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Quantum Affine Algebras

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Abstract. We classify the finite-dimensional irreducible representations of the quantum affine algebra $U_q(\hat{sl}_2)$ in terms of highest weights (this result has a straightforward generalization for arbitrary quantum affine algebras). We also give an explicit construction of all such representations by means of an evaluation homomorphism $U_q(\hat{sl}_2) \to U_q(sl_2)$, first introduced by M. Jimbo. This is used to compute the trigonometric R-matrices associated to finite-dimensional representations of $U_q(\hat{sl}_2)$.

1. Introduction

A quantum group is a Hopf algebra $U_q(\underline{a})$, depending on a parameter $q \in \mathbb{C}$, which "tends to" the universal enveloping algebra $U(\underline{a})$ of a Lie algebra $\underline{a}$ as q tends to 1. In this paper, we develop a highest weight theory for the finite-dimensional representations of $U_q(\underline{a})$ when $\underline{a}$ is the affine algebra $\hat{sl}_2$, assuming that q is not a root of unity. We also give a concrete construction of all finite-dimensional irreducible representations of $U_q(\hat{sl}_2)$. Many, but not all, of the results extend without difficulty to the case of $U_q(\hat{\underline{g}})$ with $\underline{g}$ any finite-dimensional complex simple Lie algebra.

As in the case of the quantum groups $U_q(\underline{g})$ [10], where there are 2^l irreducible representations of any given highest weight ($l = \operatorname{rank} \underline{g}$), the finite-dimensional irreducible representations of $U_q(\hat{sl}_2)$ are of 4 types depending on the choice of two signs. One of our main results (Theorem (3.5)) establishes a one-to-one correspondence between (isomorphism classes of) finite-dimensional irreducible representations of $U_q(\hat{sl}_2)$ of each type and polynomials with constant coefficient 1. A similar result was proved by Drinfel'd [4] for Yangians, which are deformations of $U(\underline{g}[t])$ (but in that case there is no question of signs).

In the classical case, the finite-dimensional irreducible representations of $\hat{\underline{g}}$ are constructed as follows [1]. One proves first that the centre of $\hat{\underline{g}}$ acts trivially on all such representations; thus, one is considering representations of the loop algebra

$L(\underline{g}) = \underline{g}[t, t^{-1}]$. For any $a \in \mathbb{C}^\times$ one has the evaluation homomorphism $\mathrm{ev}_a : L(\underline{g}) \to \underline{g}$ obtained by setting $t = a$. The pull-back by ev_a of any representation V of $\underline{g}$ is a representation $V(a)$ of $L(\underline{g})$, and every finite-dimensional irreducible representation of $L(\underline{g})$ is isomorphic to a tensor product of such evaluation representations. Moreover, a tensor product $V_1(a_1) \otimes \cdots \otimes V_r(a_r)$ is irreducible if and only if the V_i are irreducible and the a_i are distinct.

Jimbo [8] defined a quantum evaluation homomorphism $U_q(\hat{sl}_2) \to U_q(sl_2)$, so evaluation representations can still be defined. We prove that every finite-dimensional irreducible representation of $U_q(\hat{sl}_2)$ on which the centre acts trivially is a tensor product of evaluation representations; this accounts for two of the four types, and the remaining two types are obtained by twisting with a certain automorphism of $U_q(\hat{sl}_2)$. The conditions for such a tensor product to be irreducible are more subtle than in the classical case, but can be described combinatorially in terms of the polynomials associated to the factors in the tensor product. An analogous theory for Yangians was presented in [2].

Jimbo also pointed out in [7] that representations of quantum loop algebras lead to trigonometric solutions of the quantum Yang–Baxter equation. In Sect. 4, we compute the solutions associated to all the finite-dimensional irreducible representations of $U_q(\hat{sl}_2)$.

2. Quantum Affine Algebras

Throughout this paper, $q \in \mathbb{C}^\times$ is assumed not to be a root of unity.

2.1. For any integer $r > 0$, define

$$[r]_q = \frac{q^r - q^{-r}}{q - q^{-1}},$$

$$[r]_q! = \prod_{s=1}^{r} [s]_q,$$

and set $[0]_q! = 1$.

2.2. We begin with the definition of a quantum affine algebra in terms of Chevalley generators. We have modified the presentation given in [4] slightly to enable us to specialize q, and to simplify certain formulae.

Definition. *The quantum affine algebra $U_q(\hat{sl}_2)$ is the associative algebra over $\mathbb{C}$ with generators $e_i^\pm, K_i^{\pm 1}$, $i = 0, 1$, and the following relations:*

$$K_i K_i^{-1} = K_i^{-1} K_i = 1,$$

$$K_0 K_1 = K_1 K_0,$$

$$K_i e_i^\pm K_i^{-1} = q^{\pm 2} e_i^\pm,$$

$$K_i e_j^\pm K_i^{-1} = q^{\mp 2} e_j^\pm, \quad i \neq j,$$

$$[e_i^+, e_i^-] = \frac{K_i - K_i^{-1}}{q - q^{-1}},$$
$$[e_0^\pm, e_1^\mp] = 0,$$
$$(e_i^\pm)^3 e_j^\pm - [3]_q (e_i^\pm)^2 e_j^\pm e_i^\pm + [3]_q e_i^\pm e_j^\pm (e_i^\pm)^2 - e_j^\pm (e_i^\pm)^3 = 0, \quad (i \neq j).$$

Moreover, $U_q(\hat{sl}_2)$ is a Hopf algebra over $\mathbb{C}$ with comultiplication

$$\Delta(e_i^+) = e_i^+ \otimes K_i + 1 \otimes e_i^+,$$
$$\Delta(e_i^-) = e_i^- \otimes 1 + K_i^{-1} \otimes e_i^-,$$
$$\Delta(K_i) = K_i \otimes K_i,$$

and antipode

$$S(K_i) = K_i^{-1}, \quad S(e_i^+) = -e_i^+ K_i^{-1}, \quad S(e_i^-) = -K_i e_i^-.$$

Remark. According to Drinfel'd [3], this deformation of $U(\hat{sl}_2)$ is essentially characterized by the existence of a Cartan anti-involution given by:

$$\theta(e_i^\pm) = e_i^\mp, \quad i = 0, 1,$$
$$\theta(K_i) = K_i^{-1}, \quad \theta(q) = q^{-1}.$$

2.3. In [4], Drinfel'd gave a second realization of quantum affine algebras which is more convenient for the concrete construction of representations given in this paper. We recall this definition now; again we have modified it appropriately to enable us to specialize q.

Theorem. *The quantum affine algebra $U_q(\hat{sl}_2)$ is isomorphic to the associative algebra over $\mathbb{C}$ with generators $x_k^\pm$ $(k \in Z), h_k$ $(k \in Z - \{0\})$, $K^{\pm 1}$, central elements $C^{\pm 1}$ and the following relations:*

$$CC^{-1} = C^{-1}C = KK^{-1} = K^{-1}K = 1,$$
$$[h_k, h_l] = \delta_{k,-l} \frac{1}{k}[2k]_q \frac{C^k - C^{-k}}{q - q^{-1}},$$
$$Kh_k = h_k K,$$
$$Kx_k^\pm K^{-1} = q^{\pm 2} x_k^\pm,$$
$$[h_k, x_l^\pm] = \pm \frac{1}{k}[2k]_q C^{\mp (1/2)(k + |k|)} x_{k+l}^\pm,$$
$$x_{k+1}^\pm x_l^\pm - q^{\pm 2} x_l^\pm x_{k+1}^\pm = q^{\pm 2} x_k^\pm x_{l+1}^\pm - x_{l+1}^\pm x_k^\pm,$$
$$[x_k^+, x_l^-] = \frac{1}{q - q^{-1}}(C^{k-1}\psi_{k+l} - \phi_{k+l}),$$

where the ψ_k and ϕ_k are defined by the following equalities of formal power series:

$$\sum_{k=0}^{\infty} \psi_k u^k = K \exp\left((q - q^{-1}) \sum_{k=1}^{\infty} h_k u^k \right),$$

$$\sum_{k=0}^{\infty} \phi_{-k} u^{-k} = K^{-1} \exp\left((q - q^{-1}) \sum_{k=1}^{\infty} h_{-k} u^{-k} \right).$$

The isomorphism with the presentation in (2.2) *is given by*:

$$K_0 \mapsto CK^{-1}, \quad K_1 \mapsto K, \quad e_1^{\pm} \mapsto x_0^{\pm},$$
$$e_0^{+} \mapsto x_1^{-} K^{-1}, \quad e_0^{-} \mapsto C^{-1} K x_{-1}^{+}.$$

Remarks.
1. The isomorphism given in [4] is not quite correct. (For example, $\mathscr{X}_0$ and $\mathscr{Y}_1$ commute but their images do not.)
2. No explicit formula for the comultiplication is known in terms of this presentation. Partial information, sufficient for our purposes, is given in Proposition (4.4).

2.4. In the next section we shall make use of several subalgebras of $U_q(\hat{sl}_2)$ isomorphic to $U_q(sl_2)$. The quantum group $U_q(sl_2)$ is generated by elements $e^{\pm}$, $K^{\pm 1}$ with relations

$$KK^{-1} = K^{-1}K = 1,$$
$$Ke^{\pm}K^{-1} = q^{\pm 2} e^{\pm},$$
$$[e^{+}, e^{-}] = \frac{K - K^{-1}}{q - q^{-1}}.$$

The comultiplication is given by

$$\Delta(e^{+}) = e^{+} \otimes K + 1 \otimes e^{+},$$
$$\Delta(e^{-}) = e^{-} \otimes 1 + K^{-1} \otimes e^{-},$$
$$\Delta(K) = K \otimes K.$$

It is clear from the presentations (2.2) and (2.3) that $U_q(sl_2)$ is a subalgebra of $U_q(\hat{sl}_2)$ in many ways. In fact, it is easy to check that, for all $i \in Z$, the map

$$e^{+} \mapsto x_i^{+}, \quad e^{-} \mapsto C^{-i} x_{-i}^{-}, \quad K \mapsto KC^{i}$$

is a homomorphism $U_q(sl_2) \to U_q(\hat{sl}_2)$. (It follows from Proposition (4.1) that the map is injective.) Let U^i be the image of this map; note that for $i = 0, 1$, U^i is the "diagram" subalgebra of $U_q(\hat{sl}_2)$ generated by $e_{1-i}^{\pm}$ and $K_{1-i}^{\pm 1}$.

3. Finite-Dimensional Representations

In this section we state one of the main results of this paper, which gives a parametrization of the finite-dimensional irreducible representations of $U_q(\hat{sl}_2)$ in terms of polynomials with constant coefficient 1. The proof is given in this section and the next.

3.1. We begin with the following analogue of the easy half of the Poincaré–Birkhoff–Witt theorem.

Definition. *Let H (respectively $N_{\pm}$) be the subalgebras of $U_q(\hat{sl}_2)$ generated by $C^{\pm 1}, K^{\pm 1}$ and h_k for $k \neq 0$ (respectively by $x_k^{\pm}$ for all $k \in Z$).*

Remark. It is easy to see from the presentation (2.3) that H is also generated by $\{C^{\pm 1}, \psi_k, \phi_{-k} \text{ for } k \geqq 0\}$.

Proposition. *We have* $U_q(\hat{sl}_2) = N_- H N_+$.

The proof is almost the same as that for Lie algebras given in [5] (cf. [2], Proposition (1.11)).

3.2. This motivates the following definition.

Definition. *A vector Ω in a representation V of $U_q(\hat{sl}_2)$ is a highest weight vector if Ω is annihilated by x_k^+ for all $k \in Z$ and is an eigenvector of every element of H. The representation V is a highest weight representation if it is generated by a highest weight vector.*

Proposition. *Every finite-dimensional irreducible representation of $U_q(\hat{sl}_2)$ is highest weight.*

Before giving the proof of this proposition, we recall from [10] that there are exactly two irreducible representations $V_{n,\varepsilon}$, $\varepsilon = \pm 1$, of $U_q(sl_2)$ of each dimension $n+1 \geqq 1$. In fact, $V_{n,\varepsilon}$ has a basis $\{v_0, v_1, \ldots, v_n\}$ and the action is given by

$$K \cdot v_i = \varepsilon q^{n-2i} v_i, \quad e^+ \cdot v_i = \varepsilon [n-i+1]_q v_{i-1}, \quad e^- \cdot v_i = [i+1]_q v_{i+1}.$$

One can obtain $V_{n,-1}$ from $V_{n,1}$ by twisting with the automorphism of $U_q(sl_2)$ given by

$$K \mapsto -K, \quad e^+ \mapsto -e^+, \quad e^- \mapsto e^-.$$

Proof of Proposition (3.2). Let V be a finite-dimensional irreducible representation of $U_q(\hat{sl}_2)$. Assume for a contradiction that V contains no non-zero vectors annihilated by x_k^+ for all $k \in Z$. Let $0 \neq v \in V$ be any eigenvector for the action of K, say

$$K \cdot v = \lambda v, \quad \lambda \in \mathbb{C}^\times.$$

By the assumption, there is an infinite sequence of integers $k_1, k_2, k_3, \ldots$ such that the vectors $v, x_{k_1}^+ \cdot v, x_{k_2}^+ x_{k_1}^+ \cdot v, x_{k_3}^+ x_{k_2}^+ x_{k_1}^+ \cdot v, \ldots$ are all non-zero. Since they are eigenvectors of K with distinct eigenvalues $\lambda, q^2\lambda, q^4\lambda, q^6\lambda, \ldots$, they are linearly independent. This contradicts the finite-dimensionality of V.

Hence, the subspace $V_0 = \{v \in V : x_k^+ \cdot v = 0 \text{ for all } k \in Z\}$ is non-zero, and is easily seen, using the relations in (2.3), to be preserved by the action of the commuting operators K_0, K_1. Let $\Omega \in V_0$ be a simultaneous eigenvector of K_0, K_1. By considering the action of the two "diagram" subalgebras U^0, U^1 of $U_q(\hat{sl}_2)$, and using the preceding remarks, it follows that

$$K_i \cdot \Omega = \varepsilon_i q^{n_i} \Omega, \quad i = 0, 1 \tag{$*$}$$

for some $\varepsilon_i = \pm 1$ and some integers $n_0 \leqq 0, n_1 \geqq 0$. This implies that

$$K C^i \cdot \Omega = \varepsilon_1 (\varepsilon_0 \varepsilon_1)^i q^{n_1 + i(n_0 + n_1)} \Omega \tag{$**$}$$

for all i. By applying the preceding remarks to the action of U^i, it follows that the exponent of q on the right-hand side of $(**)$ must be non-negative for all i. Hence, $n_0 + n_1 = 0$ and

$$C \cdot \Omega = \varepsilon_0 \varepsilon_1 \Omega. \tag{$***$}$$

It now follows from the relations (2.3) that H acts on V by a commuting family of operators. Since H clearly preserves the subspace V_0, we may therefore assume that Ω is a simultaneous eigenvector of every element of H. Then Ω is a highest weight vector for the action of $U_q(\hat{sl}_2)$ on V, and Ω generates V because V is irreducible.

From equation (***) in the preceding proof, we obtain

Corollary. *Let V be a finite-dimensional irreducible representation of $U_q(\hat{sl}_2)$ of type $(1, 1)$. Then, C acts as 1 on V.*

3.3. Let $U_q(L(sl_2))$ denote the quotient of $U_q(\hat{sl}_2)$ by the two-sided ideal generated by the central element C. Note that $U_q(L(sl_2))$ is a Hopf algebra which is a deformation of the universal enveloping algebra of the loop algebra $L(sl_2) = sl_2[t, t^{-1}]$.

The following result, together with Corollary (3.2), shows that, as far as finite-dimensional representations are concerned, it is enough to consider representations of $U_q(L(sl_2))$.

Proposition. *For any $\varepsilon_0, \varepsilon_1 = \pm 1$, there is an algebra automorphism of $U_q(\hat{sl}_2)$ such that*

$$K_i \mapsto \varepsilon_i K_i, \quad e_i^+ \mapsto \varepsilon_i e_i^+, \quad e_i^- \mapsto e_i^-.$$

This is easily verified using the presentation (2.2). In terms of the other presentation (2.3), one can check that the automorphism is given by

$$K \mapsto \varepsilon_1 K, \quad C \mapsto \varepsilon_0 \varepsilon_1 C,$$

$$x_k^+ \mapsto \varepsilon_1 (\varepsilon_0 \varepsilon_1)^k x_k^+, \quad x_k^- \mapsto (\varepsilon_0 \varepsilon_1)^k x_k^-,$$

$$\psi_k \mapsto \varepsilon_1 \psi_k, \quad \phi_{-k} \mapsto \varepsilon_1 (\varepsilon_0 \varepsilon_1)^k \phi_{-k},$$

$$h_k \mapsto \begin{cases} h_k & \text{if } k > 0, \\ (\varepsilon_0 \varepsilon_1)^k h_k & \text{if } k < 0. \end{cases}$$

Except for the trivial case $\varepsilon_0 = \varepsilon_1 = 1$, these are *not* Hopf algebra automorphisms.

3.4. A representation V of $U_q(L(sl_2))$ is highest weight if it is generated by a vector Ω which is annihilated by the x_k^+ for all $k \in Z$ and such that

$$\psi_k \cdot \Omega = d_k^+ \Omega, \quad \phi_k \cdot \Omega = d_k^- \Omega$$

for some complex numbers d_k^+ $(k \geqq 0)$, d_k^- $(k \leqq 0)$; note that $d_0^+ d_0^- = 1$. The collection of numbers $\underline{d} = \{d_k^\pm\}$ is called the highest weight of V.

As in the case of semisimple Lie algebras, there is a universal highest weight representation $M(\underline{d})$ of $U_q(L(sl_2))$ of any given highest weight $\underline{d}$, which may be defined as the quotient of $U_q(L(sl_2))$ by the left ideal generated by $\{x_k^+$ $(k \in Z), \psi_k - d_k^+ \cdot 1$ $(k \geqq 0)$, $\phi_k - d_k^- \cdot 1$ $(k \leqq 0)\}$. Moreover, every representation of highest weight $\underline{d}$ is a quotient of $M(\underline{d})$, and $M(\underline{d})$ has a unique irreducible quotient $V(\underline{d})$.

One of our main results is the following theorem, which gives the precise condition for $V(\underline{d})$ to be finite-dimensional.

Theorem. *The irreducible highest weight representation $V(\underline{d})$ is finite-dimensional if and only if there exists a polynomial P with non-zero constant term such that*

$$\sum_{k=0}^{\infty} d_k^+ u^k = q^{\deg P} \frac{P(q^{-2}u)}{P(u)},$$

$$\sum_{k=0}^{\infty} d_{-k}^- u^{-k} = q^{\deg P} \frac{P(q^{-2}u)}{P(u)},$$

in the sense that the left-hand sides of these equations are the Laurent expansions of the right-hand sides about $u=0$ and $u=\infty$ respectively.

Remarks.
1. The polynomial P associated to any finite-dimensional representation of $U_q(L(sl_2))$ in Theorem (3.4) is unique if (for example) we normalize it so that the constant coefficient is equal to 1. We shall assume that this is done from now on.
2. A similar result holds for an arbitrary quantum loop algebra $U_q(L(\underline{g}))$, with $\underline{g}$ a finite-dimensional simple Lie algebra. To every finite-dimensional irreducible representation of $U_q(L(\underline{g}))$ is associated an l-tuple of polynomials, where $l = \operatorname{rank} \underline{g}$.

3.5. The following result is the crucial step in the proof of the "only if" part of Theorem (3.4).

For any $\xi \in U_q(L(sl_2))$, we introduce the elements

$$\xi^{(r)} = \frac{\xi^r}{[r]_q!}.$$

Proposition. *There exist elements $P_r, Q_r \in H$, $r \geqq 0$, such that:*
(i)$_r$

$$P_r \equiv (-1)^r q^{r^2} x_0^{+(r)} x_1^{-(r)} K^{-r} \pmod{UX_+},$$

$$Q_r \equiv (-1)^r q^{-r^2} x_{-1}^{+(r)} x_0^{-(r)} K^{r} \pmod{UX_+},$$

for $r>0$, and $P_0 = Q_0 = 1$;
(ii)$_r$

$$P_r = \frac{-q^r}{(q-q^{-1})[r]_q} \sum_{j=0}^{r-1} \psi_{j+1} P_{r-j-1} K^{-1},$$

$$Q_r = \frac{q^{-r}}{(q-q^{-1})[r]_q} \sum_{j=0}^{r-1} \phi_{-j-1} Q_{r-j-1} K,$$

for $r>0$;
(iii)$_r$

$$(-1)^r q^{r(r-1)} x_0^{+(r-1)} x_1^{-(r)} \equiv -\sum_{j=0}^{r-1} x_{j+1}^- P_{r-j-1} K^{r-1} \pmod{UX_+},$$

$$(-1)^r q^{-r(r-1)} x_{-1}^{+(r-1)} x_0^{-(r)} \equiv -\sum_{j=0}^{r-1} x_{-j}^- Q_{r-j-1} K^{-r+1} \pmod{UX_+},$$

for $r>0$.

Remark. In the classical limit $q \to 1$, the formulae in Proposition (3.5) appear in [1] (see Eq. (4.5) in [1], for example). In the classical case, the P_r are interpreted as the coefficients of a certain polynomial and the classical limits of the $(q-q^{-1})^{-1}\psi_r$ as the sum of the r^{th} powers of its roots. Thus, part (ii) may be interpreted as a q-analogue of Newton's formulae relating the elementary symmetric functions and the power sums.

Part (i) of the proposition can be reformulated as follows. Define

$$\mathscr{P}(u)=\sum_{r=0}^{\infty} p_r u^r, \quad \mathscr{Q}(u)=\sum_{r=0}^{\infty} Q_r u^{-r}.$$

Corollary. *We have*

$$\Psi(u)=K\frac{\mathscr{P}(q^{-2}u)}{\mathscr{P}(u)},$$

$$\Phi(u)=K^{-1}\frac{\mathscr{Q}(q^2v)}{\mathscr{Q}(u},$$

as elements of $H[[u]]$ and $H[[u^{-1}]]$ respectively.

It is now easy to prove the "only if" part of the theorem. Assume that $\dim V(\underline{d})<\infty$ and let $r\in Z_+$ be the highest weight of $V(\underline{d})$ for the action of the $U_q(sl_2)$-subalgebra U^0 of $U_q(L(sl_2))$, i.e. for the highest weight vector Ω of $V(\underline{d})$, we have

$$K\cdot\Omega=q^r\Omega.$$

From [10], it follows that the U^0-subrepresentation of $V(\underline{d})$ generated by Ω is the $(r+1)$-dimensional irreducible representation of U^0 and, in particular, that

$$(x_0^-)^{r+1}\cdot\Omega=0.$$

From Proposition (3.5) (i), it follows that

$$\mathscr{P}(u)\cdot\Omega=P(u)\Omega$$

for some polynomial

$$P(u)=\sum_{i=0}^{r}\pi_i u^i$$

of degree r. The first equation in the statement of Theorem (3.4) now follows from Corollary (3.5).

To prove the second equation in Theorem (3.4), apply $x^+_{-n-1}, n\geqq 0$, to both sides of the first congruence in Proposition (3.5) $(\mathrm{iii})_{r+1}$. Considering the action on Ω gives

$$\sum_{k=0}^{n} d^-_{-k}\pi_{r-n+k}=\sum_{k=0}^{r-n} d^+_k\pi_{r-n-k}$$

for $0\leqq n\leqq r$, and

$$\sum_{k=n-r}^{n} d^-_{-k}\pi_{r-n+k}=0$$

for $n>r$. Note that, by Proposition (3.7) $(\mathrm{ii})_{r-n}$, the right-hand side of the first equation is equal to

$$q^r q^{-2(r-n)}\pi_{r-n}.$$

Multiplying the n^{th} equation by u^{r-n} and summing from $n=0$ to ∞ then gives

$$\left(\sum_{k=0}^{\infty} d_{-k}^{-} u^{-k}\right) P(u) = q^r P(q^{-2}u)$$

as required.

This completes the proof of the "only if" part of Theorem (3.4). The proof of the converse will be given in Sect. 4.

Remark. By considering the action of the subalgebra U^1, one can prove similarly that

$$\mathcal{Q}(u)\cdot\Omega = Q(u^{-1})$$

for some polynomial Q of degree r, and that

$$\sum_{k=0}^{\infty} d_{-k}^{-} u^{-k} = q^{-r} \frac{Q(q^2 u^{-1})}{Q(u^{-1})}.$$

It follows that $Q(u) = u^r P(u^{-1})$.

We now turn to the proof of Proposition (3.5). We shall only prove the formulae involving the P_r; the Q_r case is similar. We define the P_r inductively using (ii)$_r$ and $P_0 = 1$. Then the first formula in (i)$_r$ follows immediately from that in (iii)$_r$ by multiplying on the left by x_0^+. We prove the first formula in (iii)$_r$ by induction on r, the case $r=1$ being trivial.

For the inductive step, the crucial result is the following formula:

(iv)$_r$

$$x_0^{+(r)} x_1^{-(r+1)} \equiv q^{-r} x_1^- x_0^{+(r)} x_1^{-(r)} + \frac{q^{-2r}}{q+q^{-1}} [h_1, x_0^{+(r-1)} x_1^{-(r)}] K \pmod{UX_+}.$$

Assuming this for a moment, we obtain, using (iii)$_r$ and (i)$_r$,

$$x_0^{+(r)} x_1^{-(r+1)} \equiv q^{-r} x_1^- (-1)^r q^{-r^2} K^r P_r + \frac{q^{-2r}}{q+q^{-1}} q^{-r(r-1)} \sum_{j=0}^{r-1} -[2]_q x_{j+2}^- P_{r-j-1} K^r \pmod{UX_+},$$

which gives (iii)$_{r+1}$ after some simplification.

Turning now to (iv)$_r$, we begin by proving

(v)$_r$

$$x_0^{+(r+1)} x_1^- = x_1^- x_0^{+(r+1)} + q^{-r} K x_0^{+(r)} h_1 + q^{-2r+2} K x_1^+ x_0^{+(r-1)}.$$

In fact,

$$\begin{aligned}
[x_0^{+(r+1)}, x_1^-] &= \frac{1}{[r+1]_q!} \sum_{j=0}^{r} (x_0^+)^j [x_0^+, x_1^-] (x_0^+)^{r-j} \\
&= \frac{1}{[r+1]_q!} \sum_{j=0}^{r} (x_0^+)^j K h_1 (x_0^+)^{r-j} \\
&= \frac{1}{[r+1]_q!} \sum_{j=0}^{r} q^{-2j} K (x_0^+)^j h_1 (x_0^+)^{r-j}.
\end{aligned}$$

Now

$$[h_1,(x_0^+)^r]=\sum_{i=0}^{r-1}(x_0^+)^i[h_1,x_0^+](x_0^+)^{r-i-1}$$
$$=[2]_q\sum_{i=0}^{r-1}(x_0^+)^i x_1^+(x_0^+)^{r-i-1}$$
$$=[2]_q\sum_{i=0}^{r-1}q^{-2i}x_1^+(x_0^+)^{r-1},$$

using $x_1^+x_0^+=q^2x_0^+x_1^+$. Thus,

(vi)$_r$

$$[h_1,(x_0^+)^r]=q^{-r+1}[2]_q[r]_q x_1^+(x_0^+)^{r-1}.$$

Hence,

$$[x_0^{+(r+1)},x_1^-]=\frac{1}{[r+1]_q!}\left(\sum_{j=0}^{r}q^{-2j}\right)K(x_0^+)^r h_1$$
$$+\frac{1}{[r+1]_q!}\sum_{j=0}^{r-1}q^{-2j}[2]_q q^{-r+j+1}[r-j]_q K(x_0^+)^j x_1^+(x_0^+)^{r-j-1}$$
$$=q^{-r}Kx_0^{+(r)}h_1+q^{-2r+2}Kx_1^+x_0^{+(r-1)}$$

after summing the geometric series. This proves (v)$_r$.

A similar argument using

(vi)$'_r$

$$[h_1,(x_1^-)^r]=-[2]_q[r]_q q^{-r+1}(x_1^-)^{r-1}x_2^-$$

proves

(v)$'_r$

$$x_0^+x_1^{-(r+1)}=x_1^{-(r+1)}x_0^+ +q^rKx_1^{-(r)}h_1-Kx_1^{-(r-1)}x_2^-.$$

Using (v)$_{r-1}$, we have

$$[r+1]_q x_0^{+(r)}x_1^{-(r+1)}=x_1^-x_0^{+(r)}x_1^{-(r)}+q^{-r+1}Kx_0^{+(r-1)}h_1x_1^{-(r)}+q^{-2r+4}Kx_1^+x_0^{+(r-2)}x_1^{-(r)}$$
$$=x_1^-x_0^{+(r)}x_1^{-(r)}+q^{-r+1}Kx_0^{+(r-1)}h_1x_1^{-(r)}$$
$$+\frac{q^{-r+2}}{[2]_q}K[h_1,x_0^{+(r-1)}]x_1^{-(r)}\quad(\text{using (vi)}_{r-1})$$
$$=x_1^-x_0^{+(r)}x_1^{-(r)}+q^{-r+1}Kx_0^{+(r-1)}x_1^{-(r)}h_1$$
$$+q^{-r+1}Kx_0^{+(r-1)}[h_1,x_1^{-(r)}]+\frac{q^{-r+2}}{[2]_q}K[h_1,x_0^{+(r-1)}]x_1^{-(r)}$$
$$\equiv x_1^-x_0^{+(r)}x_1^{-(r)}+q^{-r+1}Kx_0^{+(r-1)}q^{-r}K^{-1}$$
$$\cdot\{x_0^+x_1^{-(r+1)}+Kx_1^{-(r-1)}x_2^-\}+q^{-r+1}Kx_0^{+(r-1)}[h_1,x_1^{-(r)}]$$
$$+\frac{q^{-r+2}}{[2]_q}K[h_1,x_0^{+(r-1)}]x_1^{-(r)}\quad(\operatorname{mod} UX_+)$$

using (v)$'_r$. Hence,

$$([r+1]_q - q^{-1}[r]_q)x_0^{+(r)}x_1^{-(r+1)}$$

$$\equiv x_1^- x_0^{+(r)}x_1^{-(r)} + q^{-2r+1}Kx_0^{+(r-1)}x_1^{-(r+1)}x_2^- + q^{-r+1}Kx_0^{+(r-1)}[h_1, x_1^{-(r)}]$$

$$+\frac{q^{-r+2}}{[2]_q}K[h_1, x_0^{+(r-1)}]x_1^{-(r)} \pmod{UX_+}$$

$$\equiv x_1^- x_0^{+(r)}x_1^{-(r)} + q^{-2r+1}Kx_0^{+(r-1)}\left\{-\frac{[h_1, x_1^{-(r)}]}{[2]_q}q^{r-1}\right\} + q^{-r+1}Kx_0^{+(r-1)}[h_1, x_1^{-(r)}]$$

$$+\frac{q^{-r+2}}{[2]_q}K[h_1, x_0^{+(r-1)}]x_1^{-(r)} \pmod{UX_+}$$

using (vi)$'_r$. This simplifies to

$$q^r x_0^{+(r)}x_1^{-(r+1)} \equiv x_1^- x_0^{+(r)}x_1^{-(r)} + \frac{q^{-r+2}}{[2]_q}K[h_1, x_0^{+(r-1)}x_1^{-(r)}] \pmod{UX_+}$$

$$\equiv x_1^- x_0^{+(r)}x_1^{-(r)} + \frac{q^{-r}}{q+q^{-1}}[h_1, x_0^{+(r-1)}x_1^{-(r)}]K \pmod{UX_+}$$

and (iv)$_r$ is proved.

4. Evaluation Representations

4.1. In this section we define a family of type (1, 1) representations of $U_q(\hat{sl}_2)$. Their existence depends on the following result.

Proposition. *For any* $a \in \mathbb{C}^\times$, *there is a homomorphism of algebras* $\mathrm{ev}_a: U_q(\hat{sl}_2') \to U_q(sl_2)$ *such that*

$$\mathrm{ev}_a(x_k^+) = q^{-k}a^k K^k e^+,$$
$$\mathrm{ev}_a(x_k^-) = q^{-k}a^k e^- K^k,$$

for all $k \in Z$.

Proof. We first construct ev_a in terms of the Chevalley generators. Set

$$\mathrm{ev}_a(e_0^\pm) = q^{\mp 1}a^{\pm 1}e^\mp, \quad \mathrm{ev}_a(e_1^\pm) = e^\pm, \quad \mathrm{ev}_a(K_0) = K^{-1}, \quad \mathrm{ev}_a(K_1) = K.$$

To show that this defines a homomorphism $U_q(\hat{sl}_2) \to U_q(sl_2)$, one must check that the relations in (2.2) are satisfied. This is immediate except for the four quartic relations, which reduce to

$$(e^+)^3 e^- - e^-(e^+)^3 = [3]_q e^+[e^+, e^-]e^+$$

and a similar relation with + and − interchanged. This is easily verified using the defining relations of $U_q(sl_2)$. Using the isomorphism in (2.3), we find

$$\mathrm{ev}_a(C) = 1,$$
$$\mathrm{ev}_a(K) = K,$$

$$\mathrm{ev}_a(x_0^{\pm}) = e^{\pm},$$
$$\mathrm{ev}_a(x_{-1}^{+}) = qa^{-1}K^{-1}e^{+},$$
$$\mathrm{ev}_a(x_1^{-}) = q^{-1}ae^{-}K.$$

We find from these equations and the relation

$$[x_0^{+}, x_1^{-}] = h_1 K$$

that

$$\mathrm{ev}_a(h_1) = \frac{q^{-1}a(K-K^{-1})}{q-q^{-1}} - a(q-q^{-1})e^{-}e^{+}.$$

The formulae in the statement of the proposition can now be proved for $k \geqq 0$ by induction, using

$$[h_1, x_k^{\pm}] = \pm(q+q^{-1})x_{k+1}^{\pm}.$$

The proof for $k<0$ is similar.

Remark. The classical limit of ev_a is the homomorphism $L(sl_2) = sl_2[t,t^{-1}] \to sl_2$ is obtained by setting $t=a$. We refer to ev_a as an evaluation homomorphism.

4.2. Representations of $U_q(\hat{sl}_2)$ can thus be obtained by pulling back representations of $U_q(sl_2)$ by the homomorphisms ev_a. It is clear that the representations $V_{n,1}$ of $U_q(sl_2)$ lead to representations of $U_q(\hat{sl}_2)$ of type $(1,1)$, but that the $V_{n,-1}$ do not; we denote $V_{n,1}$ simply by V_n from now on.

Definition. *For any integer $n \geqq 0$ and any $a \in \mathbb{C}^{\times}$, the evaluation representation $V_n(a)$ of $U_q(\hat{sl}_2)$ is the pull-back of the representation V_n of $U_q(sl_2)$ by the evaluation homomorphism $\mathrm{ev}_a: U_q(\hat{sl}_2) \to U_q(sl_2)$.*

Remark. The representation $V_0(a)$ is trivial for all $a \in \mathbb{C}^{\times}$ but we shall see that for $n \geqq 1$, the representations $V_n(a)$ are all distinct. Moreover, the $V_n(a)$ are irreducible since ev_a is surjective. Since $\mathrm{ev}_a(C) = 1$, the $V_n(a)$ may be regarded as representations of $U_q(L(sl_2))$.

Proposition. *The action of the elements x_k^{+} on $V_n(a)$ is given by:*

$$x_k^{+} \cdot v_i = a^k q^{k(n-2i+1)}[n-i+1]_q v_{i-1},$$
$$x_k^{-} \cdot v_i = a^k q^{k(n-2i-1)}[i+1]_q v_{i+1}.$$

In particular, $V_n(a)$ is a highest weight representation with highest weight vector v_0.

Proof. This follows immediately from Proposition (4.1) and the formulae preceding the proof of Proposition (3.2).

Corollary. *The polynomial P associated to $V_n(a)$ is given by*

$$P(u) = (1-q^{n-1}au)(1-q^{n-3}au)\cdots(1-q^{-n+1}au).$$

Proof. We must compute the eigenvalues d_k^{+} of ψ_k on the highest weight vector v_0. Now, for $k>0$,

$$(q-q^{-1})^{-1}\psi_k \cdot v_0 = [x_k^{+}, x_0^{-}] \cdot v_0 = (aq^{n-1})^k [n]_q v_0,$$

while $\psi_0 \cdot v_0 = K \cdot v_0 = q^n v_0$.

Hence,

$$\sum_{k=0}^{\infty} d_k^+ u^k = q^n + (q^n - q^{-n})\frac{aq^{n-1}u}{1 - aq^{n-1}u}$$
$$= q^n \cdot \left(\frac{1 - q^{-n-1}au}{1 - q^{n-1}au}\right).$$

We must therefore check that

$$\frac{P(q^{-2}u)}{P(u)} = \frac{1 - q^{-n-1}au}{1 - q^{n-1}au},$$

which is clear.

4.3. Before studying the evaluation representations further, we pause to complete the proof of Theorem (3.4). For this, we need the following multiplicative property of the polynomials associated to the finite-dimensional irreducible representations by the "only if" part of Theorem (3.4) which we have already proved.

Proposition. *Let V and W be finite-dimensional representations of $U_q(L(sl_2))$ and assume that the tensor product $V \otimes W$ is irreducible. Let P_V, P_W and $P_{V\otimes W}$ be the polynomials* (*with constant coefficient 1*) *associated to V, W and $V \otimes W$ in Theorem* (*3.4*). *Then,*

$$P_{V\otimes W} = P_V P_W.$$

Corollary. *Let V and W be finite-dimensional representations of $U_q(L(sl_2))$. If $V \otimes W$ is irreducible, then it is isomorphic to $W \otimes V$.*

Remark. There are simple examples of finite-dimensional representations V and W of $U_q(L(sl_2))$ for which $V \otimes W$ is not isomorphic to $W \otimes V$ (see the remark at the end of Subsect. (4.8)).

4.4. The proof of Proposition (4.3) depends on the following partial description of the comultiplication of $U_q(L(sl_2))$ in terms of the presentation in (2.3). Let $X_\pm$ denote the subspaces of $U = U_q(L(sl_2))$ spanned by the $x_k^\pm$ ($k \in Z$).

Proposition. *The comultiplication Δ of U satisfies:*

(i) *modulo* $UX_+^2 \otimes UX_-$,

$$\Delta(x_k^+) \equiv x_k^+ \otimes K + 1 \otimes x_k^+ + \sum_{i=1}^{k} x_{k-i}^+ \otimes \psi_i, \quad (k \geqq 0),$$

$$\Delta(x_{-k}^+) \equiv x_{-k}^+ \otimes K^{-1} + 1 \otimes x_{-k}^+ + \sum_{i=1}^{k-1} x_{-k+i}^+ \otimes \phi_{-i}, \quad (k > 0),$$

(ii) *modulo* $UX_+ \otimes UX_-^2$,

$$\Delta(x_k^-) \equiv x_k^- \otimes 1 + K \otimes x_k^- + \sum_{i=1}^{k-1} \psi_i \otimes x_{k-i}^-, \quad (k > 0),$$

$$\Delta(x_{-k}^-) \equiv x_{-k}^- \otimes 1 + K^{-1} \otimes x_{-k}^- + \sum_{i=1}^{k} \phi_{-i} \otimes x_{-k+i}^-, \quad (k \geqq 0),$$

(iii) *modulo* $UX_+ \otimes UX_- + UX_- \otimes UX_+$,

$$\Delta(\psi_k) \equiv \sum_{i=0}^{k} \psi_i \otimes \psi_{k-i}, \quad (k \geqq 0),$$

$$\Delta(\phi_{-k}) \equiv \sum_{i=0}^{k} \phi_{-i} \otimes \phi_{-k+i}, \quad (k \geqq 0).$$

Proof. The formulae are proved by induction on k. The initial case of each of the six formulae follow from the action of Δ on the Chevalley generators given in Definition (2.2), and the isomorphism in Theorem (2.3). One then obtains

$$\begin{aligned}\Delta(h_1) &= (q-q^{-1})^{-1}\Delta(\psi_1 K^{-1}) \\ &= [\Delta(x_0^+), \Delta(x_1^-)](K^{-1} \otimes K^{-1}) \\ &= h_1 \otimes 1 + 1 \otimes h_1 - (q^2 - q^{-2})x_0^+ \otimes x_1^-.\end{aligned}$$

Using the relations

$$[h_1, x_k^\pm] = \pm(q+q^{-1})x_{k+1}^\pm,$$

the first of the formulae in parts (i) and (ii) follows. The rest of parts (i) and (ii) is proved similarly. Finally, part (iii) follows from parts (i) and (ii) using (for example) the relations

$$(q-q^{-1})[x_k^+, x_0^-] = \begin{cases} \psi_k & \text{if } k > 0, \\ -\phi_k & \text{if } k < 0. \end{cases}$$

Remark. This proposition can be formulated most simply by introducing the elements

$$\Psi(u) = \sum_{k=0}^{\infty} \psi_k u^k,$$

$$\Phi(u) = \sum_{k=0}^{\infty} \phi_{-k} u^{-k},$$

$$X_{\geqq 0}^+(u) = \sum_{k \geqq 0} x_k^+ u^k, \quad X_{<0}^+(u) = \sum_{k<0} x_k^+ u^k$$

of the Hopf algebras of formal power series $U_q(L(sl_2)) \otimes \mathbb{C}[[u]]$ and $U_q(L(sl_2)) \otimes \mathbb{C}[[u^{-1}]]$. The first formula in the proposition is equivalent to the statement that

$$\Delta(X_{\geqq 0}^+) \equiv X_{\geqq 0}^+ \otimes \Psi + 1 \otimes X_{\geqq 0}^+$$

modulo $(UX_+^2 \otimes UX_-)[[u]]$; the next three formulae can be expressed in a similar way. The first formula in part (iii) is equivalent to the statement that, modulo $(UX_+ \otimes UX_- + UX_- \otimes UX_+)[[u]]$, the element Ψ is group-like:

$$\Delta(\Psi) \equiv \Psi \otimes \Psi;$$

and similarly for Φ.

Proposition (4.3) is an obvious consequence of Proposition (4.4). For (4.4) (i) implies that the tensor product of highest weight vectors in V and W is a highest weight vector in $V \otimes W$, and the group-like property of Ψ then implies the multiplicative property of the polynomials.

A similar argument completes the proof of Theorem (3.4). Let P be any polynomial with constant coefficient 1, and let its roots be $\zeta_1,\ldots,\zeta_r$ (repeated according to multiplicity). Set $a_i=\zeta_i^{-1}$ and consider the representation $V=V_1(a_1)\otimes\cdots\otimes V_1(a_r)$. It is clear that V contains, up to a scalar multiple, a unique vector Ω of weight r (i.e. such that $K\cdot\Omega=q^r\Omega$), namely the tensor product of the highest weight vectors in each factor. It follows by a standard argument that the subrepresentation V' of V generated by Ω contains a unique maximal subrepresentation V''. By the previous argument, the finite-dimensional irreducible representation V'/V'' has associated polynomial

$$\prod_{i=1}^{r}(1-a_iu)=\prod_{i=1}^{r}(1-\zeta_i^{-1}u)=P(u).$$

This completes the proof of the "if" part of Theorem (3.4).

4.5. Recall that if V is any finite-dimensional irreducible representation of a Hopf algebra A, the vector space dual V^* can be made into a representation of A by using the antipode S:

$$(x\cdot\lambda)(v)=\lambda(S(x)\cdot v)\quad(x\in A, v\in V, \lambda\in V^*).$$

Recall also that S is a coalgebra anti-homomorphism, so that

$$(V\otimes W)^*\cong W^*\otimes V^*$$

for any two representations V and W of A.

Proposition. *The dual of the evaluation representation $V_n(a)$ is isomorphic to $V_n(q^2a)$.*

Proof. This follows from the relation

$$S\circ \mathrm{ev}_{q^2a}=\mathrm{ev}_a\circ S$$

satisfied by the antipode S of $U_q(L(sl_2))$. The relation is easily proved by checking it on the Chevalley generators of Definition (2.2).

4.6. To describe the conditions under which a tensor product of evaluation representations is irreducible, we need some simple combinatorial definitions and results.

Definition. *A non-empty finite-set of non-zero complex numbers is said to be a q-string (or simply a string) if it is of the form $\{\zeta,q^{-2}\zeta,q^{-4}\zeta,\ldots,q^{-2r}\zeta\}$ for some $\zeta\in\mathbb{C}^\times$ and some $r\in Z_+$.*

Example. The roots of the polynomial associated to an evaluation representation $V_n(a)$ form a q-string $S_n(a)$ with $\zeta=q^{n-1}a$, $r=n-1$.

4.7.

Definition. *Two q-strings S_1 and S_2 are said to be in general position if either*

(i) *$S_1\cup S_2$ is not a q-string, or*
(ii) *$S_1\subseteq S_2$ or $S_2\subseteq S_1$.*

Example. The strings $S_m(a)$ and $S_n(b)$ are *not* in general position if and only if

$$\frac{b}{a} = q^{\pm(m+n-2p+2)}$$

for some $0 < p \leqq \min\{m, n\}$.

By a set with multiplicities, we mean a set together with an assignment of a strictly positive integer to each element of the set. There is a natural definition of the union of two sets with multiplicities. Note that the roots of a polynomial form a set with multiplicities in an obvious way.

The following elementary result is left to the reader. (The proof of an equivalent result is given in [2], Proposition (3.4).)

Proposition. *Any finite set of complex numbers with multiplicities can be written uniquely as a union of q-strings, any two of which are in general position.*

4.8. The following result gives the precise condition under which a tensor product of evaluation representations is irreducible.

Theorem. *A tensor product $V_{n_1}(a_1) \otimes \cdots \otimes V_{n_r}(a_r)$ is irreducible if and only if the q-strings $S_{n_1}(a_1), \ldots, S_{n_r}(a_r)$ are in general position.*

We begin by proving this result in the case $r = 2$. We shall change notation and consider $V_m(a) \otimes V_n(b)$. By Corollary (4.3), there is no loss of generality in assuming that $m \geqq n$.

We recall from [8] that, as a representation of $U_q(sl_2)$,

$$V_m \otimes V_n \cong V_{m+n} \oplus V_{m+n-2} \oplus \cdots \oplus V_{m-n}.$$

In fact, the highest weight vector Ω_p in the component V_{m+n-2p} of $V_m \otimes V_n$ is given by

$$\Omega_p = \sum_{i=0}^{p} (-1)^i q^{i(n-i+1)} [m-p+i]_q! [n-i]_q! v_{p-i} \otimes v_i.$$

(To verify this, it is enough to check that Ω_p has the correct weight and is annihilated by e^+.) One checks that, for $p > 0, \Omega_p$ is annihilated by x_{-1}^{+} if and only if

$$\frac{b}{a} = q^{m+n-2p+2}.$$

In this case, it follows from

$$[h_{-1}, x_0^+] = (q + q^{-1}) x_{-1}^{+}$$

that $h_{-1} \cdot \Omega_p$ is also annihilated by x_0^+ and has the same weight as Ω_p; it must therefore be a scalar multiple of Ω_p. It follows easily that Ω_p is annihilated by x_k^+ and is an eigenvector of h_k for all $k < 0$. Similar arguments deal with the case $k \geqq 0$.

This proves that $V_m(a) \otimes V_n(b)$ has a subrepresentation not containing its highest component if and only if

$$\frac{b}{a} = q^{m+n-2p+2}$$

for some $0 < p \leqq n$. The tensor product has a proper subrepresentation containing the highest component if and only if its dual

$$(V_m(a)\otimes V_n(b))^* \cong V_n(q^2b)\otimes V_m(q^2a)$$

has a subrepresentation not containing the highest component. By the previous argument, this is the case if and only if

$$\frac{b}{a}=q^{-(m+n-2p+2)}$$

for some $0<p\leqq n$.

Combining these two results with the example in Subsect. (4.7), we see that $V_m(a)\otimes V_n(b)$ is reducible if and only if the strings $S_m(a)$ and $S_n(b)$ are not in general position, proving the theorem in the case $r=2$.

4.9. More detailed arguments prove the following result.

Proposition. *Let $V=V_m(a)\otimes V_n(b)$, $0<p\leqq \min\{m,n\}$. If $b/a=q^{\pm(m+n-2p+2)}$, then V has a unique proper subrepresentation W. In fact:*

(a) *If $b/a=q^{m+n-2p+2}$, we have*

$$W\cong V_{m-p}(q^{-p}a)\otimes V_{n-p}(q^pb),$$
$$V/W\cong V_{p-1}(q^{m-p+1}a)\otimes V_{m+n-p+1}(q^{-(m-p+1)}b),$$

and, as a representation of $U_q(sl_2)$,

$$W\cong V_{m+n-2p}\oplus V_{m+n-2p-2}\oplus\cdots\oplus V_{|m-n|}.$$

(b) *If $b/a=q^{-(m+n-2p+2)}$, we have*

$$W\cong V_{p-1}(q^{-(m-p+1)}a)\otimes V_{m+n-p+1}(q^{m-p+1}b),$$

$$V/W\cong V_{m-p}(q^pa)\otimes V_{n-p}(q^{-p}b),$$

and, as a representation of $U_q(sl_2)$,

$$W\cong V_{m+n}\otimes V_{m+n-2}\oplus\cdots\oplus V_{m+n-2p+2}.$$

We omit the details.

The preceding proposition admits a simple pictorial description. When the q-strings $S_m(a)$ and $S_n(b)$ are not in general position, there are two ways of producing two new strings which are in general position:

(i) The intersection of the two strings, together with the two nearest neighbour elements, is discarded.

(ii) The union of $S_m(a)$ and $S_n(b)$, regarded as sets with multiplicities, may be decomposed into the union of two strings in general position (see Proposition (4.7)). The two strings are simply the set-theoretic union and intersection of $S_m(a)$ and $S_n(b)$.

In part (a) of the proposition, W corresponds to the two strings produced by operation (i) and the quotient V/W to those produced by operation (ii). In part (b), the reverse in true.

Remark. It follows from the preceding proposition that, if $V_m(a)\otimes V_n(b)$ is reducible, it is not isomorphic to $V_n(b)\otimes V_m(a)$.

4.10. We now turn to the general case of Theorem (4.8). Suppose first that some pair of strings $S_{n_i}(a_i)$ and $S_{n_j}(a_j)$ is not in general position, and assume for a contradiction that the tensor product $V_{n_1}(a_1)\otimes\cdots\otimes V_{n_r}(a_r)$ is irreducible. It follows from Corollary (4.3) that the tensor product is unchanged, up to isomorphism, by any permutation of the factors. If we choose a permutation which leaves the i^{th} and j^{th} factors adjacent and use the results of Subsect. (4.8), we obtain the required contradiction.

The converse depends on the following

Lemma. *Suppose that the strings $S_{n_i}(a_i)$, $1\leqq i\leqq r$, are in general position and that $n_1\leqq n_2\leqq\cdots\leqq n_r$. Then $V_{n_1}(a_1)\otimes\cdots\otimes V_{n_r}(a_r)$ is generated by the tensor product of the highest weight vectors in the $V_{n_i}(a_i)$.*

Assuming this result for the moment, the proof of Theorem (4.8) is completed as follows. Let V_N, $N=\sum n_i$, be the highest component of $V=V_{n_1}(a_1)\otimes\cdots\otimes V_{n_r}(a_r)$ as a representation of $U_q(sl_2)$, and assume that the q-strings $S_{n_1}(a_1),\ldots,S_{n_r}(a_r)$ are in general position. By Corollary (4.3), we may assume that $n_1\leqq\cdots\leqq n_r$ without loss of generality. By the lemma, V has no proper subrepresentation containing V_N. On the other hand, if W is a subrepresentation of V not containing V_N, then its annihilator W° is a proper subrepresentation of

$$V^*\cong V_{n_r}(q^2a_r)\otimes\cdots\otimes V_{n_1}(q^2a_1)$$

containing its highest component. But this contradicts the lemma, since the strings $S_{n_1}(q^2a_1),\ldots,S_{n_r}(q^2a_r)$ are in general position.

The proof of the lemma is by induction on r. The case $r=2$ was proved in Subsect. (4.8). We note first that V is generated by $\Omega'\otimes v_{n_r}$, where Ω' is the tensor product of the highest weight vectors in $V'=V_{n_1}(a_1)\otimes\cdots\otimes V_{n_{r-1}}(a_{r-1})$. This is an easy consequence of Proposition (3.1) and part (i) of Proposition (4.4), together with the induction hypothesis that $V'=U\cdot\Omega'$.

We now prove, by induction on i, that $\Omega'\otimes v_i\in U\cdot\Omega$ for $1\leqq i\leqq n_r$, where $\Omega=\Omega'\otimes v_i$. For $i=0$ there is nothing to prove, and the case $i=n_r$ establishes the lemma.

Assuming that $\Omega'\otimes v_i\in U\cdot\Omega$, with $i>0$, consider the equations

$$x_k^-\cdot(\Omega'\otimes v_i)=x_k^-\cdot\Omega'\otimes v_i+\sum_{p=0}^{k-1}\psi_p\cdot\Omega'\otimes x_{k-p}^-\cdot v_i$$

for $k\geqq 1$, which follow from Proposition (4.4) (ii). Hence, using Proposition (4.2),

$$x_k^-\cdot(\Omega'\otimes v_i)=x_k^-\cdot\Omega'\otimes v_i+\sum_{p=0}^{k-1}d_{p,r-1}\,b_r^{k-p}\Omega'\otimes e^-\cdot v_i,$$

where $b_r=a_rq^{n_r-2i}$ and $d_{p,r-1}$ is the eigenvalue of ψ_p acting on Ω'. More generally, let $b_j=a_jq^{n_j}$ for $1\leqq j<p$ and let $d_{p,j}$ be the eigenvalue of ψ_p acting on the tensor product of the highest weight vectors in $V_{n_1}(a_1),\ldots,V_{n_j}(a_j)$. Then, iterating the above computation, we find

$$x_k^-\cdot(\Omega'\otimes v_i)=\sum_{j=0}^{r-1}A_{k,j}(v_0\otimes\cdots\otimes e^-\cdot v_0\otimes\cdots\otimes v_i)\quad(*)$$

(with e^- acting in the $(j+1)^{\text{th}}$ position), where

$$A_{k,j} = \sum_{p=0}^{k-1} d_{p,j} b_{j+1}^{k-p},$$

and we have set $d_{0,0} = 1$, $d_{p,0} = 0$ for $p > 0$. We shall prove that, under the hypothesis that the q-strings $S_{n_1}(a_1), \ldots, S_{n_r}(a_r)$ are in general position, the matrix $A = (A_{k,j})_{1 \leqq k \leqq r, 0 \leqq j \leqq r-1}$ is invertible. It follows from equations $(*)$ that $\Omega' \otimes v_{i+1}$ is a linear combination of the elements $x_k^- \cdot (\Omega' \otimes v_i)$ for $1 \leqq k \leqq r$, and this completes the induction step.

Our assertion is a consequence of the following formula:

$$\det A = q^{\sum_{j=1}^{r-1} n_j} \left(\prod_{j=1}^{r} b_j \right) \left(\prod_{j<k} (b_k - q^{-2n_j} b_j) \right).$$

Indeed, since the b_j are non-zero, $\det A = 0$ only if

$$b_k = q^{-2n_j} b_j$$

for some $j < k$. If $k < r$, this is equivalent to

$$a_j = q^{n_j + n_k} a_k,$$

which contradicts the fact the $S_{n_j}(a_j)$ and $S_{n_k}(a_k)$ are in general position. If $k = r$, we have

$$a_j = q^{n_j + n_r - 2i} a_r$$

again contradicting general position, since $i > 0$.

To prove the determinant formula, we first note that if $b_{j+1} = q^{-2n_j} b_j$ for some $j \geqq 1$, the j^{th} and $(j-1)^{\text{th}}$ columns of A differ only by a factor q^{n_j}.

The proof that

$$q^{n_j} A_{k,j} = A_{k,j-1}$$

proceeds by induction on k, using the relations

$$A_{k+1,j} = b_{j+1}(A_{k,j} + d_{k,j})$$

and

$$d_{k,j} = d_{k,j-1} q^{n_j} + (q^{n_j} - q^{-n_j}) A_{k,j-1},$$

which follow from the definition of $A_{k,j}$ and Proposition (4.4)(iii).

For $k = 1$, we have to prove

$$q^{n_j} d_{0,j} b_{j+1} = d_{0,j-1} b_j,$$

which follows from the fact that

$$d_{0,j} = q^{n_1 + \cdots + n_j}.$$

Assuming the result for k, we have

$$\begin{aligned} q^{n_j} A_{k+1,j} &= q^{n_j} b_{j+1} (A_{k,j} + d_{k,j}) \\ &= q^{-n_j} b_j (A_{k,j} + d_{k,j}) \end{aligned}$$

$$= q^{-n_j} b_j (A_{k,j} + d_{k,j-1} q^{n_j} + (q^{n_j} - q^{-n_j}) A_{k,j-1})$$
$$= b_j (A_{k,j-1} + d_{k,j-1})$$
$$= A_{k+1,j-1}.$$

This completes the induction step.

If $k > j$ is any pair of indices for which $b_k = q^{-2n_j} b_j$, there is a permutation σ of $\{1, 2, \ldots, r\}$ such that $\sigma(r) = r$ and $\sigma(k) = \sigma(j) + 1$. Let Ω'_σ be the result of applying σ to the factors in Ω', and define V_σ and V'_σ similarly. By the general position assumption, there is an isomorphism $V' \to V'_\sigma$ of representations of $U_q(L(sl_2))$, and we may assume that it sends Ω' to Ω'_σ. Hence, there is an isomorphism $V \to V_\sigma$ which takes $\Omega' \otimes v_i$ to $\Omega'_\sigma \otimes v_i$ for all i. It follows that

$$\{x_1^- \cdot (\Omega' \otimes v_i), \ldots, x_r^- (\Omega' \otimes v_i)\}$$

is linearly dependent if and only if

$$\{x_1^- \cdot (\Omega'_\sigma \otimes v_i), \ldots, x_r^- (\Omega'_\sigma \otimes v_i)\}$$

is linearly dependent. The first condition holds if and only if $\det A = 0$, and the second if and only if $\det A_\sigma = 0$, where A_σ is the matrix obtained by applying σ to the parameters $a_1, \ldots, a_r$ and $n_1, \ldots, n_r$. This implies that $b_k - q^{-2n_j} b_j$ is a root of $\det A$ if and only if $b_{\sigma(k)} - q^{-2n_{\sigma(j)}} b_{\sigma(j)}$ is a root of $\det A_\sigma$, which is true by the first part of the argument.

This accounts for the second product in the formula for $\det A$. The first product arises from the fact that the j^{th} column of A is divisible by b_{j+1}, and the remaining factor by counting degrees and identifying the coefficient of $b_1 b_2^2 \cdots b_r^r$ in $\det A$.

The proof of the lemma, and hence that of Theorem (4.8), is now complete.

Remark. As $q \to 1$, $\det A$ becomes a Vandermonde determinant. See [1], Sect. 4, for the analogous role played by classical Vandermonde determinants in the representation theory of affine Lie algebras.

4.11. It is now a simple matter to write down all finite-dimensional irreducible representations of $U_q(\hat{sl}_2)$.

Theorem. *Every finite-dimensional irreducible representation of $U_q(\hat{sl}_2)$ of type* $(1, 1)$ *is isomorphic to a tensor product of evaluation representations. Two such tensor products are isomorphic if and only if one is obtained from the other by permuting the factors in the tensor product.*

Proof. Let V be a finite-dimensional irreducible representation of $U_q(\hat{sl}_2)$ of type $(1, 1)$ and let P be its associated polynomial. By Proposition (4.7), the set of roots of P can be written as a union of q-strings in general position, say $S_{n_1}(a_1), \ldots, S_{n_r}(a_r)$. By Theorem (4.8), the representation

$$V_{n_1}(a_1) \otimes \cdots \otimes V_{n_r}(a_r)$$

is irreducible, and by Proposition (4.3) its associated polynomial is P. Hence,

$$V \cong V_{n_1}(a_1) \otimes \cdots \otimes V_{n_r}(a_r).$$

The last statement in the theorem follows from the fact that the decomposition of the set of roots of a polynomial into q-strings in general position is unique (up to the order of the strings).

5. Trigonometric Solutions of the QYBE

In this section, we compute the solutions of the quantum Yang–Baxter equation associated to the finite-dimensional irreducible representations of $U_q(\hat{sl}_2)$, restricting ourselves to the type $(1, 1)$ case for simplicity.

5.1. The quantum Yang–Baxter equation (QYBE) is

$$R_{12}(u-v)R_{13}(u-w)R_{23}(v-w)=R_{23}(v-w)R_{13}(u-w)R_{12}(u-v).$$

Here, $R(u)$ is a function of $u\in\mathbb{C}$ with values in $\mathrm{End}(V\otimes V)$, for some finite-dimensional vector space V, and $R_{12}=R\otimes\mathrm{id}\in\mathrm{End}(V\otimes V\otimes V)$ etc. If V is a finite-dimensional irreducible representation of $U_q(sl_2)$, there is an associated solution of the QYBE. In this section we shall compute all such solutions.

5.2. The connection between the QYBE and quantum affine algebras depends on the following observation, which follows immediately from the defining relations (2.3).

Proposition. *There is a one-parameter group of automorphisms τ_λ, $\lambda\in\mathbb{C}$, of the Hopf algebra $U_q(\hat{sl}_2)$ such that*

$$\tau_\lambda(x_k^{\pm})=e^{k\lambda}x_k^{\pm},\quad \tau_\lambda(h_k)=e^{k\lambda}h_k,\quad \tau_\lambda(K)=K,\ \tau_\lambda(C)=C.$$

Definition. *For any representation V of $U_q(\hat{sl}_2)$ and any $\lambda\in\mathbb{C}$, the pull-back of V by the automorphism τ_λ is denoted by $V(\lambda)$.*

Note that $V(\lambda)$ is not necessarily an evaluation representation. If $V_m(a)$ is an evaluation representation, we have

$$V_m(a)\cong(V_m(1))(\ln a).$$

5.3. We have seen in Theorem (4.8) that a tensor product of evaluation representations is generically irreducible. The same is true for the representations $V(\lambda)$.

Proposition. *Let V and W be finite-dimensional irreducible representations of $U_q(\hat{sl}_2)$ with highest weight vectors Ω_V and Ω_W, and let $\lambda,\mu\in\mathbb{C}$. Then:*

(i) *the tensor products $V(\lambda)\otimes W(\mu)$ are irreducible except for a finite set of values of $\lambda-\mu$ (modulo integer multiples of $2\pi i$);*

(ii) *the unique intertwining operator*

$$I(V,\lambda;W,\mu)\colon W(\mu)\otimes V(\lambda)\to V(\lambda)\otimes W(\mu)$$

which maps $\Omega_W\otimes\Omega_V$ to $\Omega_V\otimes\Omega_W$ is a rational function of $e^{\lambda-\mu}$ with values in $\mathrm{Hom}(W\otimes V, V\otimes W)$.

The proof is almost identical to that of Proposition (5.1) in [2].

Definition. *Let V be a finite-dimensional irreducible representation of $U_q(\hat{sl}_2)$. Then the R-matrix associated to V is the function $R(\lambda-\mu)$ with values in* $\mathrm{End}(V\otimes V)$ given by

$$R(\lambda-\mu)=I(V,\lambda:V,\mu)\sigma,$$

where $\sigma\in\mathrm{End}(V\otimes V)$ is the switch of the two factors.

Theorem. *The R-matrix associated to a finite-dimensional irreducible representation of $U_q(\hat{sl}_2)$ is a solution of the* QYBE.

See the proof of Theorem (5.5) in [2].

Remark. A (matrix-valued) function of $\lambda \in \mathbb{C}$ is said to be trigonometric if it is a rational function of $e^{c\lambda}$ for some $c \in \mathbb{C}^\times$. By part (ii) of the proposition, the R-matrix in the theorem is a trigonometric solution of the QYBE.

5.4. Let

$$V = V_{n_1}(a_1) \otimes \cdots \otimes V_{n_r}(a_r)$$

by any finite-dimensional irreducible representation of $U_q(\hat{sl}_2)$. The intertwining operator $I(V, \lambda; V, \mu)$ can be computed as the product of k^2 intertwining operators of the form $I(V_m, a; V_n, b)$, each of which effects an interchange of nearest neighbours.

If Ω_p is a highest weight vector for $U_q(sl_2)$ in $V_n \otimes V_m$ of weight $m+n-2p$, it is easy to see that $(e^+ \otimes 1) \cdot \Omega_p$ is also a highest weight vector. Hence, we may assume that

$$(e^+ \otimes 1) \cdot \Omega_p = \Omega_{p-1}$$

for $0 < p \leqq \min\{m, n\}$. Similarly, we may choose highest weight vectors Ω'_p in $V_m \otimes V_n$ such that

$$(e^+ \otimes 1)\Omega'_p = \Omega'_{p-1}.$$

Let $P_p : V_n \otimes V_m \to V_m \otimes V_m$ be the unique homomorphism of representations of $U_q(sl_2)$ such that

$$P_p(\Omega_p) = \Omega'_p$$

and

$$P_p(\Omega_r) = 0 \quad \text{if} \quad r \neq p.$$

Then, we can write

$$I \equiv I(V_m, a; V_n, b) = \sum_{p=0}^{\min\{m,n\}} c_p P_p$$

for some $c_p \in \mathbb{C}$.

Consider the equations

$$I(e_0^- \cdot \Omega_p) = e_0^- \cdot I(\Omega_p).$$

Using Theorem (2.3) and Proposition (4.2), this becomes

$$I((b^{-1}e^+ \otimes 1 + a^{-1}K \otimes e^+) \cdot \Omega_p) = (a^{-1}e^+ \otimes 1 + b^{-1}K \otimes e^+) \cdot I(\Omega_p).$$

Now, from Subsect. (4.8) we recall that, if $a/b = q^{m+n-2p+2}$, then $e^+ \cdot \Omega_p = 0$. Hence,

$$(K \otimes e^+) \cdot \Omega_p = -q^{m+n-2p+2}(e^+ \otimes 1) \cdot \Omega_p.$$

This gives

$$(b^{-1} - a^{-1}q^{m+n-2p+2})c_{p-1} = (a^{-1} - b^{-1}q^{m+n-2p+2})c_p,$$

and hence

$$I(V_m, a; V_n, b) = \sum_{p=0}^{\min\{m,n\}} \prod_{j=0}^{p-1} \frac{(a - bq^{m+n-2j})}{(b - aq^{m+n-2j})} P_p.$$

Theorem. *The R-matrix associated to the representation*

$$V = V_{n_1}(a_1) \otimes \cdots \otimes V_{n_r}(a_r)$$

of $U_q(\hat{sl}_2)$ *is given by*

$$R(\lambda - \mu) = \left(\sum_{i,j=1}^{r} I(V_{n_i}, e^{\lambda} a_i; V_{n_j}, e^{\mu} a_j) \right) \sigma.$$

The order of the factors in the product is such that the (i,j)*-term appears to the left of the* (i',j')*-term if and only if*

$$i > i' \quad or \quad i = i' \quad and \quad j < j'.$$

Remark. Write $x = a/b$ and clear denominators on both sides of the QYBE (5.1) satisfied by the R-matrix associated to $V_m(a) \otimes V_n(b)$. Both sides of the equation are then polynomials in x. It is clear that the constant terms, and also the highest order terms, in the R-matrix give (constant) solutions of the QYBE. These solutions have been written down by Kirillov and Reshetikhin [9].

Remark. The formula for $I(V_m, a; V_n, b)$ was first obtained by M. Jimbo in [6].

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