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Convergence of Nelson Diffusions

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Abstract. Let ψ_t, ψ_t^n, $n \geqq 1$, be solutions of Schrödinger equations with potentials form-bounded by $-\frac{1}{2}\Delta$ and initial data in $H^1(\mathbb{R}^d)$. Let P, P^n, $n \geqq 1$, be the probability measures on the path space $\Omega = C(\mathbb{R}_+, \mathbb{R}^d)$ given by the corresponding Nelson diffusions. We show that if $\{\psi_t^n\}_{n\geqq 1}$ converges to ψ_t in $H^1(\mathbb{R}^d)$, uniformly in t over compact intervals, then $\{P^n_{|\mathscr{F}_t}\}_{n\geqq 1}$ converges to $P_{|\mathscr{F}_t}$ in total variation $\forall t \geqq 0$. Moreover, if the potentials are in the Kato class K_d, we show that the above result follows from H^1-convergence of initial data, and K_d-convergence of potentials.

1. Introduction

Stochastic Quantization is an algorithm which permits to associate a diffusion process to a solution of the Schrödinger equation in such a way that the density of the process corresponds to the usual density of Quantum Mechanics (see [N] for a thorough introduction to the subject). An unpleasant characteristic of these diffusion processes is that their drift coefficients are too singular to be handled by the traditional approaches. The problem of the existence of the stochastic processes of Stochastic Mechanics was resolved by Carlen for potentials form-bounded by $-\frac{1}{2}\Delta$ and initial data in $H^1(\mathbb{R}^d)$ (see [C1, C2, C3], and Sect. 2). His existence theorem provides us a Borel probability measure on $\Omega = C(\mathbb{R}_+, \mathbb{R}^d)$, the space of the physical trajectories of the particles, such that the stochastic process $X_t(\gamma) := \gamma(t)$ is a Markov process with density $|\psi_t|^2$, and is a weak solution of the stochastic differential equation

$$X_t = X_0 + \int_0^t b(s, X_s)ds + B_t$$

with $b_t = (\Re + \Im)\nabla \log \psi_t$, as required by the stochastic quantization procedure. Carlen's theorem provides a map from the space of solutions of the Schrödinger equation to the space $\mathscr{M}_1(\Omega)$ of probability measures on Ω; it is then natural to consider the following continuity problem: let $\{\psi_t^n\}_{n\geqq 1}$ be a sequence of solutions of Schrödinger equations, possibly with different potentials V_n, and let $\{P^n\}_{n\geqq 1}$ be the corresponding sequence of probability measures given by Carlen's theorem; which convergence of the ψ_t^n's will give a convergence of the P^n's? Obviously the answer to this question depends on the choice of a topology on $\mathscr{M}_1(\Omega)$. The most used one is the metric topology given by the Prohorov metric p. This topology is the topology of weak convergence on bounded continuous functions, i.e.

$$p(P^n, P)\to 0 \Leftrightarrow \int_\Omega f dP^n \to \int_\Omega f dP \quad \forall f\in C_b(\Omega).$$

We will instead consider on $\mathscr{M}_1(\Omega)$ the stronger topology induced by the metric

$$d(P', P) := \|P' - P\| = \sup_{\{E_k\}_{k\in\mathbb{N}}} \sum_k |P'(E_k) - P(E_k)|,$$

where the sup is taken over all measurable partitions of Ω. This is the topology that $\mathscr{M}_1(\Omega)$ inherits as subset of the Banach lattice of bounded signed measures on Ω, normed with the total variation norm.

We will prove (Theorem 6.1) that if ψ_t^n converges to ψ_t in $H^1(\mathbb{R}^d)$, uniformly in t over compact intervals, then $d(P^n_{|\mathscr{F}_t}, P_{|\mathscr{F}_t})\to 0$ $\forall t\geqq 0$, and so $p(P^n, P)\to 0$. Moreover, in case the potentials are in the Kato class K_d, we will give a criterion of convergence in terms of H^1-convergence of initial data, and K_d-convergence of potentials; since $C_c^\infty(\mathbb{R}^d)$ is dense in K_d, we can then approximate physics with extremely nice potentials.

Notice that the drift coefficients b_n are typically very singular (independently of the regularity of the ψ_s^n's, they are unbounded on the nodes), it is therefore not possible to use the usual convergence theorems for diffusion processes (see [SV, Chap. 11], [Z, Theorem 5], [JS, Chap. IX, Sect. 4a], for weak convergence, and [JS, Chap. V, Sect. 4d], for convergence in variation). In fact these theorems require bounded drifts and convergence in $L^\infty_{\mathrm{loc}}(\mathbb{R}^{d+1})$. We overcome these difficulties putting together the following facts:

1) By the finite energy condition (see Sect. 2), $P_{|\mathscr{F}_T} \ll W_{|\mathscr{F}_T}$ for all T, where W is the usual Wiener measure on Ω, and $\mathscr{F}_T$ is the σ-algebra generated by trajectories up to time T;

2) By 1), every subset of $\mathbb{R}^d$ with zero (Newtonian) capacity is polar for X_t with respect to P. Moreover, introducing a suitable capacity Γ_T on subsets of $[0, T]\times\mathbb{R}^d$, every set with zero capacity Γ_T for all T is polar for $Y_t = (t, X_t)$ with respect to P;

3) For every sequence $\{\psi^n\}_{n\geqq 1}$ of function on $\mathbb{R}_+\times\mathbb{R}^d$ which are continuous from $\mathbb{R}_+$ to $H^1(\mathbb{R}^d)$, there exists a decreasing sequence of open sets $\{D_k\}_{k\geqq 1}$, with $\Gamma_T(D_k\cap[0, T]\times\mathbb{R}^d)\downarrow 0$ for all T, such that all the ψ^n's are continuous on $D_k^c\cap[0, T]\times\mathbb{R}^d$ for all k and T. Moreover if $\sup_{0\leqq t\leqq T}\|\psi_t^n - \psi_t\|_{H^1}\to 0$ for all T, then there exists a subsequence such that $\psi^{n_j}\to\psi$ pointwise and uniformly on $D_k^c\cap[0, T]\times\mathbb{R}^d$ for all k and T;

4) If P and Q are probability measures on Ω given by weak solutions of stochastic differential equations with the same constant diffusion coefficient and equal initial distribution, and with drifts b_P and b_Q such that

$$E_P \int_{\mathbb{R}_+} \|b_P\|^2(s)ds < +\infty, \qquad E_Q \int_{\mathbb{R}_+} \|b_Q\|^2(s)ds < +\infty,$$

then

$$d^2(P, Q) \leqq 2E_P \int_{\mathbb{R}_+} \|b_P - b_Q\|^2(s)ds.$$

2. Nelson Diffusion with Potentials Form-Bounded by $-\frac{1}{2}\Delta$

Let K denote the self-adjoint representation of $-\frac{1}{2}\Delta$ on $L^2(\mathbb{R}^d)$, and let V be a real-valued measurable function on $\mathbb{R}^d$ such that V is K-form-bounded, with relative bound smaller than one, i.e. $\exists a \in [0,1)$, $\exists b \geqq 0$ such that

$$|\langle \psi | V\psi \rangle_{L^2}| \leqq a\langle \psi | K\psi \rangle_{L^2} + b\langle \psi | \psi \rangle_{L^2} \quad \forall \psi \in H^1(\mathbb{R}^d).$$

We shall discuss only the case of time-independent potentials. The extension to the time-dependent case is immediate at the expense of heavier notation.

Let H be the unique self-adjoint operator associated to the sum of the quadratic forms of K and V. Such H exists by the KLNM theorem (see [RS, Theorem X.17]). Moreover one has

$$H^2(\mathbb{R}^d) \cap \mathscr{D}(V) \subset \mathscr{D}(H) \subset H^1(\mathbb{R}^d),$$

$$\langle \phi | H\psi \rangle_{L^2} = \langle \phi | K\psi \rangle_{L^2} + \langle \phi | V\psi \rangle_{L^2} \quad \forall \phi \in H^1(\mathbb{R}^d) \quad \forall \psi \in \mathscr{D}(H),$$

and

$$\|\psi\|_{H^1}^2 \leqq (1-a)^{-1}(\langle \psi | H\psi \rangle_{L^2} + (b+1)\langle \psi | \psi \rangle_{L^2}) \leqq (2(b+1)+a)(1-a)^{-1}\|\psi\|_{H_1}^2.$$

Let e^{-itH} be the one parameter unitary group generated by H. By the above relation, it follows that e^{-itH} maps $H^1(\mathbb{R}^d)$ into itself, with

$$\|e^{-itH}\|_{H^1H^1} \leqq (2(b+1)+a)(1-a)^{-1}.$$

Moreover, since

$$\lim_{t \to 0_+} \|(H+(b+1)I)^{1/2}(e^{-itH}\phi - \phi)\|_{L^2} = \lim_{t \to 0_+} \|(e^{-itH} - I)(H+(b+1)I)^{1/2}\phi\|_{L^2} = 0,$$

e^{-itH} is a continuous one parameter group of bounded linear operators on $H^1(\mathbb{R}^d)$.

By the above discussion, proceeding in the same way as in [C1], we have the following analogue of Theorem 2.1 in [C1]:

Theorem 2.1. *Let V be a K-form-bounded potential, with relative bound smaller than one, let ψ_0 be in $H^1(\mathbb{R}^d)$, and let $H = K + V$ be defined as a quadratic form. Then*
1) e^{-itH} is a continuous one parameter group of bounded linear operators from $H^1(\mathbb{R}^d)$ into $H^1(\mathbb{R}^d)$;
2) there are unique jointly measurable functions $\psi(t,x)$ and $\nabla\psi(t,x)$ such that $\psi(t,x) = e^{-itH}\psi_0(x)$, and $\nabla\psi(t,x) = \nabla e^{-itH}\psi_0(x)$;
3) defining $\varrho(t,x) := \psi(t,x)\bar{\psi}(t,x)$, and

$$u(t,x) := \Re(\nabla\psi(t,x)/\psi(t,x)),$$

$$v(t,x) := \Im(\nabla\psi(t,x)/\psi(t,x)),$$

if $\psi(t,x) \neq 0$, $u(t,x) = v(t,x) = 0$ otherwise, one has

$$\int_0^T \int_{\mathbb{R}^d} (\|u\|^2 + \|v\|^2)\varrho \, dx\, dt < +\infty \quad \forall T > 0; \qquad \text{(F.E.C.)}$$

4) $\forall f \in C_b^1(\mathbb{R}^d)$ the function $t \mapsto \int_{\mathbb{R}^d} f(x)\varrho(t,x)dx$ is differentiable, and

$$\frac{d}{dt}\int_{\mathbb{R}^d} f(x)\varrho(t,x)dx = \int_{\mathbb{R}^d} v(t,x) \cdot \nabla f(x)\varrho(t,x)dx.$$

From Theorem 2.1 it follows that the hypotheses in Theorem 4.1 in [C1] also holds for K-form-bounded potentials (see also [C2], see [C3] for uniqueness). So we have the following

Theorem 2.2. *Consider the measurable space $(\Omega, \mathscr{F})$, with $\Omega = C(\mathbb{R}_+, \mathbb{R}^d)$, $\mathscr{F}$ the Borel σ-algebra. Let u, v, and ϱ be as in Theorem 2.1, define $b := u + v$, and let $(\Omega, \mathscr{F}, \mathscr{F}_t, X_t)$ be the evaluation stochastic process $X_t(\gamma) := \gamma(t)$, with $\mathscr{F}_t = \sigma(X_s, s \leqq t)$ the natural filtration. Then there exists a unique Borel probability measure P on Ω such that:*
1) *$(\Omega, \mathscr{F}, \mathscr{F}_t, X_t, P)$ is a Markov process;*
2) *the image of P under X_t has density $\varrho(t, x)$;*

3)
$$B_t := X_t - X_0 - \int_0^t b(s, X_s) ds$$

is a P-Brownian motion.

Remark 2.3. From 3), Theorem 2.1, and 2), Theorem 2.2, it follows

$$E \int_0^T \|b(s)\|^2 ds < \infty \quad \forall T \geqq 0$$

(E denotes the expectation with respect to P), so that, by [Fö1, Proposition 2.11], or [JS, Theorem 4.23, Chap. IV] (see also [E]),

$$P_{|\mathscr{F}_T} \ll W_{|\mathscr{F}_T} \quad \forall T \geqq 0,$$

where $W := \int_{\mathbb{R}^d} W_x \varrho(0, x) dx$, and W_x is standard Wiener measure supported on Ω_x, the space of continuous paths starting at x. Moreover, by [Fö1, Proposition 2.11],

$$H_{\mathscr{F}_T}(P; W) = \tfrac{1}{2} E \int_0^T \|b(s)\|^2 ds,$$

where

$$H_{\mathscr{F}_T}(P; W) := \int_\Omega \log \frac{dP}{dW}\bigg|_{\mathscr{F}_T} dP$$

is the relative entropy of $P_{|\mathscr{F}_T}$ with respect to $W_{|\mathscr{F}_T}$ (see [Fö2, Chap. I, Sect. 3.1]), so that (F.E.C.) is a finite entropy condition. In the case of Nelson Diffusions it is called finite energy condition, since it may be written as (see [C1])

$$\int_0^T \|\nabla \psi_t\|_{L^2}^2 dt < +\infty,$$

i.e. quantum mechanical kinetic energy is integrable on $[0, T]$. Since $\sigma\left(\bigcup_{t \geqq 0} \mathscr{F}_t\right) = \mathscr{F}$ (see [SV, Sect. 1.3]), by [JS, Theorem 4.23, and Corollary 2.8, Chap. IV], if

$$E \int_{\mathbb{R}_+} \|b(s)\|^2 ds < \infty,$$

then $P \ll W$, and, in this case,

$$H(P; W) = \sup_{T \in \mathbb{R}_+} H_{\mathscr{F}_T}(P; W) = \tfrac{1}{2} E \int_{\mathbb{R}_+} \|b(s)\|^2 ds.$$

Remark 2.4. Let B be a Borel (or analytic) set such that $\operatorname{cap}(B)=0$. Here cap denotes the Choquet capacity defined for an open set B by

$$\operatorname{cap}(B):=\inf\{\|\phi\|^2_{H^1}:\phi\in H^1(\mathbb{R}^d),\ \phi\geqq\chi_B \text{ a.e.}\},$$

where χ_B is the characteristic function of B, and by

$$\operatorname{cap}(E):=\inf_{B\,\text{open},\,B\supset E}\operatorname{cap}(B)$$

for any set E. Since $\operatorname{cap}(B)=0\Leftrightarrow W_x(\tau_B<T)=0\ \forall T>0,\ \forall x\in B^c$, where τ_B denotes the first hitting time to the set B (see [Fu]), and $\operatorname{cap}(B)=0$ implies $m(B)=0$ (m denotes the Lebesgue measure), we have that $\operatorname{cap}(B)=0$ implies $W(\tau_B<T)=0$. Since $P_{|\mathscr{F}_T}\ll W_{|\mathscr{F}_T}$, capacity zero sets will be polar for the process X_t with respect to the probability measure P. Moreover, since (see [Fu])

$$\operatorname{cap}(B_k)=\|e_k\|^2_{H^1},\qquad e_k(x):=E_{W_x}(e^{-\tau_{B_k}}),$$

if $\{B_k\}_{k\geqq 0}$ is a decreasing sequence of open subsets of $\mathbb{R}^d$ such that $\operatorname{cap}(B_k)\downarrow 0$, we have $W(\tau_{B_k}<T)\downarrow 0\ \forall T>0$, and consequently $P(\tau_{B_k}<T)\downarrow 0\ \forall T>0$.

3. A Criterion for Convergence of Probability Measures

As stated in the introduction, we will be interested in the continuity of the measure P described in Theorem 2.2 with respect to the initial data and the potentials in the Schrödinger equation. Due to the singularity of the drift, we will consider first processes stopped outside a suitable subset, and prove convergence of the measures associated to the stopped processes; convergence of a subsequence $\{P^{n_j}\}_{j\geqq 1}$ to P follow then by the following

Lemma 3.1. *Let P, P^n, $n\geqq 1$, be probability measures on Ω, and let $P=\int P_x d\mu(x)$, $P^n=\int P^n_x d\mu_n(x)$ be the disintegration of such measures with respect to X_0, $\mu=P\circ X_0^{-1}$, $\mu_n=P^n\circ X_0^{-1}$. Let $\{\tau_k\}_{k\geqq 1}$ be a non-decreasing sequence of $\mathscr{F}_t$-stopping times, and let $\{P^k_x\}_{k\geqq 1}$, $\{P^{n,k}_x\}_{k\geqq 1}$ be sequences of probability measures such that, for each $k\geqq 1$, $P^k_{x|\mathscr{F}_k}=P_{x|\mathscr{F}_k}$, $P^{n,k}_{x|\mathscr{F}_k}=P^n_{x|\mathscr{F}_k}$, with*

$$\mathscr{F}_k\equiv\mathscr{F}_{\tau_k}:=\{E\in\mathscr{F}:E\cap\{\tau_k\leqq t\}\in\mathscr{F}_t\ \forall t\geqq 0\}.$$

Finally assume that, for each $k\geqq 1$, there exist subsequences $\{P^{n^k_j,k}_x\}_{j\geqq 1}$, $\{\mu_{n^k_j}\}_{j\geqq 1}$ such that

$$\lim_{j\to+\infty}\int_{\mathbb{R}^d}\|P^{n^k_j,k}_x-P^k_x\|\,d\mu_{n^k_j}(x)=0.$$

If

$$\lim_{k\to+\infty}P(\tau_k<t)=0\ \forall t>0,\quad \text{and}\quad \lim_{n\to+\infty}\|\mu_n-\mu\|=0,$$

then there exists a subsequence $\{P^{n_j}\}_{j\geqq 1}$ such that

$$\lim_{j\to+\infty}\|P^{n_j}_{|\mathscr{F}_t}-P_{|\mathscr{F}_t}\|=0\quad \forall t\geqq 0.$$

Remark 3.2. The above lemma is similar to Lemma 11.1.1 in [SV], with weak convergence replaced by convergence in variation. We don't need here any hypothesis of lower semicontinuity for the stopping times. Moreover we remark that the disintegration of measures assumed in Lemma 3.1 always exists since Ω is a Polish space (see [DM, Chap. III, nos. 70–74]).

Proof of Lemma 3.1. Let $\nu_{xn} = \nu_{xn}^+ - \nu_{xn}^-$ be the Jordan decomposition of the signed measure $\nu_{xn} := P^n_{x|\mathscr{F}_t} - P_{x|\mathscr{F}_t}$. We have

$$\|P^n_{x|\mathscr{F}_t} - P_{x|\mathscr{F}_t}\| = |\nu_{xn}|(\Omega) = \nu^+_{xn}(\Omega) + \nu^-_{xn}(\Omega) \geqq |P^n_x(E) - P_x(E)| \quad \forall E \in \mathscr{F}_t .$$

For each n, let $\{A^n_x, B^n_x\}$, $A^n_x \cup B^n_x = \Omega$, be a Hahn decomposition of Ω for ν_{xn}. Then

$$\|P^n_{x|\mathscr{F}_t} - P_{x|\mathscr{F}_t}\| = \nu^+_{xn}(A^n_x) + \nu^-_{xn}(B^n_x) .$$

Since $\{\tau_k < t\}$, and $E \cap \{\tau_k \geqq t\}$, are $\mathscr{F}_k$-measurable for each $\mathscr{F}_t$-measurable E, and $P^{n,k}_{x|\mathscr{F}_k} = P^n_{x|\mathscr{F}_k}$, we have

$$\begin{aligned} |P^n_x(E) - P_x(E)| &= |P^n_x(E \cap \{\tau_k < t\}) + P^{n,k}_x(E) - P^{n,k}_x(E \cap \{\tau_k < t\}) - P_x(E)| \\ &\leqq |P^{n,k}_x(E) - P_x(E)| + |P^{n,k}_x(E \cap \{\tau_k < t\}) - P^n_x(E \cap \{\tau_k < t\})| \\ &\leqq |P^{n,k}_x(E) - P_x(E)| + 2P^{n,k}_x(\tau_k < t) . \end{aligned}$$

Analogously we have

$$|P^k_x(E) - P_x(E)| \leqq 2P_x(\tau_k < t) ,$$

so that

$$\begin{aligned} \nu^+_{xn}(A^n_x) &= P^n_x(A^n_x) - P_x(A^n_x) \\ &\leqq |P^{n,k}_x(A^n_x) - P^k_x(A^n_x)| + 2P^{n,k}_x(\tau_k < t) + 2P_x(\tau_k < t) \\ &\leqq \|P^{n,k}_x - P^k_x\| + 2P^{n,k}_x(\tau_k < t) + 2P_x(\tau_k < t) \\ &\leqq 3\,\|P^{n,k}_x - P^k_x\| + 4P_x(\tau_k < t) . \end{aligned}$$

An analogous estimate holds for $\nu^-_{xn}(B^n_x)$. From

$$\|P^n - P\| \leqq \int_{\mathbb{R}^d} \|P^n_x - P_x\| d\mu_n(x) + \|\mu_n - \mu\| ,$$

one derives

$$\begin{aligned} \|P^n_{|\mathscr{F}_t} - P_{|\mathscr{F}_t}\| &\leqq 6 \int_{\mathbb{R}^d} \|P^{n,k}_x - P^k_x\|\, d\mu_n(x) \\ &\quad + 8 \int_{\mathbb{R}^d} P_x(\tau_k < t) d\mu_n(x) + \|\mu_n - \mu\| \\ &\leqq 6 \int_{\mathbb{R}^d} \|P^{n,k}_x - P^k_x\|\, d\mu_n(x) + 8P(\tau_k < t) + 9\,\|\mu_n - \mu\| . \end{aligned}$$

This implies that $P_{|\mathscr{F}_t}$ is a limit point of $\{P^n_{|\mathscr{F}_t}\}_{n \geqq 1}$, and our thesis follows.

In order to apply the preceding lemma to our case we must verify that, given a weak solution of a stochastic differential equation (s.d.e.), a random variable X, and a stopping time, the probability measure associated to the disintegrated (with respect to X) stopped process will be again a weak solution of a s.d.e. This is the content of the following

Lemma 3.3. *Let $(\Omega, \mathscr{F}, \mathscr{F}_t, B_t, P)$ be a weak solution of the s.d.e.*

$$X_t = X_0 + \int_0^t b(s, X_s)ds + B_t,$$

and let $P = \int P_x d\mu(x)$ be the disintegration of P with respect to the random variable X_0, $\mu = P \circ X_0^{-1}$. Let τ be a $\mathscr{F}_t$-stopping time. Define $P_x^\tau := P_x \circ X_\tau^{-1}$, with

$$X_\tau : \Omega \to \Omega, \qquad X_\tau(\gamma)(t) := X_{t \wedge \tau(\gamma)}(\gamma).$$

Then

$$B_t^{x,\tau} := X_t - x - \int_0^{t \wedge \tau} b(s, X_s)ds$$

is a P_x^τ-Brownian motion for μ-a.e. $x \in \mathbb{R}^d$.

Proof. By our hypotheses

$$M_t^f := f(X_t) - \int_0^t L_s f(X_s)ds,$$

with $L_s := \frac{1}{2}\Delta + b_s \cdot \nabla$, is a P-martingale for each $f \in C_c^2(\mathbb{R}^d)$, so that

$$\int_A M_s^f(\gamma)dP(\gamma) = \int_A M_t^f(\gamma)dP(\gamma) \qquad \forall A \in \mathscr{F}_s, \ \forall s \leqq t.$$

From the definition of disintegration of a measure it follows that $\gamma \mapsto P_{X_0(\gamma)}(\cdot)$ is a version of the conditional probability $P(\cdot|\sigma(X_0))$ (see [DM, Chap. III, no. 70]), so that, $\forall B \in \mathscr{F}_0$, $\forall s \leqq t$,

$$\int_B \int_A M_s^f(\gamma')dP_{X_0(\gamma)}(\gamma')dP(\gamma) = \int_{A \cap B} M_s^f(\gamma)dP(\gamma) = \int_{A \cap B} M_t^f(\gamma)dP(\gamma)$$
$$= \int_B \int_A M_t^f(\gamma')dP_{X_0(\gamma)}(\gamma')dP(\gamma).$$

Since $B \in \mathscr{F}_0$ is arbitrary, and $\mu = P \circ X_0^{-1}$, we have that M_t^f is a P_x-martingale for μ-a.e. $x \in \mathbb{R}^d$. From this, and the definition of P_x^τ, we have that

$$f(X_t) - \int_0^{t \wedge \tau} L_s f(X_s)ds$$

is a P_x^τ-martingale, and the lemma now follows from the equivalence between existence of solutions of martingale problem and existence of weak solutions of s.d.e.'s (see [St, Theorem 2.6, Chap. 3]).

Remark 3.4. Let

$$\tau(\gamma) = \inf\{t \geqq 0 : (t, X_t(\gamma)) \in D\}$$

be the first hitting time to the measurable set $D \subset \mathbb{R}_+ \times \mathbb{R}^d$. Suppose τ is a $\mathscr{F}_t$-stopping time, and let P_x^τ be defined as in Lemma 3.3. Let $\varrho_t^x(y)dy$ and $\varrho_t(y)dy$ be the images of P_x and P under X_t. Since $P^\tau_{x|\mathscr{F}_\tau} = P_{x|\mathscr{F}_\tau}$, we have, for each function $f \geqq 0$,

$$E_x^\tau \int_0^{t \wedge \tau} f(s, X_s)ds = E_x^\tau \int_0^t \chi_{[0,\tau)}(s) f(s, X_s)ds = \int_0^t \int_{\{\tau > s\}} f(s, X_s)dP_x ds$$
$$\leqq \int_0^t \int_\Omega \chi_{D^c}(s, X_s) f(s, X_s) f(s, X_s)dP_x ds \leqq \int_{D^c} f(s, y)d\varrho_s^x(y)dyds$$

(here E_x^τ denotes expectation with respect to P_x^τ). Since $\int \varrho_t^x(y)\varrho_0(x)dx = \varrho_t(y)$, it is easy to prove that there exists a positive constant M such that $\varrho_t^x(y) \leqq M\varrho_t(y)$ for $\varrho_0(x)dtdxdy$-a.e. $(t,x,y) \in \mathbb{R}_+ \times \mathbb{R}^{2d}$. In conclusion we have

$$E_x^\tau \int_0^{t\wedge\tau} f(s,X_s)ds \leqq M \int_{D^c} f(s,y)\varrho_s(y)dyds \quad \text{for} \quad \varrho_0(x)dx\text{-a.e.}, \qquad x \in \mathbb{R}^d.$$

In particular this implies, by Remark 2.3, that, if P is the probability measure given by Theorem 2.2, and if there exists a $T < +\infty$ such that $D^c \subset [0,T] \times \mathbb{R}^d$, then $P_x^\tau \ll W_x$ for $\varrho_0(x)dx$-a.e. $x \in \mathbb{R}^d$.

4. A Parabolic Capacity and Pointwise Behaviour of Solutions of Schrödinger Equations

As we have seen in Sect. 2, the natural space for solutions of the Schrödinger equation with K-form-bounded potentials and initial data in $H^1(\mathbb{R}^d)$ is the Banach space $(\mathscr{W}_T, \|\cdot\|_{\mathscr{W}_T})$, where

$$\mathscr{W}_T := C([0,T], H^1(\mathbb{R}^d)) \quad \text{and} \quad \|u\|_{\mathscr{W}_T} := \sup_{0\leqq t\leqq T} \|u_t\|_{H^1}.$$

We will consider real-valued functions only; considering real and imaginary parts separately, Theorem 4.3 below holds for complex-valued functions as well. We need "good" pointwise properties of functions belonging to $\mathscr{W}_T$. To this end we will introduce a sort of parabolic capacity on subsets of $[0,T] \times \mathbb{R}^d$, and we will study properties of elements of $\mathscr{W}_T$ up to sets of arbitrary small capacity. Following the general procedure in reference [AS], we define a set function on subsets of $[0,T] \times \mathbb{R}^d$,

$$\Gamma_T(E) := \inf_{\{E_k\}_{k\in\mathbb{N}},\, E_k \text{ open},\, \bigcup_k E_k \subset E} \sum_k \delta_T(E_k),$$

where, for an open set E

$$\delta_T(E) := \inf\{\|u\|^2_{\mathscr{W}_T} : u \in \mathscr{W}_T,\ u \geqq \chi_E \text{ a.e.}\}.$$

The set function Γ_T has the following properties ([AS], p. 146):

P_1: $\Gamma_T(\emptyset) = 0$;

P_2: $E \subset E' \Rightarrow \Gamma_T(E) \leqq \Gamma_T(E')$;

P_3: $\Gamma_T\left(\bigcup_k E_k\right) \leqq \sum_k \Gamma_T(E_k)$;

P_4: $\forall \varepsilon > 0 \quad \exists \delta > 0$ such that $\delta_T(E) \leqq \delta \Rightarrow \Gamma_T(E) \leqq \varepsilon$.

From the above definition it is also obvious that there exists a relation between Γ_T and the Choquet capacity defined in Remark 2.4:

Lemma 4.1.

$$\Gamma_T(E) \geqq \operatorname{cap}(E_t) \qquad \forall t \in [0,T],$$

where $E_t := \{x \in \mathbb{R}^d : (t,x) \in E\}$.

Proof. From the definitions of Γ_T, cap, and by countable subadditivity of cap (see [Fu]), it follows

$$\Gamma_T(E) \geqq \inf_{\{E_k\}_{k\in\mathbb{N}},\, E_k \text{ open},\, \bigcup_k E_k \supset E} \sum_k \operatorname{cap}((E_k)_t)$$
$$\geqq \inf_{\{E_k\}_{k\in\mathbb{N}},\, E_k \text{ open},\, \bigcup_k E_k \supset E} \operatorname{cap}\Big(\bigcup_k (E_k)_t\Big)$$
$$\geqq \inf_{A\supset E} \operatorname{cap}(A_t) = \operatorname{cap}(E_t).$$

Remark 4.2. Let D be an open subset of $\mathbb{R}_+ \times \mathbb{R}^d$. Since D is open, and $X_{(\cdot)}$ is continuous, we have

$$\{\tau_D < T\} = \bigcup_{q\in Q\cap[0,T)} \{X_q \in D_q\},$$

where Q is any dense denumerable subset of $\mathbb{R}$. By Lemma 4.1 and Remark 2.4 it follows that if $\Gamma_T(D\cap[0,T]\times\mathbb{R}^d)=0$ $\forall T\geqq 0$, then D is polar for the process $Y_t := (t, X_t)$ with respect to P. Moreover, if $\{D_k\}_{k\geqq 1}$ is a decreasing sequence of open subsets of $\mathbb{R}_+\times\mathbb{R}^d$ such that $\Gamma_T(D_k\cap[0,T]\times\mathbb{R}^d)\downarrow 0$ $\forall T\geqq 0$, then, by Lemma 4.1, $\operatorname{cap}((D_k)_t)\downarrow 0$ $\forall t\geqq 0$, and, by Remark 2.4, $W(\tau_{(D_k)_t} < T)\downarrow 0$ $\forall T>0$, $\forall t\geqq 0$. Therefore $W(\tau_{D_k} < T)\downarrow 0$ $\forall T>0$, and $P(\tau_{D_k} < T)\downarrow 0$ $\forall T>0$, by Remark 2.4.

We state now the main result of this paragraph. This result does not depend on our particular definition of Γ_T but holds for any capacity defined by means of a "good" functional space (see [AS]).

Theorem 4.3. 1) *Let u be in $\mathscr{W}_T$. Then there exists a decreasing sequence of open sets*

$$D_{T,k} \subset [0,T]\times\mathbb{R}^d$$

such that $\Gamma_T(D_{T,k})\downarrow 0$, and the restriction of u to $D^c_{T,k}\cap[0,T]\times\mathbb{R}^d$ is continuous for all k.

2) *Let $\{u_n\}_{n\geqq 1}\subset\mathscr{W}_T$ be a sequence such that $\mathscr{W}_T$-$\lim_{n\to+\infty} u_n = u\in\mathscr{W}_T$. Then there exist a decreasing sequence of open sets $D_{T,k}\subset[0,T]\times\mathbb{R}^d$ such that $\Gamma_T(D_{T,k})\downarrow 0$, and a subsequence $\{u_{n_j}\}_{j\geqq 1}$ converging pointwise and uniformly to u on $D^c_{T,k}\cap[0,T]\times\mathbb{R}^d$ for all k.*

Proof. 1) First of all we note that $\mathscr{W}_T\cap C([0,T]\times\mathbb{R}^d)$ is dense in $\mathscr{W}_T$. This can be seen considering, for each $u\in\mathscr{W}_T$, the approximating sequence of continuous functions $u_n(t,x) := (J_{1/n} * u_t)(x)$, where

$$J_{1/n}\in C_c^\infty(\mathbb{R}^d), \qquad \operatorname{supp}(J_{1/n})\subset\{x : \|x\|\leqq 1/n\},$$

is a molliflier, and then proceeding in the same way as in [LSU, Lemma 4.8, Chap. II].

From the definition of δ_T we have

$$\delta_T(\{(t,x) : |u(t,x)|>\lambda\}) \leqq \frac{1}{\lambda^2}\|u\|^2_{\mathscr{W}_T} \qquad \forall u\in\mathscr{W}_T\cap C([0,T]\times\mathbb{R}^d). \tag{$*$}$$

Now we proceed as in [AS, pp. 148–149] (see also [Fu, Theorem 3.1.3]): let

$$\{u_n\}_{n\geqq 1}\subset\mathscr{W}_T\cap C([0,T]\times\mathbb{R}^d)$$

be a sequence such that $\mathscr{W}_T$-$\lim_{n\to+\infty} u_n = u\in\mathscr{W}_T$. Since $\{u_n\}_{n\geqq 1}$ is a Cauchy sequence, by ($*$) and P_4, there exists a subsequence $\{u_{n_j}\}_{j\geqq 1}$ such that

$$\Gamma_T(A_j) < 2^{-j},$$

where the sequence of open sets $\{A_j\}_{j\geqq 1}$ is defined by

$$A_j := \{(t,x) : |u_{n_{j+1}}(t,x) - u_{n_j}(t,x)| > 2^{-j}\}.$$

If $(t,x) \in \left(\bigcup_{j\geqq J} A_j\right)^c$, then $\forall j \geqq J$, $\forall p$ we have

$$|u_{n_{j+p}}(t,x) - u_{n_j}(t,x)| \leqq \sum_{k=j+1}^{j+p} |u_{n_k}(t,x) - u_{n_{k-1}}(t,x)| \leqq 2^{-j},$$

so that $\{u_{n_j}\}_{j\geqq 1}$ uniformly converges on $\left(\bigcup_{j\geqq J} A_j\right)^c$. This implies the continuity of u on $\left(\bigcup_{j\geqq J} A_j\right)^c$. By P_3 we have

$$\Gamma_T\left(\bigcup_{j\geqq J} A_j\right) \leqq \sum_{j\geqq J} \Gamma_T(A_j) \leqq 2^{1-J}.$$

Since J is arbitrary, 1) is proven.

2) By 1), proceeding as in [Fu, Lemma 3.1.5], we have

$$\Gamma_T(\{(t,x) : |u(t,x)| > \lambda\}) \leqq \frac{1}{\lambda^2} \|u\|^2_{\mathscr{W}_T} \qquad \forall u \in \mathscr{W}_T,$$

so that, by our hypotheses, u_n converges to u in capacity, i.e.

$$\lim_{n\to+\infty} \Gamma_T(\{(t,x) : |u_n(t,x) - u(t,x)| > \varepsilon\}) = 0 \qquad \forall \varepsilon > 0.$$

Then one proceeds in essentially the same way as in 1).

5. Stopping Times and Nonattainability

We now define the stopping times we will need for the proof of our main theorem. Let ψ, ψ^n, $n \geqq 1$, be functions belonging to $\mathscr{W}_T$ $\forall T \geqq 0$. Assume

$$\lim_{n\to+\infty} \|\psi^n - \psi\|_{\mathscr{W}_T} = 0 \qquad \forall T \geqq 0,$$

and define, $\forall k \geqq 1$,

$$\tau_k^j(\gamma) := \inf\{t \geqq 0 : (t, X_t(\gamma)) \in D_k^j\} \qquad j = 1, 2,$$

where

$$D_k^1 := \{(t,x) : \|(t,x)\| > k\} \cup \{(t,x) : |\psi(t,x)| < 1/k\},$$

and the D_k^2's are the open subsets of $\mathbb{R}_+ \times \mathbb{R}^d$,

$$D_k^2 := \bigcup_{T\in\mathbb{R}_+} D_{T,k},$$

where the sets $D_{T,k}$ are given in Theorem 4.3. Define $D_k := D_k^1 \cup D_k^2$; by construction the following holds:

1) $\Gamma_T(D_k^2 \cap [0,T] \times \mathbb{R}^d) \downarrow 0 \quad \forall T \geqq 0$;
2) $\psi, \psi^n \in L^\infty(D_k^c)$ $\forall k \geqq 1$, $\forall n \geqq 1$;
3) there exists a subsequence $\{\psi^{n_j}\}_{j\geqq 1}$ such that

$$\lim_{j\to+\infty} \|\psi^{n_j} - \psi\|_{L^\infty(D_k^c)} = 0 \quad \forall k \geqq 1.$$

We remark that, since $\|\cdot\|_{\mathscr{W}_T} \leqq \|\cdot\|_{\mathscr{W}_{T'}}$, if $T \leqq T'$, the $D_{T,k}$'s may be chosen in such a way that $D_{T,k} \subseteq D_{T',k}$, if $T \leqq T'$.

In order to apply Lemma 3.1 we need to prove that τ_k^1, and τ_k^2, are $\mathscr{F}_t$-stopping times, and that $P(\tau_k^1 \wedge \tau_k^2 < T) \downarrow 0$. This is the content of the following two lemmas.

Lemma 5.1. *τ_k^1 and τ_k^2 are $\mathscr{F}_t$-stopping times.*

Proof. 1) By Remark 4.2, and [BG, Theorem 10.7, Definition 10.21], τ_k^1 is a $\mathscr{F}_t$-stopping time if D_k^1 is a "nearly Borel set," i.e. if there exist Borel sets B_k and B_k' such that

$$B_k \subset D_k^1 \subset B_k', \quad \text{and} \quad \Gamma_T(B_k' \cap B_k^c \cap [0,T] \times \mathbb{R}^d) = 0 \qquad \forall T \geqq 0 .$$

Since the class of nearly Borel sets is a σ-algebra, it will suffice to prove that $(D_k^1)^c$ is a nearly Borel set.

We have $\psi \in \mathscr{W}_T$ $\forall T \geqq 0$, so that, by Theorem 4.3, there exists a decreasing sequence of open sets $\{U_m\}_{m \geqq 1}$, $U_m \subset \mathbb{R}_+ \times \mathbb{R}^d$, $\Gamma_T(U_m \cap [0,T] \times \mathbb{R}^d) \downarrow 0$ $\forall T \geqq 0$, such that ψ is continuous on $U_m^c \cap [0,T] \times \mathbb{R}^d$ $\forall T \geqq 0$ $\forall m \geqq 1$. This implies that

$$|\psi|^{-1}[1/\mathrm{k}, +\infty) \cap \{\|(t,x)\| \leqq k\} \cap \bigcup_{m \geqq 1} U_m^c$$

is a Borel set. Since

$$\Gamma_T\Big(\bigcap_{m \geqq 1} U_m \cap [0,T] \times \mathbb{R}^d\Big) \leqq \inf_m \Gamma_T(U_m \cap [0,T] \times \mathbb{R}^d) = 0 \qquad \forall T \geqq 0 ,$$

D_k^1 is a nearly Borel set.

2) τ_k^2 is a $\mathscr{F}_t$-stopping time since D_k^2 is an open set.

Lemma 5.2. *Let ψ_t, ψ_t^n, $n \geqq 1$, be solutions of Schrödinger equations with K-form-bounded potentials with relative bounds smaller than one, and initial data in $H^1(\mathbb{R}^d)$. Suppose $\psi_t^n \to \psi_t$ in $\mathscr{W}_T$ $\forall T > 0$, define D_k^1, and D_k^2, as above and let P be the probability measure corresponding to ψ_t. Then $P(\tau_k^1 \wedge \tau_k^2 < T) \downarrow 0$ $\forall T > 0$.*

Proof. Since $P(\tau_k^1 \wedge \tau_k^2 < T) \leqq P(\tau_k^1 < T) + P(\tau_k^2 < T)$, we will prove $P(\tau_k^1 < T) \downarrow 0$ and $P(\tau_k^2 < T) \downarrow 0$ separately:

1) Let us denote by $\tau_k^{1,1}$ and $\tau_k^{1,2}$ the first hitting times to the sets

$$\{(t,x) : \|(t,x)\| > k\}, \quad \text{and} \quad \{(t,x) : |\psi(t,x)| < 1/k\}$$

respectively. Then

$$P(\tau_k^1 < T) \leqq P(\tau_k^{1,1} < T) + P(\tau_k^{1,2} < T) .$$

One has $P(\tau_k^{1,1} < T) \downarrow 0$ $\forall T > 0$ by Theorem 2.2, since this is equivalent to the non-explosion of the process $(\Omega, \mathscr{F}, \mathscr{F}_t, X_t, P)$. From Theorem 2.2 one has also $P(\tau_k^{1,2} < T) \downarrow 0$ $\forall T > 0$, since $|\psi_t|^2$ is the density of the process X_t with respect to P;

2) $P(\tau_k^2 < T) \downarrow 0$ by Remark 4.2, and the definition of D_k^2.

6. Convergence of Nelson Diffusions

We have now at our disposal all the ingredients to prove our main result:

Theorem 6.1. *Let V, V_n, $n \geqq 1$, be K-form-bounded potentials, with relative bounds smaller than one. Let H, H_n, $n \geqq 1$, be the self-adjoint operators $H = K + V$,*

$H_n=K+V_n$, defined as quadratic forms. Consider the sequence of initial data $\{\psi_0^n\}_{n\geqq 1}\subset H^1(\mathbb{R}^d)$, and let $\{\psi_t^n\}_{n\geqq 1}$ be the sequence defined by $\psi_t^n:=e^{-itH_n}\psi_0^n$. Denote by $\{P^n\}_{n\geqq 1}$ the corresponding sequence of probability measures on the path space $\Omega=C(\mathbb{R}_+,\mathbb{R}^d)$ given by Theorem 2.2. If

$$\lim_{n\to+\infty}\sup_{0\leqq t\leqq T}\|\psi_t^n-\psi_t\|_{H^1}=0 \qquad \forall T\geqq 0,$$

where $\psi_t=e^{-itH}\psi_0$, $\psi_0\in H^1(\mathbb{R}^d)$, then

$$\lim_{n\to+\infty}\|P^n_{|\mathscr{F}_t}-P_{|\mathscr{F}_t}\|=0 \quad \forall t\geqq 0,$$

where P is the probability measure corresponding to ψ_t.

Proof. Let $\tau_{D_k}=\tau_k^1\wedge\tau_k^2\equiv\tau_k$, where τ_k^1, τ_k^2, and D_k, are defined in Sect. 5, and let $P_x^k\equiv P_x^{\tau_k}$, $P_x^{n,k}\equiv P_x^{n,\tau_k}$ be defined as in Lemma 3.3. We have proven in Lemma 5.2 that $P(\tau_k<T)\downarrow 0$ $\forall T>0$. Moreover $\|\varrho_n(0,\cdot)-\varrho(0,\cdot)\|_{L^1}$ converges to zero by our hypotheses, since $\varrho_n(0,y)=|\psi_0^n(y)|^2$, by Theorem 2.2. Therefore, by Lemma 3.1, if one finds a subsequence $\{P_x^{n_j,k}\}_{j\geqq 1}$ such that

$$\lim_{j\to+\infty}\int_{\mathbb{R}^d}\|P_x^{n_j,k}-P_x^k\|\varrho_{n_j}(0,x)dx=0 \qquad \forall k\geqq 1,$$

then there exists a subsequence $\{P^{n_j}\}_{j\geqq 1}$ such that

$$\lim_{j\to+\infty}\|P^{n_j}_{|\mathscr{F}_t}-P_{|\mathscr{F}_t}\|=0 \quad \forall t\geqq 0.$$

We prove now that such subsequence exists. By [JS, Theorem 4.21, Chap. V],

$$\|P_x^{n,k}-P_x^k\|^2\leqq 16E_x^{n,k}(h_{x,\infty}^{n,k})$$

(here $E_x^{n,k}$ denotes expectation with respect to $P_x^{n,k}$), where the increasing predictable process $h_{x,t}^{n,k}$ is the Hellinger process of order $\frac{1}{2}$ between $P_x^{n,k}$ and P_x^k (see [JS, Definition 1.24, Chap. IV]). Moreover, by Lemma 3.3, there exist sets A, A_n with $\int\chi_A\varrho(0,x)dx=\int\chi_{A_n}\varrho_n(0,x)dx=1$, such that, $\forall x\in\tilde{A}_n:=A\cap A_n$,

$$B_t^k=X_t-x-\int_0^{t\wedge\tau_k}b(s,X_s)ds,$$

and

$$B_t^{n,k}=X_t-x-\int_0^{t\wedge\tau_k}b_n(s,X_s)ds$$

are Brownian motions with respect to P_x^k and $P_x^{n,k}$ respectively [since $\varrho_n(0,\cdot)\to\varrho(0,\cdot)\exists\bar{n}$ such that $\tilde{A}_n\neq\emptyset$ $\forall n\geqq\bar{n}$]. Since, by Remarks 2.3 and 3.4,

$$\tfrac{1}{2}E_x^k\int_{\mathbb{R}_+}\chi_{[0,\tau_k)}(s)\,\|b\|^2(s)ds=H(P_x^k;W_x)<+\infty,$$

$$\tfrac{1}{2}E_x^{n,k}\int_{\mathbb{R}_+}\chi_{[0,\tau_k)}(s)\,\|b_n\|^2(s)ds=H(P_x^{n,k};W_x)<+\infty,$$

by [JS, Theorem 4.23, Chap. IV], we have that the process

$$\tfrac{1}{8}\int_0^t\chi_{[0,\tau_k)}(s)\,\|b_n-b\|^2(s)ds$$

is a version of the Hellinger process $h^{n,k}_{x,t}$, $\forall x\in \tilde{A}_n$. In conclusion, by Remark 3.4,

$$\left(\int_{\mathbb{R}^d} \|P^{n,k}_x - P^k_x\|\varrho_n(0,x)dx\right)^2$$
$$\leqq 2\int_{\mathbb{R}^d}\int_{D^c_k} \|b_n(t,y)-b(t,y)\|^2\varrho^x_n(t,y)\varrho_n(0,x)dydtdx$$
$$+\int_{\tilde{A}^c_n}\|P^{n,k}_x-P^k_x\|^2\varrho_n(0,x)dx\,.$$

Since $\int \chi_{\tilde{A}^c_n}\varrho_n(0,x)dx\to 0$, we have to prove that there exists a subsequence $\{b_{n_j}\}_{j\geqq 1}$, such that

$$\lim_{j\to+\infty}\int_{D^c_k}\|b_{n_j}-b\|^2\varrho_{n_j}(t,y)dydt=0 \qquad \forall k\geqq 1\,.$$

From the definitions of b_n and ϱ_n, we have

$$\int_{D^c_k}\|b_n-b\|^2\varrho_n dydt$$
$$=\int_{D^c_k}\left\|\Re\left(\frac{\nabla\psi^n_t}{\psi^n_t}-\frac{\nabla\psi_t}{\psi_t}\right)+\Im\left(\frac{\nabla\psi^n_t}{\psi^n_t}-\frac{\nabla\psi_t}{\psi_t}\right)\right\|^2\psi^n_t-\psi^n_t dydt$$
$$\leqq 2\int_{D^c_k}\left\|\nabla\psi^n_t-\frac{\psi^n_t}{\psi_t}\nabla\psi_t\right\|^2dydt$$
$$\leqq 2\int_0^k\int_{\mathbb{R}^d}\|\nabla\psi^n_t-\nabla\psi_t\|^2dydt+2\int_{D^c_k}\frac{\|\nabla\psi_t\|^2}{\|\psi_t\|^2}\|\psi^n_t-\psi_t\|^2dydt$$
$$\leqq 2k\sup_{0\leqq t\leqq k}\|\nabla\psi^n_t-\nabla\psi_t\|^2_{L^2}+2k^3\sup_{0\leqq t\leqq k}\|\nabla\psi_t\|^2_{L^2}\|\psi^n-\psi\|^2_{L^\infty(D^c_k)}\,,$$

and we have proved the existence of a converging subsequence $\{P^{n_j}_{|\mathscr{F}_t}\}_{j\geqq 1}$ $\forall t\geqq 0$. Suppose now that the whole sequence $\{P^n_{|\mathscr{F}_t}\}_{n\geqq 1}$ does not converge. Then there exists a subsequence $\{P^{n_k}\}_{k\geqq 1}$ and an $\varepsilon>0$ such that $\|P^{n_k}_{|\mathscr{F}_t}-P_{|\mathscr{F}_t}\|>\varepsilon$ for all k. But by the above reasoning applied to the convergent sequence $\{\psi^{n_k}_t\}_{k\geqq 1}$ we get a further subsequence along which the measures converge to P, which would be a contradiction, so $\{P^n_{|\mathscr{F}_t}\}_{n\geqq 1}$ converges to $P_{|\mathscr{F}_t}$.

Remark 6.2. Since

$$E^{n,k}_x\int_{\mathbb{R}_+}\chi_{[0,\tau_k)}(s)\,\|b_n-b\|^2(s)ds<+\infty\Rightarrow P^{n,k}_x\ll P^k_x$$

(see [JS, Theorem 4.23, Chap. IV]), by Remark 3.4, we have

$$\int_{D^c_k}\|b_n-b\|^2(s,y)\varrho_n(s,y)dyds<+\infty\Rightarrow P^{n,k}_x\ll P^k_x\,,$$

so that $P^{n,k}_x\ll P^k_x$ follows from

$$b_n\in L^2([0,T]\times\mathbb{R}^d;\,\varrho_n dyds),\quad \varrho_n\in L^\infty(D^c_k),\quad\text{and}\quad b\in L^2_{\mathrm{loc}}(Z^c),$$

where $Z:=\{(t,x):\psi_t(x)=0\}$. From this we have

$$B^k_t=\int_0^{t\wedge\tau_k}(b_n-b)(s,X_s)ds+B^{n,k}_t,\qquad P^{n,k}_x\text{-a.s.},$$

so that, by [Fö1, Proposition 2.11],

$$H(P^{n,k}_x;P^k_x)=\tfrac{1}{2}E^{n,k}_x\int_{\mathbb{R}_+}\chi_{[0,\tau_k)}(s)\,\|b_n-b\|^2(s)ds\,.$$

Since

$$\|P_x^{n,k} - P_x^k\| = \int_\Omega \left|\frac{dP_x^{n,k}}{dP_x^k} - 1\right| dP_x^k \leqq H(P_x^{n,k}; P_x^k)$$

(see [Fö2, Remark 3.2]), Theorem 6.1 could be proved directly using entropy estimates.

Remark 6.3. If the sequence $\{\psi_t^n\}_{n\geqq 1}$ given in Theorem 6.1 does not converge but it is only bounded with respect to the energy norm, i.e. if

$$\sup_{n\in\mathbb{N}} \int_0^T \|\nabla\psi_t^n\|_{L^2}^2 dt < +\infty \qquad \forall T \geqq 0,$$

then, by Remark 2.3,

$$\sup_{n\in\mathbb{N}} H_{\mathscr{F}_T}(P^n; W^n) < +\infty \qquad \forall T \geqq 0,$$

where $W^n := \int W_x |\psi_0^n(x)|^2 dx$. Suppose moreover that the sequence $\{|\psi_0^n|^2 dx\}_{n\geqq 1}$ is precompact with respect to the weak-* topology on $\mathscr{M}_1(\mathbb{R}^d)$. Then, by [Z, Theorem 5], the sequence $\{P^n\}_{n\geqq 1}$ is precompact with respect to the weak-* topology on $\mathscr{M}_1(\Omega)$, and

$$H_{\mathscr{F}_T}(Q; W) < +\infty \qquad \forall T \geqq 0,$$

where $W := \int W_x d\mu(x)$, and (μ, Q) is any limit point of $\{(|\psi_0^n|^2 dx, P^n)\}_{n\geqq 1}$.

7. Convergence of Nelson Diffusions with Kato-Class Potentials

In the light of Theorem 6.1 it will be interesting to find conditions on the potentials which will guarantee the H^1-convergence of the solutions of the corresponding Schrödinger equations. To this end we now suppose that the potentials are in the Kato-class K_d, where

$$K_d := \left\{V : \limsup_{\alpha\downarrow 0} \sup_x \int_{\|x-y\|\leqq\alpha} \|x-y\|^{2-d} |V(y)| dy = 0\right\}, \qquad d \geqq 3,$$

$$K_2 := \left\{V : \limsup_{\alpha\downarrow 0} \sup_x \int_{\|x-y\|\leqq\alpha} \log\|x-y\|^{-1} |V(y)| dy = 0\right\},$$

$$K_1 := \left\{V : \sup_x \int_{\|x-y\|\leqq 1} |V(y)| dy < +\infty\right\}$$

(see [CFKS, Sect. 1.2], [Si2, Sect. A2]). We also define a K_d-norm by

$$\|V\|_{K_d} := \sup_x \int_{\|x-y\|\leqq 1} Q(x-y; d)\, |V(y)| dy,$$

where Q is the kernel in the above definition. One has the following inclusions:

$$L^p(\mathbb{R}^d) \subset L^p_{\mathrm{unif}}(\mathbb{R}^d) \subseteqq K_d \subseteqq L^1_{\mathrm{unif}}(\mathbb{R}^d),$$

with $p > d/2$ if $d \geqq 2$, $p = 2$ otherwise, where

$$L^p_{\mathrm{unif}}(\mathbb{R}^d) := \left\{V : \sup_x \int_{\|x-y\|\leqq 1} |V(y)|^p dy < +\infty\right\}$$

(see [CFKS, Sect. 1.2]), and

$$\|V\|_{K_d} \leqq \|Q\|_{L^q_{\mathrm{unif}}} \|V\|_{L^p_{\mathrm{unif}}}, \qquad 1/q + 1/p = 1.$$

By [CFKS, Sect. 1.2], if $V \in K_d$, then V is K-form-bounded, with relative bound zero, so that, when we have a sequence of potentials in K_d, we may apply Theorem 6.1. The following theorem gives us a criterion for convergence of Nelson Diffusions in terms of convergence of the physical data that generate them:

Theorem 7.1. *Let $V, V_n \in K_d$, $n \geqq 1$, $\psi_0, \psi_0^n \in H^1(\mathbb{R}^d)$, $n \geqq 1$. If P, P^n, $n \geqq 1$ are the probability measures on $\Omega = C(\mathbb{R}_+, \mathbb{R}^d)$ which correspond, according to Theorem 2.2, to $\psi_t = e^{-itH}\psi_0$, $\psi_t^n = e^{-itH_n}\psi_0^n$, $H = K + V$, $H_n = K + V_n$, and if*

$$\lim_{n \to +\infty} \|\psi_0^n - \psi_0\|_{H^1} = 0, \quad \text{and} \quad \lim_{n \to +\infty} \|V_n - V\|_{K_d} = 0,$$

then

$$\lim_{n \to +\infty} \|P^n_{|\mathscr{F}_t} - P_{|\mathscr{F}_t}\| = 0 \quad \forall t \geqq 0.$$

Proof. We will prove the case $d \geqq 3$, for the other cases the proof is analogous. Since

$$\|e^{-itH_n}\psi_0^n - e^{-itH}\psi_0\|_{H^1} \leqq \|(e^{-itH_n} - e^{-itH})\psi_0\|_{H^1} + \|e^{-itH_n}\|_{H^1, H^1} \|\psi_0^n - \psi_0\|_{H^1},$$

in order to apply Theorem 6.1 we have to prove

1) $$\sup_{n \in \mathbb{N}} \|e^{-itH_n}\|_{H^1, H^1} < +\infty,$$

2) $$\lim_{n \to +\infty} \sup_{0 \leqq t \leqq T} \|(e^{-itH_n} - e^{-itH})\psi\|_{H^1} = 0 \quad \forall \psi \in H^1(\mathbb{R}^d).$$

From the Kato-Trotter theorem (see [K], Theorem 2.16, Chap. IX) 2) is implied by 1) and 2′) there exists a complex number z, $\Im z > 0$ such that

$$\lim_{n \to +\infty} \|(H_n - zI)^{-1}\psi - (H - zI)^{-1}\psi\|_{H^1} = 0 \quad \forall \psi \in H^1(\mathbb{R}^d).$$

Let us at first show that $\forall \varepsilon > 0\ \exists \gamma_\varepsilon > 0$, $\exists n_\varepsilon > 0$ such that

$$\|(\mathrm{K} + \gamma \mathrm{I})^{-1} |\mathrm{V}_n|\,\|_{L^\infty, L^\infty} < \varepsilon \quad \forall \gamma \geqq \gamma_\varepsilon,\ \forall n \geqq n_\varepsilon.$$

We will proceed as in [CFKS, Sect. 1.2]. By [RS, Theorem IX.29], $(K + \gamma I)^{-1}$ is a convolution operator with an explicit kernel $G(x - y; \gamma)$, so that we may write, using the known properties of G (see [Sc, Theorem 3.1, Chap. 6]), and Lemma 2.6 in [Sc, Chap. 5],

$$\begin{aligned}
\|(K + \gamma I)^{-1} |V_n|\|_{L^\infty} &\leqq \sup_x \int_{\|x-y\| \leqq 1/\sqrt{\gamma}} G(x - y; \gamma) |V_n(y)| dy \\
&\quad + \sup_x \int_{\|x-y\| > 1/\sqrt{\gamma}} G(x - y; \gamma) |V_n(y)| dy \\
&\leqq c_1 \sup_x \int_{\|x-y\| \leqq 1/\sqrt{\gamma}} \|x - y\|^{2-d} |V_n(y)| dy \\
&\quad + \frac{c_2}{\sqrt{\gamma}} \sup_x \int_{\|x-y\| < 1/\sqrt{\gamma}} |V_n(y)| dy \\
&\leqq c_1 \|V_n - V\|_{K_d} + c_1 \sup_x \int_{\|x-y\| \leqq 1/\sqrt{\gamma}} \|x - y\|^{2-d} |V(y)| dy \\
&\quad + \frac{c_2}{\sqrt{\gamma}} \|V_n\|_{K_d}.
\end{aligned}$$

Since $\|V_n - V\|_{K_d} \to 0$, and $V \in K_d$, $\forall \varepsilon > 0\ \exists \gamma_\varepsilon$, $\exists n_\varepsilon$ such that

$$\|(K + \gamma I)^{-1} |V_n|\|_{L^\infty} < \varepsilon \quad \forall \gamma \geqq \gamma_\varepsilon,\ \forall n \geqq n_\varepsilon.$$

This gives the result, since $G(\cdot - y;\gamma)|V_n|$ is a positive integral kernel, and $\|A\|_{L^\infty,L^\infty} = \|A1\|_{L^\infty}$ for any A with positive integral kernel. From the above result, by duality, and by Stein interpolation theorem, proceeding in the same way as in [CFKS, Corollary 2.8], it follows that $\forall\varepsilon>0\ \exists\gamma_\varepsilon,\ \exists n_\varepsilon$ such that

$$\||V_n|^{1/2}(K+\gamma I)^{-1/2}\|_{L^2,L^2}<\varepsilon \qquad \forall\gamma\geqq\gamma_\varepsilon,\ \forall n\geqq n_\varepsilon .$$

Since

$$|\langle\psi|V_n\psi\rangle_{L^2}|\leqq\||V_n|^{1/2}(K+\gamma I)^{-1/2}\|^2_{L^2,L^2}(\langle\psi|K\psi\rangle_{L^2}+\gamma\|\psi\|^2_{L^2}),$$

we have that, $\forall n\geqq n_1$, choosing $\gamma\geqq\gamma_1$, all the V_n's are K-form-bounded with the same bound

$$a=\sup_{n\geqq n_1}\||V_n|^{1/2}(K+\gamma I)^{-1/2}\|^2_{L^2,L^2}<1\,, \qquad b=\gamma a .$$

Since

$$\|e^{-itH_n}\|_{H^1,H^1}\leqq(2(b+1)+a)(1-a)^{-1}$$

(see Sect. 2), we have that 1) holds true.

Let us now consider the operator

$$A_n(z):=(K+zI)^{-1/2}V_n(K+zI)^{-1/2} .$$

Since V_n is K-form-bounded with relative bound 0, by [CFKS, Proposition 1.3], $A_n(i\gamma)$ is a bounded operator with

$$\lim_{\gamma\to+\infty}\|A_n(i\gamma)\|_{L^2,L^2}=0 .$$

From the definition of A_n it follows, if $\gamma>0$,

$$\|A_n(i\gamma)\|_{L^2,L^2}\leqq c_3\|A_n(\gamma)\|_{L^2,L^2}\leqq c_3\|\,|V_n|^{1/2}(K+\gamma I)^{-1/2}\|^2_{L^2,L^2},$$

so that

$$\|A_n(i\gamma)\|_{L^2,L^2}<1 \qquad \forall\gamma\geqq\gamma_{1/c_3},\ \forall n\geqq n_{1/c_3},$$

and the Tiktopoulos' formula holds:

$$(H_n+i\gamma I)^{-1}=(K+i\gamma I)^{-1/2}(I+A_n(i\gamma))^{-1}(K+i\gamma I)^{-1/2}$$

(see [Si1, Sect. II.3]). Therefore we have

$$\begin{aligned}&\|((H_n+i\gamma)^{-1}-(H+i\gamma)^{-1})\psi\|_{H^1}\\ &\quad\leqq\|((I+A_n(i\gamma))^{-1}-(I+A(i\gamma))^{-1})(K+i\gamma I)^{-1/2}\psi\|_{L^2} .\end{aligned}$$

Since

$$\begin{aligned}\|A_n(i\gamma)-A(i\gamma)\|_{L^2,L^2}&\leqq c_3\|(K+\gamma I)^{-1/2}(V_n-V)(K+\gamma I)^{-1/2}\|_{L^2,L^2}\\ &\leqq c_3\||V_n-V|^{1/2}(K+\gamma I)^{-1/2}\|^2_{L^2,L^2},\end{aligned}$$

and

$$\||V_n-V|^{1/2}(K+\gamma I)^{-1/2}\|^2_{L^2,L^2}\leqq c_4\|V_n-V\|_{K_d}$$

(see [Sc, Theorem 2.2, Chap. 5, Theorem 3.1, Chap. 6]), 2') follows, and the proof of Theorem 7.1 is complete.

Remark 7.2. In Theorem 7.1 one can replace K_3 with

$$R := \left\{ V : \|V\|_R^2 := \int_{\mathbb{R}^6} \frac{|V(x)|\,|V(y)|}{\|x-y\|^2} dxdy < +\infty \right\},$$

the Banach space of Rollnik-class potentials, and Kato-convergence of potentials with convergence with respect to Rollnik norm $\|\cdot\|_R$. The proof proceeds in an analogous way, using Theorems I.21 and II.13 in [S1].

Remark 7.3. It may appear that convergence of initial data in $H^1(\mathbb{R}^d)$ be an unnecessary strong assumption; since one can disintegrate with respect to the initial distributions, one may expect that L^2-convergence be sufficient. However, suppose that, for every $\psi_0 \in H^1(\mathbb{R}^d)$, $T>0$, and for some $M>1$,

$$\sup_{n\in\mathbb{N}} \|e^{-itH_n}\|_{H^1,H^1} \leqq M \qquad \forall t \in \mathbb{R},$$

$$\lim_{n\to+\infty} \sup_{0\leqq t\leqq T} \|(e^{-itH_n} - e^{-itH})\psi_0\|_{H^1} = 0,$$

as is the case by our assumptions $\|V_n - V\|_{K_d} \to 0$. Suppose moreover that $\|\psi_0^n - \psi_0\|_{L^2} \to 0$. Then

$$\|\psi_0^n - \psi_0\|_{H^1} \to 0 \Leftrightarrow \int_0^T \|\nabla\psi_t^n - \nabla\psi_t\|_{L^2}^2 dt \to 0.$$

Indeed by our hypotheses $\psi_t^n \to \psi_t$ in energy norm is equivalent to

$$\int_0^T \|\nabla e^{-itH_n}(\psi_0^n - \psi_0)\|_{L^2}^2 dt \to 0.$$

From the group property one has

$$\begin{aligned}
\|\psi_0^n - \psi_0\|_{H^1}^2 &\leqq \|\psi_0^n - \psi_0\|_{L^2}^2 + \sup_{0\leqq t\leqq T} \|\nabla e^{-itH_n}(\psi_0^n - \psi_0)\|_{L^2}^2 \\
&\leqq \|\psi_0^n - \psi_0\|_{L^2}^2 + M^2 \inf_{0\leqq t\leqq T} \|\nabla e^{-itH_n}(\psi_0^n - \psi_0)\|_{L^2}^2 \\
&\leqq \|\psi_0^n - \psi_0\|_{L^2}^2 + \frac{M^2}{T} \int_0^T \|\nabla e^{-itH_n}(\psi_0^n - \psi_0)\|_{L^2}^2 dt \\
&\leqq \frac{M^2}{T} \int_0^T \|e^{-itH_n}(\psi_0^n - \psi_0)\|_{H^1}^2 dt \\
&\leqq M^4 \|\psi_0^n - \psi_0\|_{H^1}^2,
\end{aligned}$$

and our thesis follows.

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