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Classification of Irreducible Super-Unitary Representations of $\mathfrak{su}(p, q/n)$

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Abstract. In this paper we classify all the irreducible super-unitary representations of $\mathfrak{su}(p,q/n)$, which can be integrated up to a unitary representation of $S(U(p,q)\times U(n))$, a Lie group corresponding to the even part of $\mathfrak{su}(p,q/n)$. Note that a real form of the Lie superalgebra $\mathfrak{sl}(m/n;\mathbb{C})$ which has non-trivial super-unitary representations is of the form $\mathfrak{su}(p,q/n)(p+q=m)$ or $\mathfrak{su}(m/r,s)(r+s=n)$. Moreover, we give an explicit realization for each irreducible super-unitary representation, using the oscillator representation of an orthosymplectic Lie superalgebra.

Introduction

The theory of Lie superalgebras and their representations have come to play an important role in physics in recent years. They appear in several fields of physics such as elementary particle physics, nuclear physics, theory of supergravity and so on (cf. [20]).

Much fundamental work regarding basic classical Lie superalgebras and their finite dimensional representations has been produced by V. G. Kac ([16–18]), who classified all the finite dimensional simple Lie superalgebras. Thereafter, in mathematics, many interesting papers on these algebras and their representations have appeared.

In the early stages, mainly finite dimensional representations were studied. Irreducible representations of simple Lie superalgebras are divided into typical and atypical ones according to their central character ([18]). Finite-dimensional typical irreducible representations have many properties in common with the finite dimensional irreducible representations of simple Lie algebras ([1, Chap. II.5; 16, 18]). But atypical representations are not so easy to treat even if they are finite dimensional. Properties of atypical (finite dimensional) representations have not been studied sufficiently (cf. [7]).

In due time, infinite dimensional representations of simple Lie superalgebras become more important (e.g., [3]). At the same time, there appeared many papers on super-unitarity of the representations. It is worth noting that whether a representation is typical or atypical has little to do with its super-unitarity.

In the last few years, super-unitary representations have been studied extensively in several different ways. For example they were studied from a general point of view in [4] and explicit cases such as orthosymplectic algebras were treated in [3, 9, 10, 23], etc.

Classification of the irreducible super-unitary representations was made for some basic classical Lie superalgebras which have low ranks. For $\mathfrak{sl}(2/1;\mathbb{C})$ and its real forms $\mathfrak{sl}(2/1;\mathbb{R})$, $\mathfrak{su}(2/1)$ and $\mathfrak{su}(1,1/1)$, classification of the irreducible super-unitary representations was made in [5] and [6]. Also, classification of $\mathfrak{osp}(2/1;\mathbb{R})$ of type $B(0,1)$ was completed ([6]). In [12], irreducible super-unitary lowest weight modules of $\mathfrak{osp}(4/1)$ were classified.

Furthermore, all the irreducible super-unitary representations were classified for $\mathfrak{su}(n/1)$ in [8], and for $\mathfrak{su}(n/m)$ in [13]. In a sense, these special super-unitary algebras $\mathfrak{su}(m/n)$ correspond to compact Lie groups. So their irreducible super-unitary representations are all finite dimensional. For general orthosymplectic algebras $\mathfrak{osp}(2n/m;\mathbb{R})$, a large number of irreducible super-unitary representations were realized in explicit forms in [23]. It seems that these representations exhaust almost all of the irreducible super-unitary representations of $\mathfrak{osp}(2n/m;\mathbb{R})$. For generic super-unitary representations called "discrete series," their (super-) characters were also obtained in [21].

Recently, the authors received a preprint [14], which classifies super-unitary highest weight representations of basic classical Lie superalgebras. The method of classification in Jakobsen's paper uses Kac's determinant formula and resembles a method used in the case of semisimple Lie algebra of the Hermitian symmetric type. However, his method is completely different from ours.

Super-unitary (or star) representations of Lie superalgebras have also often been discussed in papers of mathematical physics ([9, 15, 19]). The notion of "super-unitarity" was defined under several different names and in different ways in those papers.

Here, in this paper, we study the irreducible super-unitary representations of a real form of a complex Lie superalgebra $\mathfrak{sl}(m/n;\mathbb{C})$ of type $A(m-1,n-1)$ (for the notations, see Sect. 2). The Lie superalgebra $\mathfrak{sl}(m/n;\mathbb{C})$ has two types of real forms $\mathfrak{sl}(m/n;\mathbb{R})$ and $\mathfrak{su}(p,q/r,s)$ $(p+q=m, r+s=n)$. However, $\mathfrak{sl}(m/n;\mathbb{C})$ itself and $\mathfrak{sl}(m/n;\mathbb{R})$ have no irreducible super-unitary representation except trivial ones. The real form $\mathfrak{su}(p,q/r,s)$ also has no irreducible super-unitary representation but trivial ones if p, q, r and s are all positive at the same time. So the only real forms which have non-trivial super-unitary representations are $\mathfrak{su}(p,q/n)(p+q=m)$ and $\mathfrak{su}(m/r,s)(r+s=n)$. This fact seems well-known among experts (e.g., [3, 14]). However, since we cannot find any available proof in the literature, we have given a short proof (Proposition 2.2).

The main result presented in this paper is a classification of all the irreducible super-unitary representations of $\mathfrak{su}(p,q/n)$, which can be integrated up to representations of $S(U(p,q)\times U(n))$, a Lie group corresponding to the even part (Theorem 5.3). Let us explain the method of the classification. At first, note that integrability implies they are admissible (see Lemma 1.4). Then Proposition 2.2 tells us they must be lowest or highest weight representations. So what we have to do is to determine which highest weight modules or lowest weight modules are super-unitarizable.

We imbed $\mathfrak{su}(p, q/n)$ into an orthosymplectic algebra $\mathfrak{osp}(2(p+q)N/2nN; \mathbb{R})$ $(N \geqq 1)$. Note that we indicate by $\mathfrak{osp}(2m/2n; \mathbb{R})$ the orthosymplectic algebra of type $D(n, m)$, whose even part is isomorphic to $\mathfrak{sp}(2m; \mathbb{R}) \oplus \mathfrak{so}(2n)$. Since an orthosymplectic algebra has a special super-unitary representation called oscillator representation ([22]), we get a super-unitary representation of $\mathfrak{su}(p, q/n)$ through the above imbedding. By decomposing it, we can obtain a variety of irreducible super-unitary representations. In this paper, complete decomposition is not carried out. Instead, we construct a number of primitive vectors for $\mathfrak{su}(p, q/n)$ in explicit forms. Then their weights are lowest weights of irreducible super-unitary representations. The explicit forms of primitive vectors seem to be interesting for combinatorics theory.

Finally, using the necessary conditions for super-unitarizability, we prove that the obtained super-unitarity representations indeed exhaust all of the irreducible super-unitary representations.

Our method of classification is completely different from that in [14] and we think ours is simpler. However, note that we only treat integrable super-unitary representations, while Jakobsen classifies all the irreducible super-unitary highest weight modules in [14]. Our method has one more advantage, namely it produces realizations of the representations naturally.

Let us explain each section briefly. We introduce the notion of "super-unitarity" in Sect. 1. Super-unitarity for representations of a Lie superalgebra is considered also in [6, 24, 27], etc. We also define the "admissibility" for representations in Sect. 1 and clarify the relation between integrability and admissibility.

In Sect. 2, we obtain a necessary condition for super-unitary admissible representations of $\mathfrak{su}(p, q/n)$. This condition is a very weak one and requires that all the weights of a super-unitary representation satisfy the same kind of inequality (Proposition 2.2). Roughly speaking, this inequality requires that the weights of the $\mathfrak{su}(n)$-part are to be distributed between those of the $\mathfrak{su}(p)$- and $\mathfrak{su}(q)$-parts of the $\mathfrak{su}(p, q)$-part. From this inequality, it follows that an irreducible super-unitary representation must be a lowest or highest weight representation. The *standard* positive root system Δ^+ of $\mathfrak{su}(p, q/n)$ is also introduced in Sect. 2.

In Sect. 3, we imbed $\mathfrak{su}(p, q/n)$ into $\mathfrak{osp}(2(p+q)/2n; \mathbb{R})$ in explicit forms and introduce another positive system Ψ^+ of $\mathfrak{su}(p, q/n)$. This is called the *twisted* positive system and its image, caused by the above imbedding, is the *standard* positive system for $\mathfrak{osp}(2m/2n; \mathbb{R})$.

Further, in Sect. 4.1, we explain the oscillator representation of $\mathfrak{osp}(2mN/2nN; \mathbb{R})$ which is a super-unitary lowest weight representation. An imbedding of $\mathfrak{su}(p, q/n)$ into $\mathfrak{osp}(2mN/2nN; \mathbb{R})$ is introduced in Sect. 4.2 and operators in the above representations for the root vectors of $\mathfrak{su}(p, q/n)$ are shown in Sect. 4.3. We construct a number of primitive vectors for Ψ^- in the representation space. Then each of these vectors generates the irreducible super-unitary Ψ^+-lowest weight representations (see Proposition 4.3). Thereafter we transform those Ψ^--primitive vectors into primitive vectors for the standard positive system Δ^-. So we produce realizations of irreducible super-unitary Δ^+-lowest weight representations as subrepresentations of oscillator representations (Proposition 4.5). This proposition gives a sufficient condition for super-unitarizability.

In Sect. 5, as the first step, we introduce the triplets of Young diagrams each of which represents a weight satisfying the necessary condition in Sect. 2. Then, as the second step, we translate the obtained super-unitarizable lowest weights in Proposition 4.5 into simpler forms by means of the above triplets of Young diagrams. After that, we show that these weights indeed exhaust all of the super-unitarizable lowest weights. Thus we obtain the complete list of super-unitarizable lowest weight representations in Theorem 5.3. Furthermore we produce their realizations by the imbedding $\mathfrak{su}(p,q/n) \hookrightarrow \mathfrak{osp}(2(p+q)N/2nN;\mathbb{R})$ and oscillator representations of the latter.

For highest weight representations, we can get a result in a similar way as in the case of lowest weight representations (Theorem 5.5).

1. Definitions of Super-Unitary Representations

1.1. A Definition of Super-Unitarity. There are different ways of defining super-unitary representations of a Lie superalgebra ([6, 22, 24]). They appear different, however their essence is the same.

Let (L, E) be a representation of a real Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$.

Definition 1.1 ([6, Sect. 1.3]). *(L, E) is called super-unitary if there exists a positive definite Hermitian form $\langle \cdot, \cdot \rangle$ on $E = E_{\bar{0}} \oplus E_{\bar{1}}$ such that*

(i) $\langle E_{\bar{0}}, E_{\bar{1}} \rangle = 0$,

(ii) $\sqrt{-1}L(x)$ $(x \in \mathfrak{g}_{\bar{0}})$ *is symmetric with respect to* $\langle \cdot, \cdot \rangle$, i.e., $\langle \sqrt{-1}L(x)v, w \rangle = \langle v, \sqrt{-1}L(x)w \rangle$ $(v, w \in E)$.

(iii) *There is a constant $\varepsilon = \pm 1$ depending only on (L, E) such that, if one chooses a square root j of $\varepsilon\sqrt{-1}$, then $jL(y)$ $(y \in \mathfrak{g}_{\bar{1}})$ becomes symmetric with respect to $\langle \cdot, \cdot \rangle$*, i.e., $\langle jL(y)v, w \rangle = \langle v, jL(y)w \rangle$ $(v, w \in E)$.

We call this constant ε an *associated constant.*

The notation j sometimes leads to misunderstandings because there often appear many different j's. In this definition, we respect the notation as used in the original paper. However, later on we prefer the notation $\sqrt[4]{-1}$ rather than j.

In the above definition, the choice of $j = \sqrt[4]{-1}$ seems superfluous because super-unitarity only depends on the choice of $\varepsilon = \pm 1$. So there must be a definition only using the constant ε.

We call a form $(\cdot, \cdot)$ on $E \times E$ *super-Hermitian* if it satisfies

$$(a, b) = (-)^{\deg a \deg b}\overline{(b, a)}$$

for homogeneous elements a and b in E. Here $\deg a$ means the degree of a. A super-Hermitian form is said to be *homogeneous of degree zero* if $(E_{\bar{0}}, E_{\bar{1}}) = 0$.

In [22, Definition 5.2], [24], a representation (L, E) of Lie superalgebra $\mathfrak{g}$ with the following properties is introduced. There is a super-Hermitian form $(\cdot, \cdot)$, homogeneous of degree zero, on E which satisfies

(i) $(\cdot, \cdot)$ is ε-positive definite on E, i.e., $(\cdot, \cdot)|_{E_{\bar{0}}}$ is positive definite and there is a constant $\varepsilon = \pm 1$ depending only on (L, E) such that $\varepsilon\sqrt{-1}(\cdot, \cdot)|_{E_{\bar{0}}}$ is positive definite.

(ii) The form $(\cdot,\cdot)$ is invariant under L, i.e.,

$$(L(x)v, w) + (-)^{\deg x \deg v}(v, L(x)w) = 0 \text{ for homogeneous } x \in \mathfrak{g} \text{ and } v, w \in E.$$

This representation is also called super-unitary. In fact it is easy to see that they determine the same class of representations.

1.2. Integrability and Admissibility. In this subsection, we gather some practical terms and notions for super-unitary representations, which mainly concern *integrability* of representations. The first statement is about a Lie superalgebra $\mathfrak{g}$ itself.

Definition 1.2. ([17, Sect. 2]). *A complex Lie superalgebra $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}^{\mathbb{C}}_{\bar{0}} \oplus \mathfrak{g}^{\mathbb{C}}_{\bar{1}}$ is called classical if it is simple and the adjoint representation of $\mathfrak{g}^{\mathbb{C}}_{\bar{0}}$ on $\mathfrak{g}^{\mathbb{C}}_{\bar{1}}$ is completely reducible. A real Lie superalgebra $\mathfrak{g}$ is called classical if its complexification is classical.*

In this article we only treat classical Lie superalgebras and thier super-unitary representations. If $\mathfrak{g}$ is a real Lie superalgebra which is classical, then $\mathfrak{g}_{\bar{0}}$ is reductive Lie algebra ([17, Theorem 2]). Let $\mathfrak{k}$ be a maximal compact Lie algebra in $\mathfrak{g}_{\bar{0}}$. Here we say $\mathfrak{k}$ is compact if $\exp(\operatorname{ad}\mathfrak{k})$ is compact in $\exp(\operatorname{ad}\mathfrak{g}_{\bar{0}})$. So by definition, $\mathfrak{k}$ contains the center of $\mathfrak{g}_{\bar{0}}$.

Definition 1.3. *Let (L, E) be a representation of $\mathfrak{g}$. Then we say (L, E) is admissible if its restriction $(L|_{\mathfrak{k}}, E)$ to $\mathfrak{k}$ is decomposed into a direct sum of finite dimensional irreducible representation of $\mathfrak{k}$ with finite multiplicity.*

This definition assures that an irreducible admissible super-unitary representation can be integrated at least up to $\exp(\mathfrak{g}_{\bar{0}})$ (see [26, Theorem 0.3.10], for example). Conversely, we have

Lemma 1.4 ([11, Theorem 6]). *Take an irreducible super-unitary representation (L, E) of $\mathfrak{g}$. If $(L|_{\mathfrak{g}_{\bar{0}}}, E)$ is obtained by the differentiation of a unitary representation of a reductive Lie group $G_{\bar{0}}$ with the Lie algebra $\mathfrak{g}_{\bar{0}}$, then it is admissible.*

Hence, in the following, we can assume *all the super-unitary representations are admissible.*

Lemma 1.5. *Let $\mathfrak{g}$ be a classical Lie superalgebra and (L, E) its admissible representation. Then (L, E) has a weight space decomposition with respect to a Cartan subalgebra of $\mathfrak{k}$.*

Proof. Since (L, E) is admissible, it is a direct sum of finite dimensional irreducible representations of $\mathfrak{k}$ and each of them has a weight space decomposition according to an elementary theory of Lie algebras. Therefore (L, E) itself has a weight space decomposition if we sum up weight spaces in each irreducible component of $\mathfrak{k}$. Q.E.D.

2. A Weight Condition for Unitarizable Lowest Weight Modules of $\mathfrak{su}(p, q/n)$

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a superspace of dimension (m/n) over $\mathbb{C}$, that is, $V_{\bar{0}}$ and $V_{\bar{1}}$ are complex vector spaces of dimension m and n respectively. We often denote V by $\mathbb{C}^{m,n}$. We define a super-Hermitian form $h_{(p,q/r,s)}(\cdot,\cdot)$ on V using the matrix

$J_{(p,q/r,s)}$ for $p+q=m$ and $r+s=n$:

$$J_{(p,q/r,s)} = \left(\begin{array}{cc|cc} 1_p & & & \\ & -1_q & & \\ \hline & & \sqrt{-1}1_r & \\ & & & -\sqrt{-1}1_s \end{array}\right)$$

and for $v, w \in \mathbb{C}^{m,n}$,

$$h_{(p,q/r,s)}(v,w) = {}^t v J_{(p,q/r,s)} \bar{w},$$

where we consider an element in $\mathbb{C}^{m,n}$ a column vector. Let us define a Lie superalgebra which leaves $h(\cdot,\cdot) = h_{(p,q/r,s)}(\cdot,\cdot)$ invariant. For $k \in \{\bar{0}, \bar{1}\}$, we put

$$\mathfrak{u}(p,q/r,s)_k = \{A \in \mathfrak{gl}(m/n;\mathbb{C})_k \mid h(Av,w) + (-)^{k \deg(v)} h(v,Aw) = 0 \text{ for } v,w \in V\},$$

and $\tilde{\mathfrak{g}} = \mathfrak{u}(p,q/r,s) = \mathfrak{u}(p,q/r,s)_{\bar{0}} \oplus \mathfrak{u}(p,q/r,s)_{\bar{1}}$. A subalgebra $\mathfrak{su}(p,q/r,s) = \mathfrak{u}(p,q/r,s) \cap \mathfrak{sl}(m/n;\mathbb{C})$, denoted by $\mathfrak{g}$, is a real form of $\mathfrak{sl}(m/n;\mathbb{C})$. The sets of all diagonal matrices of these Lie superalgebras become Cartan subalgebras and we write them as $\tilde{\mathfrak{h}}$ and $\mathfrak{h}$ respectively. Let $H_{k,l} = E_{k,k} + E_{l,l} \in \mathfrak{h}^{\mathbb{C}}$ for $1 \leqq k \leqq m < l \leqq m+n$, where $E_{k,l}$ is a matrix of $M(m+n;\mathbb{C})$ with (k,l)-element 1 and elsewhere 0.

Now we study the unitary representations of $\mathfrak{su}(p,q/r,s)$. However, in almost all the cases the only irreducible unitary representation is the trivial one. In fact, we have

Lemma 2.1 (cf. [5, Proposition 2.2]). *Let (π, V) be an irreducible admissible super-unitary representation of $\mathfrak{g} = \mathfrak{su}(p,q/r,s)$ with associated constant $\varepsilon = \pm 1$ and λ be any weight of V, then*

$$\varepsilon\lambda(H_{k,l}) \leqq 0 \quad \textit{for} \quad \begin{array}{l} 1 \leqq k \leqq p, m < l \leqq m+r \quad \textit{or} \\ p < k \leqq m, m+r < l \leqq m+n, \end{array}$$

and

$$\varepsilon\lambda(H_{k,l}) \geqq 0 \quad \textit{for} \quad \begin{array}{l} 1 \leqq k \leqq p, m+r < l \leqq m+n \quad \textit{or} \\ p < k \leqq m, m < l \leqq m+r. \end{array}$$

Proof. Let $\langle \cdot,\cdot \rangle$ be a Hermitian form on V which makes π super-unitary. From the definition of unitarity, we have

$$(\sqrt[4]{-1})^2 \langle \pi(X)\pi(X)v_\lambda, v_\lambda \rangle \geqq 0,$$

where $X \in \mathfrak{g}_{\bar{1}}$ and v_λ is a non-zero weight vector with weight λ. To get the first condition, we put $X = E_{k,l} + \sqrt{-1} E_{l,k} \in \mathfrak{g}_{\bar{1}}$. Then it holds that $2\pi(X)^2 = \pi([X,X]) = \pi(\sqrt{-1} H_{k,l})$ and we get

$$-\varepsilon\lambda(H_{k,l}) \geqq 0.$$

The second condition can be obtained in the same way. Q.E.D

From the above lemma, we can easily see that, if V is not trivial, then either $p = 0, m$ or $r = 0, n$. In fact, let us consider a weight $\lambda \in (\tilde{\mathfrak{h}}^{\mathbb{C}})^*$ of V, and put $\lambda_k = \varepsilon\lambda(E_{k,k})$ for $1 \leqq k \leqq m$ and $\mu_l = -\varepsilon\lambda(E_{m+l,m+l})$ for $1 \leqq l \leqq n$. Then from the lemma, we get

$$\lambda_k \leqq \mu_l \quad \text{for the first condition,}$$

and

$$\mu_l \leqq \lambda_k \quad \text{for the second condition.}$$

Therefore if $p \neq 0, m$ and $r \neq 0, n$ it holds that

$$\lambda_a \leqq \mu_b \leqq \lambda_c \leqq \mu_d \leqq \lambda_a$$

for $1 \leqq a \leqq p < c \leqq m$ and $1 \leqq b \leqq r < d \leqq n$. Thus $\lambda_k = \mu_l$ for all k, l, and the restriction of λ to $\mathfrak{h}$ is trivial.

Since $\mathfrak{su}(p, q/n, 0)$, $\mathfrak{su}(p, q/0, n)$, $\mathfrak{su}(n, 0/p, q)$ and $\mathfrak{su}(0, n/p, q)$ are all isomorphic to each other, hereafter we only consider the algebra $\mathfrak{g} = \mathfrak{su}(p, q/n, 0)$. Mainly we will write $\mathfrak{su}(p, q/n) = \mathfrak{su}(p, q/n, 0)$. Let us list a basis for $\mathfrak{su}(p, q/n)$.

a) Basis for $\mathfrak{su}(p, q/n)_{\bar{0}}$:

$$\sqrt{-1}(E_{i,i} - E_{j,j}) \quad (i < j \text{ and } i, j \in I_p = \{1, \dots, p\} \text{ or } i, j \in I_q = \{p+1, \dots, m\}),$$

$$\sqrt{-1}C, \quad \text{where} \quad C = n \sum_{i=1}^{m} E_{i,i} + m \sum_{i=m+1}^{m+n} E_{i,i},$$

$$\sqrt{-1}(E_{i,j} + E_{j,i}), \quad E_{i,j} - E_{j,i}$$
$$(i < j \text{ and } i, j \in I_p \text{ or } i, j \in I_q \textit{ or } i, j \in I_n = \{m+1, \dots, m+n\}),$$

$$\sqrt{-1}(E_{i,j} - E_{j,i}), \quad E_{i,j} + E_{j,i} \quad (i \in I_p, j \in I_q).$$

b) Basis for $\mathfrak{su}(p, q/n)_{\bar{1}}$:

$$E_{i,j} + \sqrt{-1}E_{j,i} \quad (i \in I_p, j \in I_n \text{ or } i \in I_n, j \in I_p),$$
$$E_{i,j} - \sqrt{-1}E_{j,i} \quad (i \in I_q, j \in I_n \text{ or } i \in I_n, j \in I_q).$$

We describe a positive system in the root system of $\mathfrak{su}(p, q/n)$. Let $\{e_k\}$ be a basis of $(\tilde{\mathfrak{h}}^{\mathbb{C}})^*$ such that $e_k(E_{l,l}) = \delta_{k,l}$, then one of the positive systems is given by

$\Delta_c^+ = \{e_k - e_l | k < l,\ k, l \in I_p \text{ or } k, l \in I_q \text{ or } k, l \in I_n\}$: the set of positive compact roots,

$\Delta_n^+ = \{e_k - e_l | k \in I_p, l \in I_q\}$: the set of positive non-compact roots,

$\Delta_{\bar{0}}^+ = \Delta_c^+ \cup \Delta_n^+$: the set of positive even roots,

$\Delta_{\bar{1}}^+ = \{e_k - e_l | k \in I_p \cup I_q, l \in I_n\}$: the set of positive odd roots,

$\Delta^+ = \Delta_{\bar{0}}^+ \cup \Delta_{\bar{1}}^+$: the set of positive roots.

Let us call the positive system *standard* (see Sect. 3.2 bis) and the terms highest or lowest refer to this particular positive system Δ^+. Sometimes we use the notation f_l $(l \leqq n)$ instead of e_{l+m}. If $\lambda \in (\mathfrak{h}^{\mathbb{C}})^*$ is expressed as

$$\lambda = \sum_{1 \leqq i \leqq m} \lambda_i e_i - \sum_{1 \leqq i \leqq n} \mu_i f_i, \tag{2.1}$$

then we write $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m / \mu_1, \mu_2, \dots, \mu_n)$ and call it a *coordinate expression* of λ. Note that the coordinate expression has ambiguity because it is a restriction of an element in $(\tilde{\mathfrak{h}}^{\mathbb{C}})^*$ to $\mathfrak{h}$.

Proposition 2.2. *Let (π, V) be an irreducible admissible unitary representation of $\mathfrak{g} = \mathfrak{su}(p, q/r, s)$ with associated constant $\varepsilon = \pm 1$.*

(1) *If* $p \neq 0, m$ *and* $r \neq 0, n$, *then* (π, V) *is trivial.*

(2) *For* $\mathfrak{g} = \mathfrak{su}(p, q/n) = \mathfrak{su}(p, q/n, 0)$, *there are two possibilities.*

(2a) *If* $\varepsilon = 1$, *then* (π, V) *is a highest weight representation and its highest weight* λ *satisfies*

$$\lambda_{p+1} \geqq \cdots \geqq \lambda_m \geqq \mu_n \geqq \cdots \geqq \mu_1 \geqq \lambda_1 \geqq \cdots \geqq \lambda_p.$$

(2b) *If* $\varepsilon = -1$, *then* (π, V) *is a lowest weight representation and its lowest weight* λ *satisfies*

$$\lambda_{p+1} \leqq \cdots \leqq \lambda_m \leqq \mu_n \leqq \cdots \leqq \mu_1 \leqq \lambda_1 \leqq \cdots \leqq \lambda_p.$$

Proof. (1) was proved and now we show (2a).

Since V is admissible, V has a weight space decomposition. Let v be a non-zero weight vector. From the irreducibility of V, we get $V = U(\mathfrak{g}^{\mathbb{C}})v$, where $U(\mathfrak{g}^{\mathbb{C}})$ is the universal enveloping algebra of $\mathfrak{g}^{\mathbb{C}}$. Denote

$$\mathfrak{g}_{\bar{0},n}^{\pm} = \bigoplus_{\pm\alpha \in \Delta_n^+} \mathfrak{g}_\alpha, \quad \mathfrak{g}_{\bar{1}}^{\pm} = \bigoplus_{\pm\beta \in \Delta_{\bar{1}}^+} \mathfrak{g}_\beta, \quad \text{and} \quad \mathfrak{g}_{\bar{0},c} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_c} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha$ or $\mathfrak{g}_\beta$ is a root space of $\mathfrak{g}^{\mathbb{C}}$ of root α or β respectively. Then $\mathfrak{g}_{\bar{0},c}$ is the complexification of a maximal compact subalgebra $\mathfrak{k}$ of $\mathfrak{g}_{\bar{0}}$. According to Poincaré–Birkhoff–Witt theorem, it can be written as

$$V = U(\mathfrak{g}^{\mathbb{C}})v = (\wedge \mathfrak{g}_{\bar{1}}^-)(\wedge \mathfrak{g}_{\bar{1}}^+)U(\mathfrak{g}_{\bar{0}}^{\mathbb{C}})v.$$

We write $W = U(\mathfrak{g}_{\bar{0}}^{\mathbb{C}})v$, then from Lemma 2.1 any weight λ of W will satisfy the following condition:

$$\lambda_i \leqq \mu_k \leqq \lambda_j,$$

for $1 \leqq i \leqq p < j \leqq m$ and $1 \leqq k \leqq n$. Therefore there exists a Δ_n^+-highest weight vector in W. In fact, from the above condition, there exists a non-negative integer $t(1, p+1)$ such that $(E_{1,p+1})^{t(1,p+1)}v \neq 0$ and $(E_{1,p+1})^{t(1,p+1)+1}v = 0$. There also exists a non-negative integer $t(1, p+2)$ such that

$$(E_{1,p+2})^{t(1,p+2)}(E_{1,p+1})^{t(1,p+1)}v \neq 0 \quad \text{and} \quad (E_{1,p+2})^{t(1,p+2)+1}(E_{1,p+1})^{t(1,p+1)}v = 0,$$

and so on. Then the vector

$$\{(E_{p,p+q})^{t(p,p+q)} \cdots (E_{p,p+2})^{t(p,p+2)}(E_{p,p+1})^{t(p,p+1)}\} \cdots \{(E_{1,p+q})^{t(1,p+q)} \cdots (E_{1,p+2})^{t(1,p+2)}(E_{1,p+1})^{t(1,p+1)}\}v$$

is Δ_n^+-highest. We again denote this vector by $v \in W$. Since v generates a finite dimensional representation for the compact subalgebra $\mathfrak{g}_{\bar{0},c}$, there exists an $X \in U(\mathfrak{g}_{\bar{0},c})$ such that Xv becomes a Δ_c^+-highest weight vector. The vector $w = Xv$ is also a Δ_n^+-highest weight vector because of the relation $[\mathfrak{g}_{\bar{0},n}^+, \mathfrak{g}_{\bar{0},c}] \subset \mathfrak{g}_{\bar{0},n}^+$.

If Y_1 and Y_2 are in $\mathfrak{g}_{\bar{1}}^+$, then we have

$$[Y_1, Y_2]w = (Y_1 Y_2 + Y_2 Y_1)w = 0,$$

since $[\mathfrak{g}_{\bar{1}}^+, \mathfrak{g}_{\bar{1}}^+] \subset \mathfrak{g}_{\bar{0},n}^+$. Therefore we can conclude that the action of $\wedge \mathfrak{g}_{\bar{1}}^+$ on w is well-defined. Take $p^+ \in \wedge \mathfrak{g}_{\bar{1}}^+$ with the highest weight among the elements which satisfy $p^+ w \neq 0$. Then $p^+ w$ becomes a Δ^+-highest weight vector because:

(i) If Y is a root vector in $\mathfrak{g}_{\bar{1}}^{+}$, then $Yp^{+} \in \wedge \mathfrak{g}_{\bar{1}}^{+}$ has a weight larger than that of p^{+}. So by the above definition of p^{+}, we have $Yp^{+}w = 0$.

(ii) If Y is in $\mathfrak{g}_{\bar{0},n}^{+}$, then it holds that $Yp^{+}w = p^{+}Yw = 0$, since $[\mathfrak{g}_{\bar{1}}^{+}, \mathfrak{g}_{\bar{0},n}^{+}] = (0)$.

(iii) Assume that Y is a root vector in $\mathfrak{g}_{\bar{0},c}^{+}$. We can write

$$Yp^{+}w = p^{+}Yw + [Y, p^{+}]w = [Y, p^{+}]w.$$

Since $[\mathfrak{g}_{\bar{0},c}^{+}, \mathfrak{g}_{\bar{1}}^{+}] \subset \mathfrak{g}_{\bar{1}}^{+}$, we can conclude that $[Y, p^{+}] \in \wedge \mathfrak{g}_{\bar{1}}^{+}$, which has a higher weight than that of p^{+}. Hence we have $Yp^{+}w = [Y, p^{+}]w = 0$ by the above definition of p^{+}.

Thus $p^{+}w$ becomes a Δ^{+}-highest weight vector, and V is a highest weight representation. The proof for (2b) is similar. Q.E.D.

Corollary 2.3. *Let (π, V) be an irreducible super-unitary representation of $\mathfrak{g} = \mathfrak{su}(p, q/n)$. If (π, V) can be integrated up to a representation of $S(U(p, q) \times U(n))$, then (π, V) is a highest or lowest weight representation with integral weights.*

Proof. From Lemma 1.4, (π, V) is admissible. Then it is a highest or lowest weight representation. Since the weights are obtained by the differentiation of a representation of a maximal torus in $S(U(p, q) \times U(n))$, they are integral. Q.E.D.

3. Canonical Imbedding of Super-Unitary-Algebras into Orthosymplectic Algebras

3.1. Abstract Imbedding. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a complex superspace with super-Hermitian form $h(\cdot, \cdot)$. We assume that h is of degree zero, i.e., $h(V_{\bar{0}}, V_{\bar{1}}) = h(V_{\bar{1}}, V_{\bar{0}}) = 0$. Then by $\mathfrak{u}(h)$ we denote a super-unitary algebra consisting of elements which leave $h(\cdot, \cdot)$ invariant:

$$\mathfrak{u}(h) = \{x \in \mathfrak{gl}(V) \mid h(xv, w) + (-)^{\deg x \deg v} h(v, xw) = 0\}.$$

Consider V as a real superspace and put

$$\begin{aligned} b(v, w) &= \operatorname{Im} h(v, w) \\ &= \frac{1}{2\sqrt{-1}} \{h(v, w) + (-)^{\deg v \deg w} h(w, v)\} \quad (v, w \in V). \end{aligned}$$

Then $b(\cdot, \cdot)$ is a super-skew symmetric real linear form on V.

Lemma 3.1. *Every element $x \in \mathfrak{u}(h)$ belongs to $\mathfrak{osp}(b)$ if it is considered as a real linear transformation on V.*

Proof. In fact, since x leaves $h(\cdot, \cdot)$ invariant, it leaves $\operatorname{Im} h(\cdot, \cdot)$ invariant. Q.E.D.

Note that $b(\cdot, \cdot)$ is non-degenerate if $h(\cdot, \cdot)$ is non-degenerate.

3.2. Explicit Imbedding. Let us consider the following normal form of $h(\cdot, \cdot)$ (cf. Sect. 2). Put $\dim_{\mathbb{C}} V = (m/n)$. We arrange first a basis for $V_{\bar{0}}$ then one for $V_{\bar{1}}$ and get the basis for $V = \mathbb{C}^{m,n}$. Using this basis, we define $h(\cdot, \cdot) = h_{(p,q/r,s)}(\cdot, \cdot)$ by the matrix $J_{(p,q/r,s)} (p + q = m, r + s = n)$ as in Sect. 2.

We write $\mathfrak{u}(h) = \mathfrak{u}(p, q/r, s)$ and call it a *super-unitary algebra* of type $(p, q/r, s)$. Let $\mathfrak{sl}(V)$ be a Lie superalgebra consisting of all the super-traceless matrices. Then a *special super-unitary algebra* $\mathfrak{su}(p,q/r,s)$ defined in Sect. 2 is equal to $\mathfrak{u}(p,q/r,s)\cap\mathfrak{sl}(V)$. If $m\neq n$ then the center of $\mathfrak{u}(p,q/r,s)$ is isomorphic to $\mathfrak{u}(1)$ and we have a decomposition

$$\mathfrak{u}(p,q/r,s)=\mathfrak{u}(1)\oplus\mathfrak{su}(p,q/r,s).$$

Let $\{v_i \mid 1\leqq i\leqq m+n\}$ be a standard basis for $V=\mathbb{C}^{m,n}$, whose i^{th} element is 1 and the others are all zero. Now we want to have a matrix B for super-skew symmetric form $b(\cdot,\cdot)=\operatorname{Im}h(\cdot,\cdot)$. Let $V^{\mathbb{R}}$ be a real form of V spanned by the basis mentioned above. Then, as a real vector space, V has the following basis:

$$\{\sqrt{-1}v_1,\dots,\sqrt{-1}v_p,v_{p+1},\dots,v_{p+q},v_1,\dots,v_p,\sqrt{-1}v_{p+1},\dots,\sqrt{-1}v_{p+q};$$
$$v_{m+1},\dots,v_{m+r},\sqrt{-1}v_{m+r+1},\dots,\sqrt{-1}v_{m+r+s},$$
$$\sqrt{-1}v_{m+1},\dots,\sqrt{-1}v_{m+r},v_{m+r+1},\dots,v_{m+r+s}\}.$$

Put

$$V_p^{\mathbb{R}}=\langle v_1,\dots,v_p\rangle/\mathbb{R},\qquad V_q^{\mathbb{R}}=\langle v_{p+1},\dots,v_{p+q}\rangle/\mathbb{R},$$
$$V_r^{\mathbb{R}}=\langle v_{m+1},\dots,v_{m+r}\rangle/\mathbb{R},\quad V_s^{\mathbb{R}}=\langle v_{m+r+1},\dots,v_{m+r+s}\rangle/\mathbb{R}.$$

Then the above basis is arranged in the following order:

$$\sqrt{-1}V_p^{\mathbb{R}},V_q^{\mathbb{R}},V_p^{\mathbb{R}},\sqrt{-1}V_q^{\mathbb{R}},V_r^{\mathbb{R}},\sqrt{-1}V_r^{\mathbb{R}},\sqrt{-1}V_s^{\mathbb{R}},V_s^{\mathbb{R}}.$$

For this basis the matrix B for $b(\cdot,\cdot)$ is expressed as follows:

$$B=\left[\begin{array}{cc|cc} & 1_m & & \\ -1_m & & & \\ \hline & & 1_{2r} & \\ & & & -1_{2s}\end{array}\right]$$

and

$$b(v,w)={}^t vBw\quad(v,w\in V).$$

We denote the orthosymplectic algebra $\mathfrak{osp}(b)$ for the above b by $\mathfrak{osp}(2m/2r,2s;\mathbb{R})$ or simply $\mathfrak{osp}(2m/2r,2s)$.

The above results are summarized in

Proposition 3.2. *A super-unitary algebra* $\mathfrak{u}(p,q/r,s)$ *can be canonically imbedded into* $\mathfrak{osp}(2(p+q)/2r,2s;\mathbb{R})$. *With respect to this imbedding, the commutant of* $\mathfrak{u}(p,q/r,s)$ *in* $\mathfrak{osp}(2(p+q)/2r,2s;\mathbb{R})$ *is the center* $\mathfrak{u}(1)$ *of* $\mathfrak{u}(p,q/r,s)$ *itself under the condition* $p+q\neq r+s$.

Proof. Take X in $\mathfrak{osp}(2(p+q)/2r,2s;\mathbb{R})$. Since $\mathfrak{u}(1)$ and a Cartan subalgebra of $\mathfrak{su}(p,q/r,s)$ together generate a Cartan subalgebra of $\mathfrak{osp}(2(p+q)/2r,2s;\mathbb{R})$, if X commutes with both $\mathfrak{u}(1)$ and $\mathfrak{su}(p,q/r,s)$ then X commutes with the whole Cartan subalgebra. According to the definition of Cartan subalgebra, X belongs to the Cartan subalgebra of $\mathfrak{osp}(2(p+q)/2r,2s;\mathbb{R})$. So X can be expressed as

$$X=H_{\mathfrak{u}}+H_{\mathfrak{su}}\quad(H_{\mathfrak{u}}\in\mathfrak{u}(1),H_{\mathfrak{su}}\in\text{Cartan algebra of }\mathfrak{su}(p,q/r,s)).$$

The component $H_{\mathfrak{su}}$ commutes with $\mathfrak{su}(p,q/r,s)$. Since $\mathfrak{su}(p,q/r,s)$ is simple, it must be zero. Now we can conclude that $X\in\mathfrak{u}(1)$. Q.E.D.

In the following, we only consider the super-unitary algebra $\mathfrak{u}(p,q/n,0)=\mathfrak{u}(p,q/n)$ and the orthosymplectic algebra $\mathfrak{osp}(2m/2n,0;\mathbb{R})=\mathfrak{osp}(2m/2n;\mathbb{R})$.

To exhibit the correspondence of the roots of $\mathfrak{u}(p,q/n)$ and $\mathfrak{osp}(2(p+q)/2n;\mathbb{R})$, we first prepare the notations for orthosymplectic algebras. The elements in $\mathfrak{osp}(2(p+q)/2n;\mathbb{R})$ are matrices of the form

$$\left[\begin{array}{cc|c} A & B & P \\ C & -{}^tA & Q \\ \hline -{}^tQ & {}^tP & D \end{array}\right],$$

where $A\in\mathfrak{gl}(m;\mathbb{R})$ $(m=p+q)$, B and C are symmetric, P and Q are $m\times 2n$-matrices, and D belongs to $\mathfrak{so}(2n)$. This algebra has a compact Cartan subalgebra $\mathfrak{t}$:

$$\mathfrak{t}=\left\{h=\left[\begin{array}{cc|cc} 0 & A & \multicolumn{2}{c}{0} \\ -A & 0 & & \\ \hline \multicolumn{2}{c|}{0} & 0 & B \\ & & -B & 0 \end{array}\right]\ \middle|\ A=\mathrm{diag}(a_1,a_2,\dots,a_m),\right.$$
$$\left. B=\mathrm{diag}(b_1,b_2,\dots,b_n),\ a_i,b_j\in\mathbb{R}\right\}. \tag{3.1}$$

We define $c_i\in(\mathfrak{t}^{\mathbb{C}})^*$ $(1\leqq i\leqq m)$ and $d_j\in(\mathfrak{t}^{\mathbb{C}})^*$ $(1\leqq j\leqq n)$ by putting

$$c_i(h)=\sqrt{-1}\,a_i,\quad d_j(h)=\sqrt{-1}\,b_j,$$

for $h\in\mathfrak{t}$ of the form in (3.1). Then roots are given as

$\Sigma_c^+=\{c_i-c_j\,|\,1\leqq i<j\leqq m\}\cup\{d_i\pm d_j\,|\,1\leqq i<j\leqq n\}$
: the set of positive compact roots,

$\Sigma_n^+=\{c_i+c_j\,|\,1\leqq i\leqq j\leqq m\}$: the set of positive non-compact roots,

$\Sigma_{\bar 0}^+=\Sigma_c^+\cup\Sigma_n^+$: the set of positive even roots,

$\Sigma_{\bar 1}^+=\{c_i\pm d_j\,|\,1\leqq i\leqq m,\,1\leqq j\leqq n\}$: the set of positive odd roots,

$\Sigma^+=\Sigma_{\bar 0}^+\cup\Sigma_{\bar 1}^+$: the set of positive roots.

If $\lambda\in(\mathfrak{t}^{\mathbb{C}})^*$ is of the form

$$\lambda=\sum_{1\leqq i\leqq m}\lambda_i c_i+\sum_{1\leqq i\leqq n}\mu_i d_i,$$

then we write $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_m/\mu_1,\mu_2,\dots,\mu_n)$ and call it a *coordinate expression* of λ.

Take h in a compact Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{su}(p,q/n)$:

$$h=\left[\begin{array}{cc|c} \sqrt{-1}A_p & & \\ & \sqrt{-1}A_q & \\ \hline & & \sqrt{-1}B \end{array}\right]\in\mathfrak{h},$$

where

$$A_p=\mathrm{diag}(a_1,a_2,\dots,a_p),\quad A_q=\mathrm{diag}(a_{p+1},a_{p+2},\dots,a_{p+q}),\quad B=\mathrm{diag}(b_1,b_2,\dots,b_n).$$

Then, if we denote the imbedding described above (cf. Proposition 3.2) by

$\psi, \psi(h) \in \mathfrak{osp}(2m/2n; \mathbb{R})$ is of the following form:

$$\psi(h) = \left[\begin{array}{c|c|c} 0 & \begin{matrix} A_p & \\ & -A_q \end{matrix} & \\ \hline \begin{matrix} -A_p & \\ & A_q \end{matrix} & 0 & \\ \hline & & \begin{matrix} & B \\ -B & \end{matrix} \end{array}\right] \in \mathfrak{t}.$$

Now it is easy to see that ψ maps roots

$$e_i - e_j, \quad e_i - f_j \quad \text{and} \quad f_i - f_j$$

of $\mathfrak{su}(p, q/n)$ (see Sect. 2) to the roots

$$\operatorname{sgn}(i)c_i - \operatorname{sgn}(j)c_j, \quad \operatorname{sgn}(i)c_i - d_j \quad \text{and} \quad d_i - d_j$$

of $\mathfrak{osp}(2m/2n; \mathbb{R})$ respectively, where $\operatorname{sgn}(i)$ is 1 if $1 \leqq i \leqq p$ and -1 if $p+1 \leqq i \leqq p+q$. We define a positive system Ψ^+ for $\mathfrak{su}(p, q/n)$ as

$$\Psi^+ = \{\alpha \in \Delta \mid \psi(\alpha) \in \Sigma^+\},$$

and call it the *twisted* positive system (or compatible positive system for $\mathfrak{osp}$). Recall that we called the positive system Δ^+ for $\mathfrak{su}(p, q/n)$ *standard.*

4. Primitive Vectors in the Oscillator Representation for a Special Unitary Algebra

4.1. Review of Oscillator Representations. In [22], we defined a super-unitary representation for $\mathfrak{osp}(2m/2n; \mathbb{R})$ called the oscillator representation. Let us review the construction of it briefly.

First we give the representation space of the oscillator representation ρ. Let $C^{\mathbb{C}}(r_l \mid 1 \leqq l \leqq n)$ be a Clifford algebra over $\mathbb{C}$ generated by $\{r_l \mid 1 \leqq l \leqq n\}$ with relations

$$r_i^2 = 1, \quad r_i r_j + r_j r_i = 0 \quad (1 \leqq i \neq j \leqq n).$$

We denote by $C_{\bar{0}}^{\mathbb{C}}(r_l \mid 1 \leqq l \leqq n)$ a subalgebra of $C^{\mathbb{C}}(r_l \mid 1 \leqq l \leqq n)$ generated by even products of r_j's and by $C_{\bar{1}}^{\mathbb{C}}(r_l \mid 1 \leqq l \leqq n)$ a subspace generated by odd products of r_j's. Then clearly we have

$$C^{\mathbb{C}}(r_l \mid 1 \leqq l \leqq n) = C_{\bar{0}}^{\mathbb{C}}(r_l \mid 1 \leqq l \leqq n) \oplus C_{\bar{1}}^{\mathbb{C}}(r_l \mid 1 \leqq l \leqq n)$$

and $C^{\mathbb{C}}(r_l \mid 1 \leqq l \leqq n)$ becomes a superalgebra with this $\mathbb{Z}_2$-grading.

For the representation space $F = F_{\bar{0}} \oplus F_{\bar{1}}$ of ρ, we take as follows:

$$\begin{aligned} F &= F_{\bar{0}} \oplus F_{\bar{1}}, \\ F_{\bar{0}} &= \mathbb{C}[z_k \mid 1 \leqq k \leqq m] \otimes C_{\bar{0}}^{\mathbb{C}}(r_l \mid 1 \leqq l \leqq n), \\ F_{\bar{1}} &= \mathbb{C}[z_k \mid 1 \leqq k \leqq m] \otimes C_{\bar{1}}^{\mathbb{C}}(r_l \mid 1 \leqq l \leqq n), \end{aligned}$$

where $\mathbb{C}[z_k|1 \leqq k \leqq m]$ means a polynomial algebra generated by $\{z_k|1 \leqq k \leqq m\}$.

Second, we give the operations of $\mathfrak{osp}(2m/2n;\mathbb{R})$ on F. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a superspace of dimension $(2m/2n)$ on which $\mathfrak{osp}(2m/2n;\mathbb{R})$ naturally acts. We denote a super-skew symmetric form on V by b and consider that $\mathfrak{osp}(2m/2n;\mathbb{R}) = \mathfrak{osp}(b)$. Choose the basis $\{p_k, q_k|1 \leqq k \leqq m\}$ for $V_{\bar{0}}$ such that

$$b(p_i, q_j) = -b(q_j, p_i) = \delta_{ij}, \quad b(p_i, p_j) = b(q_i, q_j) = 0,$$

and an orthogonal basis $\{r_l, s_l|1 \leqq l \leqq n\}$ for $V_{\bar{1}}$ with respect to b with length $\sqrt{2}$. Then there exists a superalgebra $C^{\mathbb{R}}(V;b)$ over $\mathbb{R}$ which is generated by $\{p_k, q_k|1 \leqq k \leqq m\} \cup \{r_l, s_l|1 \leqq l \leqq n\}$ with relations

$$p_i p_j - q_j p_i = \delta_{ij}, \quad r_i s_j + s_j r_i = 0, \quad r_i r_j + r_j r_i = 2\delta_{ij}, \quad s_i s_j + s_j s_i = 2\delta_{ij},$$

and all the other pairs of p, q, r, s commute with each other. $C^{\mathbb{R}}(V;b)$ can be considered as a Lie superalgebra in a standard way (cf. [17, Sect. 1.1]), and $\mathfrak{osp}(2m/2n;\mathbb{R})$ can be realized as a sub-Lie superalgebra in $C^{\mathbb{R}}(V;b)$ (cf. [25]). Let L be a subspace generated by second degree elements of the following form:

$$\{xy + (-)^{\deg x \deg y} yx | x, y \in \{p_k, q_k|1 \leqq k \leqq m\} \cup \{r_l, s_l|1 \leqq l \leqq n\}\}.$$

Then L becomes a sub-Lie superalgebra. An operator $\mathrm{ad}(xy + (-)^{\deg x \deg y} yx)$ preserves $V \subset C^{\mathbb{R}}(V;b)$ and the bilinear form b, and this gives an isomorphism between L and $\mathfrak{osp}(2m/2n;\mathbb{R})$. From now on, we will identify L and $\mathfrak{osp}(2m/2n;\mathbb{R})$ with each other.

The oscillator representation ρ is actually a representation of the superalgebra $C^{\mathbb{R}}(V;b)$ given by

$$\rho(p_k) = \frac{\sqrt{-1}\sqrt[4]{-1}}{\sqrt{2}}\left(z_k - \frac{\partial}{\partial z_k}\right) \otimes 1 \quad (1 \leqq k \leqq m),$$

$$\rho(q_k) = \frac{\sqrt[4]{-1}}{\sqrt{2}}\left(z_k + \frac{\partial}{\partial z_k}\right) \otimes 1 \quad (1 \leqq k \leqq m),$$

$$\rho(r_l) = 1 \otimes r_l \quad (1 \leqq l \leqq n),$$

$$\rho(s_l) = 1 \otimes \sqrt{-1}\, r_l \alpha_l \quad (1 \leqq l \leqq n),$$

where α_l is an automorphism of the Clifford algebra $C^{\mathbb{R}}(r_l|1 \leqq l \leqq n)$ which sends r_k to $(-)^{\delta_{k,l}} r_k$ $(1 \leqq k \leqq n)$. If we restrict ρ to the sub-Lie superalgebra $\mathfrak{osp}(2m/2n;\mathbb{R}) \subset C^{\mathbb{R}}(V;b)$, then ρ gives a *super-unitary* representation for $\mathfrak{osp}(2m/2n;\mathbb{R})$. For more information on ρ, see [22] and [23].

4.2. Imbedding of $\mathfrak{su}(p,q/n)$ *into* $\mathfrak{osp}(2mN/2nN;\mathbb{R})$. As in [23], we consider the following imbeddings:

$$\tilde{\psi}: \mathfrak{su}(p,q/n) \overset{\psi}{\hookrightarrow} \mathfrak{osp}(2m/2n;\mathbb{R}) \overset{\iota}{\hookrightarrow} \mathfrak{osp}(2mN/2nN;\mathbb{R}).$$

Here ι is given as follows. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be the superspace with the super-skew symmetric bilinear form $b' = b_V$ as above and $W = W_{\bar{0}}$ be a usual N-dimensional vector space with a positive definite inner product b_W. Then a superspace $V \otimes W = V_{\bar{0}} \otimes W_{\bar{0}} + V_{\bar{1}} \otimes W_{\bar{0}}$ is endowed with a super-skew symmetric bilinear form $b_{V\otimes W} = b_V \otimes b_W$. If we consider

$$\mathfrak{osp}(b_V) = \mathfrak{osp}(2m/2n;\mathbb{R})$$

and

$$\mathfrak{osp}(b_{V\otimes W}) = \mathfrak{osp}(2mN/2nN;\mathbb{R}),$$

ι is given by $\iota(A) = A \otimes 1_W$ for $A \in \mathfrak{osp}(2m/2n;\mathbb{R})$. In the matrix form, this only means

$$\iota(A) = \begin{bmatrix} a_{11}1_N & a_{12}1_N & \cdots \\ a_{21}1_N & a_{22}1_N & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \quad \text{for} \quad A = (a_{ij}).$$

Since $\mathfrak{su}(p,q/n)$ is imbedded into $\mathfrak{osp}(2m/2n;\mathbb{R})$ by ψ as in Sect. 3.2, $\mathfrak{su}(p,q/n)$ is now imbedded into $\mathfrak{osp}(2mN/2nN;\mathbb{R})$. We denote this imbedding by $\tilde{\psi}$.

Let (ρ, F) be the oscillator representation of $\mathfrak{osp}(2mN/2nN;\mathbb{R})$ so that

$$F = \mathbb{C}[z_{ij} \mid 1 \leqq i \leqq m, 1 \leqq j \leqq N] \otimes C^{\mathbb{C}}(r_{kl} \mid 1 \leqq k \leqq n, 1 \leqq l \leqq N).$$

The successive application of $\tilde{\psi}$ then ρ gives a *super-unitary* representation $\tilde{\rho} = \rho \circ \tilde{\psi}$ of $\mathfrak{su}(p,q/n)$. In the following subsections, we try to decompose this super-unitary representation $\tilde{\rho}$ of $\mathfrak{su}(p,q/n)$. Since the associated constant of ρ is $\varepsilon = -1$, an irreducible super-unitary representation for $\mathfrak{su}(p,q/n)$ which appears in $(\tilde{\rho}, F)$ is a lowest weight module. Therefore what we have to do is to find all the primitive vectors for $\tilde{\rho}$.

4.3. Operators for Root Vectors. Let X_α $(\alpha \in \Delta)$ be a non-zero root vector for a root α of $\mathfrak{su}(p,q/n)$. Then up to a non-zero constant multiple, operators $\tilde{\rho}(X_\alpha)$ are given as follows.

Root vectors for $\alpha \in \Delta^-$:

(I) $\alpha = -(e_k - f_l)(1 \leqq k \leqq p, 1 \leqq l \leqq n)$; $\displaystyle\sum_{j=1}^{N} \frac{\partial}{\partial z_{k,j}} r_{l,j}(1 + \alpha_{l,j})$,

(II) $\alpha = -(e_k - f_l)(p < k \leqq m, 1 \leqq l \leqq n)$; $\displaystyle\sum_{j=1}^{N} z_{k,j} r_{l,j}(1 + \alpha_{l,j})$,

(III) $\alpha = -(e_i - e_k)(1 \leqq i < k \leqq p)$; $\displaystyle\sum_{j=1}^{N} z_{k,j} \frac{\partial}{\partial z_{i,j}}$,

(IV) $\alpha = -(e_i - e_k)(p < i < k \leqq m)$; $\displaystyle\sum_{j=1}^{N} z_{i,j} \frac{\partial}{\partial z_{k,j}}$,

(V) $\alpha = -(e_i - e_k)(1 \leqq i \leqq p < k \leqq m)$; $\displaystyle\sum_{j=1}^{N} \frac{\partial}{\partial z_{i,j}} \frac{\partial}{\partial z_{k,j}}$,

(VI) $\alpha = -(f_k - f_l)(1 \leqq k < l \leqq n)$; $\displaystyle\sum_{j=1}^{N} r_{k,j} r_{l,j}(1 - \alpha_{k,j})(1 + \alpha_{l,j})$,

where $\alpha_{i,j}$ is an automorphism of $C^{\mathbb{C}}(r_{kl} \mid 1 \leqq k \leqq n, 1 \leqq l \leqq N)$ such that

$$\alpha_{i,j}(r_{k,l}) = \begin{cases} r_{k,l} & \text{if} \quad i \neq k \quad \text{or} \quad j \neq l, \\ -r_{i,j} & \text{if} \quad i = k \quad \text{and} \quad j = l, \end{cases}$$

for $1 \leqq k \leqq n, 1 \leqq l \leqq N$.

Root vectors for $\alpha \in \Delta^+$:

$$(\mathrm{I}+)\ \alpha = e_k - f_l (1 \leqq k \leqq p, 1 \leqq l \leqq n); \quad \sum_{j=1}^{N} z_{k,j} r_{l,j} (1 - \alpha_{l,j}),$$

$$(\mathrm{II}+)\ \alpha = e_k - f_l (p < k \leqq m, 1 \leqq l \leqq n); \quad \sum_{j=1}^{N} \frac{\partial}{\partial z_{k,j}} r_{l,j} (1 - \alpha_{l,j}),$$

$$(\mathrm{III}+)\ \alpha = e_i - e_k (1 \leqq i < k \leqq p); \quad \sum_{j=1}^{N} z_{i,j} \frac{\partial}{\partial z_{k,j}},$$

$$(\mathrm{IV}+)\ \alpha = e_i - e_k (p < i < k \leqq m); \quad \sum_{j=1}^{N} z_{k,j} \frac{\partial}{\partial z_{i,j}},$$

$$(\mathrm{V}+)\ \alpha = e_i - e_k (1 \leqq i \leqq p < k \leqq m); \quad \sum_{j=1}^{N} z_{i,j} z_{k,j},$$

$$(\mathrm{VI}+)\ \alpha = f_k - f_l (1 \leqq k < l \leqq n); \quad \sum_{j=1}^{N} r_{k,j} r_{l,j} (1 + \alpha_{k,j})(1 - \alpha_{l,j}).$$

Since the calculations are easy and elementary, we omitted them.

4.4. Description of Primitive Vectors for the Twisted Systems. At first, we consider the twisted positive system Ψ^+ (see Sect. 3.2). Then primitive vectors for Ψ^- must be killed by the operators of type (I), (II+), (III), (IV+), (V) and (VI) in Sect. 4.3.

For $1 \leqq a \leqq \min\left\{p, \frac{N}{2}\right\}$ and $1 \leqq b \leqq \min\left\{q, \frac{N}{2}\right\}$, put

$$\Lambda_a = \det(z_{k,2j-1} + z_{k,2j})_{\substack{p-a<k\leqq p,\\ 1\leqq j\leqq a}}, \qquad \bar{\Lambda}_a = \det(z_{k,2j-1} - z_{k,2j})_{\substack{p-a<k\leqq p,\\ 1\leqq j\leqq a}},$$

$$\Xi_b = \det(z_{k,2j-1} + z_{k,2j})_{\substack{p+q-b<k\leqq p+q,\\ 1\leqq j\leqq b}}, \qquad \bar{\Xi}_b = \det(z_{k,2j-1} - z_{k,2j})_{\substack{p+q-b<k\leqq p+q.\\ 1\leqq j\leqq b}}$$

For $\frac{N}{2} < a \leqq \min\{p, N\}$ we replace the element $z_{k,2j-1} \pm z_{k,2j}$ $(N - a < j \leqq a)$ in Λ_a and $\bar{\Lambda}_a$ by $z_{k,N-a+j}$. Similarly, for $\frac{N}{2} < b \leqq \min\{q, N\}$, we replace the element $z_{k,2j-1} \pm z_{k,2j} (N - b < j \leqq b)$ in Ξ_b and $\bar{\Xi}_b$ by $z_{k,N-b+j}$. We denote this situation by

$$\Lambda_a = \det(z_{k,2j-1} + z_{k,2j} | z_{k,N-a+j})_{\substack{p-a<k\leqq p,\\ 1\leqq j\leqq a}}$$

for example. We put, for $a = 0$ or $b = 0$, $\Lambda_0 = \bar{\Lambda}_0 = \Xi_0 = \bar{\Xi}_0 = 1$.

For $0 \leqq c \leqq N$, we define

$$R_c = \prod_{l=1}^{n} R_{l,c}, \quad \bar{R}_c = \prod_{l=1}^{n} \bar{R}_{l,c},$$

where $R_{l,c}$ and $\bar{R}_{l,c}$ are given by

$$\begin{cases} R_{l,c} = \prod_{j=1}^{c} (r_{l,2j-1} + r_{l,2j}) & \text{if } 1 \leqq c \leqq \frac{N}{2}, \\ R_{l,c} = \prod_{j=1}^{N-c} (r_{l,2j-1} + r_{l,2j}) \prod_{j=2N-2c+1}^{N} r_{l,j} & \text{if } \frac{N}{2} < c \leqq N, \end{cases}$$

and

$$\begin{cases} \bar{R}_{l,c} = \prod_{j=1}^{c} (r_{l,2j-1} - r_{l,2j}) & \text{if } 1 \leqq c \leqq \frac{N}{2}, \\ \bar{R}_{l,c} = \prod_{j=1}^{N-c} (r_{l,2j-1} - r_{l,2j}) \prod_{j=2N-2c+1}^{N} r_{l,j} & \text{if } \frac{N}{2} < c \leqq N. \end{cases}$$

Here we consider $R_{l,0} = \bar{R}_{l,0} = 1$. Note that they are vectors in the representation space $F = \mathbb{C}[z_{ij} | 1 \leqq i \leqq m, 1 \leqq j \leqq N] \otimes C(r_{kl} | 1 \leqq k \leqq n, 1 \leqq l \leqq N)$ of the oscillator representation ρ.

Lemma 4.1. *Let integers a, b and c satisfy the condition $a \leqq c \leqq N - b$. Then the vectors $\Lambda_a \bar{\Xi}_b R_c$ and $\bar{\Lambda}_a \Xi_b \bar{R}_c$ are primitive for Ψ^-.*

Proof. One can easily check that the vectors in the lemma are killed by the operators of type (III) and (IV+). For the other operators, we note that the following equations hold:

$$\frac{\partial}{\partial z_{k,2j-1}} \Lambda_a = \frac{\partial}{\partial z_{k,2j}} \Lambda_a \quad (j \leqq \min\{a, N-a\}), \tag{4.1}$$

$$\frac{\partial}{\partial z_{k,2j-1}} \bar{\Xi}_b = -\frac{\partial}{\partial z_{k,2j}} \bar{\Xi}_b \quad (j \leqq \min\{b, N-b\}), \tag{4.2}$$

$$r_{l,2j-1}(1 + \alpha_{l,2j-1}) R_c = -r_{l,2j}(1 + \alpha_{l,2j}) R_c \quad (j \leqq \min\{c, N-c\}), \tag{4.3}$$

$$r_{l,2j-1}(1 - \alpha_{l,2j-1}) R_c = r_{l,2j}(1 - \alpha_{l,2j}) R_c \quad (j \leqq \min\{c, N-c\}). \tag{4.4}$$

First consider an operator of type (V). It kills the vectors in the lemma. In fact, for $j \leqq \min\{a, b\}$, we have

$$\left(\frac{\partial}{\partial z_{i,2j-1}} \frac{\partial}{\partial z_{k,2j-1}} + \frac{\partial}{\partial z_{i,2j}} \frac{\partial}{\partial z_{k,2j}}\right) \Lambda_a \bar{\Xi}_b = -\frac{\partial}{\partial z_{i,2j}} \Lambda_a \frac{\partial}{\partial z_{k,2j}} \bar{\Xi}_b + \frac{\partial}{\partial z_{i,2j}} \Lambda_a \frac{\partial}{\partial z_{k,2j}} \bar{\Xi}_b = 0, \tag{4.5}$$

from Eqs. (4.1) and (4.2). For $j > \min\{a, b\}$, one of the factors $\partial \Lambda_a / \partial z$ and $\partial \bar{\Xi}_b / \partial z$ vanishes, where $\partial / \partial z$ represents an operator which appears in Eq. (4.5). More precisely, if $j > a$, then we get

$$\frac{\partial}{\partial z_{i,2j-1}} \Lambda_a = \frac{\partial}{\partial z_{i,2j}} \Lambda_a = 0.$$

Similarly, if $j > b$, then we get

$$\frac{\partial}{\partial z_{k,2j-1}} \bar{\Xi}_b = -\frac{\partial}{\partial z_{k,2j}} \bar{\Xi}_b = 0.$$

Next let us show that the operator (I) kills the vectors in the lemma. For $j \leqq \min\{a, N-c\} = \min\{a, c, N-a, N-c\}$, we have

$$\begin{aligned} &\left(\frac{\partial}{\partial z_{i,2j-1}} r_{l,2j-1}(1 + \alpha_{l,2j-1}) + \frac{\partial}{\partial z_{i,2j}} r_{l,2j}(1 + \alpha_{l,2j})\right) \Lambda_a R_c \\ &= -\frac{\partial}{\partial z_{i,2j}} \Lambda_a \cdot r_{l,2j}(1 + \alpha_{l,2j}) R_c + \frac{\partial}{\partial z_{i,2j}} \Lambda_a \cdot r_{l,2j}(1 + \alpha_{l,2j}) R_c = 0, \end{aligned} \tag{4.6}$$

from Eqs. (4.1) and (4.3). For $j > \min\{a, N-c\}$, we get

$$\frac{\partial}{\partial z}\Lambda_a = 0 \quad \text{or} \quad r(1+\alpha)R_c = 0,$$

in the same way as above. Here we denote by $\partial/\partial z$ or $r(1+\alpha)$ an operator which appears in (4.6).

For operator (II), we can use Eqs. (4.2) and (4.4) instead of (4.1) and (4.3).

Lastly we consider operator (VI). Note that the operators

$$r_{l,2j-1}(1+\alpha_{l,2j-1}) \quad \text{and} \quad r_{k,2j-1}(1-\alpha_{k,2j-1})$$

are anti-commutative. Then Eqs. (4.3) and (4.4) tell us that the vectors are killed by the operator. Q.E.D.

A *Young diagram* $Y=(\alpha_1, \alpha_2, \ldots, \alpha_l)$ is a decreasing sequence of finite non-negative integers. We put $|Y| = \sum_{i=1}^{l} \alpha_i$, $\text{depth}(Y) = \max\{r|\alpha_r \neq 0\}$, $ht(Y) = \alpha_1$, and call them *length, depth* and *height* of Y respectively. Let ${}^tY = (i_1, i_2, \ldots, i_{l'})$ be the transposed Young diagram of Y. Namely

$$i_k = \#\{j|\alpha_j \geqq k\},$$

where $\#S$ means the cardinality of the set S. Note that the operation ${}^t(\cdot)$ is involutive: ${}^t({}^tY) = Y$. Let $\mathscr{Y}$ be the set of all the Young diagrams. Define a subset $\mathscr{T}(N)$ in the direct product $\mathscr{Y} \times \mathscr{Y} \times \mathscr{Y}$ as

$$\begin{aligned}\mathscr{T}(N) = \{&(Y_A, Y_B, Z)|Z = (c_1, c_2, \ldots, c_n),\\ &ht(Y_A) \leqq \min\{p-1, N-c_1\}, ht(Y_B) \leqq \min\{q-1, c_n\}\}.\end{aligned}$$

We associate vectors v_T and $\bar{v}_T$ in F with an element $T = (Y_A, Y_B, Z) \in \mathscr{T}(N)$ as follows. For $Y_A = (\alpha_1, \alpha_2, \ldots, \alpha_{l_1})$, $Y_B = (\beta_1, \beta_2, \ldots, \beta_{l_2})$, put $I_A = {}^tY_A$, $I_B = {}^tY_B$. Write $I_A = (i_1, i_2, \ldots, i_{p-1})$, $I_B = (j_1, j_2, \ldots, j_{q-1})$ and ${}^tZ = (s_1, s_2, \ldots)$. Then the vectors v_T and $\bar{v}_T \in F$ are given by

$$R_T = \prod_{l=1}^{n} R_{l,N-c_l}, \quad \bar{R}_T = \prod_{l=1}^{n} \bar{R}_{l,N-c_l},$$

$$v_T = \prod_{i=1}^{l_1} \Lambda_{\alpha_i} \prod_{j=1}^{l_2} \bar{\Xi}_{\beta_j} \cdot R_T \quad \text{and} \quad \bar{v}_T = \prod_{i=1}^{l_1} \bar{\Lambda}_{\alpha_i} \prod_{j=1}^{l_2} \Xi_{\beta_j} \cdot \bar{R}_T.$$

Lemma 4.2. *We have for* $1 \leqq k \leqq l \leqq n$,

$$\sum_{j=1}^{N} r_{k,j} r_{l,j}(1-\alpha_{k,j})(1+\alpha_{l,j})R_T = 0,$$

$$\sum_{j=1}^{N} r_{k,j} r_{l,j}(1-\alpha_{k,j})(1+\alpha_{l,j})\bar{R}_T = 0.$$

Proof. To prove the above, we shall show that

$$\{r_{k,2j-1} r_{l,2j-1}(1-\alpha_{k,2j-1})(1+\alpha_{l,2j-1}) + r_{k,2j} r_{l,2j}(1-\alpha_{k,2j})(1+\alpha_{l,2j})\}R_T = 0$$

for any j. We gather all the terms in the product expression of R_T which concern either $r_{k,2j-1}, r_{k,2j}$ or $r_{l,2j-1}, r_{l,2j}$. Then R_T can be written as

$$R_T = \pm R(k,l,j) \cdot R'_T,$$

where $R(k,l,j)$ is one of the following forms:

$$R(k,l,j) = \begin{cases} 1, \\ r_{l,2j-1} r_{l,2j}, \\ r_{k,2j-1} r_{k,2j} r_{l,2j-1} r_{l,2j}, \\ (r_{l,2j-1} + r_{l,2j}), \\ (r_{k,2j-1} + r_{k,2j})(r_{l,2j-1} + r_{l,2j}), \\ (r_{k,2j-1} + r_{k,2j}) r_{l,2j-1} r_{l,2j}. \end{cases}$$

Now the explicit calculations show that

$$\{r_{k,2j-1} r_{l,2j-1}(1-\alpha_{k,2j-1})(1+\alpha_{l,2j-1}) + r_{k,2j} r_{l,2j}(1-\alpha_{k,2j})(1+\alpha_{l,2j})\} R(k,l,j) = 0$$

in all cases. Q.E.D.

Proposition 4.3. *For $T \in \mathscr{T}(N)$, vectors v_T and $\bar{v}_T$ are primitive for Ψ^-,* i.e., *killed by the operators $\{\tilde{\rho}(X_\alpha) | \alpha \in \Psi^-\}$. Weights of v_T and $\bar{v}_T$ are the same and given by $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m / \mu_1, \mu_2, \ldots, \mu_n)$ with*

$$\lambda_k = i_{p-k+1} + \frac{N}{2} \qquad (1 \leqq k \leqq p),$$

$$\lambda_k = -j_{m-k+1} - \frac{N}{2} \quad (p < k \leqq m),$$

$$\mu_l = c_l - \frac{N}{2} \qquad (1 \leqq l \leqq n).$$

Remark. For the coordinate expression of λ, see (2.1).

Proof. The essential part of the proof is given in Lemmas 4.1 and 4.2. Note that operators $\tilde{\rho}(X_\alpha)$ are first order differential operators except the two which are type (V) and (VI). For these operators we have

$$\tilde{\rho}(X_\alpha) v_T = \sum_{a,b} f_{a,b}(z,r) \tilde{\rho}(X_\alpha)(\Lambda_a \Xi_b R_T),$$

where $f_{a,b}(z,r)$ is a vector in F.

If the reader uses Lemma 4.1 and calculates carefully, then the reader can conclude that either

$$\tilde{\rho}(X_\alpha)(\Lambda_a \Xi_b R_T) \text{ really vanishes of } f_{a,b}(z,r) = 0$$

holds. Thus v_T is killed by $\tilde{\rho}(X_\alpha)$.

For the type (V) operator, the situation is similar because indices i and k of the partial differentials are divided into two which correspond to Λ_a and Ξ_b respectively. So they are essentially first order differential operators.

For the type (VI) operators, Lemma 4.2 tells you that v_T and $\bar{v}_T$ are killed by them.

Next we calculate the weights of v_T and $\bar{v}_T$. Note that elements in the Cartan subalgebra $\tilde{\mathfrak{h}}$ of $\mathfrak{u}(p,q/n)$ are represented by the operators as follows:

$$\tilde{\rho}(\sqrt{-1}E_{k,k}) = \operatorname{sgn}(k)\sqrt{-1}\sum_{j=1}^{N}\left\{z_{k,j}\frac{\partial}{\partial z_{k,j}} + \frac{1}{2}\right\} \quad (1 \leqq k \leqq m),$$

$$\tilde{\rho}(\sqrt{-1}E_{m+l,m+l}) = -\frac{\sqrt{-1}}{2}\sum_{j=1}^{N}\alpha_{l,j} \quad (1 \leqq l \leqq n),$$

where $\operatorname{sgn}(k)$ is defined as in Sect. 3.2 and $\alpha_{l,j}$ is the automorphism of F given in Sect. 4.1.

Since $\tilde{\rho}(\sqrt{-1}E_{k,k})$ $(1 \leqq k \leqq m)$ are only the Euler's degree operators up to constant multiple, the results are easy to get.

For $\tilde{\rho}(\sqrt{-1}E_{m+l,m+l})$ $(1 \leqq l \leqq n)$, note that

$$(\alpha_{l,2j-1} + \alpha_{l,2j})R_T = 0,$$

if $j \leqq c_l \left(c_l \leqq \frac{N}{2}\right)$ or $j \leqq N - c_l \left(c_l > \frac{N}{2}\right)$. On the other hand, we have

$$\begin{cases} \alpha_{l,j}R_T = R_T & \text{if } \; c_l \leqq \frac{N}{2} \text{ and } j > 2c_l \\ \alpha_{l,j}R_T = -R_T & \text{if } \; c_l > \frac{N}{2} \text{ and } j > 2N - 2c_l. \end{cases}$$

The above three formulas prove the result for $\tilde{\rho}(\sqrt{-1}E_{m+l,m+l})$.

The proof for $\bar{v}_T$ is similar. Q.E.D.

4.5. Description of Primitive Vectors for the Standard System. In this subsection, we give typical primitive vectors for the standard negative system Δ^- of $\mathfrak{su}(p,q/n)$. Primitive vectors for Δ^- are to be killed by the type (I–VI) operators in Sect. 4.3. As in the former subsection, we put

$$\Omega_b = \det(z_{k,2j-1} + z_{k,2j} | z_{k,N-b+j})_{\substack{p<k\leqq p+b,\\ 1\leqq j\leqq b}}$$

$$\bar{\Omega}_b = \det(z_{k,2j-1} - z_{k,2j} | z_{k,N-b+j})_{\substack{p<k\leqq p+b.\\ 1\leqq j\leqq b}}$$

For the notation in the above equations, see Sect. 4.4. Since the type (II) operators play a decisive role in the following, we have given an extra notation:

$$X_{k,l} = \sum_{j=1}^{N} z_{k,j} r_{l,j}(1 + \alpha_{l,j}).$$

Now take $T = (Y_A, Y_B, Z) \in \mathscr{T}(N)$ and define I_A and I_B as in Sect. 4.4. For $Z = (c_1, c_2, \ldots, c_n)$, put

$$m_l = \min\{m, p + c_l\}\ (1 \leqq l \leqq n), \quad \text{and} \quad X_T = \prod_{l=1}^{n}\prod_{k=p+1}^{m_l} X_{k,l}. \tag{4.7}$$

Let us consider the following vectors w_T and $\bar{w}_T$:

$$w_T = \prod_{i=1}^{l_1}\Lambda_{\alpha_i}\prod_{j=1}^{l_2}\bar{\Omega}_{\beta_j}\cdot R_T \quad \text{and} \quad \bar{w}_T = \prod_{i=1}^{l_1}\bar{\Lambda}_{\alpha_i}\prod_{j=1}^{l_2}\Omega_{\beta_j}\cdot\bar{R}_T.$$

Lemma 4.4. *The vectors $X_T w_T$ and $X_T \bar{w}_T$ are non-zero and killed by $X_{s,t}$ ($p < s \leqq m$, $1 \leqq t \leqq n$).*

Proof. Let us show that $X_T w_T$ is non-zero at first. To prove this, it is enough to see that $X_T R_T \neq 0$. Assume that m is sufficiently large. If we expand R_T into the monomials in $r_{i,j}$, then every monomial has degree

$$\sum_{l=1}^{n} (N - c_l).$$

Note that the degree of X_T is $\sum_{l=1}^{n} c_l$ because we assume m is sufficiently large.

Since the operator $r_{l,j}(1 + \alpha_{l,j})$ acts on every monomial as zero if it contains $r_{l,j}$ and as 2 otherwise, it is easy to see that, after applying X_T to every monomial, the result is of the form

$$f(z) \cdot \prod_{1 \leqq i \leqq n, 1 \leqq j \leqq N} r_{i,j}.$$

Here $f(z)$ does not vanish. In fact, if we put $d = \# \left\{ l | c_l \geqq \frac{N}{2} \right\}$, then, for example, the coefficient of

$$\prod_{l=1}^{d} \left(\prod_{j=1}^{N-c_l} z_{p+j,2j} \prod_{j=N-c_l+1}^{c_l} z_{p+j,N-c_l+j} \right) \times \prod_{l=d+1}^{n} \left(\prod_{j=1}^{c_l} z_{p+j,2j} \right)$$

in $f(z)$ is $\pm 2^{\deg X_T}$. The sign $\pm$ depends on the order of the arrangements of $r_{i,j}$'s.

If m is relatively small, then you can consider ideal operators $X_{k,l}$ ($m+1 \leqq k \leqq m'$) with ideal symbols $\{z_{k,j} | m+1 \leqq k \leqq m', 1 \leqq j \leqq N\}$ for a sufficiently large m'. Consider X_T for this m' and write it as X'_T. Then according to the above discussion, we have $X'_T R_T \neq 0$. On the other hand, we can write $X'_T = X''_T \cdot X_T$ with superfluous part X''_T. Since $X'_T R_T \neq 0$, we see that $X_T R_T \neq 0$.

Next we show that $X_{s,t} X_T R_T = 0$. If $X_{s,t}$ appears as a member of the product for X_T, then the equation holds because $X_{s,t}^2 = 0$. Thus assume that $X_{s,t}$ does not appear in X_T. Then we have $s \geqq m_t + 1$ and $m_t = p + c_t < m$ $(1 \leqq t \leqq n)$. Fix t and consider monomials in R_T which contain $\{r_{t,j} | 1 \leqq j \leqq N\}$. Then their degrees with respect to $\{r_{t,j} | 1 \leqq j \leqq N\}$ are the same, i.e. $N - c_t$ $(1 \leqq t \leqq n)$. On the other hand, the degree of X_T with respect to $\{X_{k,t} | p < k \leqq m\}$ is $m_t - p = c_t$. So we get

$$\deg X_T + \deg R_T = N$$

for any t. This means that the monomials which appear in $X_T R_T$ necessarily contain the product $\prod_{j=1}^{N} r_{t,j}$. Since it is easy to see

$$X_{s,t} \prod_{j=1}^{N} r_{t,j} = 0,$$

we have $X_{s,t} X_T R_T = 0$. Q.E.D.

Proposition 4.5. *For $T \in \mathscr{T}(N)$ vectors $X_T w_T$ and $X_T \bar{w}_T$ are non-zero and primitive for Δ^-*, i.e., *killed by the operators $\{\tilde{\rho}(X_\alpha) | \alpha \in \Delta^-\}$. Weights of $X_T w_T$ and $X_T \bar{w}_T$ are* *the same and given by* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m/\mu_1, \mu_2, \ldots, \mu_n)$ *with*

$$\lambda_k = i_{p-k+1} + \frac{N}{2} \qquad (1 \leqq k \leqq p),$$

$$\lambda_k = -j_{k-p} - s_{k-p} - \frac{N}{2} \quad (p < k \leqq m),$$

$$\lambda_l = c_l - m_l + p - \frac{N}{2} \qquad (l \leqq l \leqq n),$$

where $\{m_l\}$ *are given in (4.7).*

Proof. As in the proof of Proposition 4.3, it can be proved similarly that the vectors w_T and $\bar{w}_T$ are killed by the operators of type (I, III, IV, V) and (VI).

Since operators (I) and (III) commutes with $\{X_{k,l}\}$, they kill $X_T w_T$ *and* $X_T \bar{w}_T$. Lemma 4.4 tells us that operators (II) kill them.

Since $X_T w_T$ and $X_T \bar{w}_T$ can be treated in the same way, we only concern ourselves with $X_T w_T$ in the following.

Let us consider an operator Y of type (IV). Y satisfies either $[Y, X_{k,l}] = 0$ or $[Y, X_{k,l}] = X_{j,l}$ for $j < k$. Considering these two equations, one can conclude that Y commutes with X_T. Hence operators (IV) kill $X_T w_T$ and $X_T \bar{w}_T$.

For an operator Z of type (V), we should note $[Z, X_{k,l}] = 0$ or $[Z, X_{k,l}]$ is of type (I). Hence Z commutes with X_T or $[Z, X_T] = \sum_{Z'} X'_T Z'$ using operators $\{Z'\}$ of type (I). Now it can be easily seen that

$$ZX_T w_T = \begin{cases} X_T Z w_T = 0 & \text{if } \; [Z, X_T] = 0, \\ X_T Z w_T + \sum_{Z'} X'_T Z' w_T = 0 & \text{otherwise.} \end{cases}$$

Finally, consider an operator W of type (VI). Since $[W, X_{k,l}] = 0$ or $[W, X_{k,l}] = X_{k,t}$ for $l < t$, $[W, X_T]$ is a linear combination of the operators $X_{k,t} X_T(k, l)$, where $X_T(k, l)$ denotes the operator obtained by eliminating $X_{k,l}$ in X_T. Note that $l \neq t$. Careful check of the proof of Lemma 4.4 leads to $X_{k,t} X_T(k, l) w_T = 0$ for $l \neq 0$ for $l \neq t$. Now we have $[W, X_T] w_T = 0$ and

$$WX_T w_T = X_T W w_T + [W, X_T] w_T = 0.$$

Since the weight of $X_{k,l}$ is $e_k - f_l$, it is easy to calculate the weights of $X_T w_T$ and $X_T \bar{w}_T$, using weight operators in the proof of Proposition 4.3. Q.E.D.

5. Determination of the Unitarizable Lowest Weight Modules for $\mathfrak{su}(p, q/n)$

In this section we investigate a necessary condition for the unitarizability of irreducible lowest weight representations of $\mathfrak{su}(p, q/n)$ with respect to Δ^+. It turns out that the unitarizable lowest weight modules obtained in Sect. 4 exhaust all the irreducible unitary representations of $\mathfrak{su}(p, q/n)$ (Theorem 5.3).

Definition 5.1. *For the weight* λ *of the form in Proposition 2.2(2b), we define three Young diagrams* Y_1^λ, Y_2^λ *and* Y_3^λ:

$$Y_1^\lambda = (\lambda_p - \lambda_1, \lambda_{p-1} - \lambda_1, \dots, \lambda_2 - \lambda_1, 0),$$
$$Y_2^\lambda = (\lambda_m - \lambda_{p+1}, \lambda_m - \lambda_{p+2}, \dots, \lambda_m - \lambda_{m-1}, 0),$$
$$Y_3^\lambda = (\mu_1 - \mu_n, \mu_2 - \mu_n, \dots, \mu_{n-1} - \mu_n, 0).$$

We put $d_i = \mathrm{depth}(Y_i^\lambda)$ *for* $1 \leqq i \leqq 3$, *and* $g_1 = \mu_n - \lambda_m$, $g_2 = \lambda_1 - \mu_1$, $g_3 = \lambda_1 - \lambda_m$.

From the result of [2, Theorem 7.4], we get

Proposition 5.2. *With notations as above, the lowest weight* λ *of an irreducible super-unitary representation satisfies*

$$g_3 \geqq d_1 + d_2.$$

Proof. If (π, V) is an irreducible super-unitary representation with the lowest weight λ, then (π, V) contains a unitary representation of $\mathfrak{su}(p,q) \subset \mathfrak{su}(p,q/n)_{\bar{0}}$ with the lowest weight $(\lambda_1, \lambda_2, \dots, \lambda_m)$. Therefore an irreducible representation of $\mathfrak{su}(p,q)$ with the lowest weight $(\lambda_1, \lambda_2, \dots, \lambda_m)$ is unitarizable. Let us convert the condition in [2, Theorem 7.4] to the condition of the representations of the lowest weight type and apply that condition to the above weight. Then we get the inequality of this proposition. Q.E.D.

Now we are ready to state our main theorem.

Theorem 5.3. *Let* (π, V) *be an irreducible* Δ^+*-lowest weight representation of* $\mathfrak{su}(p,q/n)$ *with the integral lowest weight* $\lambda \in (\mathfrak{h}^{\mathbb{C}})^*$. *Conditions* (I) *and* (II) *are necessary and sufficient for* (π, V) *to be super-unitary.*

(I) *The lowest weight* λ *satisfies*

$$\lambda_{p+1} \leqq \cdots \leqq \lambda_m \leqq \mu_n \leqq \cdots \leqq \mu_1 \leqq \lambda_1 \leqq \cdots \leqq \lambda_p. \tag{5.1}$$

(II) *With notations given in Definition 5.1,* λ *satisfies condition* (5.2) *or* (5.3):

$$g_2 \geqq d_1 + q \quad \text{and} \quad g_1 \geqq d_3, \tag{5.2}$$

$$g_2 \geqq d_1 + d_2 \quad \text{and} \quad g_1 = d_3 = 0. \tag{5.3}$$

Remark 1. The above theorem, together with the highest weight version Theorem 5.5, classifies all the irreducible super-unitary representations which can be integrated up to the representations of $S(U(p,q) \times U(n))$, a Lie group corresponding to the even part $\mathfrak{su}(p,q/n)_{\bar{0}}$. In fact, Corollary 2.3 tells us that the representation is a highest or lowest weight module with integral weights. All such representations are classified by Theorem 5.3.

Remark 2. If $n = 0$ then the conditions in this theorem become

$$\text{(I)}\ \lambda_{p+1} \leqq \cdots \leqq \lambda_m \leqq \lambda_1 \leqq \cdots \leqq \lambda_p, \quad \text{(II)}\ g_3 \geqq d_1 + d_2.$$

Condition (II) above is just the condition given in Proposition 5.2, which is derived from [2, Theorem 7.4].

If $q = 0$ then the conditions become

$$\text{(I)}\ \mu_n \leqq \cdots \leqq \mu_1 \leqq \lambda_1 \leqq \cdots \leqq \lambda_m, \quad \text{(II)}\ \lambda_1 - \mu_1 \geqq d_1,$$

and if $p = 0$ then the conditions become

$$\text{(I)}\ \lambda_1 \leqq \cdots \leqq \lambda_m \leqq \mu_n \leqq \cdots \leqq \mu_1, \quad \text{(II)}\ \mu_n - \lambda_m \geqq d_3.$$

Proof. First of all, we will show that these conditions are sufficient. Let λ satisfy conditions (I) and (II) of the theorem and define Y_i^λ, d_i and g_i according to Definition 5.1. To prove the sufficiency, it is enough to show λ is the Δ^+-lowest weight for a super-unitary representation obtained in Sect. 4.5. We put $Y_A = {}^tY_1^\lambda = (a_1, a_2, \ldots, a_{l_1})$ in the notation of Sect. 4.4. Note that $\operatorname{depth}(Y_1^\lambda) = d_1 = a_1 = ht(Y_A)$ holds. We divide the proof into three cases according to the value of g_1.

(i) *The case where* $g_1 \geqq n$. In this case, we put

$$N = g_3 - n, \quad Y_B = {}^tY_2^\lambda \quad \text{and} \quad Z = Y_3^\lambda + \square(n, q + g_1 - n),$$

where $Y_1 \pm Y_2 = (a_1 \pm b_1, a_2 \pm b_2, \ldots)$ for two Young diagrams $Y_1 = (a_1, a_2, \ldots)$, $Y_2 = (b_1, b_2, \ldots)$ and $\square(k, l) = (l, l, \ldots, l)$ is a box type Young diagram with depth k. Put $T = (Y_A, Y_B, Z)$. Then T belongs to $\mathscr{T}(N)$. In fact, we have

$$c_n = q + g_1 - n \geqq q > d_2 = b_1 = ht(Y_B)$$

and

$$N - c_1 = (g_3 - n) - (g_3 - g_2 + q - n) = g_2 - q \geqq d_1 = a_1 = ht(Y_A).$$

From the definition, we have

$$\begin{aligned}
(i_1, i_2, \ldots, i_{p-1}) &= {}^tY_A = Y_1^\lambda = (\lambda_p - \lambda_1, \lambda_{p-1} - \lambda_1, \ldots, \lambda_2 - \lambda_1) \\
(j_1, j_2, \ldots, j_{q-1}) &= {}^tY_B = Y_2^\lambda = (\lambda_m - \lambda_{p+1}, \lambda_m - \lambda_{p+2}, \ldots, \lambda_m - \lambda_{m-1}) \\
(c_1, c_2, \ldots, c_n) &= Z = Y_3^\lambda + \square(n, q + g_1 - n) \\
&= (\mu_1 - \lambda_m + q - n,\ \mu_2 - \lambda_m + q - n, \ldots, \mu_n - \lambda_m + q - n),
\end{aligned}$$

because it holds that $\mu_j - \mu_n + g_1 = \mu_j - \lambda_m$. Since g_1 is greater than or equal to n, we obtain $q + g_1 - n \geqq q$, so we get $c_l \geqq q$ for all l and every m_l in (4.7) is equal to m. Therefore we get

$$(s_1, s_2, \ldots, s_q, \ldots) = {}^tZ = (\underbrace{n, n, \ldots, n}_{q}, \ldots).$$

Now we apply these values to Proposition 4.5 and get the weight of the vector $X_T w_T$. Since we have

$$i_{p+1-k} + \frac{N}{2} = \lambda_k - \lambda_1 + \frac{N}{2} \qquad (1 \leqq k \leqq p),$$

$$-j_{k-p} - s_{k-p} - \frac{N}{2} = -(\lambda_m - \lambda_k) - n - \frac{N}{2} \qquad (p < k \leqq m),$$

$$c_l - m_l + p - \frac{N}{2} = (\mu_l - \lambda_m + q - n) - m + p - \frac{N}{2} \quad (1 \leqq l \leqq n),$$

the weight is

$$(\lambda_1 + t, \lambda_2 + t, \ldots, \lambda_p + t, \lambda_{p+1} + t, \lambda_{t+2} + t, \ldots, \lambda_m + t / \mu_1 + t, \mu_2 + t, \ldots, \mu_n + t),$$

where $t = \frac{N}{2} - \lambda_1 = -\frac{N}{2} - \lambda_m - n$. This is a translation of λ by t, hence its restriction to $\mathfrak{h}^{\mathbb{C}}$ coincides with that of λ. Thus we prove that the lowest weight module with the lowest weight λ in this case is super-unitarizable.

(ii) *The case where* $0 < g_1 < n$. In this case, we put $s_j = \min\{n, \mu_n - \lambda_{p+j}\}$ for $1 \leqq j \leqq q$ and denote the Young diagram $(s_1, s_2, \ldots, s_q)$ by S. We also put

$$N = g_3 - g_1, \quad Y_B = {}^t(Y_2^\lambda + \square(q, g_1) - S) \quad \text{and} \quad Z = Y_3^\lambda + {}^tS.$$

We claim that $T = (Y_A, Y_B, Z)$ is in $\mathscr{T}(N)$. Since we have

$$\begin{aligned} Y_2^\lambda + \square(q, g_1) &= (\lambda_m - \lambda_{p+1}, \ldots, \lambda_m - \lambda_{m-1}, 0) + (\mu_n - \lambda_m, \ldots, \mu_n - \lambda_m, \mu_n - \lambda_m) \\ &= (\mu_n - \lambda_{p+1}, \ldots, \mu_n - \lambda_{m-1}, \mu_n - \lambda_m), \end{aligned}$$

if we put $(b'_1, b'_2, \ldots, b'_n, \ldots) = {}^t(Y_2^\lambda + \square(q, g_1))$ then we have $b'_j = \#\{k \mid \mu_n - \lambda_{p+k} \geqq j\}$. Note that ${}^tS = (b'_1, b'_2, \ldots, b'_n)$. Therefore it holds that

$$c_n = \#\{k \mid s_k \geqq n\} = \#\{k \mid \mu_n - \lambda_{p+k} \geqq n\} = b'_n.$$

Similarly we get

$$b_1 = \#\{k \mid \mu_n - \lambda_{p+k} - s_k \geqq 1\} = \#\{k \mid \mu_n - \lambda_{p+k} \geqq n+1\} = b'_{n+1}.$$

So T satisfies

$$c_n = b'_n \geqq b'_{n+1} = b_1 = ht(Y_B).$$

For the other condition $N - c_1 \geqq ht(Y_A)$, we calculate as follows:

$$\begin{aligned} N - c_1 &= g_3 - g_1 - c_1 \\ &= g_3 - g_1 - (ht(Y_3^\lambda) + q) \quad (\text{because } b'_1 = q) \\ &= g_2 - q \geqq d_1 = a_1 = ht(Y_A). \end{aligned}$$

Now we proceed in the same way as in case (i). We note that

$$\begin{aligned} (i_1, i_2, \ldots, i_{p-1}) &= {}^tY_A = Y_1^\lambda = (\lambda_p - \lambda_1, \lambda_{p-1} - \lambda_1, \ldots, \lambda_2 - \lambda_1), \\ (j_1, j_2, \ldots, j_{q-1}) &= {}^tY_B = Y_2^\lambda + \square(q, g_1) - S \\ &= (\mu_n - \lambda_{p+1} - s_1, \mu_n - \lambda_{p+2} - s_2, \ldots, \mu_n - \lambda_{m-1} - s_{q-1}), \\ (c_1, c_2, \ldots, c_n) &= Z = Y_3^\lambda + {}^tS = (\mu_1 - \mu_n + b'_1, \mu_2 - \mu_n + b'_2, \ldots, \mu_n - \mu_n + b'_n). \end{aligned}$$

From the definition we get $b'_1 = \cdots = b'_{g_1} = q \geqq b'_{g_1+1}$. On the other hand, it holds that $\mu_k - \mu_n = 0$ for any $d_3 + 1 \leqq k \leqq n$. Thus, from the condition $g_1 \geqq d_3$, we get

$$(c_1, c_2, \ldots, c_n) = (\mu_1 - \mu_n + q, \mu_2 - \mu_n + q, \ldots, \mu_{g_1} - \mu_n + q, b'_{g_1+1}, \ldots, b'_n).$$

Therefore we have $m_l = m$ for $1 \leqq l \leqq g_1$ and $m_l = b'_l + p$ for $g_1 < l \leqq n$ (see Eq. (4.7)). Now we apply these values to Proposition 4.5. Then we get the weight of the vector $X_T w_T$. First note that, for $g_1 < l \leqq n$, it holds that

$$\begin{aligned} c_l + p - m_l - N/2 &= b'_l + p - (b'_l + p) - N/2 \\ &= -N/2 = \mu_l - \mu_n - N/2 \quad (\text{because } \mu_l = \mu_n). \end{aligned}$$

The other coordinates of the weight of $X_T w_T$ can be calculated similarly. So we conclude that

$$(\lambda_1 + t, \lambda_2 + t, \ldots, \lambda_p + t, \lambda_{p+1} + t, \lambda_{t+2} + t, \ldots, \lambda_m + t / \mu_1 + t, \mu_2 + t, \ldots, \mu_n + t)$$

is the weight of $X_T w_T$, where $t = \frac{N}{2} - \lambda_1 = -\frac{N}{2} - \mu_n$. Thus we proved that the lowest weight module with the lowest weight λ in this case is also super-unitarizable.

(iii) *The case where* $g_1 = 0$. We keep the notations used above. Then N, Y_B and Z used above become

$$N = g_3 = g_2, \quad Y_B = {}^t(Y_2^\lambda - S) \quad \text{and} \quad Z = {}^tS,$$

because of the condition $g_1 = 0$. It can be similarly proved that $T = (Y_A, Y_B, Z)$ is in $\mathscr{T}(N)$ and the weight of $X_T w_T$ is equal to λ as an element in $(\mathfrak{h}^{\mathbb{C}})^*$.

Thus we proved that a lowest weight module with the lowest weight λ, where λ satisfies conditions (I) and (II), is super-unitarizable.

To prove the necessary condition, we consider two cases where $\mu_1 - \lambda_m$ is zero or not.

When $\mu_1 - \lambda_m$ is zero, we must have $g_1 = 0$ and $g_2 = g_3$. Then, from Proposition 5.2, λ satisfies condition (5.3) of Theorem 5.3. For the case where $\mu_1 - \lambda_m$ is not zero, we prove the necessarity in the next lemma. Q.E.D.

Lemma 5.4. *If* $\mu_1 \neq \lambda_m$ *then the condition in Theorem 5.3 is a necessary condition for* (π, V) *to be super-unitary.*

Proof. Let v_λ be a lowest weight vector in V and put

$$v_k = \pi(E_{m,m+k}) \cdots \pi(E_{m,m+1}) v_\lambda \quad \text{for} \quad 1 \leqq k \leqq n-1.$$

We claim that v_k is equal to zero if and only if $\pi(E_{m+1,m}) \cdots \pi(E_{m+k,m}) v_k$ is equal to zero. In fact, the following three statements are equivalent:

(i) $\pi(E_{m+1,m}) \cdots \pi(E_{m+k,m}) v_k \neq 0$,

(ii) $v_k \neq 0$,

(iii) $v_\lambda \in U(\mathfrak{g}^{\mathbb{C}}) v_k$.

It is clear that (i) implies (ii). Since V is irreducible, (iii) follows from (ii).

So let us show that (iii) implies (i). According to the Poincaré–Birkhoff–Witt theorem, it can be written as

$$U(\mathfrak{g}^{\mathbb{C}}) v_k = U(\mathfrak{g}_{\bar{0}}^+)(\wedge \mathfrak{g}_{\bar{1}}^+)(\wedge \mathfrak{g}_{\bar{1}}^-) U(\mathfrak{g}_{\bar{0}}^-) U(\mathfrak{h}^{\mathbb{C}}) v_k,$$

where

$$\mathfrak{g}_{\bar{0}}^\pm = \bigoplus_{\pm\alpha \in \Delta_{\bar{0}}^+} \mathfrak{g}_\alpha.$$

Since v_k is a $\Delta_{\bar{0}}^+$-lowest weight vector, we can omit the term $U(\mathfrak{g}_{\bar{0}}^-) U(\mathfrak{h}^{\mathbb{C}})$ without loss of generality. Thus we assume that v_λ is of the form

$$v_\lambda = \sum_j u_j X_j Y_j v_k,$$

where $u_j \in U(\mathfrak{g}_{\bar{0}}^+)$, $X_j \in \wedge \mathfrak{g}_{\bar{1}}^+$ and $Y_j \in \wedge \mathfrak{g}_{\bar{1}}^-$. If $u_j X_j$ is not scalar, then the weight of $Y_j v_k$ is lower than λ, thus the vector $Y_j v_k$ vanishes. Therefore we get

$$v_\lambda = \sum_j Y_j v_k.$$

If we consider the difference between the weights of v_k and v_λ carefully, we see that Y_j is of the form:

$$Y_j = s\pi(E_{m+1,m}) \cdots \pi(E_{m+k,m}),$$

where s is a constant. Therefore we have

$$v_\lambda = s\pi(E_{m+1,m})\cdots\pi(E_{m+k,m})v_k,$$

hence s is not zero. Now statement (i) holds.

An easy calculation tells us

$$\pi(E_{m+1,m})\cdots\pi(E_{m+k,m})v_k = \prod_{j=1}^{k}(\lambda_m - \mu_j + j - 1)v_\lambda.$$

So we conclude that $v_k = 0$ if and only if $\mu_j - \lambda_m \neq j - 1$ for any j satisfying $1 \leqq j \leqq k$.

If v_k is not zero, then its weight $\lambda + ke_m - (f_1 + \cdots + f_k)$ must satisfy condition (2b) of Proposition 2.2. Comparing the m^{th} component and the $(m+n)^{\text{th}}$ component of the weight, we get $g_1 = \mu_n - \lambda_m \geqq k$.

If $g_1 \geqq n-1$, then the second half of condition (5.2) is obvious. So we can assume $1 \leqq g_1 \leqq n-2$. The above argument tells us that if $g_1 < k$ then v_k vanishes. In particular, we have $v_{g_1+1} = 0$. Then there exists an integer $1 \leqq j \leqq g_1 + 1$ such that

$$\mu_j - \lambda_m = j - 1 (\leqq g_1).$$

Since $\mu_j - \lambda_m \geqq \mu_n - \lambda_m = g_1$, the above j must be $g_1 + 1$. Thus we get $\mu_{g_1+1} - \lambda_m = g_1 = \mu_n - \lambda_m$ so $\mu_{g_1+1} = \mu_n$. This means $d_3 \leqq g_1$ and we show that the second half of condition (5.2) in Theorem 5.3.

Next we put

$$w_k = \pi(E_{m-k+1,m+1})\cdots\pi(E_{m,m+1})v_\lambda \quad \text{for} \quad 1 \leqq k \leqq m-1.$$

Then we can prove as above that $w_k \neq 0$ holds if and only if $\lambda_j - \mu_1 \neq m - j$ for any j satisfying $m-k+1 \leqq j \leqq m$. This condition is trivial for $p+1 \leqq m$, so we have $w_k \neq 0$ for $1 \leqq k \leqq q$. Note that the weight of w_k is $\lambda + (e_{m-k+1} + \cdots + e_m) - kf_1$. If w_k does not vanish then $g_2 = \lambda_1 - \mu_1 \geqq k$ according to arguments similar to those above. Therefore $g_2 \geqq q$ holds and if g_2 is greater than $m-2$, the condition $g_2 \geqq d_1 + q$ is obvious. So we can assume $q \leqq g_2 \leqq m-2$ without loss of generality. The same arguments as above lead us to equation $\lambda_{m-g_2} = \lambda_1$. From this it follows that $d_1 \leqq g_2 - q$ and this is the first half of condition (5.2) in Theorem 5.3. Q.E.D.

We get similar results for the highest weight representations. Let $\lambda = (\lambda_1, \ldots, \lambda_m/\mu_1, \ldots, \mu_n)$ be an element of $(\tilde{\mathfrak{h}}^{\mathbb{C}})^*$ satisfying the condition

$$\lambda_{p+1} \geqq \cdots \geqq \lambda_m \geqq \mu_n \geqq \cdots \geqq \mu_1 \geqq \lambda_1 \geqq \cdots \lambda_p.$$

We define three Young diagrams Y_1^λ, Y_2^λ and Y_3^λ:

$$Y_1^\lambda = (\lambda_1 - \lambda_p, \lambda_1 - \lambda_{p-1}, \ldots, \lambda_1 - \lambda_2, 0),$$

$$Y_2^\lambda = (\lambda_{p+1} - \lambda_m, \lambda_{p+2} - \lambda_m, \ldots, \lambda_{m-1} - \lambda_m, 0),$$

$$Y_3^\lambda = (\mu_n - \mu_1, \mu_n - \mu_2, \ldots, \mu_n - \mu_{n-1}, 0).$$

We put $d_i = \operatorname{depth}(Y_i^\lambda)$ for $1 \leqq i \leqq 3$, and $g_1 = \lambda_m - \mu_n, g_2 = \mu_1 - \lambda_1, g_3 = \lambda_m - \lambda_1$.

Then we have

Theorem 5.5. *Let* (π, V) *be an irreducible* Δ^+*-highest weight representation of* $\mathfrak{su}(p, q/n)$ *with the highest weight* $\lambda \in (\mathfrak{h}^{\mathbb{C}})^*$. *Conditions* (I) *and* (II) *are necessary and sufficient for* (π, V) *to be super-unitary.*

(I) *The lowest weight* λ *satisfies*

$$\lambda_{p+1} \geqq \cdots \geqq \lambda_m \geqq \mu_n \geqq \cdots \geqq \mu_1 \geqq \lambda_1 \geqq \cdots \geqq \lambda_p. \tag{5.4}$$

(II) λ *satisfies condition (5.5) or (5.6)*:

$$g_2 \geqq d_1 + q \quad \text{and} \quad g_1 \geqq d_3, \tag{5.5}$$

$$g_2 \geqq d_1 + d_2 \quad \text{and} \quad g_1 = d_3 = 0, \tag{5.6}$$

Proof. We can construct the oscillator representations with the associated constant $\varepsilon = 1$ in the same way as in [22]. Note that these oscillator representations are highest weight modules. Then the sufficient condition follows from arguments similar to those used in the case of the lowest weight representations. On the other hand, we can obtain the necessary condition in a way similar to those used in the proof of Lemma 5.4. Q.E.D.

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