots above are on the roots of G which are not roots of H.

Since $T + T_H = T_G + T_{SO}$, and this is an identity as a tensor product of conformal field theories as well, all properties of the theory may be computed. The central charge of the theory is given by $c = c_G + c_{SO} - c_H$. Since $c_H = kD_G/(k+g)$, where D_G is the dimension of G [23], we find

$$c = \frac{D_G - D_{H_1} - D_{H_2} - \cdots D_{H_n}}{2} + \frac{kD_G}{k+g} - \sum_{i=1}^{m} \frac{D_{H_i}(k + g - \frac{1}{2}h_i\theta_i^2)}{k+g}, \tag{2.4}$$

where for a $U(1)$ factor we take $\theta^2 = h = 0$.

Using the strange formula of Freudenthal–De Vries,

$$\frac{\rho^2}{g} = \frac{D_G \theta^2}{24}. \tag{2.5}$$

The central charge may be written more compactly as

$$\tfrac{1}{3}c(k+g) = (k+g)\mathrm{Dim}(\Delta_{G/H}) - 4\rho_G(\rho_G - \rho_H), \tag{2.6}$$

where ρ_G and ρ_H are half the sum of positive roots for G and H respectively. We used the fact that $\rho_G \rho_H = \rho_H^2$ which is easy to prove since ρ obeys $\rho\alpha_i = \frac{1}{2}\alpha_i^2$ for all the simple roots.

The dimension and $U(1)$ charge of the field $\Phi_\lambda^{\Lambda,a}$ can be read from Eq. (2.1),

$$\Delta = \frac{\Lambda(\Lambda + 2\rho_G) - \lambda(\lambda + 2\rho_H)}{2(k+g)} + \frac{a^2}{2} + n, \quad q = -2sa + \frac{2}{k+g}(\rho_G - \rho_H)\lambda + 2m. \tag{2.7}$$

s is the spinor weight of the $SO(n)$ algebra, n and m are some integers. As noted earlier, there are infinitely many different fields which correspond to the decomposition Eq. (2.1) for a given Λ, λ and a. The integers n and m are different for each of these fields. For particular fields in the theory, these integers may be computed using the actual dimension of the other fields appearing in Eq. (2.1), but there is no simple formula for these integers, in general.

Recall from Sect. (1) that all the fields in an $N = 2$ conformal field theory obey $\Delta \geqq |q|/2$, where the chiral fields are the ones for which the equality holds.

As discussed in Sect. (1) the operator product algebra of the chiral fields gives rise to a finite dimensional associative algebra. We shall now turn to the derivation of these algebras in the framework of coset $N = 2$ theories which will be described in this section and the next one. This result was announced in ref. [24].

To find the chiral fields we could, in principle, use the relation $\Delta = Q/2$ along with Eq. (2.7) for the dimensions. There is a problem, however, associated to the fact that there is no good way to determine, in general, the integers appearing in Eq. (2.7). Thus, we will need to make a direct study of the fields in the theory.

The method we will use to derive all the chiral fields is a direct extension of that used in ref. [8] to derive a subset of the chiral fields. The chiral fields obey,

$$G_-(\zeta)C(z) = O\left(\frac{1}{\zeta - z}\right), \tag{2.8}$$

$$G_+(\zeta)C(z) = O(1). \tag{2.9}$$

For the coset $N=2$ theories the superpartners of the energy-momentum tensor, $G_\pm$, are given by Eq. (2.2). Note that the fields $G_\pm$ are singlets of H, i.e., under the decomposition into H the fields $G_\pm$ give the unit field in the H current algebra. Now, a chiral field C is obtained be a decomposition of the field $\hat{C}$ in the $G \times SO(n)$ current algebra,

$$\hat{C} = C \times h, \tag{2.10}$$

where h is some field in the H current algebra. The fact that $G_\pm$ decompose into the unit operator in H implies that Eqs. (2.8–2.9) are equivalent to the following equations

$$G_-(\zeta)\hat{C}(z) = O\left(\frac{1}{\zeta - z}\right), \tag{2.11}$$

$$G_+(\zeta)\hat{C}(z) = O(1), \tag{2.12}$$

which are equations purely in the $G \times SO(n)$ current algebra.

Let us make the following ansatz for the chiral fields. Assume that the chiral fields are of the form

$$C = \rho^{-a_1}\rho^{-a_2}\cdots\rho^{-a_n}g, \tag{2.13}$$

where the a_i are different positive roots of the simple algebra G and g is some field in the Hilbert space of the $\hat{G}$ current algebra. Let us also assume at first, for simplicity, that $H = U(1)^r$, where r is the rank of G. Under the ansatz Eq. (2.13) each of the terms in $G_\pm$ must obey the conditions for chirality, Eqs. (2.8–2.9), separately. This is due to fermion number conservation for each of the adjoint fermions.

Thus, each term in the expressions for $G_\pm$ gives a separate condition. Let us analyze the condition for the terms of the type $G_{\alpha,\beta,\gamma} = f_{-\alpha,\beta,\gamma}\rho^{-\alpha}\rho^{\beta}\rho^{\gamma}$, where α, β and γ are three positive roots. The requirement that $f_{-\alpha,\beta,\gamma} \neq 0$ is equivalent to $\alpha = \beta + \gamma$. Denote by N_α the number of times that the field $\rho^{-\alpha}$ appears in the expression for C, Eq. (2.13). N_α is either zero or one for each of the positive roots. From the operator product of the free fermions, $\rho^{\alpha}(\zeta)\rho^{-\alpha}(z) = \dfrac{1}{\zeta - z}$ and $\rho^{\alpha}(\zeta)\rho^{\alpha}(z) = O(\zeta - z)$, it follows that the operator product

$$G_{\alpha\beta\gamma}(\zeta)C(z) = O[(\zeta - z)^{N_\beta + N_\gamma - N_\alpha}]. \tag{2.14}$$

Similarly, the degree of the singularity of $G^\dagger_{\alpha\beta\gamma}$ is $-N_\beta - N_\gamma + N_\alpha$. It follows that a necessary condition for the field C to be chiral is,

$$1 \geqq N_\beta + N_\gamma - N_{\beta+\gamma} \geqq 0, \quad \text{for all} \quad \beta, \gamma, \beta + \gamma \in \Delta_+, \tag{2.15}$$

where we denoted by Δ_+ the set of positive roots of G. This set of equations appears to be rather cumbersome, but in fact, as we shall now show there is a very simple and beautiful solution to it. Let us define

$$N_{-\beta} = 1 - N_\beta, \tag{2.16}$$

where β is any positive root. It is easy to check that with this definition, Eq. (2.15) holds for all the roots of G and not just the positive ones. Define now

$$\tilde{\Delta} = \{\gamma \in \Delta \mid N_\gamma = 0\}. \tag{2.17}$$

The set $\tilde{\Delta}$ has the properties. 1) For every $\gamma \in \Delta$ either $\gamma \in \tilde{\Delta}$ or $-\gamma \in \tilde{\Delta}$. 2) If $\beta, \gamma \in \tilde{\Delta}$ then $\beta + \gamma \in \tilde{\delta}$. This follows from Eq. (2.15) and is, in fact, equivalent to it.

Let us call an element $\alpha \in \tilde{\Delta}$ indecomposable if it cannot be written as $\alpha = \beta + \gamma$, where β and $\gamma \in \tilde{\Delta}$. Denote the set of indecomposable elements by Δ_s. From property (1) and the definition, it is clear that every root γ can be written as $\gamma = \sum n_i \alpha_i$, where $\alpha_i \in \Delta_s$ and the n_i are either all non-negative or all non-positive. From property (2) it follows that the elements of Δ_s are linearly independent. Thus Δ_s is a basis for the root system.

A standard theorem in the theory of Lie algebras (see, e.g., ref. [25]) gives a description for the set of basis of the root system. Every basis is obtained by the action of some element of the Weyl group, $w \in W(G)$, on a standard basis. Thus, $\tilde{\Delta} = w^{-1}(\Delta_+)$, where w is some element of the Weyl group. Different Weyl elements give different basis, since the Weyl group acts transitively on the Weyl chambers, and thus the solutions of Eq. (2.15) are in a 1–1 correspondence with the Weyl group. For any root α, $N_\alpha = 0$ iff $w(\alpha) > 0$. To summarize, the chiral fields C must be of the form,

$$C = \rho^{-w} g, \tag{2.18}$$

where

$$\rho^{-w} = \prod_{\substack{\alpha > 0 \\ w(\alpha) < 0}} \rho^{-\alpha}. \tag{2.19}$$

and w is any element of the Weyl group. Different elements of the Weyl group give rise to different fields in the $G \times SO(n)$ current algebra.

Let us turn now to the first term in $G_\pm$. These are the terms of the type $p^{-\alpha} J_\alpha$, where α is a positive (for G_+) or negative (for G_-). If $\alpha > 0$ is a root such that $w(\alpha) > 0$, then from Eqs. (2.11–2.12) it follows that

$$J_\alpha(\zeta) g(z) = O(1), \quad J_{-\alpha}(\zeta) g(z) = O\left(\frac{1}{\zeta - z}\right). \tag{2.20}$$

If $w(\alpha) < 0$ then

$$J_\alpha(\zeta) g(z) = O\left(\frac{1}{\zeta - z}\right), \quad J_{-\alpha}(\zeta) g(z) = O(1). \tag{2.21}$$

Since the elements of the CSA are obtained as the commutators $[J^\alpha, J^{-\alpha}]$ it follows that for all the currents of G, J^a, we have

$$J^a(\zeta) g(z) = O\left(\frac{1}{\zeta - z}\right). \tag{2.22}$$

Thus g is a primary field of the G current algebra, denoted by G^Λ_λ, where Λ is a weight at level $k, k \geqq \Lambda\theta$, where θ is the highest root, and λ is some weight in the representation with the highest weight Λ. Now, the conditions Eqs. (2.20–2.21) are equivalent to $J_\alpha G^\Lambda_\lambda = 0$ iff $w(\alpha) > 0$. Since the representation is invariant under the action of the Weyl group this implies that $\alpha + w(\lambda) \notin L(\Lambda)$ for every positive root $\alpha > 0$, and $w(\lambda)$ is the highest weight of the representation, $\lambda = w^{-1}(\Lambda)$.

We conclude that the chiral fields are all obtained from the fields,

$$C^\Lambda_w = \rho^{-w} G^\Lambda_{w^{-1}(\Lambda)}, \tag{2.23}$$

in the $G \times SO(n)$ current algebra, where $w \in W(G)$ and Λ is an integrable highest weight. For each such Λ and w the field C_w^Λ is chiral.

Due to the quotient by the H current algebra, some of the fields C_w^Λ give rise to identical chiral fields in the $G \times SO(n)/H$ theory.

Let us recall some facts about field identification in current algebra (see ref. [22] for a discussion and further references). Let σ be an automorphism of the extended Dynkin diagram of the affine Lie algebra $\hat{G}$. The simple roots of $\hat{G}$ are the simple roots of G, denoted by $\alpha_1, \alpha_2, \ldots, \alpha_r$, where r is the rank of the algebra, along with the root $\alpha_0 = \delta - \theta$, where θ is the highest root and δ is a null vector, $\delta v = 0$ for any root vector v. The external diagram automorphism is some permutation of the simple roots $\sigma(\alpha_i) = \alpha_{p(i)}$ which preserves the scalar products, $\sigma(\alpha_i)\sigma(\alpha_j) = \alpha_i \alpha_j$.

Now, the dual to the root space is the weight space generated by the fundamental weights of the simple algebra G, along with the weight Λ_0. These obey $\Lambda_i \alpha_j = \delta_{ij}\alpha_i^2/2$ for $i, j = 0, 1, \ldots, r$. Thus also under the external automorphism σ the fundamental weights transform by the same permutation of the indices which holds for the simple roots, $\sigma(\Lambda_i) = \Lambda_{p(i)}$. If $\hat{\lambda}$ is a weight of $\hat{G}$ then $\hat{\lambda}\delta = k$ is the level of the weight (we normalized $\theta^2 = 2$). It follows that the weight $\hat{\lambda}$ can be written as

$$\hat{\lambda} = (k - \theta\lambda) + \sum_{i=1}^{r} n_i \Lambda_i, \tag{2.24}$$

where $\lambda = \sum_{i=1}^{r} n_i \Lambda_i$ is the finite algebra part of the weight. A weight is integrable iff all the coefficients appearing in Eq. (2.24) are non-negative integers, $n_i \in Z$ and $k \geqq \theta\lambda$.

Under the action of the external automorphism σ the weight $\hat{\lambda}$ transforms as

$$\sigma(\hat{\lambda}) = (k - \lambda\theta)\Lambda_{p(0)} + \sum_{i=1}^{r} n_i \Lambda_{p(i)}. \tag{2.25}$$

If σ is such an external automorphism then σ gives rise to a change of basis of the finite algebra G. Let $\beta_i = \alpha_i$ for $i = 1, 2, \ldots, r$ and $\beta_0 = -\theta$. Then this change of basis is $\hat{\sigma}(\beta_i) = \beta_{p(i)}$. Since scalar products are preserved, this is an automorphism of the finite algebra and it follows that $\hat{\sigma}$ is equal to a product of some element of the Weyl group w_σ (which affects this change of basis) times a possible external diagram automorphism of the finite algebra G. An external automorphism, σ, for which $\hat{\sigma} = w_\sigma \in W(G)$ (i.e., it does not contain a finite diagram automorphism) are called proper, and these are the only ones that interest us.

For proper external automorphisms it follows that

$$\sigma(\lambda) = \sigma(0) + w_\sigma(\lambda), \tag{2.26}$$

where $\sigma(0)$ obeys $\sigma(0) = k\Lambda_i$ and $\Lambda_i \theta = 1$. Such weights are called minimal fundamental weights and for each such weight there is one proper external automorphism. The external automorphism group is isomorphic to the center of the group G, or equivalently to the weight lattice modulo the long root lattice. The same group is also obtained from the Weyl transformations w_σ. We thus have an embedding of the center group in the Weyl group given by $\sigma \to w_\sigma$. The external automorphisms may be described as a twist of the algebra $\hat{G}$ by an element of the weight lattice, which may be affected continuously [26].

For example, take $SU(n) \approx A_{n-1}$. The Dynkin diagram of $\hat{A}_{n-1}$ is a circle with n points. The proper automorphisms are rotations of this circle, forming a Z_n group. The finite weight $\sum_{i=1}^{n-1} m_i \Lambda_i$ at level k transforms under the automorphism σ to $\sigma(\lambda)=(k-m_1-m_2-\cdots m_{n-1})\Lambda_1+m_1\Lambda_2+m_2\Lambda_3+\cdots+m_{n-2}\Lambda_{n-1}$. The other proper automorphisms are powers of this one. We have $w_\sigma = w_{\alpha_1} w_{\alpha_2} \cdots w_{\alpha_{n-1}}$, where w_β is a Weyl reflection by the root β.

Consider now the fields of the G current algebra theory at level k. As was discussed in ref. [19] one can decompose the currents into n free bosons and "parafermions,"

$$h_i(z) = \frac{2i\sqrt{k}}{\alpha_i^2} \vec{\alpha}_i \partial_z \vec{\phi}, \tag{2.27}$$

$$x_\alpha(x) = \sqrt{\frac{2k}{\alpha^2}}\, c_\alpha \psi_\alpha e^{i\vec{\alpha}\vec{\phi}/\sqrt{k}}, \tag{2.28}$$

where h_i are the currents of the Cartan subalgebra, x_α are the others and we are using the Chevaley basis. The c_α are cocycle factors. The highest weight states of the G current algebra may be written as

$$G_{\lambda\bar{\lambda}}^{\Lambda\bar{\Lambda}} = e^{i(\lambda\phi+\bar{\lambda}\bar{\phi})/\sqrt{k}}\, \Phi_{\lambda,\bar{\lambda}}^{\Lambda,\bar{\Lambda}}. \tag{2.29}$$

We also have $\psi_\alpha = \Phi_{\alpha,0}^{0,0}$. The parafermionic fields obey

$$\Phi_{\lambda,\bar{\lambda}}^{\Lambda,\bar{\Lambda}} = \Phi_{\lambda+\beta,\bar{\lambda}}^{\Lambda,\bar{\Lambda}} = \Phi_{\lambda,\bar{\lambda}+\beta}^{\Lambda,\bar{\Lambda}}, \tag{2.30}$$

where β is any element of the lattice kM_L, where M_L is the lattice spanned by the long roots. In addition,

$$\Phi_{\lambda+\sigma(0),\bar{\lambda}+\sigma(0)}^{\sigma(\Lambda),\sigma(\bar{\Lambda})} = \Phi_{\lambda,\bar{\lambda}}^{\Lambda,\bar{\Lambda}}, \tag{2.31}$$

for any proper external automorphism, σ.

Let us return now to the chiral fields C_w^Λ. We will prove that the field identification in the parafermionic theory implies that,

$$C_w^\Lambda = C_{w_\sigma w}^{\sigma(\Lambda)}, \tag{2.32}$$

for any proper automorphism σ and for any Λ and w. To prove Eq. (2.32) let us decompose it into parafermions. Also, we can bosonize the fermions ρ^α (for $\alpha > 0$),

$$\rho^\alpha = e^{i\phi_\alpha}, \quad \rho^\alpha \rho^{-\alpha} = i\partial_z \phi, \tag{2.33}$$

where ϕ_α is a canonical free boson.

Let us first prove,

Lemma (2.1). *For any root $\beta > 0$ and any proper external automorphism σ: $\beta\sigma(0) = k$ if and only if $w_\sigma^{-1}(\beta) < 0$.*

Proof. First, we can compute $\sigma(-w_\sigma^{-1}(\sigma(0))) = \sigma(0) - w_\sigma(w_\sigma^{-1}(\sigma(0))) = 0$. Thus,

$$w_\sigma^{-1}(\sigma(0)) = -\sigma^{-1}(0). \tag{2.34}$$

Assume now that $\beta\sigma(0) = k$. Then, from Eq. (2.34), $k = -\beta w_\sigma(\sigma^{-1}(0)) = -w_\sigma^{-1}(\beta)\sigma^{-1}(0)$. Since $\sigma^{-1}(0)$ is a fundamental weight, any root that has a negative scalar product with it is a negative root and, in particular, $w_\sigma^{-1}(\beta) < 0$.

To prove the other direction, it is enough to show that if $\beta\sigma(0)=0$ then $w_\sigma^{-1}(\beta)>0$. Assume then that $\beta\sigma(0)=0$. Note that $\beta>0$ implies that $\theta-\beta>0$, where θ is the highest root. Since $\theta\sigma(0)=k$ we have $(\theta-\beta)\sigma(0)=k$. From the first part above as applied to $\theta-\beta$, we find $w_\sigma^{-1}(\theta-\beta)<0$, or $w_\sigma^{-1}(\beta)>w_\sigma^{-1}(\theta)$ (with no equality). From the definition $w_\sigma^{-1}(\theta)=-\alpha_i$, where α_i is some simple root. Thus $w_\sigma^{-1}(\beta)>0$, completing the proof.

We need also a second lemma,

Lemma (2.2). *Denote by* $\vec{\chi}=\sum_{\alpha>0}\vec{\alpha}\phi_\alpha$. *For any proper automorphism* σ *and any Weyl transformation* w,

$$\rho^{-w_\sigma w}=\rho^{-w}\exp\{-iw(\vec{\chi})\sigma^{-1}(0)/k\}. \tag{2.35}$$

Proof. Let us compare both sides of Eq. (2.34) for each positive root α. There are three possibilities.

1) $w(\alpha)\sigma^{-1}(0)=k$. This implies $w(\alpha)>0$ (since $\sigma^{-1}(0)$ is a fundamental weight). From Lemma (1), $w_\sigma w(\alpha)<0$. Thus ρ^{-w} does not contain $\rho^{-\alpha}$ but the exponent on the right-hand side (of Eq. (2.35)) does contain $\rho^{-\alpha}$. From $w_\sigma w(\alpha)<0$ the left-hand side also contains $\rho^{-\alpha}$.

2) $w(\alpha)\sigma^{-1}(0)=-k$. Similar to case (1), we now have $w(\alpha)<0$ and $w_\sigma w(\alpha)>0$. Thus $\rho^{-\alpha}$ does not appear in the left-hand side. Since $\rho^{-\alpha}$ appears in ρ^{-w} and the exponent now contains ρ^{α}, on the right-hand side, we now have $\rho^{-\alpha}\rho^{\alpha}=1$ and both sides do not contain $\rho^{-\alpha}$.

3) $w(\alpha)\sigma^{-1}(0)=0$. We now have two further possibilities using Lemma (1): a) $w(\alpha)>0$ and $w_\sigma w(\alpha)>0$. b) $w(\alpha)<0$ and $w_\sigma w(\alpha)<0$. In case (a) both sides do not contain $\rho^{-\alpha}$ and in case (b) both sides do contain it.

From the three cases above we see that the two sets of roots appearing in the left-hand side and right-hand side of Eq. (2.35) are identical, proving this equation.

Let us return now to Eq. (2.32). From the definition of C,

$$C^{\sigma(\Lambda)}_{w_\sigma w}=\rho^{-w_\sigma w}G^{\sigma(\Lambda)}_{w^{-1}w_\sigma^{-1}\sigma(\Lambda)}. \tag{2.36}$$

From the proof of Lemma (1), $w_\sigma^{-1}\sigma(0)=-\sigma^{-1}(0)$, implying that

$$G^{\sigma(\Lambda)}_{w^{-1}w_\sigma^{-1}\sigma(\Lambda)}=G^{\sigma(\Lambda)}_{w^{-1}(\Lambda-\sigma^{-1}(0))}, \tag{2.37}$$

where we used Eq. (2.26). We can now decompose G into parafermions according to Eq. (2.29),

$$G^{\sigma(\Lambda)}_{w^{-1}(\Lambda-\sigma^{-1}(0))}=\exp\{i\vec{\phi}w^{-1}(\Lambda-\sigma^{-1}(0))/\sqrt{k}\}\,\Phi^{\Lambda}_{\sigma^{-1}(0)-w^{-1}\sigma^{-1}(0)+w^{-1}(\Lambda)}, \tag{2.38}$$

where we used Eq. (2.31). Since $\sigma^{-1}(0)=k\Lambda_i$, where Λ_i is a fundamental weight, $\sigma^{-1}(0)-w^{-1}\sigma^{-1}(0)\in kM_L$, where M_L is the long root lattice. Thus using Eq. (2.30),

$$G^{\sigma(\Lambda)}_{w^{-1}w_\sigma^{-1}\sigma(\Lambda)}=\exp\{i\vec{\phi}w^{-1}(\Lambda-\sigma^{-1}(0))/\sqrt{k}\}\,\Phi^{\Lambda}_{w^{-1}(\Lambda)}. \tag{2.39}$$

Combining now Eq. (2.39) with Lemma (2) it follows that,

$$\begin{aligned}C^{\sigma(\Lambda)}_{w_\sigma w}&=[\rho^{-w}\Phi^{\Lambda}_{w^{-1}(\Lambda)}\exp\{i\vec{\phi}w^{-1}(\Lambda)/\sqrt{k}\}]\\&\quad\cdot\left[\exp\left\{-i\left(\vec{\phi}\sqrt{k}+\sum_{\alpha>0}\vec{\alpha}\phi_\alpha\right)w^{-1}\sigma^{-1}(0)/k\right\}\right]\end{aligned} \tag{2.40}$$

To complete the proof of Eq. (2.32) we need only to show that the exponent in Eq. (2.40) is equal to 1 modulo H.

To see this let us write the currents of H $(=U(1)^r)$ in terms of those of $G \times SO(n)$. Using the parafermions Eq. (2.27) and Eq. (2.33), these may be written as

$$h_\alpha = i\sqrt{k}\vec{\alpha}\partial_z\vec{\phi} + \sum_{\beta>0} \vec{\alpha}\vec{\beta}\partial_z\phi_\beta = i\vec{\alpha}\partial_z\left(i\sqrt{k}\vec{\phi} + \sum_{\beta>0} \vec{\beta}\phi_\beta\right). \tag{2.41}$$

Thus the free bosons $i\sqrt{k}\vec{\phi} + \sum_{\beta>0} \beta\phi_\beta$ are precisely the ones that vanish modulo H. But this is what appears in the exponent in Eq. (2.40) and thus we proved Eq. (2.32).

3. The Case of Nonabelian H

Let us turn now to the case when H is not abelian. We shall assume that $\operatorname{rank}(H) = \operatorname{rank}(G) = r$, and that H is a diagram subgroup of G, i.e., H is the subgroup spanned by a set of simple roots $\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_m}$, where $m \leqq r$, along with the elements of the Cartan subalgebra. As in Sect. (2), we shall make the ansatz that a general chiral field is of the form,

$$C = \rho^{-a_1}\rho^{-a_2}\cdots\rho^{-a_n}G, \tag{3.1}$$

where G is a field in the G current algebra and the a_i are now positive roots of G which are not roots of H, $\alpha_i \in \Delta_G^+ - \Delta_H^+$ (which we denote by $\Delta_{G/H}$).

The same analysis as before for the term $\rho^{-\alpha}J_\alpha$ shows that,

$$\begin{aligned} &J_\alpha(\zeta)G(z) = O(1), && \text{if} \quad 0 < \alpha \neq a_i \quad \text{or} \quad \alpha = -\alpha_i, \\ &J_\alpha(\zeta)G(z) = O\left(\frac{1}{\zeta - z}\right) && \text{otherwise}. \end{aligned} \tag{3.2}$$

It follows that for any current of the G algebra $J^a(\zeta)G(z)$ is at most singular as $O\left(\frac{1}{\zeta - z}\right)$ and thus G is a primary field. Let us denote it then by G_λ^Λ, where Λ is the highest weight and λ is some weight in the representation.

Now, the currents of H act trivially on the fields of $G \times SO(n)$ (modulo H). Let α be any root of H. The associated current is

$$J_\alpha = J_\alpha^{(G)} + i\sum_{\beta,\gamma} f_{\alpha\beta\gamma}\rho^\beta\rho^\gamma. \tag{3.3}$$

We can form an equivalent chiral field to C by writing the operator product of J_α and C and picking up, say, the most singular term, which is of order $O\left(\frac{1}{\zeta - z}\right)$. We get the field C' which is a sum of fields in which each of the a_i is going to $\alpha_i + \alpha$ (provided $\alpha_i + \alpha \neq a_j$) or λ goes to $\lambda + \alpha$. Continuing to form operator products in this fashion, we arrive at a unique equivalent field of the form,

$$\tilde{C} = \rho^{-b_1}\rho^{-b_2}\cdots\rho^{-b_n}\Phi_\gamma^\Lambda, \tag{3.4}$$

where $b_i - a_i \in H$ and $\gamma - \lambda \in H$. In addition, for any positive root of H, $\alpha \in \Delta_H^+$, $b_i + \alpha \in \Delta_G$ implies that $\rho^{-(b_i+\alpha)}$ appears in $\tilde{C}$, and that $\gamma + \alpha$ is not a vector in the representation $L(\Lambda)$. This field is the highest weight vector with respect to the finite algebra H. This also shows that for any chiral field C we can form an equivalent one by adding roots of H to any of the a_i and λ.

Let us thus assume that C is a highest weight of H. Consider the action of $G_\pm$ on C. From the second term, of the type $f_{\alpha\beta\gamma}\rho^\alpha\rho^\beta\rho^\gamma$, we get, as before, the condition

$$1 \geqq N_\alpha + N_\beta - N_{\alpha+\beta} \geqq 0, \tag{3.5}$$

where now $\alpha, \beta, \alpha + \beta \in \Delta_{G/H}$. Now define $N_\alpha = 0$ if $\alpha \in \Delta_H^+$. Equation (3.5) is still valid, since if $\alpha \in \Delta_{G/H}^+$ and $\beta \in H$ then $\alpha + \beta \in \Delta$ implies that $N_{\alpha+\beta} = 1$ as we assumed that C is a highest weight vector of H. Thus $N_\alpha + N_\beta - N_{\alpha+\beta} = 1$ and Eq. (3.5) holds. In other words, we can extend the map N to the whole of Δ_G while preserving this equation.

Thus, as in the case of an abelian H, it follows that N is obtained from some Weyl transformation $w \in W(G)$. We have $N_\alpha = 0$ iff $w(\alpha) > 0$. In addition, if $\alpha \in \Delta_H^+$ then $w(\alpha) > 0$. We denote $\rho^{-a_1}\rho^{-a_2}\cdots\rho^{-a_n} = \rho^{-w}$.

From the first term, of the type $\rho^{-\alpha}J_\alpha$, we have: 1) if $\alpha \in \Delta_H$ then $\lambda + \alpha \notin L(\Lambda)$. 2) If $\alpha \in \Delta_{G/H}^+$ and $\pm w(\alpha) > 0$ then $\lambda \pm \alpha \notin L(\Lambda)$. Thus for any positive roots of G, β, we have $w(\lambda) + \alpha \notin L(\Lambda)$ implying that $\lambda = w^{-1}(\Lambda)$.

To summarize, we have shown that all the chiral fields are of the form

$$C_w^\Lambda = \rho^{-w} G_{w^{-1}(\Lambda)}^\Lambda, \tag{3.6}$$

where Λ is an integrable highest weight and w is an element of the Weyl group.

Now, not all $w \in W(G)$ will give rise to different chiral fields, due to the identification modulo H. We will now show that if $w_h \in W(H)$ then $C_w^\Lambda = C_{ww_h}^\Lambda$, and these are all the identifications due to the quotient by H.

We first need a lemma,

Lemma (3.1). *Any Weyl transformation $w \in W(H)$ permutes the positive roots in $\Delta_{G/H}$.*

Proof. Recall that we assumed that H is a diagram subgroup. Denote the simple roots of H by $\alpha_1, \alpha_2, \ldots, \alpha_m$. Since the Weyl group is generated by reflections by the simple roots, w_{α_i}, it is enough to prove the lemma for these. If $\beta \in \Delta_{G/H}$ then $\beta = \sum_{i=1}^{r} s_i \alpha_i$, where the s_i are non-negative integers and some s_i for $i > m$ is positive. Now, $w_{\alpha_i}(\beta) = \sum_{i=1}^{m} s_i' \alpha_i + \sum_{i=m+1}^{r} a_i \alpha_i$. Since some s_i is positive it follows that $w(\alpha_i) > 0$.

Using the lemma we find

$$\rho^{-ww_h} = \prod_{\substack{\delta > 0 \\ ww_h(\delta) < 0}} \rho^{-\delta} = \prod_{\substack{\delta > 0 \\ w(\delta) < 0}} \rho^{-w_h^{-1}(\delta)}. \tag{3.7}$$

But $w_h^{-1}(\delta) = \delta + \alpha$, where $\alpha \in \Delta_H$, and as discussed earlier adding roots of H leaves the chiral field invariant modulo H. Thus,

$$\rho^{-ww_h} = \rho^{-w}. \tag{3.8}$$

Similarly, for the second term,

$$G_{w_h^{-1}w^{-1}(\Lambda)}^\Lambda = G_{w^{-1}(\Lambda)+\alpha}^\Lambda, \tag{3.9}$$

where $\alpha\in\Delta_H$. It follows that, modulo H,

$$C_w^\Lambda = C_{ww_h}^\Lambda. \tag{3.10}$$

It remains to show that $C_{w_1}^\Lambda = C_{w_2}^\Lambda$ implies that $w_2 w_1^{-1}\in W(H)$.

Lemma (3.2). *Suppose that* $w\in W(G)$ *obeys* $w(\delta)>0$ *for all* $\delta\in\Delta_{G/H}^+$. *Then* $w\in W(H)$.

Proof. Suppose that $w\in W(G)$ obeys the assumption of the lemma. Denote by $\Delta_H' = \{\delta\in\Delta_H \mid w(\delta)>0\}$. Δ_H' is a positive root system for H and thus $\Delta_H' = w_0^{-1}(\Delta_H^+)$ for some $w_0\in W(H)$. Consider now the Weyl transformation ww_0^{-1}. If $\delta\in\Delta_{G/H}^+$ then $ww_0^{-1}(\delta)>0$ (since w_0^{-1} permutes the positive roots in $\Delta_{G/H}^+$). If $\delta\in\Delta_H^+$ then from the definition $ww_0^{-1}(\delta)>0$. Thus for all $\delta\in\Delta_G$ we have $ww_0^{-1}(\delta)>0$. But the Weyl group acts transitively on the basis, implying that $ww^{-1}=1$, which proves the lemma.

Thus, each $w\in W(G)$ may be decomposed uniquely as $w=ur$, where $r\in W(H)$ and u preserves the positivity of the roots of H. From Eq. (3.10) it follows that we can replace w by u. Then different u's would give different fields, just as in the abelian case, since these are in one to one correspondence with the basis of G which contain Δ_H^+.

Our discussion so far was independent of the spectrum of the theory, or the particular modular invariant used to describe it. The same discussion holds also for the right movers and for the left movers separately (except for the field identifications which must be carried simultaneousy for the left and right movers).

For the sake of completeness let us digress here on the spectrum of quotient conformal field theories. As discussed in ref. [22], the field identifications and the restriction on the weights play a crucial role in the spectrum. Specifically, let G and H be arbitrary algebra and its subalgebra. Then the fields in the theory, as in Eq. (2.1), are given by Φ_λ^Λ. Similarly, one defines the string functions, which are the characters over each such block of fields, $c_\lambda^\Lambda(\tau)$ (see the discussion of modular invariance in Sect. (1)). Under modular transformations $c_\lambda^\Lambda(\tau)$ transforms as the product of G characters times a product of the complex conjugate of H characters. Let $N_{\Lambda,\bar\Lambda}$ and $M_{\lambda,\bar\lambda}$ be arbitrary modular invariants for the G and H conformal field theories, respectively [27]. Then the partition function of the G/H conformal field theory is given by

$$Z=\sum_{(\Lambda,\lambda)\in A} N_{\Lambda,\bar\Lambda} M_{\lambda,\bar\lambda} c_\lambda^\Lambda c_{\bar\lambda}^{\bar\Lambda *}, \tag{3.11}$$

where A is the set of cosets (Λ,λ), where $\Lambda-\lambda$ is in the root lattice of G (or the appropriate generalization of this condition in case of non-simple G according to the decomposition of representations of the finite algebras; this is the condition $C(\Lambda,\lambda)$ of ref. [22]) modulo the field identifications of the theory, the action of σ. The sum above is understood in terms of picking one arbitrary representative from each coset. The choice of representative is immaterial due to the field identifications – all the representatives describe the same field. Due to the fact that $N_{\sigma(\Lambda),\sigma(\bar\Lambda)} = N_{\Lambda,\bar\Lambda}$ for any modular invariant N of any algebra well and for any Λ and $\bar\Lambda$ [9], it follows that the partition function Eq. (3.11) is indeed well defined on the cosets.

The partition function Z defined in Eq. (3.11) is modular invariant. To see this note that it can be written as

$$Z=\frac{1}{a^2}\sum_{\Lambda,\lambda,\sigma,\rho,\bar{\Lambda},\bar{\lambda}} e^{2\pi i\Lambda_\sigma(\Lambda-\lambda)} c_{\rho(\lambda)}^{\rho(\Lambda)} c_{\bar{\lambda}}^{\bar{\Lambda}*} N_{\Lambda,\bar{\Lambda}} M_{\lambda,\bar{\lambda}}, \tag{3.12}$$

where a is the order of the field identification group. This is since the exponent gives a delta function which is the condition $C(\Lambda,\lambda)$ and since $c_{\sigma(\lambda)}^{\sigma(\Lambda)}(\tau)=c_\lambda^\Lambda(\tau)$. Modular invariance can be now shown using the explicit expressions for the modular transformations (see ref. [27, 22] for details). In particular, $S:\tau\to-\frac{1}{\tau}$ simply exchanges σ and ρ, implying the invariance of the sum over both. The spectrum, as read from Eq. (3.12), is closed under the operator product algebra since the fusion rules [27] obey $\sigma(\lambda)\times\mu=\lambda\times\sigma(\mu)=\sigma(\lambda\times\mu)$, where "$\times$" stands for the fusion rules, λ and μ are any integrable weights and σ is any external automorphism [19]. Thus the fusion rules of the quotient theory are the same as the original one, in terms of representatives modulo σ. Hence this spectrum represents a closed operator algebra if the original invariants N and M were so. It follows that the spectrum of the most general G/H theory is given by the set of cosets described above. Note, in particular, that this holds also when σ have fixed points when acting on the pairs $(\Lambda,\lambda)^2$.

An important choice for the modular invariants N and M is the left–right symmetric one, $N_{\Lambda,\bar{\Lambda}}=\delta_{\Lambda,\bar{\Lambda}}$ and $M_{\lambda,\bar{\lambda}}=\delta_{\lambda,\bar{\lambda}}$. From hereon we shall assume this invariant, unless otherwise specified. This choice of modular invariant will be referred to as the principal theory.

The entire discussion can now be summarized in the following theorem.

Theorem (3.1). *The chiral fields in a quotient, G/H, $N=2$ conformal field theory with the left–right symmetric modular invariant are given by $C_w^\Lambda=\rho^{-w}\bar{\rho}^{-w}G_{\lambda,\lambda}^{\Lambda,\Lambda}$ mod H, where $w\in W(G)$ and $\lambda=w^{-1}\Lambda$. In addition, $C_w^\Lambda=C_{ww_h}^\Lambda=C_{w_\sigma w}^{\sigma(\Lambda)}$. In other words, the chiral algebra is in a 1–1 correspondence with pairs (Λ,w), where Λ is an integrable highest weight and $w\in W(G)/W(H)$ (right quotient), identified by the action of (σ,w_σ).*

Actually, our proof of this theorem assumes that the chiral fields are given by the general form Eq. (2.13). This ansatz needs to be justified. To do so [24] we can utilize the connection between the problem of finding the chiral field and the harmonic forms of a Lie algebra cohomology. In the Ramond sector, the superconformal generators, $G_0^\pm$ become a Lie algebra derivative with respect to some Borel subalgebra of the affine $\hat{G}$ and its dual with respect to the killing form. The ansatz Eq. (2.13) is equivalent to assuming that the fields are in the ground state of the fermionic theory. This cohomology problem was studied by mathematicians in connection with the cohomology of loop groups [28]. When translated into this context, it implies that the chiral fields are given by replacing

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2 In case there are no such fixed points we can replace the partition function, Eq. (3.12), with the sum over all (Λ,λ) and divide by a. (This is the partition function given in ref. [22].) In case there are fixed points, it is incorrect to do so; in particular, the result would not be modular invariant

each of the objects by its corresponding affine counterpart. In particular the chiral fields are of the form C_w^Λ, where w is in the affine Weyl group and the appropriate definition for ρ^w is given. Decomposing w into $T \times W$, where W is the Weyl group of G and T is the translation subgroup, it can be seen that the translation is irrelevant modulo H, and thus indeed the ansatz Eq. (2.13) is justified.

It is straightforward to generalize this theorem to other modular invariants, as well to the anti-chiral algebra. Note that the complex conjugate of a chiral field is an anti-chiral one. For a general modular invariant, the chiral fields are of the type $C_{w,\bar{w}}^{\Lambda,\bar{\Lambda}} = C_w^\Lambda \bar{C}_{\bar{w}}^{\bar{\Lambda}}$, where Λ and $\bar{\Lambda}$ are the left and right highest weights and w and $\bar{w}$ are two elements of $W(G)/W(H)$, appearing in the spectrum a number of times according to the modular invariant used. In addition, $C_{w,\bar{w}}^{\Lambda,\bar{\Lambda}} = C_{w_\sigma w, w_\sigma \bar{w}}^{\sigma(\Lambda),\sigma(\bar{\Lambda})}$ for every proper external automorphism σ. A similar result holds for the anti-chiral fields where for the right movers we take the complex conjugate of the corresponding chiral field.

It can be seen that in the case that the left-right symmetric modular invariant is used for H, $H_{\lambda,\bar{\lambda}} = \delta_{\lambda,\bar{\lambda}}$, the field $C_{\max}$, discussed in Sect. (1), is in the spectrum of the theory. This field is given (up to field identifications) by C_w^0, where w is the longest element in W_1,[3] i.e., the element that reflects all the roots in $\Delta_{G/H}$, which is of length $\dim \Delta_{G/H}$ and $\rho - w^{-1}\rho = \rho - \rho_H$. It is not hard to compute, using Eq. (3.20), that indeed for this field $Q = c/3$. It follows that, irrespective of the modular invariant used for G, this field will be in the spectrum of the theory and the transposition operation of Sect. (1) can be defined. Thus also the Poincaré polynomials are always dual obeying Eq. (1.16).

Assume then that $M_{\lambda,\bar{\lambda}} = \delta_{\lambda,\bar{\lambda}}$. If the invariant used for $G, N_{\Lambda,\bar{\Lambda}}$, is not the left-right symmetric one, then the chiral fields are still given by C_w^Λ each appearing $N_{\Lambda,\Lambda}$ times. This follows from the fact that the equation $w(\Lambda+\rho)-\rho = w_1(\Lambda_1+\rho)-\rho$ has as the only solution $w = w_1$ and $\Lambda = \Lambda_1$ (this is fairly standard, see e.g., [29]). If the modular invariant used for G is also the left-right symmetric one, $M_{\Lambda,\bar{\Lambda}} = \delta_{\Lambda,\bar{\Lambda}}$, it follows that there are no anti-chiral fields. The relevant equation in this case is, $w(\Lambda+\rho)-\rho = -w_1(\Lambda+\rho)+\rho \bmod \Delta_H$ and this has no solutions.

When w_σ acts without fixed points on the pairs (Λ, w), the number of chiral fields for the principal theory is

$$\frac{|W(G)|\, N_k^G}{|Z| \times |W(H)|}, \tag{3.13}$$

where $|W(G)|$ is the number of elements in the Weyl group, $|Z|$ is the number of elements in the center and N_k^G is the number of integrable highest weights at level k. When Z acts with fixed points on the pair (Λ, w), the number of chiral fields is strictly greater than the number above.[4]

Let us now turn to the decomposition into H of the fields C_w^Λ of the $G \times SO(n)$ current algebra. As was discussed earlier these fields are primary fields of the diagonal H subalgebra. In addition, the foregoing discussion shows that the field

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[3] This observation is due to R. Cohen

[4] Equation (3.13) was independently derived as a lower bound on the number of chiral fields in ref. [30], where also the Poincaré polynomials of the level one h.s.s. simply laced theories, Eq. (3.26), are described

C_w^Λ is a highest weight vector under H if and only if $w(h) > 0$ for all $h \in \Delta_H^+$, and that there is exactly one such h in each coset of $W(G)/W(H)$. We shall denote the set of such Weyl elements by $W\left(\frac{G}{H}\right)$. Let us assume that $w \in W\left(\frac{G}{H}\right)$. For such a w the field C_w^Λ decomposes according to Eq. (2.1) as

$$C_w^\Lambda = K_w^\Lambda H^\lambda, \tag{3.14}$$

where λ is the weight of C_w^Λ under the diagonal subalgebra (since, C is the highest vector), and H^λ is the field in the H current algebra with highest weight λ and weight λ. The weight λ is easily computed from that of the fermions and the G primary field,

$$\lambda = w^{-1}(\Lambda) + \sum_{\substack{\alpha \in \Delta_{G/H} \\ w(\alpha) < 0}} \alpha. \tag{3.15}$$

Since $w(h) > 0$ for $h \in \Delta_H^+$, the sum above can be taken over all positive roots of G. It is easy to see that,

$$\sum_{\substack{\alpha > 0 \\ w(\alpha) < 0}} \alpha = \rho - w^{-1}(\rho), \tag{3.16}$$

since

$$2w^{-1}(\rho) = \sum_{\alpha > 0} w^{-1}(\alpha) = \sum_{\substack{\beta > 0 \\ w(\beta) > 0}} \beta - \sum_{\substack{\beta > 0 \\ w(\beta) < 0}} \beta, \tag{3.17}$$

where we substituted $\beta = w^{-1}(\alpha)$. Thus,

$$\lambda = w^{-1}(\Lambda + \rho) - \rho. \tag{3.18}$$

We can now compute the dimension and charge of the field K_w^Λ from Eq. (3.15). We find,

$$\Delta = \frac{n(w)}{2} + \frac{\lambda(\rho - \hat{\rho})}{k + g}, \quad Q = 2\Delta, \tag{3.19}$$

and we see that the fields K are indeed chiral. $n(w)$ is the number of positive roots $\alpha > 0$ for which $w(\alpha) < 0$. The number $n(w)$ is also equal to the length of w, denoted by $l(w)$, which is the minimal number of reflections by the simple roots from which w is composed (e.g., ref. [25]).

As discussed in Sect. (1), a convenient way to summarize the various grades in a graded algebra is the Poincaré polynomial, Eq. (1.15).

Consider the h.s.s. family of theories, where $\alpha, \beta \in \Delta_{G/H}$ implies that $\alpha + \beta \notin \Delta_{G/H}^+$. These pairs of G and H may be characterized by the fact that the central charge $c = 0$ for $k = 0$. Assume also that G is simply laced. Now, the fact that $c = 0$ for $k = 0$ implies that for this value of k all the $U(1)$ charges computed from Eq. (3.19) must vanish. This implies that,

$$n(w) = -\frac{2(w^{-1}(\rho) - \rho)(\rho - \hat{\rho})}{g}. \tag{3.20}$$

Thus for the h.s.s. family Eq. (3.19) can be written as,

$$Q = \frac{l(w)g}{k + g} + \frac{w^{-1}(\Lambda)(\rho - \hat{\rho})}{k + g}. \tag{3.21}$$

Consider, now the case of $k=1$. In this case, since all the level one weights of G are minimal (as G is simply laced) the field identification may be implemented by assuming that $\Lambda=0$ (we are also assuming the left–right symmetric modular invariant). Thus, for $k=1$ the $U(1)$ charges are,

$$q=\frac{l(w)}{g+1}, \quad \text{for} \quad w\in W(G), \quad w(h)>0. \tag{3.22}$$

Denoting by $u=t^{1/(g+1)}$ the Poincaré polynomial becomes

$$P_{G/H}(u)=\sum_{w\in W(G/H)} u^{l(w)}. \tag{3.23}$$

Define now the following polynomial,

$$P_G=\sum_{w\in W(G)} u^{l(w)}, \tag{3.24}$$

for any semi-simple Lie algebra. Note now that any $w\in W(G)$ may be decomposed uniquely as $w'w_h$, where $w\in W(H)$ and $w'(h)>0$. (The arguments are similar to the ones described for Lemma (3.2).) This decomposition is unique and goes both ways. Thus there is one-to-one map from such pairs of Weyl transformation to the Weyl group. In addition, it is easy to see that $l(w)=l(w')+l(w_h)$ (use the fact that $l(w)=n(w)$ and the fact that w_h permutes the positive roots not in H). Thus we have proved,

$$P_{G/H}(u)=\frac{P_G(u)}{P_H(u)}. \tag{3.25}$$

Finally, there is a simple formula for $P_G(u)$ which is closely related to the cohomology of the manifold G/H [31],

$$P_G(u)=\prod_i \frac{1-u^{m_i}}{1-u}, \tag{3.26}$$

where the m_i are the exponents of the Lie algebra. Using Eq. (3.26) we can compute the Poincaré polynomial of the simply laced $k=1$ h.s.s. cases.

4. Dihedrality and Statistical Models

Consider the theory

$$T(k;n_1,n_2,\ldots,n_l)=\frac{U(n_1+n_2+\cdots+n_l)_k}{U(n_1)\times U(n_2)\cdots\times U(n_l)}, \tag{4.1}$$

at some level k. This is the most general theory which can be derived from $SU(N)$, and is obtained by the deletion of l roots of $SU(n_1+n_2+\cdots+n_l)$, i.e., the roots $\alpha_{n_1},\alpha_{n_1+n_2},\ldots,\alpha_{n_1+n_2+\cdots+n_{l-1}}$. This class of theories have many nice properties that make its investigation easier, and thus provides a good testing ground. In addition, most of the lower central charge theories come from $SU(N)$, and thus play a bigger role in compactification.

One property is that if $\gcd(n_1,n_2,\ldots,n_l,k)=1$ then the theory Eq. (4.1) has no fixed points (see the discussion in Sect. (6)). If $\gcd(n_1,n_2,\ldots,n_l)=1$ then there are no fixed points for any k, and these are the only such theories, simplifying considerably their investigation.

Another, rather amazing property, is that k can be assumed to be equal to zero, $k=0$, without any loss of generality. This is due to the following equivalence of theories,

$$\frac{U(n_2+n_3+\cdots+n_l)_{n_1}}{U(n_2)\times U(n_3)\times\cdots\times U(n_l)}=\frac{U(n_1+n_2+\cdots n_l)_0}{U(n_1)\times U(n_2)\times\cdots\times U(n_l)}, \tag{4.2}$$

where the first theory is at level n_1 and the second is at level 0. We shall prove this identity (for any $n_i\geqq 0$) below. As we shall show this identity implies the dihedrality of the theories of this type. Namely, the freedom of permuting $n_1, n_2, \ldots, n_l$,

$$T(n_1;n_2,n_3,\ldots,n_l)=T(n_{p(1)};n_{p(2)},n_{p(3)},\ldots,n_{p(l)}), \tag{4.3}$$

where p is any dihedral permutation, generated by either a rotation, $p(i)=i+s \bmod l$ for some s, or a reflection $p(i)=p(l+2-i)$ for $2\leqq i\leqq l$.

First it is not hard to calculate the central charges of both theories in Eq. (4.1). The results are the same,

$$c=\frac{3S_3}{S_1},\quad \text{where}\quad S_r=\sum_{i_1<i_2\cdots<i_r} n_{i_1}n_{i_2}\cdots n_{i_r}. \tag{4.4}$$

We shall establish the equivalence of theories Eq. (4.2) by describing a 1–1 map from the chiral fields of the one theory to the other, which preserves the $U(1)$ charges. Let $R=SU(n_1+n_2+\cdots+n_l)$, $G=SU(n_2+n_3+\cdots+n_l)$, $H_1=SU(n_1)$ and $H=SU(n_2)\times SU(n_3)\cdots SU(n_l)$. As Lie algebras, $R\supset H_1\times G$, $G\supset H$. According to Theorem (3.1) the fields of the theory $\frac{R}{H_1\times H}$ at level $k=0$ are given by C_w^0, where $w\in W\left(\frac{R}{H_1\times H}\right)$ is the set of Weyl transformations of R, which leaves as positive the positive roots of $H_1\times H$. The $U(1)$ charge, is given according to Eq. (3.19), by

$$(k+g)Q_w=(k+g)l(w)-2(\rho_R-\rho_{H_1}-\rho_H)\sum_{\substack{\alpha>0\\ w(\alpha)<0}}\alpha, \tag{4.5}$$

where we used Eq. (3.16), $k=n_1$ and $g=n_2+n_3+\cdots n_l$.

The first step is to decompose w and we follow closely the proof of Lemma (3.2), in which such a decomposition was established. Consider the set,

$$\Theta_w=\{\alpha\in G\,|\,w(\alpha)>0\}, \tag{4.6}$$

where we use $\alpha\in G$ to denote any of the roots of G. Θ_w is a positive roots system for G and thus, $\Theta_w=r^{-1}(\Delta_G)$, for some $r\in W(G)$, where Δ_G denotes the positive roots of G. Since w preserved the positivity of H, it follows also that $r\in W\left(\frac{G}{H}\right)$. Define the Weyl element $u=w\rho^{-1}\in W(R)$. Acting upon a positive root of G,β, we find $u(\beta)=wr^{-1}(\beta)>0$, since $r^{-1}(\beta)\in\Theta_w$. Thus, $u\in W\left(\frac{R}{H_1\times G}\right)$ and we found a unique decomposition, $w = ur$,

$$W\left(\frac{R}{H_1 \times H}\right) = W\left(\frac{G}{H_1 \times G}\right) \times W\left(\frac{G}{H}\right). \tag{4.7}$$

Obviously this decomposition is 1–1, since r is defined uniquely from w (using the property that the Weyl group permutes the Weyl chambers, acting transitively) and onto.

Consider the set of positive roots that w reflects, $\Phi_w = \{\alpha \in R | \alpha > 0 \ w(\alpha) < 0\}$. These can be separated to two disjoint subsets. The first subset consists of the positive roots of G which are negated by r, denoted by Φ_r. Other roots of G are not reflected by w, since r maintains the positivity of the roots of G. The second subset consists of positive roots of $R - G$. Since r permutes the positive roots of $R - G$ among each other, due to Lemma (3.1), it follows that these roots are of the form $r^{-1}(\Phi_u)$, where Φ_u is the set of positive roots which are negated by u. It follows that

$$l(w) = l(u) + l(r), \tag{4.8}$$

$$\Phi_w = \Phi_u \cup \Phi_r, \quad \Phi_u \cap \Phi_r = \emptyset, \tag{4.9}$$

where Eq. (4.8) follows from Eq. (4.9) since the length is equal to the number of positive roots which are negated.

Let us compute the charge Q, Eq. (4.5), in this decomposition. Equations (4.8–4.9) imply that Q_w splits into two terms, $Q_w = Q_u + Q_r$, where

$$(k+g)Q_r = (k+g)l(r) - 2(\rho_R - \rho_{H_1} - \rho_H) \sum_{\alpha \in \Phi_r} \alpha, \tag{4.10}$$

$$(k+g)Q_u = (k+g)l(u) - 2r(\rho_R - \rho_{H_1} - \rho_H) \sum_{\alpha \in \Phi_u} \alpha. \tag{4.11}$$

Now if $\alpha \in \Phi_r$ then $(\rho_R - \rho_G)\alpha = 0$, since α is a sum of simple roots of G and each of these has a product 1 with both ρ_G and ρ_R. Similarly, $\alpha\rho_{H_1} = 0$. Thus we can recast Eq. (4.10) as

$$(k+g)Q_r = (k+g)l(r) - 2(\rho_G - \rho_H)(\rho_G - r^{-1}(\rho_G)). \tag{4.12}$$

Consider now the second term Q_u. First, note that

$$2(\rho_R - \rho_{H_1} - \rho_G) = (k+g)\Lambda_R^k, \tag{4.13}$$

where Λ_R^i denotes the i^{th} fundamental weight of R. For a proof see Sect. (6), Eq. (6.16). Now each of the roots $\alpha \in \Phi_w$ is of the form $\alpha_k + \cdots$, where the "$\cdots$" refers to other simple roots. Any of the roots of R contain α_k either once or none at all (since $g_i = 1$ for $SU(N)$). If it does not contain α_k, then it is a root of $H_1 \times G$, contradicting the fact that u preserves the positivity of these roots. Since the number of roots in Φ_u is $l(u)$, and r permutes the positive roots of $R - G$, we find,

$$(k+g)l(u) = 2r(\rho_R - \rho_{H_1} - \rho_G) \sum_{\alpha \in \Phi_u} \alpha. \tag{4.14}$$

Combining Eq. (4.11) with Eq. (4.14), leads to

$$(k+g)Q_u = -2r(\rho_G - \rho_H) \sum_{\alpha \in \Phi_u} \alpha. \tag{4.15}$$

Let $\lambda = \sum_{\alpha \in \Phi_u} \alpha$, which is a weight in R. λ may be decomposed as $\lambda = \lambda_1 - \Lambda_u$, where Λ_u is some weight of G and λ_1 is some weight such that $\lambda_1 \alpha_i = 0$ for $i > k$, (where α_i stands for the i^{th} simple root). Explicitly,

$$\Lambda_u = - \sum_{i=k+1}^{g+k} (\lambda \alpha_i) \Lambda_G^{i-k}, \tag{4.16}$$

where Λ_G^i is the i^{th} fundamental weight of G. Since $r(\rho_G - \rho_H)$ contains only roots of G, it follows that

$$(k+g) Q_u = 2(\rho_G - \rho_H) r^{-1}(\Lambda_u). \tag{4.17}$$

Combining Eqs. (4.12) and (4.17) we arrive at the following form for the $U(1)$ charge,

$$(k+g)Q = (k+g) l(r) + 2(\rho_G - \rho_H)[r^{-1}(\Lambda_u + \rho_G) - \rho_G]. \tag{4.18}$$

In this equation, Λ_u is a weight of $G = SU(g)$ and $r \in W\left(\frac{G}{H}\right)$.

Let us compare this expression to the $U(1)$ charge of the G/H theory at level k. Again, from Eq. (3.19),

$$(k+g)Q = (k+g) l(r) + 2(\rho_G - \rho_H)[w^{-1}(\Lambda + \rho_G) - \rho_G], \tag{4.19}$$

where as before $r \in W\left(\frac{G}{H}\right)$ and Λ is an integrable highest weight of $G = SU(g)$. Thus, the two expressions Eq. (4.18) and Eq. (4.19) are identical, with the identification $w = r$ and $\Lambda = \Lambda_u$. This is the map of chiral fields between the two theories, which preserves the $U(1)$ charges.

It remains to show that Λ_u so defined is an integrable highest weight at level k. This will be done by explicitly calculating it.

The positive roots of R are given by $\varepsilon_i - \varepsilon_j$, where $1 \leqq i < j \leqq k+g$ and ε_i stands for an orthonormal set of unit vectors, $\varepsilon_i \varepsilon_j = \delta_{ij}$. The positive roots of H_1 or G are of the same form where, respectively, $1 \leqq i < j \leqq k$ and $k+1 \leqq i < j \leqq k+g$. The Weyl group of $SU(N)$ consists of all the permutations $\varepsilon_i \to \varepsilon_{p(i)}$, where p is any permutation. Thus $W\left(\frac{R}{H_1 \times G}\right)$ consists off all the permutations $p(i) \in S_{k+g}$ for $1 \leqq i \leqq k+g$, which obey (since it must preserve the positivity of the roots of H_1 and G),

$$\begin{gathered} 1 \leqq p(1) < p(2) < \cdots < p(k) \leqq k+g, \\ 1 \leqq p(k+1) < p(k+2) < \cdots < p(k+g) \leqq k+g. \end{gathered} \tag{4.20}$$

Let $a_i = p(i)$. Then $1 \leqq a_1 < a_2 < \cdots < a_k \leqq k+g$. Each such ascending series defines uniquely a permutation obeying Eq. (4.20) and thus also an element of $W\left(\frac{R}{H_1 \times G}\right)$. Hence, we shall denote an element of $W\left(\frac{R}{H_1 \times G}\right)$ by $a = [a_1, a_2, \ldots, a_k]$, where the a_i is such a series.

Each of the positive roots reflected by a are of the form $\varepsilon_i - \varepsilon_j$, where $i \leqq k$, $j \geqq k+1$ and $p(j) \leqq a_i$. Since the $p(j)$ also defines an ascending series, the latter condition will hold iff $j \leqq t_i$, where $t_i = a_i + k - i$. It follows that the total number of such reflected roots is

$$l(u) = \sum_{i=1}^{k} a_i - i, \tag{4.21}$$

and that the sum over the reflected roots is,

$$\sum_{\substack{\alpha > 0 \\ u(\alpha) < 0}} \alpha = \sum_{i=1}^{k} \sum_{j=k+1}^{a_i+k-i} \varepsilon_i - \varepsilon_j. \tag{4.22}$$

Now, $\varepsilon_i - \varepsilon_j = \Lambda^{j-k-1} - \Lambda^{j-k}$, up to a weight of R which vanishes when multiplied by the roots of G, and where we have conveniently defined, $\Lambda^0 = \Lambda^g = 0$ (this, if you will, is the affine weight Λ^0). We can now sum up the string of roots

$$\sum_{j=k+1}^{a_i+k-i} \varepsilon_i - \varepsilon_j = \sum_{j=k+1}^{a_i+k-i} \Lambda^{j-k-1} - \Lambda^{j-k} = -\Lambda^{a_i-i}, \tag{4.23}$$

and thus

$$\Lambda_u = \sum_{i=1}^{k} \Lambda^{a_i-i}. \tag{4.24}$$

The projection of λ over the weights of H_1, because of the symmetry of exchanging H_1 and G, is given by the same expression with a_i replaced b_s, where $b_s = p(s+k)$.

Now, the map that we have defined is not 1–1 as it stands. This is due to the fact that we have not taken into account the field identifications. As explained in Sect. (2) the chiral fields are C_w^Λ modulo the field identification, $C_w^\Lambda = C_{w_\sigma w}^{\sigma(\Lambda)}$, where σ is any external automorphism. We need to show that the map so defined is one to one and onto between the **cosets** modulo this identification.

Let us note then the action of w_σ on $w \in W\left(\frac{R}{H_1 \times G}\right)$. The elements of $W\left(\frac{R}{H_1 \times G}\right)$ are described by permutations $p(i)$ for $1 \leqq i \leqq k+g$, as explained above, which obey the ascending chain condition, $p(i) < p(j)$, where $i < j$ and i, j are in the same subgroup, H_i. In this explicit notation, the decomposition that we defined earlier, $w = ur$, can be described as,

$$u = [a_1, a_2, \dots, a_k] = [p(1), p(2), \dots, p(k)],$$

and r correspond to the permutation $q(i)$ of $k < i \leqq k+g$ such that $q(i) < q(j)$ for $i < j$ in the same subgroup, H_i, and $pq(i) < pq(j)$, for all $k < i < j \leqq k+g$.

The action of w_σ^s on w, the element $w_\sigma^i w$, is given by the permutation $p(i) + s \bmod k+g$. What does that do to the decomposition of w? Consider for simplicity the case $s = 1$. There are two possibilities.

Possibility (1). If $p^{-1}(k+g) \leqq k$ then we get precisely the same decomposition as before, $w \to (\Lambda_u, r)$. That r does not change is clear since it depends only on the order of the roots of G and this stays the same. As of u, it does change, but Λ_u does not, since $\Lambda^0 = \Lambda^g$ and the set $a_i - i$ stays the same, except for this substitution.

Possibility (2). If $p^{-1}(k+g) > k$ then clearly $a_i - i < g$ for all $1 \leqq i \leqq k$. Then $a_i \to a_i + 1$ implies that $\Lambda_u \to \sigma(\Lambda_u)$, where σ stands here for the automorphism of

G. Similarly, q is rotated and so $r \to w_\sigma r$. Thus we find precisely the field identification of the theory $\frac{G}{H}$.

Now, our claim is that the map we have defined is 1–1 and onto from $W\left(\frac{R}{H_1 \times G}\right) \bmod \{w_\sigma\}$ to the pairs (Λ_u, r) modulo the action of σ. Consider then a representative from the first set. We can choose a representative such that $a_i - i < g$ for all i by using rotations of type (1). It is clear from Eq. (4.24) that Λ_u is an integrable highest weight at level k. The map $u \to \Lambda_u$ is also 1–1 and onto since if Λ is an integrable highest weight, it can be written uniquely as $\sum_{i=1}^{k} \Lambda^{b_i}$, where $0 \leqq b_1 \leqq b_2, \ldots \leqq b_k < g$. Define the $a_i = b(i) + i$ and $u = [a_i]$. Then $\Lambda = \Lambda_u$ in a unique fashion. Choosing different representatives, for which $a_i - i < g$, would lead to rotations of type (2) giving precisely the field identification of the theory G/H. Thus the map we have constructed is well-defined, 1–1 and onto, between the two sets of cosets and we have proved,

Theorem (4.1). *There is a one to one map from the chiral fields of the two theories $T(n_1; n_2, \ldots, n_l)$ to $T(0; n_1, n_2, \ldots, n_l)$, where the n_i are any non-negative integers. This maps respects the $U(1)$ charges and is given by the Weyl group decomposition described above.*

In particular, the two theories will have the same Poincaré polynomials. Presumably, this map is a graded algebras isomorphism, namely, it respcts the multiplication of the chiral fields. Albeit, in view of the fact that the structure constants of the theories in question are not, in general, known, it is hard to establish this directly.

Since the Weyl group of $SU(N)$ consists of permutations, and since the level, k, can be assumed to be equal to zero without any loss of generality, a purely combinatorial formulation can be given for all the theories in this class. The permutations $p \in W\left(\frac{R}{H_1 \times G}\right)$ obviously correspond to the different orderings of l type of objects, where we have n_1 of type $1, n_2$ of type two, etc. To simplify the notation, let us assume that $l = 3$ and that there are three objects, n_1 red balls (say), n_2 blue ones and n_3 green ones. Let $p \in W\left(\frac{R}{H_1 \times G}\right)$ be some permutation. We then locate the red balls in the $p(i)$ places $1 \leqq i \leqq n_1$, the blue ones in $p(i)$ for $n_1 + 1 \leqq i \leqq n_1 + n_2$ and the green ones at the places $p(i)$ for $n_1 + n_2 + 1 \leqq i \leqq n_1 + n_2 + n_3$. Quite evidently, this is a one to one and onto map between the various arrangements of these balls on a line and $W\left(\frac{R}{H_1 \times G}\right)$.

In view of the field identifications by w_σ, which is a rotation of the balls, they should actually be considered as arranged on a circle, and two configurations which can be rotated to each other are one and the same. Our configuration space is thus the different arrangements of these balls on a circle. We can now define the "energy" of each configuration as a measure of its derivation from the ordered state where the balls are lumped in three groups ($p(i) = i$). The energy is the $U(1)$

charge $Q = (n_1 + n_2 + n_3)Q_w$, which using Eq. (4.5) can be shown to be,

$$Q = -n_2 l(\text{red, green}) + n_3 l(\text{red, blue}) + n_1 l(\text{blue, green}). \tag{4.25}$$

Here the l's are defined as follows. Open the circle at some arbitrary point to a line. Now count the number of green balls that are lower than red balls, by summing up the number of green balls that are to the left of each red ball. This number is denoted by l(red, green). (This is the minimal length of a permutation needed to arrange all the green balls to the right of the red ones.) Similarly define the other l's.

The energy Q, so defined, is, of course, a function of the arrangements on the circle. If we pull, say, a green ball from the exteme right to the extreme left, we add n_1 to l(red, green) and n_2 to l(blue, green). The change in the energy is then, $\Delta Q = -n_2(n_1) + n_3(0) + n_1(n_2) = 0$, showing that Q is invariant. Similarly, Q is easily seen to be invariant under the pulling of red or blue balls. Thus Q is indeed a function on the circular configurations. In fact, this property completely determines the coefficients in Eq. (4.25) and it is thus the unique function of this type.

We can now define the partition function as the sum over all configurations at the temperature $\beta = -\log t$. We have

$$Z(t) = \sum_{\text{config.}} e^{-\beta Q} = \sum_{\text{config.}} t^Q. \tag{4.26}$$

Note that the partition function $Z(t)$ is precisely the Poincaré polynomial of the theory, $P(t)$.

Thus we have translated the problem of understanding these theories to a very simply stated statistical mechanics problem on the circle. One can then ask questions about the zeros of the partition function, grand canonical partition functions (summing over the n_i with some chemical potentials) and thermodynamic limits (the limit of an infinite number of balls). In fact, these are precisely the questions that we will deal with in much of this paper. As explained in Sect. (1) the zero's of $P(t)$ encode the algebraic information about it. The grand canonical partition function is the polynomial generating function introduced in Sect. (6), and the thermodynamic limit corresponds to taking $k \to \infty$, a valuable limit in which the algebra becomes a free polynomial algebra.

This picture can be drawn equally well for $l > 3$ by the introduction of l distinct types of balls. The energy Q then assumes the form

$$Q = \sum_{s=1}^{l} \sum_{t=s+1}^{l} m_{st} l(s,t), \tag{4.27}$$

where $l(s,t)$ is the number of t balls to the left of s-type ones. It is not hard to see that Q will be cyclically invariant if and only if

$$\sum_t m_{st} n_t = 0, \tag{4.28}$$

for any s and we defined $m_{st} = -m_{ts}$. A given m_{st} is a solution of Eq. (4.28) if and only if it is of the form

$$m_{st} = \sum_u K_{stu} n_u, \tag{4.29}$$

where K_{stu} is any fully antisymmetric tensor[5]. For each such tensor we would have a cyclically invariant Q. The coefficients m_{st} that correspond to the $U(1)$ charge Q can be calculated from Eq. (4.5) and we find that they correspond to the most symmetric case of these, $K_{stu} = 1$ if $s < t < u$ and is antisymmetric in the three indices. Explicitly,

$$m_{st} = \sum_{i=1}^{s-1} n_i - \sum_{i=s+1}^{t-1} n_i + \sum_{i=t+1}^{l} n_i, \tag{4.30}$$

for $1 \leqq s < t \leqq l$ and $m_{ts} = -m_{st}$.

The rather formal proof that we have given above for Theorem (4.1) can be understood, quite intuitively, in terms of these "colorful" balls. We leave the details as an exercise.

We finally come to the dihedrality of the theory, Eq. (4.3). This follows simply from the field identification. Under w_σ, w cyclically permutes, $p(i) \to p(i) + s \bmod (k+g)$. Thus, the very same permutations correspond to $T(n_1, n_2, \ldots, n_l)$ and $T(n_2, n_3, \ldots, n_l, n_1)$. It follows that these theories have an identical configuration space and identical polynomials. Similarly, since $T(0; n_1, n_2, n_3, \ldots, n_l) = T(n_1, n_2, \ldots, n_l)$ (Theorem (4.1)), it follows that we can reflect $n_i \to n_{l+2-i}$ for $2 \leqq i \leqq l$. (It is the same algebra, with the same roots deleted.) Thus we have proved the identity stated in the beginning of this section.

Theorem (4.2). *We have the following equivalence of theories (in terms of their Poincaré polynomials),*

$$\frac{U(n_2 + n_3 + \cdots + n_l)_{n_1}}{U(n_2) \times U(n_3) \times \cdots \times U(n_l)} \approx \frac{U(n_{p(2)} + n_{p(3)} + \cdots + n_{p(l)})_{n_{p(1)}}}{U(n_{p(2)}) \times U(n_{p(3)}) \times \cdots \times U(n_{p(l)})}, \tag{4.31}$$

where p is any diehedral permutation, generated by a rotation $p(i) = i + 1$ or a reflection $p(i) = l + 2 - i$ for $2 \leqq i \leqq l$.

The isomorphism map between the chiral algebras of the theories is given by the composition of two of the isomorphisms described in Theorem (1). Namely, multiply w and Λ (represented by the corresponding Weyl element), then rotate the result by w_σ to the appropriate power and, finally, decompose back according to a different subgroup. In general, this two step isomorphism becomes quite complicated, while each of the steps is relatively simple.

The case of $l = 3$ of Theorem (4.2) was conjectured in ref. [21], whereas a case of this theorem for $n_3 = n_4 = \cdots = n_l = 1$ was proved in ref. [32].

Both Theorem (4.1) and Theorem (4.2) hold for all l and all $n_l > 0$. If $\gcd(n_i) \neq 1$ there are fixed points of the action of σ. However, since we established the isomorphism as maps on cosets, this does not matter.

[5] To see this note that if $V_{i_1, i_2, \ldots, i_n}$ is a fully anti-symmetric tensor, i.e., an element of the exterior algebra $\wedge V$, or a form, then we can define a boundary operator, $\partial V = \sum_{i_1} n_{i_1} V_{i_1, i_2, \ldots, i_n}$. It is easy to check that $\partial^2 = 0$ and thus we defined a complex. Equation (4.28) is the statement that m is a closed form, while Eq. (4.29) implies that it is exact. Define, as usual, the cohomology group as closed forms modulo the exact ones. Then this statement is equivalent to the second cohomology group being trivial. By a direct calculation it is not hard to establish that all the cohomology groups are trivial provided that $n_i \neq 0$

Note that although the central charge is completely symmetric in the n_i's (Eq. (4.4)), permutations other than the diehedral ones described above, would, indeed, lead to different theories. This can be seen by the direct calculation of the Poincaré polynomials. The polynomials turn out to be different for theories related by permutations other than the diehedral ones, showing that these are distinct theories. At a given l there are $\frac{l!}{2l}=\frac{1}{2}(l-1)!$ inequivalent theories ($l \geqq 3$) for every generic choice of n_i's. Thus at $l=3$ there is a unique theory, three at $l=4$ and at $l=9$ there are 20, 160 inequivalent theories, all with the same central charge!

5. Dihedral Polynomials

The typical polynomials that one anticipate in a scalar field theory have the form (see Sect. (1)),

$$P(t)=\prod_{r=1}^{n}\frac{(1-t^{d-m_r})}{(1-t^{m_r})}, \tag{5.1}$$

where g is the degree of the potential and the m_r are a set of exponents which are the degrees of the generators. It will prove useful to introduce a short-hand notation for any polynomial of the type Eq. (5.1). Let $q(x)\in Z((x))$, where by $Z((x))$ we denote all the Laurent polynomials with coefficients in Z of which only a finite number is nonzero. For every such Laurent polynomial we can associate a rational function with integral coefficients as follows. Suppose

$$q(x)=\sum_{r=-N}^{N} a_r x^r, \tag{5.2}$$

then define $P_q(t)$ to be the rational function,

$$P_q(t)=\prod_{r=-N}^{N}(1-t^r)^{a_r}. \tag{5.3}$$

We shall call $q(x)$ the degree series of the rational function $P_q(t)$. It is also useful to define the "dimension" of the series $q(x)$ as a map from $Z((x))$ to Q (the field of rational numbers),

$$D(q)=\prod_{r=-N}^{N} r^{a_r}. \tag{5.4}$$

It is clear that the map $q\to P_q(t)$ is onto the field of rational functions over Z whose zeros and poles are all primitive roots of unity, or zero. This map is almost one-to-one, in the following sense. Let q be some degree series. Then $P_q(t)\neq 0$ may be written as

$$P_q(t)=(-1)^s t^k\prod_{r=1}^{N}(1-t^r)^{a_r}, \tag{5.5}$$

where k is the order of the zero at $t=0$ (equal to the sum of the negative power terms of q), s is the sum of the negative coefficients and a_r is the sum of the coefficients of r and $-r$. Clearly, there might be some ambiguity, in case $k\neq 0$, in reading q from Eq. (5.5). However, k and a_r are uniquely determined from $P_q(t)$. The proof is a generalization of the polynomial case [12], and proceeds by induction. k is the order of $t=0$ and thus it is enough to consider the case of $k=0$. If $P_q(t)$ is of the form Eq. (5.5) then all the roots and poles are primitive roots of unity. Clearly, for n large enough, $P_q(t)$ has no poles or zeros which are m primitive roots, with $m>n$. Let n be the biggest integer such that $P_q(t)$ has a primitive n^{th} root of unity as a zero or a pole, with degree a_n. Multiplying both sides of Eq. (5.5) by $(1-t^n)^{-a_n}$ lowers the value of n. Repeating this step proves the assertion. The ambiguity associated with $k \neq 0$ will not be of a problem in our subsequent discussion, as we shall be dealing with $P_q(t)$ of the form Eq. (5.1) where it does not arise.

An evident but very useful property of the maps defined above is,

Lemma (5.1). *If q_1 and q_2 are two Laurent polynomials then*

$$P_{q_1+q_2}=P_{q_1}P_{q_2}, \tag{5.6}$$

$$D_{q_1+q_2}=D_{q_1}D_{q_2}. \tag{5.7}$$

Proof. Immediate from the definition.

Suppose that $q(x)$ is such that $q(1)=0$. This implies that the sum of positive and negative coefficients is equal. We can then write the polynomial $P_q(t)$ as,

$$P_q(t)=\frac{\prod_{i=1}^{r}(1-t^{n_i})}{\prod_{i=1}^{r}(1-t^{m_i})}, \tag{5.8}$$

and thus the function $P_q(t)$ is regular at $t=1$. (Similarly, if $q(1)$ is nonzero, $P(t)$ has a zero or a pole according to the sign of $q(1)$ whose degree is $|q(1)|$.) We can compute $P_q(1)$ by dividing both the denominator and numerator by $(1-t)^r$ and expanding,

$$P_q(1)=\frac{\prod_{i=1}^{r}(1+t+t^2+\cdots+t^{n_i-1})}{\prod_{i=1}^{r}(1+t+t^2+\cdots+t^{m_i-1})}. \tag{5.9}$$

Substituting $t=1$ we find,

$$P_q(1)=D(q). \tag{5.10}$$

In case P_q is the Poincaré series of a graded algebra, then $P_q(1)$ is the dimension of the algebra over the base field, and thus this dimension is given by $D(q)$ (hence the name of this map). In particular, in this case $D(q)$ must be a non-negative integer. Similarly, if a Poincaré polynomial of a graded algebra is given in the form,

$$P(t)=\sum_{i=1}^{s}P_{q_s}(t), \tag{5.11}$$

then the dimension of the algebra is

$$P(1)=\sum_{s=1}^{n}D(q_s), \tag{5.12}$$

where for simplicity we assumed that $q_s(1) = 0$ for all s. Note that each of the terms separately is not necessarily a positive integer and that the P_{q_s} itself need not be a polynomial.

We want to use the notation introduced above to investigate the following question. Consider the $N = 2$ superconformal field theory,

$$\frac{SU(n+m)_k}{SU(n) \times SU(m) \times U(1)}. \tag{5.13}$$

We know from Sect. (4) that this theory is dihedral. Namely, it remains the same under any permutation of k, n and m. Suppose the Poincaré polynomials of the associated chiral algebras are given by expressions of the type Eq. (5.1). Then this expressions must be dihedral as well, namely, invariant under a permutation of k, m and n. As we will now show, this fact alone is almost enough to determine the degree polynomials completely and hence also the Poincaré polynomials.

It is enough to supplement it with the following: assume also that the set of degrees obey the duality relation

$$\{m_i\} = \{n + m - m_i\} \tag{5.14}$$

and that the exponents m_i are positive integers. In that case, since $d = k + g$ and $g = n + m, P(t)$ assumes the form

$$P(t) = \prod_{i=1}^{r} \frac{(1 - t^{m_i + k})}{(1 - t^{m_i})}. \tag{5.15}$$

The reason for this assumption is simple. For all hermitian symmetric spaces (h.s.s.) theories the conformal field theory at $k = 0$ is trivial, $c = 0$. Thus at $k = 0$, $P(t) = 1$, which is possible only if this duality holds.

Define the degree polynomial $q(n, m)$ by

$$q(n, m)(x) = \sum_i x^{m_i}. \tag{5.16}$$

The duality property Eq. (5.14) is equivalent to,

$$q(1/x) = x^{-n-m} q(x). \tag{5.17}$$

Then the denominator of Eq. (5.15) is given by $P_{q(n,m)}$ and the numerator is given by $P_l(t)$, where $l(x) = x^k q(x)$. Thus, the polynomial $P(t)$ is

$$P(t) = P_u(t), \quad \text{where} \quad u(x) = (x^k - 1)q(x). \tag{5.18}$$

The dihedrality of permuting n, m and k implies that the Poincaré polynomials are the same:

$$P^{n,m,k}(t) = P^{m,n,k}(t) = P^{n,k,m}(t), \tag{5.19}$$

where $P^{n,m,k}$ is the polynomial of $SU(n+m)/SU(m) \times SU(n)$. Let also $q_{n,m}(x)$ stand for the degree polynomial of this algebra. Since the Poincaré polynomial determines completely the degree polynomial (using also the positivity of the degrees), as discussed earlier, it follows that

$$(1 - x^k) q_{n,m}(x) = (1 - x^k) q_{m,n} = (1 - x^n) q_{m,k}. \tag{5.20}$$

Equation (5.20) determines completely the degree polynomial $q_{n,m}$ for all n and m up to a constant polynomial, which is independent of n and m. To see this substitute $k=1$ in Eq. (5.20). We get $(1-x)q_{n,m}=(1-x^n)q_{1,m}$. Repeating again we find,

$$q_{n,m}=\frac{(1-x^n)(1-x^m)}{(1-x)^2}q_{1,1}(x). \tag{5.21}$$

Using now the duality of q, $x^{n+m}q(1/x)=q(x)$ we find $q_{1,1}=x$. Thus we determined completely the allowed exponents,

$$q_{n,m}=\frac{x(1-x^n)(1-x^m)}{(1-x)^2}. \tag{5.22}$$

It is easy to see that all the coefficients in Eq. (5.22) are indeed non-negative. An equivalent way to write this equation is

$$q_{n,m}=\sum_{i=1}^{n}\sum_{j=1}^{m}x^{i+j-1}, \tag{5.23}$$

allowing us to write the Poincaré polynomials Eq. (5.23) in the form

$$P^{n,m,k}(t)=\prod_{i=1}^{n}\prod_{j=1}^{m}\frac{(1-t^{k+i+j-1})}{(1-t^{i+j-1})}. \tag{5.24}$$

By the preceding discussion, the Poincaré polynomial $P^{n,m,k}(t)$ is given by $P_q(t)$, where

$$q(x)=-\frac{x(1-x^k)(1-x^n)(1-x^m)}{(1-x)^2} \tag{5.25}$$

and is thus manifestly symmetric in n, m and k and the dihedrality is explicit. Thus we have shown that a Poincaré polynomial of the form Eq. (5.15) is dihedral if and only if it is identical to $P^{n,m,k}(t)$ defined in Eq. (5.24).

One might wish to relax some of the assumptions made. In particular, all the arguments can be seen to hold if some of the degrees $\{m_i\}$ appearing in Eq. (5.15) are allowed to be negative. In this case the proof is exactly the same and we arrive at the conclusion,

Theorem (5.1). *Assume that a family of polynomials $P^{n,m,k}$ is given in the form Eq. (5.15) and $P^{n,m,k}=P_q(t)$, where q is the degree polynomial. Then this family is dihedral in the sense that $P^{n,m,k}=P^{m,n,k}=P^{k,m,n}$ if and only if the degree polynomial q is*

$$q(x)=(x^k-1)p_{m,n}(x), \tag{5.26}$$

where $p_{m,n}(x)=\sum_i x^{m_i}$ is the degrees generating polynomial and is of the form

$$p_{m,n}(x)=\frac{x(1-x^n)(1-x^m)}{(1-x)^2}s(x), \tag{5.27}$$

where $s(x)$ is an arbitrary polynomial. The degrees $\{m_i\}$ will permute among each other under the reflection $m_i\to m+n-m_i$ if and only if $s(x)=s(1/x)$. The number *of generators is given by* $nms(1)$ *and the sum of the degrees is*

$$\sum_i m_i = p'_{nm}(1). \tag{5.28}$$

If $s(x) = s(1/x)$ *then* $s'(1) = 0$ *and the sum of the degrees is* $\frac{1}{2}s(1)nm(n+m)$*. If all the degrees are positive integers, then* $s(x) = a$*, where* a *is an arbitrary positive integer*[6].

It is possible to cast our answer for the dihedral polynomial for

$$\frac{SU(n+m)}{SU(n) \times SU(m) \times U(1)}$$

in a root system form such that it would be evident how to generalize it to any G/H. Note that the number of generators appearing in this formula is equal to the number of positive roots in $\Delta_{G/H}$. Furthermore, it is not hard to see that the exponent generating polynomial $q(x) = \sum_{i=1}^{n} \sum_{j=1}^{m} x^{i+j-1}$ can be written as,

$$q(x) = \sum_{\alpha \in \Delta_{G/H}} x^{\rho\alpha}. \tag{5.29}$$

In other words, the degrees are given by $\rho\alpha$, where α runs over all the positive roots which are in G but not in H, and the gradation is given by $k+g$.

Written in this form, it is obvious how to generalize Eq. (5.29) to all theories. Simply replace $\Delta_{G/H}$ with the roots of the corresponding theory. The Poincaré polynomial of the theory G/H graded at $k+g$ will then be given by,

$$P_{G/H}(t) = \prod_{\alpha \in \Delta_{G/H}} \frac{(1 - t^{k+g-\alpha\rho})}{\cdot(1 - t^{\alpha\rho})}. \tag{5.30}$$

Most importantly, for all G and H this polynomial has the correct central charge. The central charge can be read from the degree of the polynomial, as explained in Sect. (1), and we find

$$\frac{1}{3}c(k+g) = \sum_{\alpha \in \Delta_{G/H}} k + g - 2\alpha\rho = (k+g)\mathrm{Dim}(\Delta_{G/H}) - 4\rho_G(\rho_G - \rho_H), \tag{5.31}$$

which is precisely the formula for the central charge that was derived in Sect. (2), Eq. (2.6).

In the special case where $\frac{G}{H}$ is an h.s.s. theory, since $\Lambda_\sigma \alpha = 1$ iff $\alpha \in \Delta_{G/H}$, where Λ_σ is the minimal fundamental weight that was deleted and α is any positive root of G, it follows that the polynomials Eq. (5.30) can be written as

$$P(t) = \prod_{\alpha \in \Delta_G} \frac{1 - t^{(\Lambda+\rho)\alpha}}{1 - t^{\rho\alpha}}, \tag{5.32}$$

[6] Taking $a = s(x) \neq 1$ corresponds to replacing $P(t)$ by $P(t)^a$ a freedom that always exists, but that does not concern us in our discussion since it will result in an incorrect central charge. This would be the Polynomial of a tensor product of a identical copies of the theory, which is, obviously, dihedral

where $\Lambda = k\Lambda_\sigma$. A well known result due to Weyl is that $P(t)$ is the character of $L(\Lambda)$ graded by ρ (the so-called principal gradation) (see, for example, [29], Proposition 10.10, where a generalization of this to all Kac–Moody algebras is described), noted in this context in ref. [30],

$$P(t) = \sum_{\lambda \in L(\Lambda)} t^{(\Lambda - \lambda)\rho}, \tag{5.33}$$

where the coefficient of t^a is the number of vectors in $L(\Lambda)$ of height a. It follows, in particular, that $P(t)$ is a polynomial in this case. (This is not necessarily so in the non h.s.s. case.) Another consequence is that in the h.s.s. case, $P(t)$ is a unimodal polynomial, i.e., if the positive coefficients of $P(t)$ are denoted by $d_1, d_2, \dots, d_m$ then $d_i = d_{m-i}$ and the series increases up to $d_{[(m+1)/2]}$, for any Λ [33]. This is not hard to see since $P(t)$ is, up to a factor of $t^{-\Lambda\rho}$, the character of $L(\Lambda)$ as an $SU(2)$ representation, where the $SU(2)$ algebra is generated by the sum of all positive or all negative roots. It follows that $d_i - d_{i-1}$ is the number of times the representation with weight $(m+1)/2 - i$ appears in $L(\Lambda)$ and is thus non-negative, for $i \leqq (m+1)/2$.

6. Polynomial Generating Functions

Let us return now to the graded chiral algebras of Sect. (2–3). We arrived there at the following expression for the $U(1)$ charges (grades) of the fields in these algebras,

$$Q_w^\Lambda = l(w) + \frac{2[w^{-1}(\Lambda + \rho) - \rho](\rho_G - \rho_H)}{k+g}, \tag{6.1}$$

where Λ is an integrable highest weight, and $w \in W\left(\frac{G}{H}\right)$ is an element of the Weyl group of G, $W(G)$, under which all the positive roots of H remain positive. For each such Λ and w there is precisely one field in the principal theory modulo the field identifications. The number of fields is then given by (in case there are no fixed points), Eq. (3.13)

$$N = \frac{|W(G)|}{|W(H)|Z} N_k^G, \tag{6.2}$$

where N_k^G is the number of integrable highest weight fields at level k and Z is the order of the center group.

Let us consider the Poincaré polynomial of some reductive pair (G, H) at level k, graded by $k+g$ and denote it by $P_k^{G/H}(t)$ or succinctly by $P_k(t)$,

$$P_k^{G/H}(t) = \sum_{\Lambda, w} t^{(k+g)Q_w^\Lambda}. \tag{6.3}$$

It is useful to introduce a polynomial generating function for all such polynomials at an arbitrary level,

$$P^{G/H}(z, t) = \sum_{k=0}^{\infty} P_k^{G/H}(t) z^k. \tag{6.4}$$

We shall derive an explicit expression for the generating function $P(z, t)$.

An integrable weight at level k can be written in the form

$$\Lambda = \sum_{i=1}^{n} l_i \Lambda_i, \tag{6.5}$$

where Λ_i are the fundamental weights of G and n is the rank of the algebra. The l_i obey the integrability condition $k \geqq \sum g_i l_i$. It is convenient to adjoin also the affine weight $l_0 \Lambda_0$, where $l_0 = k - \sum g_i l_i$, and the g_i are defined by

$$\theta = \sum_{i=1}^{n} \frac{2g_i}{\alpha_i^2} \alpha_i. \tag{6.6}$$

Define also $g_0 = 1$. θ stands for the highest root. The sum $\sum g_i = g$ is the dual Coxeter number.

The expression for the charge Q of the field C_w^Λ, Eq. (6.1), can now be written in the form

$$Q_w^\Lambda = \frac{s_w + \sum_{i=0}^{n} a_w^i l_i}{k + g}, \tag{6.7}$$

where $n = \mathrm{rank}(G)$ and the "exponents" a_w^i are defined by

$$a_w^i = l(w) g_i + 2(\rho - \rho_H) w^{-1}(\Lambda_i), \tag{6.8}$$

and

$$s_w = g l(w) - 2(\rho - \rho_H)[\rho - w^{-1}(\rho)]. \tag{6.9}$$

We can now compute

$$P(z, t) = \sum_{\Lambda, w \in W_1, k} z^k t^{(k+g) Q_w^\Lambda} = \sum_{w \in W_1} \sum_{l_0, l_1, \dots, l_n} t^{s_w} z^{g_1 l_1 + g_2 l_2 + \dots + l_n g_n} t^{a_1^w l_1 + a_2^w l_2 + \dots + a_n^w l_n}. \tag{6.10}$$

The sum factorizes into a product of geometric series and we find an expression for the polynomial generating function,

$$P(z, t) = \sum_{w \in W_1} \frac{t^{s_w}}{\prod_{i=0}^{n} (1 - z^{g_i} t^{a_i^w})}. \tag{6.11}$$

The case that G/H is an hermitian symmetric space corresponds to a reductive pair (G, H) such that H is obtained from the diagram of G by a removal of one node which is minimal, $\Lambda_\sigma = \sigma(0)$ for some automorphism σ of the extended Dynkin diagram. A fundamental weight Λ_i is minimal if and only if $g_i = 1$ and the simple root α_i is a long root. Recall from Sect. (2) that an external automorphism can be written as

$$\sigma(\lambda) = \sigma(0) + w_\sigma(\lambda), \tag{6.12}$$

where w_σ is an element of the Weyl group. Furthermore, ρ at level g is invariant under any external automorphism, $\sigma(\rho) = \rho$. To prove this note that ρ at level g corresponds to the affine "sum of positive roots" defined by the equation

$$\hat{\rho}_i \alpha_i = \tfrac{1}{2} \alpha_i^2, \quad \text{for} \quad i = 0, 1, \dots, n, \tag{6.13}$$

and is thus invariant under any permutation of the affine simple roots α_i which preserves the scalar products, proving that $\sigma(\rho)=\rho$. Using Eq. (6.12) we get

$$\rho = w_\sigma(\rho) + g\Lambda_\sigma, \tag{6.14}$$

or $\rho - w_\sigma(\rho) = g\Lambda_\sigma$. Recall from Sect. (3), Eq. (3.16) that

$$\rho - w_\sigma(\rho) = \sum_{\substack{\alpha>0 \\ w_\sigma^{-1}(\alpha)<0}} \alpha = \sum_{\substack{\alpha>0 \\ \alpha\Lambda_\sigma=1}} \alpha, \tag{6.15}$$

where we used Lemma (2.1). The sum on the right-hand side of Eq. (6.15) ranges, however, precisely over all positive roots that are in G but not in H, since every such root is of the form $\beta = \alpha_\sigma + \cdots$ (since $g_i = 1$) and is thus equal to $2(\rho - \rho_H)$. We have proved,

$$2(\rho - \rho_H) = g\Lambda_\sigma. \tag{6.16}$$

It follows also that

$$2(\rho - \rho_H)(\rho - w^{-1}(\rho)) = g\Lambda_\sigma \sum_{\substack{\alpha>0 \\ w(\alpha)<0}} \alpha = gl(w), \tag{6.17}$$

where we used Eq. (3.16). This is a more direct proof for Eq. (3.20). Thus $s_w = 0$ for every $w \in W\left(\frac{G}{H}\right)$ and it follows that

Lemma (6.1). *When G/H is an hermitian symmetric space, $s_w = 0$ and the generating function assumes the form*

$$P(z,t) = \sum_{w\in W_1} \prod_{i=0}^{n} \frac{1}{1 - z^{g_i} t^{a_w^i}}. \tag{6.18}$$

The $U(1)$ charges can be written for this family as

$$Q_w^\Lambda = \frac{kl(w) + g\Lambda_\sigma w^{-1}(\Lambda)}{k+g}. \tag{6.19}$$

Picking the z independent term in Eq. (6.11) we find the Poincaré polynomial of the $k=0$ theory,

$$P_0(t) = \sum_{w\in W_1} t^{s_w}, \tag{6.20}$$

or, the s_w are the charges of the $k=0$ theory. In the hermitian symmetric space case the central charge vanishes when $k=0$ and $P_0(t)$ becomes a constant integer. This represents the trivial algebra with only the unit field, appearing $\left|W\left(\frac{G}{H}\right)\right|$ times due to an over counting.

In fact, as discussed in Sects. (2–3), the fields C_w^Λ and $C_{w_\sigma w}^{\sigma(\Lambda)}$ are one and the same, i.e., we need to identify fields under the action of the external automorphisms. We have assumed in our discussion that there are no fixed points of this action. If there are then Eq. (6.11) needs to be modified as explained in Sect. (3). In case there are no fixed points, the correct expression for $P(z,t)$ is obtained by dividing it by Z, the order of the center. Further, it will be shown that $s_{w_\sigma w} = s_w$ and

$a^i_w = a^{\sigma^{-1}(i)}_{w_\sigma w}$, Eq. (6.32), and thus the terms in Eq. (6.11) associated with w and $w_\sigma w$ are the same. Assuming no fixed points is equivalent to demanding a faithful action of w_σ on W_1 and thus the sum can simply range over a set of representatives of $W_1 \bmod \{w_\sigma\}$, where the quotient is on the left.

It is not hard to extend the formalism described above to allow for fixed points. This will be demonstrated through an example calculation of the polynomial generating function. Consider thus $\dfrac{SO(5)}{SO(3)\times U(1)}$ which is an h.s.s. theory. The simple roots of $SO(2n+1)\approx B_n$ are $\varepsilon_i-\varepsilon_{i+1}$, where $1\leqq i<n$ along with ε_n. The Weyl group of $SO(2n+1)$ has $2^n n!$ elements, which are the signed permutations of the ε_i; the most general element being of the form $w(\varepsilon_i)=s_i\varepsilon_{p(i)}$, where $s_i=\pm 1$ and p is any permutation. Thus the number of cosets in $W\left(\dfrac{SO(2n+1)}{SO(2n-1)}\right)$ is $2n$. It is not hard to see that there is only one non-trivial external automorphism, $w_\sigma=\delta_1$, where δ_i is the Weyl transformation that only flips the sign of ε_i. The elements of $W\left(\dfrac{SO(2n+1)}{SO(2n-1)}\right)$ are given by either p_r or $p_r\delta_1$, where $p_r=w_r w_{r-1}\cdots w_1$ and $0\leqq r<n$. Since $\delta_1 p_r = p_r\delta_2$ if $r>0$ it follows that w_σ exchanges the two cosets p_0 and $p_0\delta_1$ (since $p_0=1$) and fixes all the other cosets, $w_\sigma p_r = p_r$ and $w_\sigma p_r\delta_1 = p_r\delta_1$, for $r>1$.

Let us compute the polynomial generating function for $n=2$. It is enough to compute a^i_w for the three Weyl elements $1, w_1$ and $w_2 w_1$, for which $l(w)$ is $0,1,2$, respectively. For this theory, $g_i=1$, $g=3$, $2(\rho-\rho_H)=g\Lambda_1$, $\Lambda_1=\varepsilon_1$ and $\Lambda_2=\frac{1}{2}(\varepsilon_1+\varepsilon_2)$. Substituting into Eq. (6.8) we find that

$$a^i_1=(0,3,\tfrac{3}{2}),\quad a^i_{w_1}=(1,1,\tfrac{5}{2}),\quad a^i_{w_2w_1}=(2,2,\tfrac{1}{2}). \tag{6.21}$$

We can now repeat the calculation of the polynomial generating function Eq. (6.10), taking into account the fixed points. For the first coset, $w=1$, the calculation is unchanged since it is not a fixed point of w_σ and thus, as before, $P(z,t)=(1-z)^{-1}(1-zt^6)^{-1}(1-zt^3)^{-1}$. (We have replaced t with t^2 so as to make all the powers integral.) For the coset w_1 we find

$$P(z,t)=\left(\frac{1}{2}\sum_{l_1,l_2,l_3}+\frac{1}{2}\sum_{l_1=l_2,l_3}\right)z^{l_1+l_2+l_3}t^{2l_1+2l_2+5l_3}, \tag{6.22}$$

where the first sum is the usual one, dividing by 2 in view of the action of σ on Λ (which exchanges l_1 with l_2) and the second sum adds half for every fixed point so as to make the total equal to one for these as well. It is now easy to sum Eq. (6.22) and we find $P(z,t)=[(1-zt^2)(1-z^2t^4)(1-zt^5)]^{-1}$. Similarly we get for the third coset $P(z,t)=[(1-zt^4)(1-z^2t^8)(1-zt)]^{-1}$. The generating function of the theory is the sum of the contributions from the three cosets,

$$\begin{aligned}P(z,t)=&\frac{1}{(1-z)(1-zt^6)(1-zt^3)}\\&+\frac{1}{(1-zt^2)(1-z^2t^4)(1-zt^5)}+\frac{1}{(1-zt^4)(1-z^2t^8)(1-zt)}.\end{aligned} \tag{6.23}$$

This concludes the example.

Note the striking similarity between the expressions that we obtained for the "bi-graded" Poincaré polynomial, $P(z,t)$, and a classical Poincaré polynomial. Recall from invariant theory (e.g., ref. [14]) that for any group G of matrices acting on a vector space we can define the invariant polynomial algebra $S(V)^G$ defined as all the polynomials in $\mathrm{Dim}(V)$ variables invariant under the action of G. The Poincaré polynomial of such an algebra is according to Molien's Theorem [14] of the form

$$P(z)=\sum_{g\in G}\det(I-zg)^{-1}=\sum_{g\in G}\prod_i\frac{1}{1-z\lambda_i}, \tag{6.24}$$

where the λ_i's are the eigenvalues of g in the representation acting on V. If we interpret $t^{a^i_w}$ as the eigenvalue of $W\left(\frac{G}{H}\right)$ we see that the expression we have derived in the h.s.s. case, Eq. (6.18), is identical to Eq. (6.24). This clearly suggests that $P(z,t)$ might be interpreted as the Poincaré polynomial of a symmetric polynomial algebra. Namely, take all the polynomials generated by the Λ_i graded at the level g_i and interpret these as tensor products which are invariant under the action of W_1. This is consistent since k is additive under such products, hence respecting the z-grading. This polynomial cohomology will then have as the Poincaré series the polynomial generating functions that we have computed. It is thus natural to try and identify it with our bigraded chiral algebra.

The set of exponents a^i_w obeys the following property,

$$\sum_{i=0}^{n} a^i_w=s_w+2\rho(\rho-\rho_H), \tag{6.25}$$

which follows their definition, Eq. (6.8), using $\sum g_i=g$ and $\sum\Lambda_i=\rho$. It follows that the generating function $P(z,t)$ can be written as

$$P\left(\frac{1}{z},\frac{1}{t}\right)=(-1)^{n+1}z^g t^{2\rho(\rho-\rho_H)}\sum_{w\in W_1}\frac{1}{\prod_{i=0}^{n}(1-z^{g_i}t^{a^i_w})}. \tag{6.26}$$

As discussed in Sect. (1), the Poincaré polynomial is dual,

$$P_k\left(\frac{1}{t}\right)=t^{-(k+g)c/3}P_k(t), \tag{6.27}$$

implying that,

$$P(z,t)=\sum_{k=0}^{\infty}z^kP_k(1/t)t^{(k+g)M-4\rho(\rho-\rho_H)}=\sum_{k=0}^{\infty}P_k(1/t)(zt^M)^k t^{gM-4\rho(\rho-\rho_H)}, \tag{6.28}$$

and we proved

$$P(z,t)=t^{gM-4\rho(\rho-\rho_H)}P\left(zt^M,\frac{1}{t}\right). \tag{6.29}$$

The field identification,

$$C^{\Lambda}_w=C^{\sigma(\Lambda)}_{w_\sigma w}, \tag{6.30}$$

discussed in Sect. (2), implies, in particular, that the $U(1)$ charges of the two fields in Eq. (6.30) must be equal. Owing to Eq. (6.7) this is equivalent to

$$s_{w_\sigma w} + a^{\sigma^{-1}(i)}_{w_\sigma w} = s_w + a^i_w, \tag{6.31}$$

for all w and i. Summing Eq. (6.31) over i and using Eq. (6.25), we find

$$s_{w_\sigma w} = s_w, \quad a^{w_\sigma w}_{\sigma^{-1}(i)} = a^w_i. \tag{6.32}$$

In the h.s.s. case, since $s_w = 0$, comparing Eq. (6.18) with Eq. (6.26) we find that the generating function has the property

$$A{:}P(z,t) = (-1)^{n+1} z^{-g} t^{-(1/2)gM} P\left(\frac{1}{z}, \frac{1}{t}\right), \tag{6.33}$$

where $M = \mathrm{Dim}(\Delta_{G/H})$ and we used Eq. (6.16) and the fact that $2\Lambda_\sigma \rho = M$ (since for every $\alpha > 0$, $\alpha \in \Delta_{G/H}$ iff $\alpha\Lambda_\sigma = 1$).

This property may be thought of as a generalization of the Poincaré duality for infinite dimensional graded algebra and is common for many "simple" algebras. For example every Gorenstein algebra obey this [34, 13]. Every Cohen–Macaulay integral domain with this property is a Gorenstein algebra [13]. Thus the simplicity of the polynomial generating function $P(z,t)$ strongly suggests that a simple bigraded algebraic structure arises. As discussed earlier, it is natural to identify this structure as some invariant version of a tensor algebra which owing to Lemma (6.1) must be a relatively simple algebra.

In the h.s.s. case, since $gM = 4\rho(\rho - \rho_H)$, the Poincaré duality Eq. (6.29) assumes the form,

$$B{:}P(z,t) = P\left(zt^M, \frac{1}{t}\right). \tag{6.34}$$

The two transformations, A and B, generate a $Z_2 \times Z_2$ group of "bi-dualities," $A^2 = B^2 = 1$ and $AB = BA$.

Let us consider now the theories of the type $\dfrac{SU(m+n)}{SU(m) \times SU(n) \times U(1)}$. First, in case $G \approx SU(N)$, we can write the exponents in a nicer form, for any subgroup H. This is since for $SU(N)$ all the fundamental weights can be written as $\Lambda_i = \sigma^i(0)$, where σ is the generating external automorphism (which generates a cyclic group of order N, Z_N). Thus, we can use Eq. (6.32) to write,

$$a^i_w = a^0_{w^i_\sigma w} = l(w^i_\sigma w). \tag{6.35}$$

Thus in the case that $G = SU(N)$ the generating function may be written as

$$P(z,t) = \sum_{w \in W_1} \prod_{i \bmod N} [1 - zt^{l(w^i_\sigma w)}]^{-1}. \tag{6.36}$$

We shall be mostly concerned with the h.s.s. family, where $H = SU(n) \times SU(m) \times U(1)$ and $G = SU(n+m)$, which is obtained by the deletion of precisely one weight from the Dynkin diagram of $SU(n+m)$. Recall from Sect. (4) the structure of the Weyl group W, in this case. As described there the simple roots of $SU(n+m)$ are given by $\varepsilon_i - \varepsilon_{i+1}$, where $\{\varepsilon_i\}$ is an orthonormal set of unit vectors, and $1 \leqq i \leqq n+m$. The elements of $W(G)$ are given by the permutations $w(\varepsilon_i) = \varepsilon_{p(i)}$, where p is any permutation. The elements of $W(G/H)$ correspond to permutations such that $p(i) < p(j)$ for $1 \leqq i < j \leqq n$ or $n < i < j \leqq n + m$. Thus, $W\left(\frac{G}{H}\right)$ is in a one to one correspondence with the sequences $w = [a_1, a_2, \ldots, a_n]$ such that $1 \leqq a_1 < a_2 < \cdots < a_n \leqq n + m$, where $a_i = p(i)$. The length of the Weyl element w is, Eq. (4.21),

$$l(w) = \sum_{i=1}^{n} a_i - i. \tag{6.37}$$

We can compute $w_\sigma w$. w_σ corresponds to the cyclic rotation, $p(i) = i + 1 \bmod n$, $p = (1, 2, 3, \ldots, n + m)$. For any integer x, denote, by $\{x\}$, the integer i such that $1 \leqq i \leqq n + m$ and $i = x \bmod (n + m)$. Let $w = [a_1, a_2, \ldots, a_n]$ be some element of $W\left(\frac{G}{H}\right)$. Then $w_\sigma^s w$ is given by $[a_i + s]$. In particular, $l(w_\sigma w)$ is

$$-\frac{1}{2}n(n+1) + \sum_i \{a_i + s\} = -\frac{1}{2}n(n+1) - (n+m)k + \sum_i a_i, \tag{6.38}$$

where k is the number of times we "wrapped" around with the a_i's.

Now assume that $\gcd(n, m) = 1$. Then $a_w^s = a_w^t$ implies that $s = t \bmod (n + m)$, since if $a_w^s = a_w^t$, where $w = [a_1, a_2, \ldots, a_n]$, Eq. (6.38) gives

$$-\tfrac{1}{2}n(n+1) + sn + \sum a_i = -\tfrac{1}{2}n(n+1) + tn + \sum a_i \bmod (n+m), \tag{6.39}$$

implying that indeed $i = j \bmod (n + m)$. Thus, there are no fixed points for any k if $\gcd(m, n) = 1$. This is easily generalized for $l > 3$ (in the terminology of Sect. (4)) showing that there are no fixed points iff $\gcd(n_1, n_2, \ldots, n_l) = 1$.

As $a_w^i = l(w_\sigma^i w)$, where $l(w)$ is the number of positive roots of $\Delta_{G/H}$ that become negative roots under the action of w, we find that $0 \leqq a_w^i \leqq M$, where $M = nm$ is the number of elements in $\Delta_{G/H}$[7].

It follows that for the family $\frac{SU(n+m)}{SU(n) \times SU(m) \times U(1)}$, we can take as a common denominator in Eq. (6.36), $\prod_{d=0}^{M} (1 - zt^d)$, showing that,

Lemma (6.2). *For the family of* h.s.s. *theories* $\frac{SU(n+m)}{SU(n) \times SU(m) \times U(1)}$, *the generating function assumes the form,*

$$P(z, t) = \frac{\sum_{j=0}^{J} E_j(t) z^j}{\prod_{d=0}^{M} (1 - zt^d)}, \tag{6.40}$$

where $M = nm$, $J = M + 1 - g$, and the $E_j(t)$ are some polynomials. From the bi-duality of the generating function, Eqs. (6.33–6.34), it follows that the polynomials

[7] For any G and $H, a_w^i \geqq 0$ and $s_w \geqq 0$. To see this, apply the formula for the $U(1)$ charge, Eq. (6.7), to the weight $s\Lambda_i$, where s is very large. Taking $k = 0$ shows that $s_w \geqq 0$

$E_j(t)$ obey

$$E_0(t) = S, \quad E_J(t) = St^{(1/2)M(M+1-g)}, \tag{6.41}$$

$$A: E_j(t) = t^{(1/2)M(M+1-g)} E_{M+1-g-j}\left(\frac{1}{t}\right), \tag{6.42}$$

$$B: E_j(t) = E_j\left(\frac{1}{t}\right) t^{jM}, \tag{6.43}$$

where $S = |W(G/H)|$ is the number of cosets in $W(G)$ modulo $W(H)$.

Proof. Described above. It remains only to show the properties listed. That $E_0 = \left|W\left(\frac{G}{H}\right)\right|$ is clear by substituting $z = 0$ in Eq. (6.36). The rest of the proof follows as in Sect. (7), Theorem (7.2).

The only known series of theories which are scalar field theories at an arbitrary level k, is the series $\frac{SU(N+1)}{SU(N) \times U(1)}$ [8]. The generators are of degrees $\{1, 2, \ldots, N\}$. The Poincaré polynomials agree precisely with the ones we computed using dihedrality alone, Eq. (5.24),

$$R_k^N(t) = \prod_{i=1}^{N} \frac{(1 - t^{k+i})}{(1 - t^i)}. \tag{6.44}$$

The generating polynomial $P(z, t)$, Eq. (6.36), in this case assumes the form

$$P(z, t) = \prod_{i=0}^{N} (1 - zt^i)^{-1}, \tag{6.45}$$

since there is only one right coset of $W_1 \bmod \{w_\sigma\}$, and no fixed points. We thus arrive at the identity,

$$\prod_{i=0}^{N} (1 - zt^i)^{-1} = \sum_k z^k \prod_{i=1}^{N} \frac{(1 - t^{k+i})}{(1 - t^i)}. \tag{6.46}$$

The identity, Eq. (6.46), aside from being very pretty[8], can be used to derive closed formulas for the Poincaré polynomial $P(t)$ at any k for many other theories. Consider for example $\frac{SU(5)}{SU(3) \times SU(2) \times U(1)}$ which is an h.s.s. theory. In this case there are no fixed points and precisely two inequivalent elements of the Weyl group that are needed to be considered. These are 1 and w_2 (a reflection by the second simple root, that was deleted). We find,

$$a_1^i = (0, 3, 6, 4, 2) \quad \text{and} \quad a_{w_2}^i = (1, 4, 2, 5, 3). \tag{6.47}$$

[8] This identity is known in mathematics and goes back to Cauchy. It is called the q-binomial theorem. See, for example, ref. [35]

Thus, using Eq. (6.36),

$$\begin{aligned} P(z,t) &= \frac{1}{(1-z)(1-zt^3)(1-zt^6)(1-zt^4)(1-zt^2)} \\ &\quad + \frac{1}{(1-zt)(1-zt^4)(1-zt^2)(1-zt^5)(1-zt^3)} \\ &= \frac{2 - z(1+t+t^5+t^6) + 2z^2t^6}{\prod_{i=0}^{6}(1-zt^i)}. \end{aligned}$$

Using, Eq. (6.46), we find for the k^{th} term of $P(z,t)$, the Poincaré polynomial $P_k(t)$,

$$P_k(t) = 2R_k^7 - (1+t+t^5+t^6)R_{k-1}^7 + 2t^6R_{k-2}^7, \tag{6.49}$$

and so

$$\begin{aligned} P_k(t) &= \frac{\prod_{i=1}^{4}(1-t^{k+i})}{\prod_{i=1}^{6}(1-t^i)} \\ &\quad \cdot \left[2(1-t)(1-t^5) - \frac{t^{k-1}(1-t^3)(1-t^4)(1-t^5)}{(1-t)(1-t^2)} + \frac{t^{2k+3}(2+t-t^6-2t^7)}{(1-t)}\right]. \end{aligned} \tag{6.50}$$

As we can see the answer, even in this case, is quite complicated. It displays some sort of an algebraic structure, as evidenced by the many primitive roots of unity that appear as zeros and poles, yet a big chunk of the answer does not seem to have much of an explanation. Certainly the theory is not a scalar field theory, except for $k=1$.

In order to make any further progress we need to cast the Poincaré polynomials that we have computed in a form that would make their algebraic structure more transparent. This we will do in the next section.

7. The Resolution Series

Let us go back to the canonical form for an SFT Poincaré Polynomial in the h.s.s. case. In this case $P(t)$ assumes the form (Sects. (1, 5)),

$$P_k(t) = \sum_{s=0}^{S} \prod_{i=1}^{M} \frac{(1-t^{k+m_i^s})}{(1-t^{m_i^s})}. \tag{7.1}$$

Note that we generalized this expression to include more than one term. The meaning of this generalization will become clear later in this section. We shall further assume that the exponents $\{m_i\}$ are dual, $\{m_i\} = \{g - m_i\}$ for some g and that the sum $\sum_i m_i^s = R$ is independent of s. We will also assume that $P_k(t) = 0$ for

$-1 \geqq k \geqq 1-g$, which is satisfied if $m_i = 1, 2, \ldots, g-1$ are part of the exponent set. The exponents, m_i can be positive or negative and, of course, $m_i^s \neq 0$.

We start with a theorem.

Theorem (7.1). *Let $P(t)$ be any polynomial given as above. Then $R = \frac{1}{2}Mg$. Define as usual the generating function*

$$P(z,t) = \sum_{k=0}^{\infty} z^k P_k(t).$$

Then $P(z,t)$ obeys the bi-duality property (Eqs. (6.33–6.34)). Namely,

$$A: P(z,t) = (-1)^{M+1} z^{-g} t^{-R} P\left(\frac{1}{z}, \frac{1}{t}\right), \tag{7.2}$$

and

$$B: P(z,t) = P\left(z t^M, \frac{1}{t}\right). \tag{7.3}$$

Proof. Property B is the consequence of the Poincaré duality of $P_k^s(t)$ which is easy to establish from the explicit expression Eq. (1). The proof of property B is very similar to that given in Sect. (6) for the G/H generating functions and we shall omit the details. This property can of course be generalized to the non-h.s.s. cases.

Consider then property A. For simplicity consider first the case $S = 0$, where there is only one polynomial. Then we can compute

$$\begin{aligned} P(z,t) &= \sum_{k=0}^{\infty} z^k \prod_{i=1}^{M} \frac{(1-t^{k+m_i})}{(1-t^{m_i})} = \sum_{k=0}^{\infty} z^k \prod_{i=1}^{M} \left[\frac{1-(1/t)^{m_i-k-g}}{1-(1/t)^{-m_i}}\right] \\ &= (-1)^M t^{-R} z^{-g} \sum_{k'=-g}^{-\infty} \left(\frac{1}{z}\right)^{k'} \prod_{i=1}^{M} \frac{[1-(1/t)^{k'+m_i}]}{[1-(1/t)^{m_i}]}, \end{aligned} \tag{7.4}$$

where we used the assumptions on the exponents. Now, Eq. (4) looks precisely like the desired A duality, except for one thing, the sum on the right-hand side ranges over $k = -g$ to $-\infty$ instead of $k = 0$ to ∞. Since $P_k(t) = 0$, for $-1 \geqq k \geqq 1-g$, we can change the range of the last summation from $\sum_{k'=-g}^{\infty}$ to $\sum_{-1}^{\infty}$. So, it is enough to show that

$$-\sum_{k=-1}^{-\infty} z^k P_k(t) = \sum_{k=0}^{\infty} z^k P_k(t), \tag{7.5}$$

for any z, where this identity is understood in the sense of analytic continuation of the series, each defined from its range of convergence.

We can expand $P_k(t)$,

$$P_k(t) = \prod_{i=1}^{M} \frac{(1-t^{k+m_i})}{(1-t^{m_i})} = \prod_{i=1}^{M} (1-t^{m_i})^{-1} \sum_{d=0}^{M} (-1)^d N_d(t) t^{kd}, \tag{7.6}$$

where the $N_d(t)$ are the polynomials

$$N_d(t) = \sum_{\substack{i_1, i_2, \ldots, i_d \\ i_1 < i_2 < \cdots < i_d}} t^{m_{i_1} + m_{i_2} + \cdots + m_{i_d}}. \tag{7.7}$$

Using the expansion Eq. (7.6), Eq. (7.5) assumes the form,

$$\sum_d \frac{N_d(t)}{(1 - zt^d)} = -\sum_d \frac{z^{-1}t^{-d}N_d(t)}{(1 - z^{-1}t^{-d})}, \tag{7.8}$$

where we summed the series on each side. This is evidently correct, completing the proof.

Since the bi-duality is valid term by term, it follows also for $S > 0$, if it holds for $S = 0$.

The fact that $R = \frac{1}{2}Mg$ is required for the consistency, $AB = BA$. Otherwise we would find $P(z,t) = t^a P(z,t)$ with a nonzero a when combining the two dualities, implying that $P(z,t) = 0$.

Consider now the application theorem (7.1) to the Poincaré polynomials of Sect. (5). From there, Eq. (5.30), in the h.s.s. case,

$$P_k(t) = \prod_{\alpha \in \Delta_{G/H}} \frac{(1 - t^{k + \rho\alpha})}{(1 - t^{\rho\alpha})} = \prod_{i=1}^{M} \frac{(1 - t^{k + m_i})}{(1 - t^{m_i})}, \tag{7.9}$$

where $M = \mathrm{Dim}(\Delta_{G/H})$, $m_i = \alpha^i \rho$, and we indexed the roots of $\Delta_{G/H}$ in some irrelevant manner. It is not hard to check that the exponents m_i are dual $\{m_i\} = \{g - m_i\}$ and that $M_i = 1, 2, \ldots, g-1$ are part of the exponents, and so $P_k(t) = 0$ for $-1 \geqq k \geqq 1 - g$. Furthermore $R = \sum_i m_i = 2\rho(\rho - \rho_H) = \frac{1}{2}gM$. We thus find that the generating function

$$P(z,t) = \sum_{k=0}^{\infty} z^k \prod_{\alpha \in \Delta_{G/H}} \frac{(1 - t^{k + \alpha\rho})}{(1 - t^{\alpha\rho})}, \tag{7.10}$$

obeys the same bi-duality relation that we found for the actual generating functions of the theory Eqs. (7.2–7.3) with precisely the same exponents and we start to see a firm connection between the two.

Let us continue now our investigation of the generating functions of the type (7.10). The next step is to give a more explicit expression for these. This is described in the next theorem.

Theorem (7.2). *Let $P(z,t)$ be as in Theorem (7.1). Then it can be written as*

$$P(z,t) = \frac{\sum_{j=0}^{J} z^j E_j(t)}{\prod_{l=0}^{M} (1 - zt^l)}, \tag{7.11}$$

where $J = M + 1 - g$ and $E_j(t)$ are some polynomials. Furthermore, the polynomials $E_j(t)$ obey

$$E_0(t) = S + 1, \quad E_J(t) = (S + 1)t^{(1/2)M(M+1-g)}, \tag{7.12}$$

$$A: E_j(t) = t^{(1/2)M(M+1-g)} E_{M+1-g-j}\left(\frac{1}{t}\right), \tag{7.13}$$

$$B: E_j(t) = E_j\left(\frac{1}{t}\right) t^{jM}. \tag{7.14}$$

Proof. Let us first assume for simplicity $S=0$. We can expand $P_k(t)$, as in the proof of Theorem (7.1), Eqs. (7.6–7.7), and then evaluate the series,

$$P(z,t)=\prod_{i=1}^{M}(1-t^{m_i})^{-1}\sum_{k=0}^{\infty}z^k\sum_{d=0}^{M}(-1)^dN_d(t)t^{kd}$$
$$=\prod_{i=1}^{M}(1-t^{m_i})^{-1}\sum_{d=0}^{M}(-1)^d\frac{N_d(t)}{1-zt^d}. \tag{7.15}$$

Taking a common denominator leads to

$$P(z,t)=\prod_{i=1}^{M}(1-t^{m_i})^{-1}\prod_{d=0}^{M}(1-zt^d)^{-1}\sum_{j=0}^{J}Q_j(t)z^j, \tag{7.16}$$

where the $Q_j(t)$ are some polynomials and J is some integer.

In case $S>0$ we will get precisely the same kind of expression, Eq. (7.16), since each of the terms separately gives this. Now, since $P(t)$ is a polynomial, there must be no poles at $t=c$ in the generating function Eq. (7.16), for any complex number c. This implies that $Q=\prod_{i=1}^{M}(1-t^{m_i})$ divides each of the $Q_j(t)$ and $E_j(t)=Q_j(t)/Q(t)$ is a polynomial. Thus,

$$P(z,t)=\frac{\sum_{j=0}^{J}E_j(t)z^j}{\prod_{d=0}^{M}(1-zt^d)}, \tag{7.17}$$

which proves the first part of the theorem.

Now, it is not hard to compute E_0 by tracing back the steps in the proof above. It follows that

$$E_0(t)=\frac{\sum_d(-1)^dN_d(t)}{\prod_{i=1}^{M}(1-t^{m_i})}=1, \tag{7.18}$$

where we used Eq. (7.7). Similarly if $S>0$ we find $E_0=S+1$.

We can now apply Theorem (7.1) to $P(z,t)$. Using Eqs. (7.2, 7.17)

$$P\left(\frac{1}{z},\frac{1}{t}\right)=\frac{\sum_{j=0}^{J}E_j\left(\frac{1}{t}\right)z^{-j}}{\prod_{d=0}^{M}(1-z^{-1}t^{-d})}=(-1)^{M+1}z^{M+1}t^{(1/2)M(M+1)}\frac{\sum_{j=0}^{J}E_j\left(\frac{1}{t}\right)z^{-j}}{\prod_{d=0}^{M}(1-zt^d)}. \tag{7.19}$$

From the A duality, Eq. (7.2), we find,

$$P\left(\frac{1}{z},\frac{1}{t}\right)=(-1)^{M+1}z^gt^RP(z,t)=(-1)^{M+1}z^gt^R\frac{\sum_{j=0}^{J}E_j(t)z^j}{\prod_{d=0}^{M}(1-zt^d)}. \tag{7.20}$$

Comparing the two expressions Eq. (7.19) and Eq. (7.20) proves property A, Eq. (7.13).

Property B follows from the B-duality of $P(z,t)$. From Eq. (7.17)

$$P\left(zt^M,\frac{1}{t}\right)=\frac{\sum_{j=0}^{J} E_j\left(\frac{1}{t}\right)z^j t^{Mj}}{\prod_{s=0}^{M}(1-zt^s)}. \tag{7.21}$$

Comparing with Eq. (7.3) proves property B, Eq. (7.14).

Since the coefficients of z^j in the denominator of Eq. (7.17) vanish when $j<0$, the A duality implies that $E_j=0$ for $j>M+1-g$ and $J=M+1-g$ as stated in the theorem. Finally, combining the A and B dualities acting on E_0 leads to the expression for E_J, completing the proof.

A number of corollaries follow from the theorem.

Corollary (7.1). *Consider the module V over the ring of polynomials $P(t)$, the polynomials in t, spanned by the $P(z,t)$ polynomials of the form Eq. (7.1), which satisfy the assumptions of Theorem (1). Then* $\mathrm{Dim}(V)\leqq[\frac{1}{2}(M-g+1)]$, *(where we denote by $[x]$ the maximal integer j such that $j\leqq x$). In particular, if $M<g-1$ then* $\dim(V)=0$ *and there are no such polynomials. If $M=g-1$ then* $\dim(V)=1$ *and there is a unique such polynomial, which is the polynomial of* $\frac{SU(g+1)}{SU(g)\times U(1)}$, *$R_k^g$ defined in Eq. (6.44).*

Proof. If $v(z,t)\in V$ is a such a polynomial it follows from Theorem (2) that $v=\prod_{d=0}^{M}(1-zt^d)^{-1}\sum_{j=0}^{J}E_j^v(t)z^j$. In addition the AB duality implies that $E_j(t)=E_{M+1-j}(t)$ up to t to some power. Thus, at most $[\frac{1}{2}(M-g+1)]$ such polynomials can be linearly independent over $P(t)$. If $M<g-1$ the AB duality implies that there does not exist a polynomial satisfying the assumptions of Theorem (7.1). If $M=g-1$ we have $J=0$ and $E_0=1$ (or a positive integer), from Theorem (7.2), and thus $P(z,t)$ is given by R_k^g, Eq. (6.44).

The inverse of Theorem (7.2) can also be stated. Namely, given a $P(z,t)$ of the form Eq. (7.11) one can calculate $P_k(t)$. This is done as in Sect. (6) by utilizing the identity (6.46). We find

$$P(z,t)=\frac{\sum_{j=0}^{J} z^j E_j(t)}{\prod_{l=0}^{M}(1-zt^l)}=\left[\sum_{j=0}^{J}E_j(t)z^j\right]\left[\sum_{k=0}^{\infty}z^k R_k^M(t)\right], \tag{7.22}$$

where

$$R_k^M(t)=\prod_{i=1}^{M}\frac{(1-t^{k+i})}{(1-t^i)}. \tag{7.23}$$

Thus, $P_k(t)$ is given by

$$P_k(t)=\sum_{j=0}^{J}E_j(t)R_{k-j}^M(t). \tag{7.24}$$

This identity provides also a simple way to compute $E_j(t)$ from $P_k(t)$, since it can be inverted into a recursion relation for $E_j(t)$,

$$E_j(t) = P_j(t) - \sum_{h=0}^{j-1} E_h(t) P_{j-h}(t), \quad \text{for} \quad j = 0, 1, \ldots, J. \tag{7.25}$$

This recursion relation shows that the polynomials $E_j(t)$ have only integral coefficients. More importantly, we can show:

Lemma (7.1). *Suppose $P_k(t)$ and $M_k(t)$ are two polynomials of the type Eq. (1). Then if $P_k(t) = M_k(t)$ for $k = 0, 1, \ldots, s$, where $s \geqq \frac{1}{2}(M-g)$, then $P_k(t) = M_k(t)$ for all $k \geqq 0$. Furthermore, if $N(z,t)$ is of the form Eq. (7.11) with arbitrary coefficients $E_j(t)$, and it obeys the bi-duality property, Eqs. (7.2) and (7.3), then the above holds also for $N_k(t)$.*

Proof. This follows directly from the recursion relation, Eq. (7.25). Let $D_k(t)$ stand for $P_k(t) - M_k(t)$ for the first case, and $P_k(t) - N_k(t)$ for the second. In both cases, $D_k(t)$ assumes the form Eq. (7.11),

$$D(z,t) = \frac{\sum_{j=0}^{J} E_j(t) z^j}{\prod_{l=0}^{M} (1 - zt^l)}, \tag{7.26}$$

and $D_k(t) = 0$ for $j = 0, 1, \ldots, J$, where we used the bi-duality Eqs. (7.13–7.14). The coefficients $E_j(t)$ can now be computed from the recursion relation Eq. (7.25), and we find $E_j(t) = 0$ for all j. Thus, $D(z,t) = 0$ and $D_k(t) = 0$ for all k, which proves the lemma.

Let us return now to the h.s.s. family of theories of type $\dfrac{SU(n+m)}{SU(n) \times SU(m) \times U(1)}$. In Lemma (6.2) it was proved that if $\gcd(n,m) = 1$ the polynomial generating functions of the theory can be written in the form,

$$P(z,t) = \frac{\sum_{j=0}^{J} E_j(t)}{\prod_{d=0}^{M} (1 - zt^d)}, \tag{7.27}$$

where $M = nm$ and $J = nm - m + 1$. This is precisely the same form that we obtained here for polynomials of the type Eq. (7.1) and in particular for the dual polynomials of Sect. 5, Eq. (5.24). Furthermore, since in both cases the generating function obeys the bi-duality property, the coefficients E_j obey properties A and B, Eqs. (7.13–7.14).

So the two polynomials would be equal, if and only if, their $E_j(t)$'s agree, for $0 \leqq j \leqq \frac{1}{2}(M - J + 1)$.

In addition, according to Lemma (7.1), if we can find enough polynomials of the type Eq. (1) then we might be able to express the generating function in terms of these.

To see how this works, let us first discuss an example. For the theory of $\dfrac{SU(5)}{SU(3) \times SU(2) \times U(1)}$ we found that the generating function is given by Eq. (6.48),

$$P(z,t)=\frac{E_0^p+zE_1^p+E_2^pz^2}{\prod_{i=0}^{6}(1-zt^i)},\tag{7.28}$$

where $E_0=2$, $E_1=-1-t-t^5-t^6$ and $E_2=2t^6$.

The dihedral polynomial $P_k(t)$, Eq. (5.24), is in this case,

$$Q_k(t)=\prod_{i=1}^{3}\prod_{j=1}^{2}\frac{(1-t^{k+i+j-1})}{(1-t^{i+j-1})}.\tag{7.29}$$

According to Theorem (7.2) it is given in the same form Eq. (7.28) with the coefficients $E_0^Q=1$, $E_2^Q=t^6$. E_1^Q can be easily computed from the recursion relation Eq. (7.25), and we find, $E_1^Q=Q_1-R_1^7=t^2+t^3+t^4$. Now consider the polynomial,

$$L_k(t)=\frac{(1-t^{k-1})(1-t^{k+6})}{(1-t^{-1})(1-t^6)}\times\prod_{j=1}^{4}\frac{(1-t^{k+j})}{(1-t^j)}.\tag{7.30}$$

Again it is of the form Eq. (7.1), with the same g and M, and $L_j(t)=0$ for $-1\geqq j\geqq -4$, since $1,2,3,4$ are part of the exponent set. So again $L(z,t)$ is of the form Eq. (7.1) and the coefficients can be similarly computed from the recursion relation Eq. (7.25). We find that $E_0^L=1$ and

$$E_1^L=L_1(t)-R_1^7(t)=-R_1^7(t)=-\frac{(1-t^7)}{(1-t)}.\tag{7.31}$$

It follows that $D_k(t)=P_k(t)-Q_k(t)-L_k(t)$ will have both E_0 and E_1 equal to zero! Thus we have proved that for all k, $D_k(t)=0$,

$$P_k(k)=\prod_{i=1}^{3}\prod_{j=1}^{2}\frac{(1-t^{k+i+j-1})}{(1-t^{i+j-1})}+\frac{(1-t^{k-1})(1-t^{k+6})}{(1-t^{-1})(1-t^6)}\times\prod_{j=1}^{4}\frac{(1-t^{k+j})}{(1-t^j)}.\tag{7.32}$$

This is our first example of a "resolution series." Note that the polynomial $L_k(t)$ is of an SFT type, with the same central charge, **but**, with some of the degrees being negative.

Now, how do we go about generalizing Eq. (7.32), to all n and m? Differently put, from the close similarity in structure that we have found, it is natural to suspect that the Poincaré polynomial of the theories is always given by an expression of the form

$$P(z,t)=\sum_{w\in W_1}\prod_{i=0}^{n+m}(1-zt^{a_w^i})^{-1}=\sum_{k=0}^{\infty}z^kP_k(t),\tag{7.33}$$

where

$$P_k(t)=\sum_{s=0}^{S}\prod_{i=1}^{mn}\frac{(1-t^{k+m_i^s})}{(1-t^{m_i^s})},\tag{7.34}$$

and our main problem is to figure out the set of exponents m_i^s. This we do by invoking the dihedrality of exchanging n,m and k. We know that $P_k(t)$ is dihedral. We thus assume that each of the polynomials appearing in Eq. (7.34) is dihedral as well, or that this equation is dihedral term-by-term. Then we can use the powerful results of Sect. (5), where it has shown that dihedrality alone can almost fix the allowed exponents. There we found that a set of polynomials of the type Eq. (7.34) would be dihedral if and only if for all n and m, for which $P_q(t)$ is defined, the degrees m_i are given by

$$q_s(x)=\sum_i x^{m_i^s}=\frac{x(1-x^m)(1-x^n)}{(1-x^2)}Q_s(x), \tag{7.35}$$

where $Q_s(x)$ is a fixed polynomial which obeys $Q_s(x)=Q_s(1/x)$, ensuring that $\{m_i^s\}=\{g-m_i^s\}$. Further, we want the number of exponents to be equal to nm implying that $Q_s(1)=1$. Now, the sum of the exponents may be computed by

$$\sum_i m_i^s=q_s'(1). \tag{7.36}$$

Since $Q_s(x)=Q_s(1/x)$ it follows that $Q_s'(1)=0$ and

$$\sum_i m_i^s=\tfrac{1}{2}nm(n+m). \tag{7.37}$$

Finally, we want $q_s(x)$ to have as coefficients only non-negative integers and the constant term to be equal to zero, since otherwise we will get a contradiction in Eq. (7.35).

All these conditions severely restrict the possible polynomials $Q_s(x)$. We can calculate these simply by studying examples. Assuming that the polynomial $L_k(t)$ is dihedral we can immediately generalize it to all n and m, by substituting $n=3$ and $m=2$ in Eq. (7.35) and calculating $Q_1(x)$. We find

$$Q_1(x)=\frac{(1+x^3)^2}{x^2(1+x)^2}, \tag{7.38}$$

and thus obtain a first "deficiency" polynomial for all n and m. Continuing now for a different n and m we find a second deficiency polynomial, which again can be dihedralized, and so forth. The situation turns out to be the simplest for $n=2$ and any m, in which case we find that

$$Q_s(x)=x^{-2s}\frac{(1+x^{2s+1})^2}{(1+x)^2}, \tag{7.39}$$

for any integral s, gives all the exponents.

Thus we have found:

Conjecture. For all n and m such that $\gcd(n,m)=1$, the Poincaré polynomial of the theory $\dfrac{SU(n+m)}{SU(n)\times SU(n)\times U(1)}$ is equal to

$$P_k(t)=\sum_{s=0}^{S}a_sP_q^s(t), \tag{7.40}$$

where $P_q^s(t)$ is

$$P_q^s(t)=\prod_{i=1}^{mn}\frac{(1-t^{m_i^s+k})}{(1-t^{m_i^s})}, \tag{7.41}$$

and S is equal to the number of cosets in $W(G/H)$ mod $\{w_\sigma\}$, which is $\frac{(n+m-1)!}{n!m!}$, a_s are some integral coefficients and the q_s are some degree series, $q_s(x)=\sum_i x^{m_i}$.

For $n=2$ the coefficients $a_s=1$ and the degree series is

$$q_{n,m,s}(x)=\sum_i x^{m_i}=x^{1-2s}\frac{(1-x^n)(1-x^m)}{(1-x^2)}\frac{(1+x^{2s+1})^2}{(1+x)^2}. \tag{7.42}$$

For $n=2$ if n is odd $S=\frac{1}{2}(n-1)$. If n is even and k is odd, using dihedrality we find $k=2S+1$.

For $n,m,k>2$ the situation is more complicated. By dihedrality, the exponents $q_{n,m,s}$ above are part of the exponent set, in this case, but there are a few more. The coefficients a_s are not equal in general to 1. An example will be given below.

We shall prove the following theorem regarding this conjecture.

Theorem (7.3). *Let n and m be some fixed integers. If the conjecture above holds for k such that $-n-m+1\leqq k\leqq\frac{1}{2}(nm+1-n-m)$ then it holds for all k.*

Proof. The polynomials $P_q^s(t)$ are of the form Eq. (7.1) and obey all the assumptions of Theorem (7.1), as we checked above: the number of exponents is fixed at nm, the sum of the exponents is constant by the calculation above and $P_k(t)$ vanishes for $1-g\leqq k\leqq -1$ by the assumption of the theorem.[9] The set of exponents is dual under g, $\{m_i\}=\{g-m_i\}$, since $s(x)=s(1/x)$ (Sect. (5)). Owing to Lemma (6.2) the Poincaré polynomial of the theory $\frac{SU(n+m)}{SU(n)\times SU(m)}$ is also of this form. Thus we can apply Lemma (7.1) and the theorem follows.

The number of chiral fields in the theories $\frac{SU(n+m)}{SU(n)\times SU(m)\times U(1)}$ (we shall assume $\gcd(n,m,k)=1$, so there are no fixed points) is given by (from Eq. (3.13))

$$\frac{|W(G)|}{Z|W(H)|}N_k=\frac{(n+m+k-1)!}{n!m!k!}, \tag{7.43}$$

where N_k is the number of integrable weights of $SU(n+m)$ at level k.

The number of fields in the corresponding polynomial

$$P_k(t)=\prod_{i=1}^{M}\frac{(1-t^{k+m_i})}{(1-t^{m_i})} \tag{7.44}$$

is given by Eq. (5.12),

$$P_k(1)=\prod_{i=1}^{M}\frac{(k+m_i)}{m_i}=D(p), \tag{7.45}$$

where

$$p(x)=(x^k-1)\sum_{i=1}^{M}x^{m_i}. \tag{7.46}$$

[9] It can be verified that $1,2,\ldots,g-1$ are part of the exponent set for all the exponents that we have computed, and it is probably so in general. Hence $P_k(t)=0$ for $0>k\geqq 1-g$

Thus, comparing the dimensions of both sides of Eq. (7.41), we find the remarkable identity,

Corollary (7.2). *Fix some n and m. Let $p_s(x) = (x^k - 1)q_s(x)$. where q_s is as in the conjecture above. Then*

$$\sum_{s=0}^{S} D(p_s) = \frac{(n+m+k-1)!}{n!k!m!}, \tag{7.47}$$

where S is as in the conjecture.

Example. Take the simple case of $n = m = 2$, k odd. Then $q_s(x) = x^{1-2s} + 2x^2 + x^{3+2s}$ and the above corollary translates into the series

$$\sum_{s=0}^{(k-1)/2} \frac{(k+2)^2(k+1-2s)(k+3+2s)}{4(1-2s)(3+2s)} = \frac{(k+1)(k+2)(k+3)}{4}. \tag{7.48}$$

Even this simplest identity is quite hard to prove directly. This can be done by eliminating the common factor of $(k+2)^2/4$, taking a second difference of both sides with respect to k, and using the fact that

$$-4\sum_{s=a}^{b} \frac{1}{(1-2s)(3+2s)} = \sum_{s=a}^{b} \frac{1}{2s-1} - \frac{1}{2s+3} = \frac{1}{2a-1} + \frac{1}{2a+1} - \frac{1}{2b+1} - \frac{1}{2b+3}, \tag{7.49}$$

as all the middle terms cancel.

A more complicated example is the case of $n = 3$ and $m = 4$. Here we have five sets of exponents ($S = 4$) which are listed in Table (1), along with s_s.

The exponents $q_{n,m,s}(x)$ can be observed to have many intriguing properties. First, $q_{n,m,s}$ has only primitive roots of unity as its zeros and poles, and thus it is of the form

$$q_{n,m,s}(x) = \prod_{i=1}^{r} \frac{(1-x^{e^i})}{(1-x^{f_i})}, \tag{7.50}$$

where c_i and f_i are some exponents of the exponents!

Next, it can be seen that $q_{n,m,s}$ is always given by some gradation of the positive roots $\Delta_{G/H}$,

$$q(x) = \sum_{\alpha \in \Delta_{G/H}} x^{\alpha\lambda}, \tag{7.51}$$

for some λ in the roots lattice of G. Further, we always find that $\lambda^2 = \rho^2$.

Table 1. 3×4 exponents

a_s	Exponents
1	$1, 2, 2, 3, 3, 3, 4, 4, 4, 5, 5, 6$
1	$-1, 1, 1, 2, 3, 3, 4, 4, 5, 6, 6, 8$
2	$-2, -1, 1, 2, 2, 3, 4, 5, 5, 6, 8, 9$
2	$-3, -2, -1, 1, 2, 3, 4, 5, 6, 8, 9, 10$
-1	$-5, -2, -1, 1, 2, 3, 4, 5, 6, 8, 9, 12$

Example. For the exponent $q_{n,m,s}$ with $s=1$, Eq. (7.42), the gradation is given by $\lambda=\rho+\theta-\alpha_\sigma$, where α_σ is the deleted root, for any n and m. In the case of $n=2$, all the gradations that appear can be seen to come from the affine Weyl group, $W(\hat{G})$. Namely, $\lambda=w(\rho)$, where w is some element of the affine Weyl group. E.g., for the first deficiency polynomial, $\lambda=\rho+\theta-\alpha_\sigma=w_{\alpha_0}w_{\alpha_\sigma}(\rho)$. The elements w calculated are then seen to be the same for any n and m. Thus, finding a general formula for these would enable to write down the entire resolution series in a closed form, similar to the one given for the first term, Eq. (5.30). Since $W(H)$ permutes $\Delta_{G/H}$, taking w or hw, where $h\in W(H)$ results in the same exponent, through a different gradation. Attempts that have been made to identify the precise elements of $W(\hat{G})\bmod W(H)$ that give rise to the exponents have not been successful, so far. w may be decomposed as $t_\beta w_1$, where $w_1\in W(G/H)$ and t_β is some translation. Then w_1 can be seen to correspond to the cosets of $W(G/H)\bmod\{w_\sigma\}$. However, the required translations remain mysterious.

8. Discussion

In this paper we have described the calculation of the chiral algebras in the framework of rational $N=2$ superconformal field theory. This offers a large testing ground for the study of these algebras, which is central in the connection between $N=2$ string theory and complex geometry. In view of the interrelations among conformal field theories, the classification of such $N=2$ theories, through their geometrical nature, is a long step towards the classification of all rational conformal field theories, whereas in the context we have topological tools that are not available elsewhere.

The emerging picture in the detailed study that we have made here is rather complex, and as yet incomplete. We chose as the first objects to study the Poincaré polynomials of the chiral algebras. As explained in the introduction, these encode the algebraic information, where the degrees of the generators and relations can be read off, in many instances. It turns out that the generic polynomials are not of the complete intersection type, and thus, in particular, the generic theory is not a scalar field theory.

However, one regular feature did arise in all the theories studied, and this is the resolution series. In Sect. (7) we have shown that the Poincaré polynomials of all the theories of type $\dfrac{SU(n)_k}{SU(n)\times SU(m)\times U(1)}$, where $\gcd(n,m)=1$ can be described as a sum of scalar field theory type polynomials with all except for the first one containing negative exponents. This is a highly nontrivial statement on the structure of these algebras.

This is by no means special to this family, but is, in fact, a common feature of all studied $N=2$ superconformal field theories. The reason that we choose to explore this in detail in the context of $SU(n)$ is the many simplifying properties of this class of theories, such as the dihedrality, proved in Sect. (4) and explored in detail in Sect. (5), and the lack of fixed points, which enables us to deduce the polynomials of one theory from another.

Let us discuss an assortment of random examples of the resolution series. The Poincaré polynomial of the h.s.s. theory $\frac{SO(5)_4}{SO(3) \times U(1)}$ is given by

$$P(t) = P_0(t) + \prod_{i=1}^{3} \frac{1 - t^{2k+m_i}}{1 - t^{m_i}}, \tag{8.1}$$

where $P_0(t)$ is the generic polynomial Eq. (5.30) (corresponding to $m_i = 2, 3, 4$), $m_i = -2, 3, 8$ and $k = 4$. Note that these exponents are given by the gradation of $\Delta_{G/H}$ by $\lambda = \rho + \theta - \alpha_\sigma$ (see the discussion at the end of Sect. (7)) and thus this polynomial is precisely the generalization of the first deficiency polynomial to this case. This suggests that once the gradations that give rise to the exponents are identified, this will immediately generalize to, at least, all the h.s.s theories.

Another example is $\frac{SU(3)_2}{U(1)^2}$. Here we find,

$$P(t) = P_0(t) + 2\frac{1 - t^{k+3-m}}{1 - t^m}, \tag{8.2}$$

where $P_0(t)$ is the generic polynomial Eq. (5.30) ($m_i = 1, 1, 2$), $m = -1$ and $k = 2$.

The examples above, as well as most of the discussion in this paper, were for the principal theories, with the left–right symmetric modular invariants. The resolution series can be seen also in the context of other modular invariants. An instance of this is $\frac{SU(3)}{SU(2) \times U(1)}$, where we can take any modular invariant for the $SU(3)$ factor. If we keep the left–right symmetric resolutions for the $SU(2) \times U(1)$ factor, then the chiral fields are given by C_w^Λ, now appearing each $N_{\Lambda,\Lambda}$ times, where N is the $SU(3)$ invariant used in the defining the partition function, Eq. (3.12). Thus we can attach a Poincaré polynomial to each modular invariant of $SU(3)$. Denote by $S_{l_1,l_2} = N_{\Lambda,\Lambda}$ where $\Lambda = l_1\Lambda_1 + l_2\Lambda_2$ and Λ_i are the fundamental weights. Then the polynomial is

$$P_1(t) = \sum_{\substack{l_1,l_2 \geq 0 \\ l_1 + l_2 \leq k}} S_{l_1,l_2} t^{2l_1 + l_2}. \tag{8.3}$$

Every modular invariant can be assumed to contain the field $C_{\max}$ and so the polynomial $P(t)$ is dual, $P(t) = t^{2k}P(1/t)$. Similarly, one can define the polynomial for $\frac{SU(3)}{U(1)^2}$ with analogous properties,

$$P_2(t) = \sum_{\substack{l_1,l_2 \geq 0 \\ l_1 + l_2 \leq k}} S_{l_1,l_2}(t^{2l_1 + 2l_2} + t^{2l_1 + k + 1}), \tag{8.4}$$

and $P(t) = t^{3k+1}P(1/t)$. These two Poincaré dualities give strong constraints on the multiplicities of the left–right symmetric representations which can appear in a modular invariant. This system of equations is enough to determine completely many of the multiplicities. It is clear that a classification of these Poincaré polynomials will be more or less equivalent to the classification of the modular invariants. This can of course be generalized to other Lie algebras.

An instance of this is the exceptional solution at $k=5$ whose diagonal representations are $(0,0),(2,0),(3,0),(2,1),(2,2),(5,0),(3,2)$ along with the conjugate representations of the non-real ones[10]. Substituting into Eq. (8.3) we find

$$P_1(t)=1+t^2+t^3+2t^4+2t^5+2t^6+t^7+t^8+t^{10} \tag{8.5}$$

which is evidently dual. Now, this polynomial is also an example of a resolution series since it can be written as

$$P_1(t)=P_0(t)+\frac{1-t^{k+g-m}}{1-t^m}, \tag{8.6}$$

where $P_0(t)$ is the generic polynomial of the left–right symmetric theory, Eq. (5.30), $k=5$, $g=3$ and $m=-1$. This example serves to demonstrate also the importance of this problem to the classification of rational conformal field theories, in general.

Thus, the resolution series defines for us a *notion of relatives among conformal field theories*. Namely, we can declare two theories to be relatives if they have identical central charges and their Poincaré polynomials are equal only for primitive roots of unity or zero. At a first glance this may seem as a bizarre criteria. However, the algebraic significance of the polynomials, as elucidated in Sect. (1), along with the many examples discussed, convince us that it is a correct one.

This can be seen even in the context of the minimal series, which is the case of $\dfrac{SU(2)_k}{U(1)}$ with any modular invariant. The minimal models have been classified in ref. [1]. As discussed in ref. [17] these theories are related to Arnold's

Table 2. Minimal series polynomials

Type	c	Potential	Polynomial	$P-P_A$
A_k	$\dfrac{3(k-1)}{k+1}$	z^{k+1}	$\dfrac{1-t^k}{1-t}$	0
D_k	$\dfrac{3(k-2)}{k-1}$	$z^{k-1}+y^2z$	$\dfrac{(1-t^{2k-4})(1-t^k)}{(1-t^2)(1-t^{k-2})}$	$-t\dfrac{(1-t^{2k-2})(1-t^{k-4})}{(1-t^2)(1-t^{k-1})}$
E_6	$\dfrac{15}{6}$	z^4+y^3	$\dfrac{(1-t^8)(1-t^9)}{(1-t^3)(1-t^4)}$	$\dfrac{(1-t^2)(1-t^5)(1-t^{12})}{(1-t^{-1})(1-t^4)(1-t^6)}$
E_7	$\dfrac{8}{3}$	y^3+yz^3	$\dfrac{(1-t^{12})(1-t^{14})}{(1-t^4)(1-t^6)}$	$\dfrac{(1-t^3)(1-t^5)(1-t^8)(1-t^{18})}{(1-t^{-1})(1-t^4)(1-t^6)(1-t^9)}$
E_8	$\dfrac{14}{5}$	z^5+y^3	$\dfrac{(1-t^{24})(1-t^{20})}{(1-t^6)(1-t^{10})}$	$\dfrac{(1-t^5)(1-t^9)(1-t^{14})(1-t^{30})}{(1-t^{-1})(1-t^6)(1-t^{10})(1-t^{15})}$

[10] A level by level classification of all $SU(3)$ modular invariants has been pursued in collaboration with G. Harel. It can be checked explicitly that the physical invariants indeed obey the properties discussed. This exceptional invariant has been found in this work

classification of 0-modal singularities. We find two series (A_k and D_k) related to the A and D algebras, and three exceptional solutions, E_6, E_7 and E_8. The singularities are in 1–1 correspondence with the modular invariants of $SU(2)$. In Table (2) we list the potentials and the Poincaré polynomials of each of the theories. The last column corresponds to the difference of the polynomial of each of the D and E theories with the polynomial of the A theory at the same central charge, denoted by $P - P_A$.

Rather strikingly, it turns out that the deficiency polynomials $P - P_A$ have only zero and primitives roots of unity as their zeros. In other words, they are all of the form $tP_q(t)$, where q is some degree series. Thus all the theories are related to the A series in the precise sense defined above.

This brings us to an important, but rather complex, question. Namely, what is the algebraic and geometrical significance of the resolution series. Clearly, the two questions are connected and understanding the algebraic structure would lead also to the correct geometrical interpretation. At this stage we can only speculate on this. One possibility is that the deficiency polynomials correspond to ghost system conformal field theories that are needed to be added to the theory. The fact that these polynomials have the scalar field theory structure, with some of the exponents being negative is very encouraging in this direction, hinting that one may be able to consider the ghost systems as scalar field theories with some of the fields having negative dimensions.

Alternatively, one may be able to think of the deficiency polynomials as the result of imposing extra constraints on the theory. There are clear signs that this is what is taking place. To see it, consider the limit of $k \to \infty$ of the $\dfrac{SU(n+m)_k}{SU(n) \times SU(m) \times U(1)}$ theories. In this limit all the relations in the chiral algebra become null, and the algebra becomes a free polynomial algebra with $n + m - 1$ generators whose grades are $l(w^i_\sigma)$ (see the discussion in Sect. (6)). The generic polynomial $P_0(t)$, Eq. (5.30), also becomes a free polynomial algebra, but the number of generators is, in this case, nm. Thus the two can be equal only if $nm = n + m - 1$, which is solved either for $n = 1$ or $m = 1$. In other cases, the Krull dimensions of the algebras are different, the difference being $nm - n - m + 1$. Thus, we would need precisely this number of additional equations in the naive theory $P_0(t)$ in order to get the actual chiral algebra. The deficiency polynomials express precisely this reduction of the dimensionality through either the addition of ghosts or extra imposed constraints. This picture is also consistent with geometrical considerations. Work on this question is currently in progress and will be reported elsewhere.

It is hoped that the results described in this paper will form a basis for a study of the possibilities in string theory, and the classification of rational and non-rational conformal field theory. The theories studied here are certainly very intricate in structure and offer considerable complexity. Yet, we hope to have demonstrated that the complete understanding of $N = 2$ string theory is not beyond reach.

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The same reasons which permit us to extend our possibilities of action in some cases will condemn us to importance in others.

– R. THOM, *Catastrophe Theory: Its Present State and Future Perspectives* (1974)

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