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2+1 Gravity for Genus >1

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Abstract. In [1] we analysed the algebra of observables for the simple case of a genus 1 initial data surface Σ^2 for 2+1 De Sitter gravity. Here we extend the analysis to higher genus. We construct for genus 2 the group of automorphisms H of the homotopy group π_1 induced by the mapping class group. The group H induces a group D of canonical transformations on the algebra of observables which is related to the braid group for 6 threads.

1. Introduction

In [1, 2] we have derived the algebra of observables for quantum gravity in 2+1 dimensions, when the spatial hypersurfaces are genus 1 Riemann surfaces [3], namely tori. The cases without [1] and with [2] a cosmological constant were discussed, and in the case of the De Sitter theory, it was shown that the quantum algebra of gauge invariant quantities, i.e. observables, is trivially related to the quantum group $SU(2)_q$ [4].

In this paper the analysis is extended to higher genus. The classical algebra of observables is explicitly defined and calculated for genus $g=2$. There are at least two isomorphic and independent sets of observables with corresponding isomorphic symplectic structures. Identities satisfied by traces of $SL(2,R)$ matrices used in the representations of the fundamental group $\pi_1(\Sigma^2, B)$, where B is the base point on the initial data surface Σ^2, are used systematically and pose no problem at the classical level. Similarly an additional set of identities follows from the relator of π_1 which fixes the genus to be exactly g. It is not yet clear which role these combined sets of identities should play at the quantum level.

In Sect. 2 we review notations and conventions and set the stage for the calculation of the algebra A of observables for genus $g \geqq 2$. In Sect. 3 this algebra is discussed in detail. In Sect. 4 we discuss the role and the relationships among the mapping-class group, the braid group and their representations in terms of canonical transformations on A. Relevant formulas are presented in the Appendix.

2. Previous Results

The Lagrangian density for pure gravity with cosmological constant Λ in $2+1$ dimensions is the Chern-Simons 3-form [2]:

$$\alpha/8\,(d\omega^{AB}-\tfrac{2}{3}\omega^{A}_{T}\cap\omega^{TB})\cap\omega^{CD}\varepsilon_{ABCD} \tag{2.1}$$

with $A,B,\ldots=0,1,2,3$, $\eta_{AB}=(-1,1,1,k)$ and $\Lambda=\frac{1}{3}k\alpha^{-2}$. The 1-forms ω^{AB} are the De Sitter spin connections:

$$\omega_{AB}=\begin{vmatrix}\omega^{ab} & e^{a}/\alpha \\ -e^{b}/\alpha & 0\end{vmatrix} \tag{2.2}$$

with $a,b=0,1,2$, $\sqrt{k}=1$ ($k=1$, de Sitter), $\sqrt{k}=i$ ($k=-1$, anti-de Sitter) and $\varepsilon_{abc3}=-\varepsilon_{abc}$. The action (2.1) leads to the Poisson brackets:

$$[\omega^{AB}_{i}(x),\omega^{CD}_{j}(y)]=k\alpha^{-1}\varepsilon_{ij}\varepsilon^{ABCD}\delta^{2}(x-y), \tag{2.3}$$

$i,j=1,2$, $x,y\in\Sigma^{2}$ and field equations:

$$R^{AB}=d\omega^{AB}-\omega^{AT}\cap\omega^{B}_{T}=0 \tag{2.4}$$

implying that space-time is locally de Sitter. The constraints are given by the vanishing of the spatial components R^{AB}_{ij} of the curvatures. The action (2.1) is invariant under $SO(3,1)$ or $SO(2,2)$.

As argued by Witten [5] one should solve the constraints in (2.4) exactly before quantisation. In [2] this was achieved by considering the quantum representation $\Psi:\pi_1(\Sigma^2,B)\to G$, where $G=SO(3,1)$ or $SO(2,2)$ [and their corresponding spinor groups $SL(2,C)$ or $SL(2,R)\otimes SL(2,R)$] of $\pi_1(\Sigma^2,B)$. The algebra of observables is generated by the traces $a(X)$ of the representation $\Psi(X)$, with the gauge invariance $a(X)=a(Y^{-1}XY)$, $X,Y\in\pi_1$.

The fundamental group π_1 of a surface Σ^2 of genus g admits a presentation with the generators:

$$U_1,V_1,U_2,V_2\ldots U_g,V_g \tag{2.5}$$

and the single relator normally given as:

$$U_1V_1U_1^{-1}V_1^{-1}U_2V_2U_2^{-1}V_2^{-1}\ldots U_gV_gU_g^{-1}V_g^{-1}=1. \tag{2.6}$$

In Sect. 3 we shall use an equivalent form for this relator consistent with the definitions of the paths U_iV_i, $i=1,2$ appearing in Fig. 1.

The constraints are satisfied by writing:

$$dS^{\pm}=\Delta^{\pm}S^{\pm}, \tag{2.7}$$

where $\Delta^{\pm}$ are the upper/lower spinor components of $\Delta(x)=\frac{1}{4}\omega^{AB}(x)\gamma_{AB}$. The $\pm$ in S and (2.7) refer to the 2×2 irreducible decompositions of the representations (see [2] for details). Let $S^{\pm}(\varrho),S^{\pm}(\sigma)$ be elements of $SL(2,R)$ obtained by integrating the connection S along paths ϱ,σ in Σ^2, with base points A,B, having a single intersection at the point P. Their Poisson Brackets can be derived from those of the ω^{AB}, (2.3).

$$\begin{aligned}[][S^{\pm}(\varrho)^{\beta}_{\alpha},S^{\pm}(\sigma)^{\tau}_{\gamma}]&=\pm i(s/2\alpha\sqrt{k})(-S^{\pm}(\varrho)^{\beta}_{\alpha}S^{\pm}(\sigma)^{\tau}_{\gamma}\\&\quad+2S^{\pm}(\varrho_2\sigma_1)^{\tau}_{\alpha}S^{\pm}(\sigma_2\varrho_1)^{\beta}_{\gamma}),\\ [S^{+}(\varrho),S^{-}(\sigma)]&=0,\end{aligned} \tag{2.8}$$

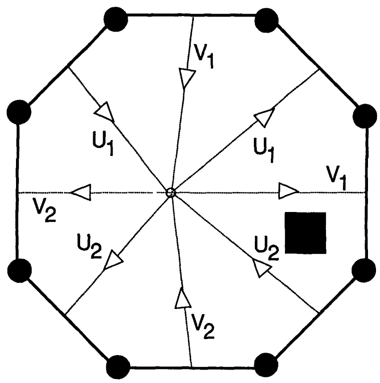

Fig. 1. Octagon with identified sides showing two holes of a surface of arbitrary genus. The black square is an obstruction leading into the remainder of the surface

where the intersection number $s=s(\sigma,\varrho)=-s(\varrho,\sigma)$, and from now on we set $s=1$ for an orientation as in Fig. 1 and (2.8) for example for the pairs U_1, V_1 or U_2V_2. The paths $\sigma=\sigma_2\sigma_1$, $\varrho=\varrho_2\varrho_1$, and σ_1,σ_2 are the initial and final open ended paths joining B to P and P to B, similarly for ϱ with base point A.

The integrated connections $S^{\pm}$ are not gauge invariant but their traces are. Define now for a generic closed path τ the traces:

$$c^{\pm}(\tau)=\tfrac{1}{2}S_{\alpha}^{\pm\alpha}(\tau). \tag{2.9}$$

Comments. 1) Note that if $\delta=\sigma_1^{-1}\varrho_1$ is the open path from A to B:

$$c^{\pm}(\sigma_1\sigma_2\varrho_1\varrho_2)=c^{\pm}(\sigma_2\varrho_1\varrho_2\sigma_1)=c^{\pm}(\sigma\sigma_1^{-1}\varrho_1\varrho_2\sigma_1)=c^{\pm}(\sigma\varrho'),$$

where σ and $\varrho'=\delta\varrho\delta^{-1}$ share the same base point B in Σ^2. This means that σ,ϱ' identify elements U, V of the homotopy group $\pi_1(\Sigma^2,B)$ based on B. We write then $c^{\pm}(U)$, $c^{\pm}(V)$, $c^{\pm}(UV)$ in place of $c^{\pm}(\sigma)$, $c^{\pm}(\varrho')=c^{\pm}(\varrho)$, $c^{\pm}(\sigma\varrho')$, etc. for every group element. We then trace (2.8) and find:

$$\begin{gathered}[c^{\pm}(V),c^{\pm}(U)]=\pm i/(2\alpha\sqrt{k})(-c^{\pm}(V)c^{\pm}(U)+c^{\pm}(UV)),\\ [c^{+}[V],c^{-}[U]]=0.\end{gathered} \tag{2.10}$$

For $k=1$ the $c^{\pm}$ are mutually complex conjugate, if $k=-1$ they are real and independent. From now on we choose the case $+$ with $k=-1$ and drop any explicit reference to the $\pm$ sign.

2) For matrices $\in SL(2,R)$ the identity holds:

$$c(U)c(V)=\tfrac{1}{2}(c(VU)+c(VU^{-1})), \tag{2.11}$$

and therefore (2.10) can be also written as:

$$[c(V),c(U)]=i/(4\alpha\sqrt{k})(c(UV)-c(UV^{-1})). \tag{2.12}$$

3) By abuse of language we use as argument of $c(V)$ either an element $V\in\pi_1$ or its image $v=S(V)\in SL(2,R)$ induced by the connection.

Under this form the algebra of traces is given an infinite Lie algebra L structure subject to non-linear constraints (2.11). These constraints generate an ideal in the enveloping algebra of L. By using repeatedly (2.11) we can compute recursively all the $c(W)$, $W \in \pi_1$, starting from a finite set of traces. For instance all traces in the subgroup generated by U_1, V_1, where U_1 and V_1 refer to the meridian and parallel around a given hole, say the first, in Σ^2, can be computed from $c(U_1)$, $c(V_1)$, $c(U_1 V_1)$. It is therefore reasonable to assume that we can derive a representation for L once we know an appropriate representation for the above set of traces.

Set $c(U)=u$, $c(V)=v$, $c(UV)=c(VU)=t$ and consider (2.10):

$$[u,v] = -(1/2\alpha)(t-uv) \tag{2.13}$$

and cyclical permutations of t, u, v. We set $uv - vu = (u,v) = i\hbar[u,v]$ and symmetrise the uv product. The symmetrised commutator is therefore:

$$(u,v) = -iz(t - \tfrac{1}{2}(uv+vu)), \tag{2.14}$$

where $z = \hbar/2\alpha = -2\tan(\frac{1}{2}\theta)$. $|\theta| < \pi$, or alternatively:

$$e^{\frac{1}{2}i\theta}uv - e^{-\frac{1}{2}i\theta}vu = 2i\sin\theta/2t \text{ and cyclical}. \tag{2.15}$$

In [2] t, u, v were each rescaled by $\cos(\frac{1}{2}\theta)$, irrelevant in the $\hbar \to 0$ limit of (2.15) since it corresponds to changes of the order of $\hbar^2$. The algebra (2.14) is not a Lie algebra but is trivially related, as shown in [2], to the Lie algebra of the quantum $SU(2)_q$ groups [3]. It admits the central element:

$$F^2 = \cos^2\tfrac{1}{2}\theta + 2e^{i\frac{1}{2}\theta}uvt - e^{i\theta}(u^2+t^2) - e^{-i\theta}v^2. \tag{2.16}$$

In the classical limit $\hbar \to 0$ ($\theta \to 0$) the values of F^2 classify the representations of the subalgebra (2.15). $F^2=0$ corresponds to the trace of the relator (2.6) for $g=1$.

3. The Classical Algebra

For $g \geqq 2$ the arguments in the previous section hold for the subalgebra generated by the elements U_m, V_m, respectively meridian and parallel of the m^{th} hole. In particular the brackets (2.8) are still valid for any two paths in Σ^2 with a single intersection. However we must include elements which are generic products of the above generators in any combination and arising from different holes. At this point it is of vital importance to establish a suitable notation.

Consider the subgroup of π_1 generated by U_1, V_1, U_2, V_2, i.e. the paths around any 2 holes and traces of products of these paths of the form:

$$c(U_1^{n_0} V_1^{n_1} U_2^{n_2} V_2^{n_3}), \tag{3.1}$$

where $n_0, n_1, n_2, n_3 = 0, 1$. Any other trace can be reduced to a polynomial in traces of the form (3.1) by repeated use of the trace identities in the Appendix. With two exceptions we identify the elements (3.1) by A_n, where n is the number $n_0 + 2n_1 + 4n_2 + 8n_3$ or $n_0 n_1 n_2 n_3$ in binary form.
These exceptions are

$$A_3 = -c(U_1 V_1) + 2c(U_1)c(V_1) = c(U_1 V_1^{-1}),$$
$$A_{12} = -c(U_2 V_2) + 2c(U_2)c(V_2).$$

Some examples of the general rule are $A_4 = c(U_2)$, $A_9 = c(U_1 V_2)$, $A_{11} = c(U_1 V_1 V_2)$. We use italics for the images in $SL(2,R)$ of elements $\in \pi_1$, i.e. $u_1 = S(U_1)$, etc.

Given two indices n, p define $n\,XOR\,p = n_0\,XOR\,p_0$, $n_1\,XOR\,p_1$, $n_2\,XOR\,p_2$, $n_3\,XOR\,p_3$, where $0\,XOR\,0 = 1\,XOR\,1 = 0$, $1\,XOR\,0 = 0\,XOR\,1 = 1$. The binary degree $d(P)$ of a generic product P in the A's is given by setting $d(A_n) = n$, $d(\text{number}) = 0$ and by the recursion rule $d[PQ] = d(P)\,XOR\,d(Q)$. As a useful check we remark that all polynomials appearing in the following are homogeneous in the binary degree and that all equations are satisfied if we set $A_i = 1$, $[A_i, A_k] = 0$ for all i, k.

The 105 brackets of these 15 elements were calculated by direct geometrical methods. We do not report all the details but try to explain our reasoning and method. We made ample use of (2.8) and of the representation of a compact surface of genus g by means of a polygon of $4g$ sides suitably identified, each side corresponding to a factor in the relator (2.6). We denote by A the ring of polynomials in the A_i endowed with their brackets. It is convenient to perform calculations on an octagon which represents explicitly the first 2 holes and possesses an obstruction (see Fig. 1) which represents the remaining holes and thus applies to any $g \geqq 2$. The convention used in Fig. 1 for the paths U_1, V_1, U_2, V_2 leads to the relator:

$$V_1^{-1} U_1 V_1 U_1^{-1} V_2^{-1} U_2 V_2 U_2^{-1} \tag{3.2}$$

which should be set equal to the identity if there were no obstruction and $g = 2$. After calculation of each bracket, using (2.8), the paths are reassembled, then we trace and simplify using the trace properties of $SL(2, R)$ matrices. The final result is best displayed by the complete hexagon appearing in Fig. 2. Also we omit the factor $-1/2\alpha$. To each element A_i we associate the i^{th} line of the hexagon. If the lines i, j have no point in common then the corresponding paths are homotopic to non-intersecting paths and:

$$[A_i, A_j] = 0, \tag{3.3}$$

for example

$$[A_1, A_5] = [A_3, A_{12}] = 0.$$

If the sequence of lines i, k, j forms a triangle and runs clockwise around its perimeter then we have $i = k\,XOR\,j$, $k = j\,XOR\,i$, $j = i\,XOR\,k$, the corresponding paths intersect once and:

$$[A_i, A_k] = A_i A_k - A_j. \tag{3.4}$$

For example:

$$[A_1, A_2] = A_1 A_2 - A_3.$$

Finally there are pairs of diagonal lines, say n, p, which intersect at one point P inside the hexagon. These correspond to traces of paths which have 2 intersections. Let n have end points P_1, P_3 and p have end points P_2, P_4 and let $i_{ab} = i_{ba}$ the line connecting the points P_a and P_b. If we connect the points P_a $a = 1 \dots 4$ in all possible ways we obtain a quadrilateral with diagonals n, p. We may always choose a convention such that the triples $i_{12}, i_{23}, i_{31} = n$ and $i_{42} = p, i_{23}, i_{34}$ run clockwise (see Fig. 3). We have then:

$$[A_n, A_p] = 2A_{i_{12}} A_{i_{34}} - 2A_{i_{23}} A_{i_{14}}. \tag{3.5}$$

The six elements $A_{12} A_{23} A_{13} A_{14} A_{24} A_{34}$ generate a subalgebra of A, there are 15 such subalgebras corresponding to the 15 quadrilaterals contained in Fig. 2. For

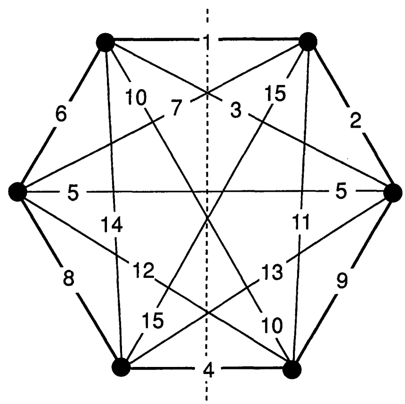

Fig. 2. Diagram showing the combinatorial rules for the Poisson brackets used in the text

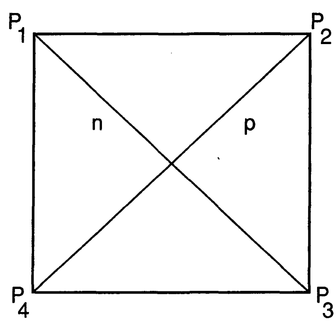

Fig. 3. Diagram used for double intersections

example:

$$[A_5 A_{10}] = 2A_9 A_5 - 2A_3 A_{12}.$$

One can show that it is impossible to eliminate completely the double intersections by considering traces of the form (3.1) and reversing the sign of any n_i, that is, one can only change the intersection number by ± 2.

The algebra A endowed with the above brackets cannot be identified immediately with the algebra of classical observables. The variables $A_1 \dots A_{15}$ are not algebraically independent since they are gauge invariant functions of the $SL(2, R)$ matrices $u_1 = S(U_1)$, $v_1 = S(V_1)$, $u_2 = S(U_2)$, $v_2 = S(V_2)$ which represent the elements U_1, V_1, U_2, V_2. Each matrix is determined by 3 real parameters giving a total of 12 independent elements from which we must subtract 3 corresponding to the gauge freedom. This reduction, from 15 to 9, in the dimension of the above algebra is achieved through the introduction of an ideal $R(A) \subset A$ generated by a basic set of trace identities. These identities can be classified as follows:
1) Generic 2×2 real matrices can be considered as vectors $\in R^4$ endowed with a scalar product $u \cdot v = \mathrm{trace}(uv)$. Consider now the 5 elements $1, u_1, v_1, u_2, v_2$. The Gram determinant of their scalar products can be evaluated as polynomial of the

traces A_i, $i\in I$, where $I=\{1,2,3,4,5,6,8,9,10,12)$ by repeated use of the characteristic equation $u^2-2c(u)u+1=0$, $u\in SL(2,R)$ and must vanish:

$$\mathrm{Det}=\begin{vmatrix} 1 & A_1 & A_2 & A_4 & A_8 \\ A_1 & 2A_1^2-1 & (2A_1A_2-A_3) & A_5 & A_8 \\ A_2 & (2A_1A_2-A_3) & 2A_2^2-1 & A_5 & A_{10} \\ A_4 & A_5 & A_6 & 2A_4^2-1 & (2A_4A_8-A_{12}) \\ A_8 & A_9 & A_{10} & (2A_4A_8-A_{12}) & 2A_8^2-1 \end{vmatrix}=0. \tag{3.6}$$

Equation (3.6) yields an identity among the 10 traces A_i, $i\in I$ which reduces the number of independent variables to 9, as expected and defines an algebraic variety $\mathrm{Det}=0$ within the original phase space. The explicit form of Det is extremely complicated and we see no way to extend it directly into the quantum theory. Moreover the brackets do not close on the subset A_i, $i\in I$ and in order to get a consistent quantum theory we must include traces of higher order. Notice that $d(\mathrm{Det})=0$.

2) If we replace $S(U_1)$ etc. by their inverses we see that the A_i, $i\in I$ considered above, i.e. traces of products of less that 3 matrices, are unchanged, whereas the remaining 5 traces, namely A_7, A_{11}, A_{13}, A_{14}, A_{15} are changed into traces of products of the same matrices in reversed order. Traces of products ordered as in (3.1) are called cyclic whereas the transformed traces obtained by replacing each matrix with its inverse, or equivalently by reversing the order of the paths in (3.1) are called anticyclic and denoted $B_1 \dots B_{15}$ respectively. We call the traces A_i, $i\in I$ selfcyclic. The set of anticyclic traces can be expressed in terms of polynomials in the set of cyclic traces. For example the cyclic element $A_7=c(U_1V_1U_2)$ is related to its anticyclic partner $B_7=c(U_2V_1U_1)$ by:

$$A_7+B_7=2(-A_4A_3+A_2A_5+A_1A_6), \tag{3.7}$$

$$\begin{aligned} A_7B_7=&A_1^2+A_2^2+A_3^2+A_4^2+A_5^2+A_6^2-2A_3A_5A_6,\\ &+4A_1A_2A_5A_6-2A_1A_2A_3-2A_1A_4A_5-2A_2A_4A_6-1 \end{aligned} \tag{3.8}$$

as can be checked explicitly. Equations (3.7) and (3.8) together determine a quadratic equation for either A_7 or B_7 in terms of the A_i, $i\in l$. Similar equations exist for the pairs A_iB_i, $i=11, 13, 14$. Since these identities are always symmetric in A and B they are invariant under the map $\gamma: A_i \leftrightarrow B_i$, with $\gamma^2=$identity map. The mirror map γ is not an automorphism of A and therefore there are at least two inequivalent sets of brackets. Neither the above derived quadratic equations for A_i, $i=7$, 11, 13, 14 nor the equation $\mathrm{Det}=0$ form a convenient basis for $R(A)$. The identity:

$$\begin{aligned} W=&A_{15}-A_4A_{11}-A_2A_{13}-A_1A_{14}-A_7A_8-A_3A_{12}\\ &-A_6A_9+A_5A_{10}+2A_2A_4A_9+2A_1A_6A_8=0 \end{aligned} \tag{3.9}$$

has many properties which make it a suitable candidate for the quantum theory. In order to display these properties we must use the automorphisms $D(n)n=1 \dots 15$ of A defined in the Appendix and discussed in detail in the next section. By explicit computation we can prove that:

$$\begin{gathered}((W,A_n),A_n)=(A_n^2-1)W,\\ D(n)W-A_nW-(W,A_n)=0,\\ D^{-1}(n)W-A_nW+(W,A_n)=0,\\ n=1,2,4,8,7,11,13,14,\\ D(m)W=W,\quad (W,A_m)=0,\\ m=3,5,6,9,10,12.\end{gathered} \tag{3.10}$$

We conjecture that a complete basis for $R(A)$ is given by $D(n)W$. It can be checked that W and its images under the maps $D(n)$, $n=1,2,4,8$ are linear in the cyclic traces $A_7A_{11}A_{13}A_{14}A_{15}$. The ensuing relations are not independent by virtue of $\mathrm{Det}=0$ and we cannot determine rationally the cyclic traces from the selfcyclic ones. We can use these relations in order to determine 4 cyclic traces as rational functions of the remaining one. In order to obtain this we must solve a quadratic equation. The two possible signs of the square root in this equation correspond to cyclic and anticyclic traces. Since the generic $D(n)$ is not linear in the cyclic traces we conjecture that they yield precisely the missing quadratic relation and that the whole basis contains also implicitly the $\mathrm{Det}=0$ relation. In the classical theory all the identities among traces can be proved explicitly by replacing the traces with their explicit expressions in terms of matrices in $SL(2,R)$. In the quantum theory we know of no convenient way to setup commutation relations which would take into account the constraints and there is no way to check directly the validity of the trace identities. We can however extend all properties deduced so far for W to the quantum domain and obtain a consistent formulation for the theory. We denote by $O=A/R(A)$ the subalgebra of physical observables generated by loops around holes 1,2. Thus is appears that the above brackets define a symplectic structure on an algebraic variety of dimension 9 whose complete structure is yet to be completely elucidated.

3) If we restrict ourselves to genus 2 then we must take into account more identities. The homotopy group π_1 is generated by U_1, V_1, U_2, V_2 with the single relator given by (2.6):

$$V_1^{-1}U_1V_1U_1^{-}\,V_2^{-1}U_2V_2U_2^{-1}=1\,.$$

This last identity must be adjoined to the ones discussed in 1) and 2). Since we deal with gauge invariant quantities we must express it as a set of equivalent relations among the traces A_i defined above which would bring the dimension of phase space down to 6, the number of real moduli on a surface of genus 2. We have not investigated in any detail this particular case.

4. The Automorphisms of A

The algebra A presented in the previous section is highly symmetric. We have studied its automorphisms at both the level of the traces $A_1 \ldots A_{15}$ and at the level of π_1 and its representations on $SL(2,R)$.

The maps $D(n)$ mentioned in Sect. 3 can be constructed in the following manner. As an example consider the canonical transformation generated by $G=\Theta^2/2$, with

$A_3 = \cosh\Theta$, through the differential equation:

$$dO(t)/dt = [O(t), G], \tag{4.1}$$

where O is a generic function on $A \cdot A_i(t)$ denotes the transform of $A_i = A_i(0)$. Let:

$$\Omega(t) = (\sinh\Theta)^{-1} \begin{vmatrix} \sinh(\Theta(1-t)) & +\sinh(\Theta t) \\ -\sinh(\Theta t) & \sinh(\Theta(1+t)) \end{vmatrix}. \tag{4.2}$$

One can show that:

$$\begin{vmatrix} A_p(t) \\ A_m(t) \end{vmatrix} = \Omega(t) \begin{vmatrix} A_p \\ A_m \end{vmatrix}, \tag{4.3}$$

where p, m is a pair from $(1,2)$, $(5,6)$, $(13,14)$, $(9,10)$ forming a triangle with 3 as in Fig. 2. For arbitrary t, $\Omega(t)$ is a transcendental function of A_3 which reduces to a polynomial for integer t:

$$\Omega(1) = \begin{vmatrix} 0 & 1 \\ -1 & 2A_3 \end{vmatrix}, \qquad \Omega(n) = \Omega(1)^n. \tag{4.4}$$

A similar derivation can be performed for A_7, A_{11}, A_{15}, which originate from paths having double intersections with A_3. Their transformations are considerably more complicated but are still polynomial in the A's. They are reported in the Appendix. Therefore we take for $D(3)$ the above canonical transformation with $t=1$. The full set $D(n)$, $n = 1 \dots 15$ can be derived similarly. To each A_n we therefore associate the map $D(n)$ which leaves invariant A_n and all O such that $[O, A_n] = 0$.

We denote by D the group of maps generated by the $D(n)$. The same hexagon appearing in Fig. 2 can be used to classify the identities satisfied by the $D(n)$ just as we did for the A_n. In particular if the lines i, j do not intersect we have:

$$[D(i), D(j)] = 0. \tag{4.5}$$

If the lines i, j, k run clockwise around a triangle of Fig. 2 we have:

$$D(i)D(j)D(i) = D(j)D(i)D(j)$$

and cyclical

$$D(i)D(j) = D(j)D(k) = D(k)D(i). \tag{4.6}$$

For doubly intersecting paths we find more complicated relations which follow directly from the ones quote above and will not be quoted here. By using these identities we can express all $D(n)$ in terms of a subset of 5 elements only, say $D(8)$, $D(6)$, $D(1)$, $D(2)$, $D(9)$, i.e. the sides of the hexagon with the 6th missing. The exclusion of $D(4)$ is purely conventional and does not reflect any breaking of the hexagonal symmetry.

We set $\zeta_1 = D(8)$, $\zeta_2 = D(6)$, $\zeta_3 = D(1)$, $\zeta_4 = D(2)$, $\zeta_5 = D(9)$ and verify from (4.5), (4.6) that:

$$\begin{aligned} \zeta_i\zeta_j &= \zeta_j\zeta_i \quad \text{if} \quad |i-j| \geqq 2, \qquad 1 \leqq i, j \leqq 5, \\ \zeta_i\zeta_{i+1}\zeta_i &= \zeta_{i+1}\zeta_i\zeta_{i+1}, \qquad 1 \leqq i \leqq 4, \end{aligned} \tag{4.7}$$

which are satisfied by the elements of $B(6)$, the braid group of order 6. In particular the element ζ_i corresponds to the element of $B(6)$ which exchanges the braids $i, i+1$. It follows that D yields a representation of $B(6)$.

The maps $D(n)$ can be lifted to maps $H(n)$ on the homotopy group π_1 and the explicit formulae for $H(i)$, $i=1,2,6,8,9$ are given in the Appendix. These maps leave the relator (3.2) invariant and reduce to the maps $D(n)$ on A. They satisfy the same identities (4.5), (4.6) as the $D(n)$ and generate a group H of homomorphisms of π_1 which is induced by the mapping class group M.

The map $Sf: A_n \to A_{sf(n)}$, (see Appendix) is an automorphism of A which acts as a rotation by $\pi/3$ of the hexagon in Fig. 2 and whose cube Ex simply exchanges the labelling of the holes 1, 2.

Let $rf(n_0 n_1 n_2 n_3) = (n_0\, XOR\, n_3) n_1 (n_2\, XOR\, n_1) n_3$, and define the map $Rf: A_n \to A_{rf(n)}$, then Rf is an *anti*automorphism of A, i.e.:

$$(A_{rf(n)}, A_{rf(p)}) = -Rf((A_n, A_p)) \quad \text{with} \quad Rf^2 = \text{Identity}, \tag{4.8}$$

and can be associated to a reflection of the hexagon along the dashed line. It can be easily seen that Rf and Sf can be generated by the maps $D(n)$ so that we may restrict our discussion to the $D(n)$.

Apart from their own relevance Sf, Ex, and Rf are extremely useful in checking and deriving further properties of A. The sixth powers of sf and Sf are of course identity maps.

It is natural to identify elements of A and O which lie on the same orbit of the group D since this would correspond to consider as part of the gauge also those diffeomorphisms of Σ^2 which are not connected to the identity map. The identities (3.9) guarantee that the brackets close on O.

5. Outlook and Conclusions

The algebra O and the related groups B, H, M can be generalized in many obvious ways and some of these look very promising for future extensions of the theory to arbitrary genus >2.

In particular consider the algebras $A(n)$ associated to complete n-gons formed by n points joined by $n(n-1)/2$ lines and including $n(n-1)(n-2)/6$ triangles and where the brackets are defined as a straightforward generalization of the rules given in Sect. 3. For $n<6$ these algebras are isomorphic to subalgebras of A and in particular $A(3)$ contains the triple associated with a triangle (genus 1) and we could have used any $A(5) \subset A$ in order to generate O at the price of less symmetry and elegance in the formalism.

This hierarchy of nested subalgebras will be relevant in the construction and discussion of the quantum theory and associated representations. The group D can be extended similarly to a hierarchy $D(n)$.

The discussion of classical $2+1$ gravity on an initial data hypersurface Σ^2 of genus 2 is now almost complete. As pointed out by many authors [7] the definition of a time variable remains an outstanding and interesting problem. In many ways our discussion of $2+1$ gravity reminds us of that of a rotator where we choose not to discuss angular variables and work with angular momenta only. We recall that indeed the algebra $A(3)$ appearing for genus 1 is related to the quantum group $SU(2)_q$ [2]. We have not reached any similar conclusions for the algebra $A(6)$ nor are aware of any work where a definite analog for variables has been established for quantum groups.

The quantisation of the algebra A will be presented elsewhere [8]. All the key features present at the classical level will be implemented quantum mechanically.

In particular the Dehn canonical maps $D(n)$ will be represented by unitary operators on the physical states.

Appendix

The various maps used in the text are defined as:

$$sf(1)=2,\ sf(2)=9,\ sf(3)=11,\ sf(4)=8,\ sf(5)=10,\ sf(6)=1,$$
$$sf(7)=3,\ sf(8)=6,\ sf(9)=4,\ sf(10)=15,\ sf(11)=13,\ sf(12)=14,$$
$$sf(13)=12,\ sf(14)=7,\ sf(15)=5.$$
$$Sf=\{A_k\to A_{sf(k)},\ \text{for } k=1\ldots 15\},$$
$$Sf^5=1=\text{identity map. } Sf^3=Ex,$$
$$Rf=\{A_1\to A_1,\ A_2\to A_5,\ A_3\to A_7,\ A_4\to A_4,\ A_5\to A_5,\ A_6\to A_2,\ A_7\to A_3,\ A_8\to A_9,\ A_9\to A_8,\ A_{10}\to A_{15},\ A_{11}\to A_{14},\ A_{12}\to A_{13},\ A_{13}\to A_{12},\ A_{14}\to A_{11},\ A_{15}\to A_{10}\},$$
$$Rf^2=1,\ RfEx=ExRf.$$

The map Rf can be used to deduce $(D(n))^{-1}$ since:

$$RfD(n)(Rf)^{-1}=(D(rf(n)))^{-1}.$$

In place of Rf one could use equally well $SfRf(Sf)^{-1}$ or $(Sf)^{-1}RfSf$.

The map Sf is an element of D since:

$$Sf=(D(8)D(6)D(1)D(2)D(9))^{-1}.$$

Here follow explicit definitions for $D(n)$.

$$D(1)=\{A_1\to A_1,\ A_2\to A_3,\ A_7\to A_6,\ A_{15}\to A_{14},\ A_4\to A_4,\ A_5\to A_5,\ A_8\to A_8,\ A_{12}\to A_{12},\ A_{11}\to A_{10},\ A_9\to A_9,\ A_{13}\to A_{13},\ A_3\to -A_2+2A_1A_3,\ A_6\to -A_7+2A_1A_7,\ A_{10}\to -A_{11}+2A_1A_{10},\ A_{14}\to -A_{15}+2A_1A_{14}\}.$$

$D(2)=(Sf)^{-1}D(1)Sf$ and more generally $D(sf(n))=(Sf)^{-1}D(n)Sf$. From this relation we can calculate $D(2)$, $D(9)$, $D(4)$, $D(8)$, $D(6)$.

Also:

$$D(3)=\{A_1\to A_2,\ A_3\to A_3,\ A_9\to A_{10},\ A_5\to A_6,\ A_4\to A_4,\ A_8\to A_8,\ A_{12}\to A_{12},\ A_{13}\to A_{14},\ A_2\to -A_1+2A_3A_2,\ A_{10}\to -A_9+2A_3A_{10},\ A_{14}\to -A_{13}+2A_3A_{14},\ A_6\to -A_5+2A_3A_5,\ A_7\to A_7-2A_2A_5-2A_1A_6+4A_2A_3A_5,\ A_{11}\to A_{11}-2A_2A_9-2A_1A_{10}+4A_2A_3A_{10},\ A_{15}\to A_{15}-2A_1A_{14}-2A_2A_{13}+4A_2A_3A_{14}\}.$$

From $D(3)$ and Sf we can calculate $D(n)$, $n=7, 11, 12, 13, 14$.

$$\begin{aligned}D(5)=\{&A_5\to A_5, A_6\to A_3,\ A_7\to A_2,\ A_9\to A_{12},\ A_{13}\to A_8,\\ &A_1\to A_1,\ A_4\to A_4,\ A_3\to -A_6+2A_5A_3, A_2\to -A_7+2A_5A_2,\\ &A_8\to -A_{13}+2A_5A_8, A_{12}\to -A_9+2A_5A_{12},\\ &A_{10}\to A_{10}-2A_6A_{12}-2A_3A_9+4A_5A_{12}A_3\\ &A_{11}\to A_{11}-2A_2A_9-2A_7A_{12}+4A_5A_2A_{12}\\ &A_{14}\to A_{14}-2A_8A_6-2A_3A_{13}+4A_5A_3A_8\\ &A_{15}\to A_{15}-2A_7A_8-2A_2A_{13}+4A_5A_2A_8\}\,.\end{aligned}$$

From $D(5)$ and Sf we calculate $D(10)$, $D(15)$.

All maps $H(n)$ can be generated by the following 5:

$$H(1)=\{V_1\to U_1^{-1}V_1\}\,,$$

$$H(2)=\{U_1\to U_1V_1\}\,,$$

$$\begin{aligned}H(6)=\{&U_1\to V_1^{-1}U_2^{-1}V_1U_2U_1U_2V_1,\ V_1\to V_1^{-1}U_2^{-1}V_1U_2V_1,\\ &U_2\to V_1^{-1}U_2V_1,\ V_2\to V_1^{-1}U_2^{-1}V_2U_2^{-1}V_1^{-1}U_2V_1\}\,,\end{aligned}$$

$$H(8)=\{U_2\to U_2V_2\}\,,$$

$$\begin{aligned}H(9)=\{&U_1\to U_1^{-1}V_2^{-1}U_1V_2U_1,\ V_1\to U_1^{-1}V_2^{-1}V_1,\\ &U_2\to U_2V_2U_1,\ V_2\to U_1^{-1}V_2U_1\}\,.\end{aligned}$$

The identities (4.5), (4.6) can be used profitably to calculate the missing $D(n)$ or $H(n)$.

We list next some useful trace identities valid for elements $x, y, z, u\in SL(2,R)$:

$$y+y^{-1}=2c(y)1\,,$$

$$c(xy)+c(xy^{-1})=2c(x)c(y)1\,,$$

$$c(x^2)=2c(x)x-1\,,$$

$$c(xyz)+c(xzy)=2(c(xy)c(z)+c(yz)c(x)+c(xz)c(y)-2c(x)\,c(y)c(z)\,,)$$

$$\begin{aligned}c(xyzu)=&\,c(x)c(yzu)+c(y)c(xzu)+c(z)c(xyu)+c(u)c(xyu)\\ &+c(zu)c(xy)+c(ux)c(yz)-c(yu)c(xz)-2c(zu)c(x)c(y)\\ &-2c(xy)c(u)c(z)-2c(ux)c(y)c(z)-2c(yz)c(u)c(x)\\ &+4c(x)c(y)c(z)c(u)\,.\end{aligned}$$

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