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Quantum q-White Noise and a q-Central Limit Theorem

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Abstract. We establish a connection between the Azéma martingales and certain quantum stochastic processes with increments satisfying q-commutation relations. This leads to a theory of q-white noise on q-$*$-bialgebras and to a generalization of the Fock space representation theorem for white noise on $*$-bialgebras. In particular, quantum Azéma noise, q-interpolations between Fermion and Boson quantum Brownian motion and unitary evolutions with q-independent multiplicative increments are studied. It follows from our results that the Azéma martingales and the q-interpolations are central limits of sums of q-independent, identically distributed quantum random variables.

1. Introduction

The Azéma martingales $(X_t)_{t \geq 0}$ are square integrable martingales with sample paths which are right continuous and have left limits such that

$$\mathrm{d}[X, X]_t = (q-1) X_{t\text{-}} \mathrm{d}X_t + \mathrm{d}t$$

where $[X, X]_t$ denotes the quadratic variation of X_t and where q is a real parameter. For $q = 0$, J. Azéma proved the existence of such a process in [6]. Then Emery [14] constructed X_t for arbitrary q and proved that, for $|q| \leq 1$, X_t has the chaos completeness property. The processes X_t were the first examples of martingales with this property which are not classical stochastic processes with independent increments. The quantum stochastic integral equation

$$L_t = (q-1) \int_0^t L_\tau \mathrm{d}\Lambda_\tau + A_t \tag{1.1}$$

was treated by Parthasarathy in [28]. The integrators Λ_t and A_t are the preservation and annihilation processes of quantum stochastic calculus in the sense of

Hudson and Parthasararthy [20] on Boson Fock space. It was shown in [28] that, for $|q| \leq 1$, the process $L_t^* + L_t$ of operators gives rise to a commuting family of self-adjoint operators and that, in the vacuum state, this process is the Azéma martingale. For $q = 1$ the sum $L_t^* + L_t$ is standard Brownian motion on Fock space whereas the pair (L_t, L_t^*) is Boson standard quantum Brownian motion as introduced by Cockcroft and Hudson [11]. Moreover, if J_{st} denotes the second quantization of the linear operator on $L^2(\mathbb{R}_+)$ which maps a function f to the function $f\chi_{[0,s)} + qf\chi_{[s,t)} + f\chi_{[t,\infty)}$ then for $q = -1$ the sum $L_t^* J_{0t} + J_{0t}L_t$ is the Clifford process [10] and the pair $(J_{0t}L_t, L_t^* J_{0t})$ is Fermion quantum Brownian motion on Boson Fock space (see [21]). We have for $\check{L}_t = J_{0t}L_{t,q^{-1}}$

$$\check{L}_t = \int_0^t J_{0\tau}\,\mathrm{d}A_\tau \tag{1.2}$$

which makes sense for $q \in \mathbb{R}$. The processes $(\check{L}_t, \check{L}_t^*)$ form an interpolation between Fermion and Boson quantum Brownian motion as q varies from -1 to $+1$. This also fits into the framework of [24]. In the language of [24] the process $\check{L}_t$ arises from the bounded cocycle given by the function on $\mathbb{R}_+ \times \mathbb{R}_+$ which is equal to q on $\{(s,t) : s < t\}$ and equal to 1 on $\{(s,t) : s > t\}$; cf. Definition 5.2. of [24].

The present paper arose from the discovery that, in a sense, the "quantum Azéma noise" (L_t, L_t^*) and the "quantum q-Brownian motion" $(\check{L}_t, \check{L}_t^*)$ can be regarded as processes with "q-additive" increments; see [34]. More precisely, let L_{st} be the solution of $(s \leq t)$

$$L_{st} = (q-1)\int_s^t L_{s\tau}\,\mathrm{d}A_\tau + (A_t - A_s)\,.$$

Then for $r \leq s \leq t$

$$L_{rt} = L_{rs}J_{st} + L_{st}\,. \tag{1.3}$$

The corresponding equation for

$$\check{L}_{st} = \int_s^t J_{s\tau}\,\mathrm{d}A_\tau$$

is

$$\check{L}_{rt} = \check{L}_{rs} + J_{rs}\check{L}_{st}\,. \tag{1.4}$$

Quantum Azéma noise and quantum q-Brownian motion are examples of an independent, stationary increment process (or white noise) on a $*$-bialgebra in the sense of [3] if we choose an appropriate $*$-bialgebra; see [34]. In both cases, the underlying $*$-algebra is the one generated by x and x^* and a hermitian generator y with the relation $xy = qyx$. For the quantum Azéma noise the comultiplication is defined by

$$\Delta x = x \otimes y + \mathbf{1} \otimes x\,, \tag{1.5}$$

$$\Delta y = y \otimes y \tag{1.6}$$

and for quantum q-Brownian motion (1.5) is replaced by

$$\Delta x = x \otimes \mathbf{1} + y \otimes x . \tag{1.7}$$

Equations (1.1) and (1.3) and (1.2) and (1.4) reflect the coalgebra structures given by (1.5) and (1.6) and (1.7) and (1.6) respectively. Moreover, if $q \neq 0$ we may treat both cases simultaneously by adding another generator y^{-1} satisfying $yy^{-1} = y^{-1}y = 1$. Then xy^{-1} satisfies (1.7) if x satisfies (1.5) and $y^{-1}x$ satisfies (1.5) if x satisfies (1.7).

The next observation is that the process

$$B_{st} = L_{st} J_{t,\infty}$$

has additive increments, i.e.

$$B_{rt} = B_{rs} + B_{st} .$$

The same is true for

$$\check{B}_{st} = J_{0s} \check{J}_{st} .$$

Moreover, we have for $0 \leq s \leq t \leq s' \leq t'$,

$$\begin{aligned} B_{s't'} B_{st} &= q B_{st} B_{s't'} , \\ B_{s't'} B^*_{st} &= q B^*_{st} B_{s't'} , \end{aligned} \tag{1.8}$$

and

$$\begin{aligned} \check{B}_{st} \check{B}_{s't'} &= q \check{B}_{s't'} \check{B}_{st} , \\ \check{B}_{s't'} \check{B}^*_{st} &= q \check{B}^*_{st} \check{B}_{s't'} . \end{aligned} \tag{1.9}$$

Notice that, for real q, if we take the adjoint of both sides of the second relations of (1.8) and (1.9), we obtain the same relations but with the time order of the intervals (s, t) and (s', t') reversed. Thus, in the case of real q, the second relations of (1.8) and (1.9) hold for all s, t, s', t' with $(s, t) \cap (s', t') = \emptyset$.

Relations (1.9) follow also from Proposition 5.6. of [24] if we apply it to the above mentioned bounded cocycle. They were treated in [8] where a realization was given with the help of a "q-scalar product" in Fock space. If we also take into account the fact that for both B_t and $\check{B}_t = \check{L}_t$ the additive increments factorize in the vacuum state for disjoint intervals, we have examples B_t and $\check{B}_t$ of (left and right) q-white noise in the sense of this paper.

However, there are a lot of other examples. For a general "commutation factor," again denoted by q, we introduce the notion of a left and a right q-$*$-bialgebra in a natural way. We reduce q-$*$-bialgebras to ordinary $*$-bialgebras (by which we mean q-$*$-bialgebras with trivial q) by adding a generator with certain commutation relations and by defining an appropriate comultiplication on the enlarged $*$-algebra. Using this procedure, we obtain the result that q-white noise on a q-$*$-bialgebra can always be reduced to white noise on an associated $*$-bialgebra. Thus the representation theorem for white noise on an ordinary $*$-bialgebra [35] yields a representation theorem for q-white noise on q-$*$-bialgebras which, on the other hand, is a generalization of the ordinary representation theorem. In other words, a q-white noise can be realized as a solution of quantum stochastic differential equations.

As a consequence, we get a characterization (up to a canonical equivalence) of all families $(B_t)_{t \geq 0}$ (or $(\check{B}_t)_{t > 0}$) of operators on a pre-Hilbert space D such that $B_{st} = B_t - B_s$ (or $\check{B}_{st} = \check{B}_t - \check{B}_s$) satisfy the relations (1.8) [or (1.9)] if we also require the existence of a "stationary product state" in D. The B_t (or $\check{B}_t$) are given by equations of a type similar to (1.1) [or (1.2)] on a Bose Fock space. This generalizes the results of [33] to the q-case. Since q can be any complex number, there are many ways of interpolation between the Fermion ($q = -1$) and the Boson ($q = 1$) case. In all cases, the q-interpolations satisfy "q-canonical commutation relations"; cf. [28, 24]. The operators B_t (or $\check{B}_t$) form an additive q-white noise which plays the same fundamental role as in the case $q = 1$, in the sense that an arbitrary q-white noise is the solution of an additive q-white noise quantum stochastic integral equation.

The product property of the state is a translation of the classical independent stationary increment property to the non-commutative (quantum) case. It was used in [22] as the basic axiom for quantum white noise. However, an additional algebraic independence condition (like (1.8) or (1.9); see also [23]) seems to be necessary to have a realization of white noise on Fock space and to have a quantum Ito's formula.

As for additive white noise, the characterization of unitary white noise (which is called "unitary evolution with independent, stationary increments" in [32]) can be generalized to the case of q-independence by applying the general theory to a q-version of the non-commutative coefficient algebra of the unitary group; for the definition and structure of the ordinary non-commutative coefficient algebra see [40, 3, 17]. Using our methods, it can be shown that unitary q-white noise arises from ordinary unitary white noise by multiplication with the second quantization of a multiplication by powers of q operator on Fock space.

The left and right q-$*$-bialgebras associated with $\check{B}_t$ and B_t respectively are q-versions of the tensor $*$-bialgebra which plays an important role in the theory of infinitely divisible representations of Lie algebras [26, 38]. The tensor $*$-bialgebra is used in [15, 41] to formulate an algebraic central limit theorem; cf. also [13, 19] for an analytic version of this theorem. The limiting functionals of [15, 41] are moment functionals of quasi-free (or gaussian) states in the Boson [15] and Fermion [41] case. One would like to identify the distributions of $\check{B}_t$ and B_t as central limits. We show that, at least as far as moments are concerned, this can be done by using a $\frac{1}{\sqrt{n}}$-normalization as in the Bose and Fermi case. The proof rests on a general limit theorem for graded coalgebras; cf. [30].

Another interpolation between Boson and Fermion Brownian motion based on "free independence" is described in [9]. A central limit theorem for the interpolating distributions in this case is established in [36].

One of the basic constructions used in this paper (the triplet (D, η, ϱ) in Sect. 3) generalizes a construction for groups which is part of the well known Araki-Woods embedding theorem [4, 5, 18, 29, 37]; see also [31].

The paper is organized as follows. In Sect. 2 we introduce the notion of q-$*$-bialgebras and show how they can be reduced to ordinary $*$-bialgebras. We also treat the questions of positivity connected with the convolution product. In Sect. 3 the representation theorem for general q-white noise is proved. In Sect. 4 and 5 we treat the special cases of additive and of unitary q-white noise. Section 6 contains the central limit theorem for graded coalgebras, and, finally, in Sect. 7 we apply this theorem and our theory of q-white noise to prove the q-central limit theorem.

2. q-$*$-Bialgebras

The vector space tensor product $\mathscr{A} \otimes \mathscr{A}$ of a $*$-algebra $\mathscr{A}$ with itself is turned into a $*$-algebra in the usual way by setting

$$(a \otimes b)\,(a' \otimes b') = aa' \otimes bb' \tag{2.1}$$

and

$$(a \otimes b)^* = a^* \otimes b^* \tag{2.2}$$

for $a, a', b, b' \in \mathscr{A}$. This can be generalized to $\mathbb{Z}_2$-graded $*$-algebras ($a \mapsto a^*$ is required to be even) by putting in the scalar factors $(-1)^{\varepsilon(b)\varepsilon(a')}$ and $(-1)^{\varepsilon(a)\varepsilon(b)}$ on the right-hand sides of (2.1) and (2.2) respectively, where a, a', b, b' are homogeneous and $\varepsilon(a)$ denotes the degree of a. The trivial graduation yields the non-graded case above. Now let us assume that $\mathscr{A}$ is $\mathbb{Z}$-graded. For a complex number $q \neq 0$ we build in $q^{\varepsilon(b)\varepsilon(a')}$ on the right-hand side of (2.1) to obtain an associative algebra structure on $\mathscr{A}$. However, if we build in $q^{\varepsilon(a)\varepsilon(b)}$ on the right-hand side of (2.2) the map $a \otimes b \mapsto (a \otimes b)^*$ is not self-inverse unless $|q| = 1$. This problem can be overcome as follows. We assume that $\mathscr{A}$ carries another $\mathbb{Z}$-graduation which also turns $\mathscr{A}$ into a graded algebra but with the property $\mathrm{d}(a^*) = -\mathrm{d}(a)$ instead of evenness of $a \mapsto a^*$ (here $\mathrm{d}(a)$ denotes the degree of a with respect to the second graduation). Then we put in $q^{\mathrm{d}(b)\varepsilon(a')}$ and $q^{-\mathrm{d}(b)\varepsilon(a)}$ on the right-hand sides of (2.1) and (2.2) respectively, and one easily checks that this defines a $*$-algebra structure onf $\mathscr{A} \otimes \mathscr{A}$ for arbitrary real $q \neq 0$. This setting can be generalized for arbitrary "commutation factors" q, and then the general case of a complex scalar $q \neq 0$ is included.

We proceed with the general theory. An involutive semi-group is a semi-group Λ with an involution, that is a selfinverse mapping $\lambda \mapsto \lambda^*$ on Λ such that

$$(\lambda_1\lambda_2)^* = (\lambda_2)^*(\lambda_1)^*$$

for all λ_1, $\lambda_2 \in \Lambda$. Let Λ and Γ be two abelian, involutive semi-groups (written multiplicatively!). Let $\mathbb{C}^*$ denote the multiplicative group of non-zero complex numbers. We say that

$$q : \Lambda \times \Gamma \to \mathbb{C}^*$$

is a Λ-Γ-commutation factor if

$$\begin{aligned} q(\lambda, \gamma_1\gamma_2) &= q(\lambda, \gamma_1)q(\lambda, \gamma_2)\,, \\ q(\lambda_1, \lambda_2, \gamma) &= q(\lambda_1, \gamma)q(\lambda_2, \gamma)\,, \\ q(\lambda^*, \gamma^*) &= \overline{q(\lambda, \gamma)}^{-1} \end{aligned}$$

for all $\lambda, \lambda_1, \lambda_2 \in \Lambda$ and $\gamma, \gamma_1, \gamma_2 \in \Gamma$.

In the applications we will always be in one of the following situations. Λ will be the cyclic group $\mathbb{Z}_\nu = \mathbb{Z}/\nu\mathbb{Z}$ of order $\nu \in \mathbb{N}$ or the integers $\mathbb{Z}$ itself (now written additively). We put $\mathbb{Z}_\infty = \mathbb{Z}$. The involution for Λ is the taking of inverses. Γ will be a direct product $\mathbb{Z}_{\nu_1} \times \mathbb{Z}_{\nu_2}$ with ν_1, $\nu_2 \in \mathbb{N} \cup \{\infty\}$, and the involution of $\mathbb{Z}_{\nu_1} \times \mathbb{Z}_{\nu_2}$ is given by

$$(n_1, n_2)^* = (n_1, -n_2)\,,$$

$n_1 \in \mathbb{Z}_{\nu_1}$, $n_2 \in \mathbb{Z}_{\nu_2}$. The following is straightforward.

2.1. Proposition. *Let $v, v_1, v_2 \in \mathbb{N} \cup \{\infty\}$ and consider Λ and Γ as above. Let q_1 be a real number not equal to 0 and let q_2 be a complex number of modulus 1 such that $q_1^v = q_2^v = 1$ if $v < \infty$, $q_1^{v_1} = 1$ if $v_1 < \infty$, and $q_2^{v_2} = 1$ if $v_2 < \infty$. Then the mapping*

$$(m, (n_1, n_2)) \mapsto q_1^{mn_1} q_2^{mn_2}$$

is a Λ-Γ-commutation factor. Conversely, any Λ-Γ-commutation factor q is of this form, and one can choose $q_1 = q(1, (1,0))$ and $q_2 = q(1, (0,1))$. Moreover, if $v, v_1, v_2 > 1$, q_1 and q_2 are uniquely determined by q. □

A complex number $q \neq 0$ generates a cyclic (multiplicative) subgroup of $\mathbb{R}\backslash\{0\}$ which is isomorphic to some $\mathbb{Z}_{v(q)}$, $v(q) \in \mathbb{N} \cup \{\infty\}$. Moreover, q gives rise to a Λ-Γ-multiplication factor with $v = v(q)$, $v_1 = v(|q|)$ and $v_2 = v(q/|q|)$ if we set $q_1 = |q|$ and $q_2 = q/|q|$. We use the same letter q for the complex number and for the corresponding commutation factor!

Proposition 2.1 gives a classification of all commutation factors for the case when Λ and Γ are of the form $\mathbb{Z}_v$. For the only involutions on $\mathbb{Z}_v$ are the identity and the taking of inverses, and one has to distinguish the two cases that the involutions in Λ and Γ are of the same or of different type which correspond to $v_1 = 1$ or $v_2 = 1$. If the involutions are of the same type the commutation factors are given by complex numbers of modulus 1, and if the involutions are of different type they are given by real numbers not equal to 0. The need for two different involutions arises only if one wants to use commutation factors which are not of modulus 1.

We return to the general theory. Let $\mathscr{A}$ be a Λ-Γ-graded algebra, that is $\mathscr{A}$ is a Λ-graded and a Γ-graded algebra; we write

$$\mathscr{A} = \bigoplus_{\lambda \in \Lambda} \mathscr{A}_{(\lambda)}$$

and

$$\mathscr{A} = \bigoplus_{\gamma \in \Gamma} \mathscr{A}^{(\gamma)} .$$

We denote by $\mathrm{d}(a)$ and $\varepsilon(a)$ the grades of an element $a \in \mathscr{A}$ (assumed homogeneous) relative to the Λ and Γ graduations respectively. In the sequel, if we write $\mathrm{d}(a)$ or $\varepsilon(a)$ the element a is always understood to be homogeneous without further mention. We call $\mathscr{A}$ a Λ-Γ-graded $*$-algebra if $\mathscr{A}$ is a $*$-algebra and

$$\mathrm{d}(a^*) = \mathrm{d}(a)^* \quad \text{and} \quad \varepsilon(a^*) = \varepsilon(a)^* .$$

The proof of the following proposition is a straightforward computation; see also [7].

2.2. Proposition. *Let q be a Λ-Γ-commutation factor and let $\mathscr{A}$ be a Λ-Γ-graded $*$-algebra. There is a unique Λ-Γ-graded $*$-algebra structure $\mathscr{A} \otimes_q \mathscr{A}$ on the algebraic graded vector space tensor product $\mathscr{A} \otimes \mathscr{A}$ such that*

$$(a \otimes b)\,(a' \otimes b') = q(\mathrm{d}(a'), \varepsilon(b))^{-1} aa' \otimes bb' ,$$
$$(a \otimes b)^* = \overline{q(\mathrm{d}(a), \varepsilon(b))} a^* \otimes b^* .$$

□

We obtain a Λ-Γ-commutation factor $\tilde{q}$ from q if we set

$$\tilde{q}(\gamma, \lambda) = q(\lambda, \gamma)^{-1}.$$

If we regard $\mathcal{A}$ as a Γ-Λ-graded $*$-algebra we obtain another $*$-algebra structure on $\mathcal{A} \otimes \mathcal{A}$ which we denote by $\mathcal{A} \otimes^q \mathcal{A}$.

For $\Lambda = \Gamma$ the trivial group, we have $q = 1$ and we have the usual algebraic tensor product of algebras. If $\Lambda = \Gamma = \mathbb{Z}_2$ with the trivial involution and $q(m,n) = \mp 1$ depending on whether both m and n are odd or not, we have the graded tensor product of the $\mathbb{Z}_2$-graded $*$-algebra $\mathcal{A}$ with itself. In both cases, $\mathcal{A} \otimes_q \mathcal{A} = \mathcal{A} \otimes^q \mathcal{A}$.

We will embed $\mathcal{A} \otimes_q \mathcal{A}$ and $\mathcal{A} \otimes^q \mathcal{A}$ into an ordinary tensor product of $*$-algebras. Let $\mathcal{A}$ be a Λ-Γ-graded $*$-algebra. Denote by $\mathbb{C}\Gamma$ the semi-group algebra of Γ which means that $\mathbb{C}\Gamma$ consists of formal finite linear combinations of semi-group elements with the multiplication given by the semi-group multiplication. $\mathbb{C}\Gamma$ becomes a $*$-algebra if we extend the involution on Γ to an involution on $\mathbb{C}\Gamma$. Denote by $\mathcal{A}_{\Gamma,q} = \mathcal{A}_\Gamma$ the $*$-algebra obtained from the free $*$-algebra product $\mathcal{A} \star \mathbb{C}\Gamma$ (see [7, Chap. III]) by dividing by the ideal in $\mathcal{A} \star \mathbb{C}\Gamma$ generated by the elememts

$$a\gamma - q(\mathrm{d}(a), \gamma)\gamma a$$

for $a \in \mathcal{A}$ and $\gamma \in \Gamma$. The $*$-algebra $\mathcal{A}_\Gamma$ becomes Λ-Γ-graded if we set $\mathrm{d}(\gamma) = \varepsilon(\gamma) = \mathbf{1}$ for all $\gamma \in \Gamma$. We have

2.3. Lemma. *The equation*

$$\imath(a \otimes b) = a\varepsilon(b) \otimes b \tag{2.3}$$

determines a graded, injective $*$-algebra homomorphism

$$\imath : \mathcal{A} \otimes_q \mathcal{A} \to \mathcal{A}_\Gamma \otimes \mathcal{A}_\Gamma .$$

This is also true if we replace $\mathcal{A} \otimes_q \mathcal{A}$ by $\mathcal{A} \otimes^q \mathcal{A}$ and (2.3) *by*

$$\imath(a \otimes b) = a \otimes \varepsilon(a)b . \tag{2.4}$$

Proof. We have

$$\mathcal{A} \otimes \mathcal{A} = \bigoplus_{\gamma \in \Gamma} \mathcal{A} \otimes \mathcal{A}^{(\gamma)}.$$

The injective linear mapping from $\mathcal{A} \otimes \mathcal{A}^{(\gamma)}$ to $\mathcal{A}_\Gamma \otimes \mathcal{A}_\Gamma$ given by right multiplication by $\gamma \otimes \mathbf{1}$ uniquely determines a graded, injective linear mapping $\imath$ from $\mathcal{A} \otimes \mathcal{A}$ to $\mathcal{A}_\Gamma \otimes \mathcal{A}_\Gamma$ satisfying (2.3). We show that $\imath$ is a $*$-algebra homomorphism. We have

$$\begin{aligned}
\imath((a \otimes b)\,(a' \otimes b')) &= q(\mathrm{d}(a'), \varepsilon(b))^{-1} aa'\varepsilon(b)\varepsilon(b') \otimes bb' \\
&= a\varepsilon(b)a'\varepsilon(b') \otimes bb' \\
&= \imath(a \otimes b)\imath(a' \otimes b')
\end{aligned}$$

and

$$\imath((a \otimes b)^*) = \overline{q(\mathrm{d}(a), \varepsilon(b))} a^* \varepsilon(b^*) \otimes b^* = \varepsilon(b)^* a^* \otimes b^* = (\imath(a \otimes b))^*,$$

where we made use of the relations in $\mathcal{A}_\Gamma$. □

We come to the notion of a q-$*$-bialgebra. Let $\mathcal{B}$ be a Λ-Γ-graded $*$-algebra and suppose that $\mathcal{B}$ is also coalgebra with comultiplication Δ and counit δ. If Δ and δ are graded $*$-algebra homomorphisms where we consider on $\mathcal{B} \otimes \mathcal{B}$ the $*$-algebra structure $\mathcal{B} \otimes_q \mathcal{B}$, then $\mathcal{B}$ is called a left q-$*$-bialgebra. If $\mathcal{B} \otimes_q \mathcal{B}$ is replaced by $\mathcal{B} \otimes^q \mathcal{B}$ we say that $\mathcal{B}$ is a right q-$*$-bialgebra. Notice that every right

q-$*$-bialgebra is a left $\tilde{q}$-$*$-bialgebra and vice versa. A graded linear operator on a q-$*$-bialgebra $\mathscr{B}$ is called an antipode if it is the inverse of the identity with respect to the convolution product of linear operators on $\mathscr{B}$.

2.4. Theorem. *Let $\mathscr{B}$ be a left q-$*$-bialgebra. Then the equations*

$$\Delta_\Gamma \lceil \mathscr{B} = \imath \circ \Delta, \qquad \Delta_\Gamma \gamma = \gamma \otimes \gamma \tag{2.5}$$

uniquely determine a graded $*$-algebra homomorphism

$$\Delta_\Gamma : \mathscr{B}_\Gamma \to \mathscr{B}_\Gamma \otimes \mathscr{B}_\Gamma$$

and Δ_Γ is a comultiplication. The equations

$$\delta_\Gamma \lceil \mathscr{B} = \delta, \qquad \delta_\Gamma(\gamma) = 1$$

uniquely determine a graded $*$-algebra homomorphism

$$\delta_\gamma : \mathscr{B}_\Gamma \to \mathbb{C}$$

and δ_Γ is a counit. Thus $\mathscr{B}_\Gamma$ forms an ordinary $*$-bialgebra with comultiplication Δ_Γ and counit δ_Γ. Moreover, if S is an antipode of $\mathscr{B}$ and if Γ is an involutive group then the equations

$$S_\Gamma(b) = \varepsilon(b)^{-1} S(b), \qquad S_\Gamma(\gamma) = \gamma^{-1} \tag{2.6}$$

determine a linear mapping S_Γ on $\mathscr{B}_\Gamma$ and S_Γ is an antipode. If $\mathscr{B}$ is a right q-$*$-bialgebra everything remains valid if we replace (2.3) *by* (2.4) *and* (2.6) *by*

$$S_\Gamma(b) = S(b)\varepsilon(b)^{-1}.$$

Proof. By Lemma 2.4. there is a unique graded $*$-algebra homomorphism

$$\tilde{\Delta} : \mathscr{B} \star \mathbb{C}\Gamma \to \mathscr{B}_\Gamma \otimes \mathscr{B}_\Gamma$$

satisfying (2.5) (if we replace Δ_Γ by $\tilde{\Delta}$). We show that $\tilde{\Delta}$ vanishes on $b\gamma - q(\mathrm{d}(b), \gamma)\gamma b$. Let

$$\Delta b = \sum b_{1i} \otimes b_{2i}.$$

Then

$$\begin{aligned}\tilde{\Delta}(b\gamma) &= q(\mathrm{d}(b_{1i}), \gamma) q(\mathrm{d}(b_{2i}), \gamma)(\gamma \otimes \gamma)\Big(\sum b_{1i}\varepsilon(b_{2i}) \otimes b_{2i}\Big) \\ &= q(\mathrm{d}(b), \gamma)\tilde{\Delta}(\gamma b).\end{aligned}$$

Next we show that Δ_Γ is coassociative. We have

$$\begin{aligned}((\Delta \otimes \mathrm{id}) \circ \Delta)(b) &= (\Delta \otimes \mathrm{id})\Big(\sum b_{1i}\varepsilon(b_{2i}) \otimes b_{2i}\Big) \\ &= \sum b_{1i}\varepsilon(b_{2i})\varepsilon(b_{3i}) \otimes b_{2i}\varepsilon(b_{3i}) \otimes b_{3i} \\ &= ((\mathrm{id} \otimes \Delta) \circ \Delta)(b).\end{aligned}$$

We show that δ_Γ is a counit. But

$$((\mathrm{id} \otimes \delta_\Gamma) \circ \Delta_\Gamma)(b) = \sum b_{1i}\varepsilon(b_{2i})\delta(b_{2i}) = \sum b_{1i}\delta(b_{2i}),$$

because δ vanishes on $\mathscr{B}^{(\gamma)}$ unless γ is the unit element of Γ. For the same reason δ_Γ is a left counit. Now let S be an antipode for $\mathscr{B}$. We claim that

$$S(bc) = q(\mathrm{d}(c), \varepsilon(b))^{-1} S(c) S(b).$$

This follows from the fact that both

$$b \otimes c \mapsto S(bc)$$

and

$$b \otimes c \mapsto q(\mathrm{d}(c), \varepsilon(b))^{-1} S(c) S(b)$$

define an inverse of the multiplication in $\mathscr{B}$ with respect to the convolution algebra structure of $L(\mathscr{B} \otimes_q \mathscr{B}, \mathscr{B})$, where we define the coalgebra structure on $\mathscr{B} \otimes_q \mathscr{B}$ in the natural way by

$$b \otimes c \mapsto \sum q(\mathrm{d}(c_{1j}), \varepsilon(b_{2i})) b_{1i} \otimes c_{1j} \otimes b_{2i} \otimes c_{2j} .$$

We have that

$$\begin{aligned} \varepsilon(bc)^{-1} S(bc) &= q(\mathrm{d}(c), \varepsilon(b))^{-1} \varepsilon(b)^{-1} S(c) S(b) \\ &= \varepsilon(c)^{-1} S(c) \varepsilon(b)^{-1} S(b) \end{aligned}$$

and

$$\tilde{S}(b) = \varepsilon(b)^{-1} S(b)$$

defines an anti-homomorphism from $\mathscr{B}$ into $\mathscr{B} \star \mathbb{C}\Gamma$. We extend $\tilde{S}$ to an antihomomorphism on the whole of $\mathscr{B} \star \mathbb{C}\Gamma$ by setting $\tilde{S}(\gamma) = \gamma^{-1}$. Next we have

$$\tilde{S}(b\gamma) = \gamma^{-1} \varepsilon(b)^{-1} S(b) = q(\mathrm{d}(b), \gamma) \varepsilon(b)^{-1} S(b) \gamma^{-1} = q(\mathrm{d}(b), \gamma) \tilde{S}(\gamma b),$$

so that $\tilde{S}$ gives rise to an anti-homomorphism S_Γ on $\mathscr{B}_\Gamma$. Finally, we show that S_Γ is an antipode. We have

$$(\mathrm{id} \star S_\Gamma)(b) = \sum b_{1i} \varepsilon(b_{2i}) \varepsilon(b_{2i})^{-1} S(b_{2i}) = \delta(b) \mathbf{1}$$

and

$$(S_\Gamma \star \mathrm{id})(b) = \sum \varepsilon(b_{2i})^{-1} \varepsilon(b_{1i})^{-1} S(b_{1i}) b_{2i} = \varepsilon(b)^{-1} \delta(b) = \delta(b) \mathbf{1} . \qquad \square$$

Since 1-parameter convolution semi-groups of states will play an important role, we are interested in the question under which conditions on a linear functional ψ on a q-$*$-bialgebra $\mathscr{B}$ the convolution exponentials $\exp_\star t\psi$ are states (i.e. positive, normalized linear functionals) for all $t \in \mathbb{R}_+$. For $*$-bialgebras this holds if and only if ψ is conditionally positive which means

$$\begin{aligned} &\psi(1) = 0, \\ &\psi(b^*) = \overline{\psi(b)} \quad \text{for all} \quad b \in \mathscr{B}, \\ &\psi(b^* b) \geq 0 \quad \text{for all} \quad b \in \operatorname{Kern} \delta . \end{aligned} \tag{2.7}$$

For the more general case of $\mathbb{Z}_2$-graded $*$-bialgebras one has to restrict oneself to even linear functionals to get the same result. All rests on the question under which conditions the convolution product ot two positive linear functionals is again positive. For $*$-bialgebras this holds in general. For $\mathbb{Z}_2$-graded $*$-bialgebras one of the functionals has to be even. Let us denote by $\mathscr{B}^*_{(e)}$ the convolution subalgebra of $\mathscr{B}^*$ consisting of the linear functionals that vanish on $\mathscr{B}_{(\lambda)}$, $\lambda \in \Lambda$, inless $\lambda = e$, e the unit element of Λ. We call the elements of $\mathscr{B}^*_{(e)}$ even. Clearly, if Λ is trivial $\mathscr{B}^*_{(e)} = \mathscr{B}^*$ and if $\Lambda = \mathbb{Z}_2$ our notion of even linear functionals coincides with the usual one.

2.5. Proposition. *Let $\mathscr{B}$ be a left (right) q-*-bialgebra and let φ and ψ be positive linear functions on $\mathscr{B}$. If φ is even (ψ is even) then $\varphi \star \psi$ is again positive.*

Proof. Let $\mathscr{B}$ be a left q-*-bialgebra and let φ and ψ be positive with φ even. Then

$$(\varphi \star \psi)(b^* b) = \sum q(\mathrm{d}(b_{1i}^* b_{1j}), \varepsilon(b_{2i})^*)^{-1} \varphi(b_{1i}^* b_{1j}) \psi(b_{2i}^* b_{2j}) \geq 0. \qquad \square$$

Next we show that $\mathscr{B}^*$ can be embedded into $(\mathscr{B}_\Gamma)^*$ in such a way that positivity is preserved for even linear functionals.

2.6. Proposition. *Let φ be a linear functional on the left q-*-bialgebra $\mathscr{B}$. There is a uniquely determined linear functional φ_Γ on $\mathscr{B}_\Gamma$ satisfying*

$$\varphi_\Gamma(b\gamma) = \varphi(b) \tag{2.8}$$

for all $b \in \mathscr{B}$, $\gamma \in \Gamma$. Moreover, the mapping $\varphi \mapsto \varphi_\Gamma$ from $\mathscr{B}^*$ to $(\mathscr{B}_\Gamma)^*$ is a graded algebra homomorphism. The restriction of this homomorphism to $\mathscr{B}^*_{(e)}$ preserves positivity and conditional positivity. The theorem holds also for right q--bialgebras if* (2.8) *is replaced by*

$$\varphi_\Gamma(\gamma b) = \varphi(b).$$

Proof. Let $(b_i)_{i \in I}$ be a vector space basis of the left *-bialgebra $\mathscr{B}$. Then $(b_i\gamma)_{i \in I, \gamma \in \Gamma}$ forms a vector space basis of $\mathscr{B}_\Gamma$. We define the linear mapping

$$r_\Gamma : \mathscr{B}_\Gamma \to \mathscr{B}$$

by

$$r_\Gamma(b_i\gamma) = b_i.$$

Then

$$(r_\Gamma \otimes r_\Gamma) \circ \Delta_\Gamma = \Delta \circ r_\Gamma$$

which means that r_Γ is a coalgebra homomorphism. This gives the first part of the proposition. Now let φ be even and positive. Then

$$\varphi_\Gamma\left(\left(\sum_{i,\gamma} \alpha_{i\gamma} b_i \gamma\right)^* \left(\sum_{i'\gamma'} \alpha_{i'\gamma'} b_{i'} \gamma'\right)\right) = \sum_{i,\gamma,i',\gamma'} \bar{\alpha}_{i\gamma} \alpha_{i'\gamma'} \varphi(b_i^* b_{i'}) \geq 0,$$

because

$$\varphi_\Gamma(\gamma b \gamma') = q(\mathrm{d}(b), \gamma)^{-1} \varphi_\Gamma(b \gamma \gamma') = \varphi(b).$$

Similarly, it can be shown that φ_Γ is conditionally positive if $\varphi \in \mathscr{B}^*_{(e)}$ is. $\square$

Remark. By Theorem 2.4. and Proposition 2.6., the diagrams

$$\begin{tikzcd} \mathscr{B} \arrow[r] \arrow[d, "\Delta"] & \mathscr{B}_\Gamma \arrow[d, "\Delta_\Gamma"] \\ \mathscr{B} \otimes \mathscr{B} \arrow[r, "\iota"] & \mathscr{B}_\Gamma \otimes \mathscr{B}_\Gamma \end{tikzcd}$$

and

$$\begin{tikzcd} \mathscr{B} \arrow[d, "\Delta"] & \mathscr{B}_\Gamma \arrow[l, "r_\Gamma"'] \arrow[d, "\Delta_\Gamma"] \\ \mathscr{B} \otimes \mathscr{B} & \mathscr{B}_\Gamma \otimes \mathscr{B}_\Gamma \arrow[l, "r_\Gamma \otimes r_\Gamma"'] \end{tikzcd}$$

commute.

The following theorem is now an easy corollary of the preceding proposition and of Theorem 3.5. of [31] which it generalizes.

2.7. Theorem. *For an even linear functional ψ on a (left or right) q-*-bialgebra the following conditions are equivalent*
(i) *ψ is conditionally positive*
(ii) $\exp_\star t\psi$ *is a state for all $t \geq 0$.* $\square$

3. q-White Noise

We introduce some quantum probabilistic language. Let $\mathscr{B}$ be a Λ-Γ-graded $*$-algebra and $\mathscr{A}$ a $*$-algebra, and let Φ be s tate on $\mathscr{A}$. A homomorphism j from $\mathscr{B}$ to $\mathscr{A}$ is called a random variable on $\mathscr{B}$ over $(\mathscr{A}, \Phi)$ if the state $\varphi = \Phi \circ j$ on $\mathscr{B}$ is even. We say that φ is the distribution of j. For a Λ-Γ-commutation factor q an n-tuple $(j_1, \ldots, j_n)$, $n \in \mathbb{N}$, of random variables over the same $(\mathscr{A}, \Phi)$ is said to be left q-independent if

$$j_k(b) j_l(c) = q(\mathrm{d}(b), \varepsilon(c)) j_l(c) j_k(b)\,, \tag{3.1}$$

and if

$$\Phi(j_1(b_1) \ldots j_n(b_n)) = \varphi_1(b_1) \ldots \varphi_n(b_n) \tag{3.2}$$

for b, $c \in \mathscr{B}$, $k < l$, and for $b_1, \ldots, b_n \in \mathscr{B}$, where φ_k is the distribution of j_k. If we replace (3.1) by

$$j_k(b) j_l(c) = q(\mathrm{d}(c), \varepsilon(b))^{-1} j_l(c) j_k(b)$$

then $(j_1, \ldots, j_n)$ is said to be right q-independent. If $\mathscr{B}$ is a left q-*-bialgebra the convolution product $j_1 \star \ldots \star j_n$ of random variables $j_1, \ldots, j_n$ is again a random variable if $(j_1, \ldots, j_n)$ is left q-independent, and the distribution of $j_1 \star \ldots \star j_n$ is equal to the convolution product $\varphi_1 \star \ldots \star \varphi_n$ of the distributions φ_k of j_k. The same holds if we replace "left" by "right." A quantum stochastic process on $\mathscr{B}$ in the sense of Acccardi, Frigerio, and Lewis [2] is a family $(j_k)_{k \in I}$ of random variables on $\mathscr{B}$ over some fixed $(\mathscr{A}, \Phi)$. We call a quantum stochastic process $(j_{st})_{0 \leq s \leq t}$ on a left q-*-bialgebra $\mathscr{B}$ a q-independent, stationary increment process (or a q-white noise) if the following conditions are fulfilled
(a) (increment property)

$$\begin{aligned} j_{rs} \star j_{st} &= j_{rt} \quad \text{for} \quad r \leq s \leq t \\ j_{tt} &= \delta 1 \end{aligned}$$

(b) (q-independence of increments)
for all $n \in \mathbb{N}$ and $t_1 \leq \ldots \leq t_{n+1}$ the n-tuple $(j_{t_1 t_2}, \ldots, j_{t_n t_{n+1}})$ of random variables is left q-independent
(c) (stationary of increments)
the distribution φ_{st} of j_{st} only depends on the difference $t - s$, i.e. $\varphi_{st} = \varphi_{0,t-s} = \varphi_{t-s}$
(d) (weak continuity)

$$\varphi_t(b) \to \delta(b) \quad \text{as} \quad t \downarrow 0 \quad \text{for all} \quad b \in \mathscr{B}\,.$$

If $\mathscr{B}$ is a right q-*-bialgebra we replace "left" in (b) by "right."

Our definition of q-independent, stationary increment processes (which, for trivial graduations, was already given in [3]) splits into two parts. The first is the algebraic q-commutation relations between the increments. The second is the "classical probabilistic" condition

$$\Phi(j_{t_1t_2}(b_1)\dots j_{t_nt_{n+1}}(b_n)) = \varphi_{t_2-t_1}(b_1)\dots\varphi_{t_{n+1}-t_n}(b_n) \tag{3.3}$$

for all $n \in \mathbf{N}$, $t_1 \leq \dots \leq t_{n+1}$, $b_1, \dots, b_n \in \mathscr{B}$, which we express by saying that Φ is a stationary product state for the process (j_{st}).

Although a right q-*-bialgebra is a left $\tilde{q}$-*-bialgebra, in general a right q-white noise is not a left $\tilde{q}$-white noise, because the generator of a q-white noise is required to be even with respect to the Λ-graduation whereas the generator of a $\tilde{q}$-white noise is required to be even with respect to the Γ-graduation. However, in the special case when both graduations agree the two notions coincide in the sense that a right q-white noise can be regarded as a left $\tilde{q}$-white noise.

It is not difficult to see that (a) and (3.3) imply that $\{\varphi_t : t \in \mathbb{R}_+\}$ forms a 1-parameter convolution semi-group of states on $\mathscr{B}$. There are a couple of results which are analogous to the case when Λ and Γ are trivial. We state them without proof.

3.1. Theorem. *The* 1*-parameter convolution semi-group* $\{\varphi_t\}$ *of even states associated to a q-white noise on a (left or right) q-*-bialgebra* $\mathscr{B}$ *is pointwise differentiable at* $t = 0$. *Moreover,*

$$\varphi_t(b) = (\exp_\star t\psi)(b),$$

where

$$\psi(b) = \frac{\mathrm{d}}{\mathrm{d}t}\varphi_t(b)\varepsilon|_{t=0},$$

and the linear functional ψ *is even and conditionally positive.* □

We call ψ the generator of the process j_{st}.

3.2. Theorem. (i) *Two q-independent, stationary increment processes on a (left or right) q-*-bialgebra are equivalent in the sense of Accardi, Frigerio and Lewis if and only if they have the same generator.*
(ii) *For a given even conditionally positive linear functional on a (left or right) q-*-bialgebra* $\mathscr{B}$ *there exists a q-independent, stationary increment process on* $\mathscr{B}$ *with generators* ψ. □

The next theorem shows how q-white noise on a left q-*-bialgebra can be reduced to white noise on a *-bialgebra.

3.3. Theorem. *Let* ψ *be an even conditionally positive linear functional on a left q-*-bialgebra* $\mathscr{B}$. *Let* $j_\Gamma(s,t)$ *be a white noise on the *-bialgebra* $\mathscr{B}_\Gamma$ *with generator* ψ_Γ. *Then*

$$j_{st}(b) = j_\Gamma(0,s)(\varepsilon(b))j_\Gamma(s,t)(b),$$

$b \in \mathscr{B}$, $s \leq t$, *form a q-white noise on* $\mathscr{B}$ *with generator* ψ.

Proof. One checks that j_{st} satisfies the axioms of q-white noise and that its generator is equal to ψ. □

Formally, for right q-*-bialgebras

$$j_{st}(b) = j_\Gamma(s,t)(b)j_\Gamma(t,\infty)(\varepsilon(b))$$

is the right thing to look at. As $j_\Gamma(t,\infty)(\varepsilon(b))$, in general, makes no sense, we need some additional information on j_Γ which is given by the representation theorem for processes with independent, stationary increments. It is established in [33, 32] for special cases. In [16] it is given for the case of bounded operators. Finally, in [35] the general case is treated. We give a brief review of some of the results of [35].

Let $\mathscr{B}$ be a $*$-algebra and let $\delta : \mathscr{B} \to \mathbb{C}$ be a $*$-algebra homomorphism. A linear functional ψ on $\mathscr{B}$ is called conditionally positive if it satisfies (2.7). For a given conditionally positive linear functional ψ on $\mathscr{B}$ we quotient $\mathscr{B}$ by the null space of the sesquilinear form

$$(b, c) \mapsto \psi((b - \delta(b)\mathbf{1})^* (c - \delta(c)\mathbf{1}))$$

to obtain a pre-Hilbert space D whose completion we denote by H. Let $\eta : \mathscr{B} \to D$ be the canonical mapping. A $*$-representation ϱ of $\mathscr{B}$ on D is given by

$$\varrho(b)\eta(c) = \eta(b(c - \delta(c)\mathbf{1})) .$$

This construction generalizes a crucial part of the well known Araki-Woods embedding theorem for groups [4, 5, 18, 29, 37]; see [31] for the case of cocommutative bialgebras which include groups and tensor algebras. We call (D, η, ϱ) the triplet associated with ψ. Now let $\mathscr{B}$ be a $*$-bialgebra and let ψ be a conditionally positive linear functional on $\mathscr{B}$. Denote by $\mathscr{F}_H$ the Bose Fock space over the Hilbert space $L^2(\mathbb{R}_+, H)$ of H-valued square-integrable functions on $\mathbb{R}_+$. Then $\mathscr{F}_H$ is the direct integral

$$\int_{\mathscr{S}} H^{\otimes \# \omega} \mu(\mathrm{d}\omega)$$

where $(\mathscr{S}, \mu)$ is the symmetric measure space of the measure space $(\mathbb{R}_+, \lambda)$, λ Lebesgue measure, (see [18]) and $\#\omega$ denotes the number of elements of the finite subset ω of $\mathbb{R}_+$. Thus $\mathscr{F}_H$ consists of measurable functions F on $\mathscr{S}$ where $F(\omega) \in H^{\otimes \# \omega}$ with

$$\int_{\mathscr{S}} \|F(\omega)\|^2 \mu(\mathrm{d}\omega) < \infty.$$

Denote by $\mathscr{D}$ the dense linear subspace of $\mathscr{F}_H$ consisting of functions F with the properties

(a) There exists a bounded subset K_F of $\mathbb{R}_+$ such that $F(\omega) = 0$ unless $\omega \subset F_F$.

(b) There exists a finite-dimensional linear subspace E_F of D such that $F(\omega) \in (E_F)^{\otimes \# \omega}$.

(c) There exists a constant $C_F \in \mathbb{R}_+$ such that $\|F(\omega)\| \leq C^{\# \omega}$.

The quantum stochastic integral equation

$$j_{st}(b) = \delta(b)\,\mathrm{id} + \int_s^t \Bigg(\sum_i j_{s\tau}(b_{1i})\,(\mathrm{d}A_\tau^*(\eta(b_{2i})) + \mathrm{d}\Lambda_\tau(\varrho(b_{2i}) - \delta(b_{2i})\,\mathrm{id})$$

$$+ \mathrm{d}A_\tau(\eta((b_{2i})^*)) + \psi(b_{2i})\mathrm{d}\tau) \Bigg),$$

has a unique solution on $\mathscr{D}$ for all $b \in \mathscr{B}$, and, in the vacuum state, the j_{st} form a white noise on the $*$-bialgebra $\mathscr{B}$ with generator equal to ψ. Using the kernel method (see [25, 27]), a solution of the integral equations can be given explicitly [35].

We want to apply the representation theorem to our case of q-white noise. For a linear operator T on D and $s, t \in [0, \infty]$, $s < t$, we denoze by $J_{st}(T)$ the linear operator on $\mathscr{D}$ given by the kernel

$$k(\sigma, \tau, \varrho) = \begin{cases} \prod_{r \in \tau} \lambda(r)\,(T - \mathrm{id}) & \text{if} \quad \sigma = \varrho = \emptyset \quad \text{and} \quad \tau \subset \omega \cap [s, t] \\ 0 & \text{otherwise}. \end{cases}$$

Here $\lambda(r)\,(T)$, $r \in \omega$, denotes the operator on $D^{\otimes \# \omega}$ which acts on the r^{th} factor like T; see [35]. We call $J_{st}(T)$ the (s, t)-second quantization of T.

Let ψ be an even conditionally positive linear functional on a q-$*$-bialgebra $\mathscr{B}$ and let (D, η, ϱ) and $(D_\Gamma, \eta_\Gamma, \varrho_\Gamma)$ be the triplets associated to ψ and ψ_Γ respectively. Clearly, $\eta(b) \mapsto \eta_\Gamma(b)$ is an isomorphism from D to D_Γ. We identify D and D_Γ, and, in this sense, $\eta(b) = \eta_\Gamma(b)$ and $\varrho(b) = \varrho_\Gamma(b)$ for $b \in \mathscr{B} \subset \mathscr{B}_\Gamma$.

3.4. Proposition. *Let ψ be an even conditionally positive linear functional on a (left or right) q-$*$-bialgebra $\mathscr{B}$, and let $j_\Gamma(s, t)$ be the realization of the white noise on $\mathscr{B}$ with generator ψ_Γ on Fock space. Then $j_\Gamma(\gamma)$, $\gamma \in \Gamma$, is the (s, t)-second quantization $J_{st}(\varrho_\Gamma(\gamma))$ of $\varrho_\Gamma(\gamma)$. Moreover,*

$$\varrho_\Gamma(\gamma)\eta(b) = q(\mathrm{d}(b), \gamma)^{-1}\eta(b)\,. \tag{3.4}$$

If, in addition, Λ is a group and $\lambda^* = \lambda^{-1}$ for all $\lambda \in \Lambda$ then D splits into the direct sum

$$D = \bigoplus_{\lambda \in \Lambda} \eta(\mathscr{B}_{(\lambda)})$$

of orthogonal linear subspaces $\eta(\mathscr{B}_{(\lambda)})$ of D.

Proof. Since $\Delta_\Gamma(\gamma) = \gamma \otimes \gamma$ and $\eta_\Gamma(\gamma) = 0$, $\psi_\Gamma(\gamma) = 0$, we deduce from Proposition 2 of [35] that $j_\Gamma(s, t)\,(\gamma) = J_{st}(\varrho_\Gamma(\gamma))$. We have

$$\begin{aligned} &\|(\varrho_\Gamma(\gamma) - q(\mathrm{d}(b), \gamma)^{-1})\eta(b)\|^2 \\ &= \|\eta(\gamma b) - q(\mathrm{d}(b), \gamma)^{-1}\eta(b)\|^2 \\ &= \psi_\Gamma(b^* \gamma^* \gamma b) + \varepsilon |q(\mathrm{d}(b), \gamma)\varepsilon|^{-2}\psi(b^* b) - 2\Re\{q(\mathrm{d}(b), \gamma)^{-1}\psi_\Gamma(b^* \gamma^* b)\} \\ &= 0\,, \end{aligned}$$

where we used the relations in $\mathscr{B}_\Gamma$ and the fact that ψ is even. Now assume that $\lambda^* = \lambda^{-1}$. Then

$$\langle \eta(b), \eta(c) \rangle = \psi((b - \delta(b)\mathbf{1})^*(c - \delta(b)\mathbf{1})) = 0$$

if $\mathrm{d}(b^* c) = \mathrm{d}(b)^{-1}\mathrm{d}(c) \neq e$, because ψ is even. Since $\eta(\mathscr{B}) = D$ and $\mathscr{B} = \bigoplus_{\lambda \in \Lambda} \mathscr{B}_{(\lambda)}$ the remaining part of the proposition follows. □

Thus in the case when Λ is a group and the involution on Λ is the taking of inverses we have a very simple description of $\varrho_\Gamma(\gamma)$, $\gamma \in \Gamma$. Namely, D splits into orthogonal eigenspaces $D_{(\lambda)}$ of $\varrho_\Gamma(\gamma)$ and the eigenvalue for $D_{(\lambda)}$ is equal to $q(\gamma, \lambda)^{-1}$. We have also shown that for a given ψ the operator $\varrho_\Gamma(\gamma)$ is well-defined by (3.4). We now come to the

3.5. Representation Theorem for q-White Noise. *Let ψ be an even conditionally positive linear functional on a left q-$*$-bialgebra $\mathscr{B}$ and let (D, ϱ, η) be the triplet* *associated to ψ. Then the quantum stochastic integral equation*

$$\begin{aligned} j_{st}(b) = \delta(b) + \int_s^t \Big(\sum_i j_{s\tau}(b_{1i}) J_{0\tau}(\varrho_\Gamma(\varepsilon(b_{2i}))) \\ \times (\mathrm{d}A_\tau^*(\eta(b_{2i})) + \mathrm{d}\Lambda_\tau(\varrho(b_{2i}) - \delta(b_{2i})\,\mathrm{id}) \\ + \mathrm{d}A_\tau(\eta(b_{2i}^*)) + \psi(b_{2i})\mathrm{d}\tau) \Big). \end{aligned} \tag{3.5}$$

has a unique solution on $\mathscr{D}$ for all $b \in \mathscr{B}$. In the vacuum state, the j_{st} form a q-white noise on $\mathscr{B}$ with generator equal to ψ. For a right q--bialgebra $\mathscr{B}$ the equations for the associated process j_Γ on $\mathscr{B}_\Gamma$ are ($b \in \mathscr{B}$)*

$$\begin{aligned} j_\Gamma(s,t)(b) = \delta(b) + \int_s^t \Big(\sum_i j_\Gamma(s,\tau)(b_{1i}) \\ \times (q(\mathrm{d}(b_{2i}), \varepsilon(b_{1i}))^{-1} \mathrm{d}A_\tau^*(\eta(b_{2i})) \\ + \mathrm{d}\Lambda_\tau(\varrho(\varepsilon(b_{1i}))\varrho(b_{2i}) - \delta(b_{2i})\,\mathrm{id}) \\ + \mathrm{d}A_\tau(\eta((b_{2i})^*)) + \psi(b_{2i})\mathrm{d}\tau) \Big). \end{aligned} \tag{3.6}$$

In the vacuum state,

$$j_{st}(b) = j_\Gamma(s,t)(b) J_{t,\infty}(\varrho_\Gamma(\varepsilon(b))) \tag{3.7}$$

form a q-white noise on $\mathscr{B}$ with generator ψ.

Proof. For a q-*-bialgebra let $j_\Gamma(s,t)(b)$ be the solution of

$$\begin{aligned} j_\Gamma(s,t)(b) = \delta(b) + \int_s^t \Big(\sum_i j_\Gamma(s,\tau)(b_{1i}\varepsilon(b_{2i})) \\ \times (\mathrm{d}A_\tau^*(\eta(b_{2i})) + \mathrm{d}\Lambda_\tau(\varrho(b_{2i}) - \delta(b_{2i})\,\mathrm{id}) \\ + \mathrm{d}A_\tau(\eta((b_{2i})^*)) + \psi(b_{2i})\mathrm{d}\tau) \Big). \end{aligned}$$

Then by Theorem 3.3. and Proposition 3.4.

$$j_{st}(b) = J_{0s}(\varrho_\Gamma(\varepsilon(b))) j_\Gamma(s,t)(b)$$

is a realization of q-white noise with generator ψ. We multiply the above integral equation by $J_{0s}(\varrho_\Gamma(\varepsilon(b))) = j_\Gamma(0,s)(\varepsilon(b))$ and use

$$J_{0s}(\varrho_\Gamma(\varepsilon(b))) j_\Gamma(s,\tau)(b_{1i}\varepsilon(b_{2i})) = j_{st}(b_{1i}) J_{0\tau}(\varrho_\Gamma(\varepsilon(b_{2i})))$$

to obtain (3.5). For the right case, we use Proposition 3.4, the homomorphism property of ϱ, and the facts that η is a 1-cocycle and that ψ is even to arrive at formula (3.6). One checks, like in Theorem 3.3 for the left case, that (3.7) is a q-white noise with generator ψ. □

In the right case, a calculation shows that (3.6) can be rewritten in the form

$$
\begin{aligned}
j_{st}(b) &= \delta(b) + \int_s^t \Big(\sum_i j_{s\tau}(b_{1i}) J_{0,\infty}(\varrho_\Gamma(\varepsilon(b_{2i}))) J_{0\tau}(\varrho_\Gamma(\varepsilon(b_{2i})^{-1})) \\
&\quad \times (\mathrm{d}A_\tau^*(\eta_\Gamma(\varepsilon(b_{2i})^{-1} b_{2i})) + \mathrm{d}\Lambda_\tau(\varrho_\Gamma(\varepsilon(b_{2i})^{-1} b_{2i}) - \delta(b_{2i})\,\mathrm{id}) \\
&\quad + (\mathrm{d}A_\tau(\eta(b_{2i}^*)) + \psi(b_{2i})\mathrm{d}\tau) \Big),
\end{aligned} \tag{3.8}
$$

where the integrals are to be understood in the obvious way.

4. Additive *q*-White Noise

The most fundamental white noise is the one where the increments are additive; cf. [33]. The underlying $*$-bialgebra is the tensor algebra.

Let V be a vector space which splits into two linear subspaces V_+ and V_-, i.e. $V = V_+ \oplus V_-$. Assume that there is a conjugation on V (that is an antilinear, selfinverse mapping $v \mapsto v^*$) such that V_+ is mapped into V_-. The tensor algebra

$$T(V) = \bigoplus_{n=0}^{\infty} V^{\otimes n}$$

($V^{\otimes 0} = \mathbb{C}$) becomes a $*$-algebra if we extend the conjugation on V to an involution on $T(V)$ (in the only possible way). $T(V)$ can also be described as the free $*$-algebra generated by indeterminates v_k, $k \in I$, where $\{v_k : k \in I\}$ forms a vector space basis of V_+. In other words, $T(V)$ is the $*$-algebra of non-commutative polynomials in v_k and v_k^*, $k \in I$. If we assign to every homogeneous polynomial in $T(V)$ its degree this gives an $\mathbb{N}$-graduation of $T(V)$ and $T(V)^{[n]} = V^{\otimes n}$ is the linear subspace of homogeneous elements of degree n. Of course, any $\mathbb{N}$-graduation gives a $\mathbb{Z}$-graduation if we put the linear subspaces of the $\mathbb{Z}$-graduation equal to $\{0\}$ for negative degrees. Moreover, if we assign to a monomial the difference between the number of v_k's and the number of v_k^*'s occurring this defines another $\mathbb{Z}$-graduation of $T(V)$ and

$$T(V)_{[m]} = \bigoplus_{k=\varepsilon(m)}^{\infty} \bigoplus_{\substack{(l_1,\dots,l_k)\in\{\pm 1\} \\ l_1+\dots+l_k=m}} V_{l_1} \otimes \dots \otimes V_{l_k}$$

is the linear subspace of homogeneous elements of degree m. For $\nu \in \mathbb{N} \cup \{\infty\}$ we put

$$T(V)^{[n],\nu} = \bigoplus_{k=n \bmod \varphi} T(V)^{[k]}$$

and

$$T(V)_{[m],\nu} = \bigoplus_{l=m \bmod \nu} T(V)_{[l]}$$

to obtain the $\mathbb{Z}_\nu$-graduations

$$T(V) = \bigoplus_{n\in\mathbb{Z}_\nu} T(V)^{[n],\nu} = \bigoplus_{m\in\mathbb{Z}_\nu} T(V)_{[m],\nu}$$

of $T(V)$. For $\nu, \nu_1, \nu_2 \in \mathbb{N} \cup \{\infty\}$ the algebra $T(V)$ becomes a Λ-Γ-graded $*$-algebra with $\Lambda = \mathbb{Z}_\nu$, $\Gamma = \mathbb{Z}_{\nu_1} \times \mathbb{Z}_{\nu_2}$ and the involutions as in Proposition 2.1.

Let q be in $\mathbb{C}^*$. Equip $T(V)$ with the Λ-Γ-graduation as above with $\nu = \nu(q)$, $\nu_1 = \nu(|q|)$, $\nu_2 = \nu(q/|q|)$; see the remark following Proposition 2.1. Then q gives rise to a Λ-Γ-commutation factor which we also denote by q. If we set

$$\Delta(v) = v \otimes \mathbf{1} + \mathbf{1} \otimes v$$

and $\delta(v) = 0$ for $v \in V$ then Δ and δ extend, in a unique way, to algebra homomorphisms

$$\Delta : T(V) \to T(V) \otimes_q T(V)$$

and

$$\delta : T(V) \to \mathbb{C},$$

and we obtain a left q-$*$-bialgebra. If we replace $T(V) \otimes_q T(V)$ by $T(V) \otimes^q T(V)$ we have a right q-$*$-bialgebra. An antipode of $T(V)$ is, in both cases, given by $S(v) = -v$, $v \in V$.

We describe $T(V)_{\Gamma,q}$ for some special choices of q. In the left case with q real and $|q| \neq 1$, the $*$-bialgebra $T(V)_{\mathbb{Z},q}$ is obtained from $T(V)$ by adding generators y and y^{-1} with the relations

$$yy^{-1} = y^{-1}y = \mathbf{1}, \tag{4.1}$$

$$y^* = y; (y^{-1})^* = y^{-1}, \tag{4.2}$$

$$v_k y = q y v_k . \tag{4.3}$$

If q is complex with $|q| = 1$ but not a root of unity, (4.2) is repaced by $y^* = y^{-1}$. If q is a root of unity we have these latter relations plus the relation $y^\nu = 1$. The comultiplication of $T(V)_{\Gamma,q}$ is given by

$$\Delta_\Gamma(v) = v \otimes \mathbf{1} + y \otimes v . \tag{4.4}$$

In the right case, (4.4) is replaced by

$$\Delta_\Gamma(v) = v \otimes y + \mathbf{1} \otimes v .$$

The antipode S_Γ of $T(V)_{\Gamma,q}$ is given by

$$S_\Gamma(v) = y^{-1} v .$$

Notice that, in the left and right case with q real and $|q| \neq 1$, $v \in V$ and y generate the sub-$*$-bialgebra $T(V)_{\mathbb{N},q}$ of $T(V)_{\mathbb{Z},q}$. This $*$-bialgebra also makes sense for $q = 0$, we denote it by $T(V)_{\mathbb{N},0}$.

For a pre-Hilbert space $\mathscr{D}$ we denote by $H(\mathscr{D})$ the $*$-algebra of linear operators F on $\mathscr{D}$ with the properties that $\operatorname{dom}(F^*) \supset \mathscr{D}$ and $F^*\mathscr{D} \subset \mathscr{D}$. If $F = (F_k)_{k \in I}$ is a family of elements in $H(\mathscr{D})$ then there is a $*$-representation j_F of $T(V)$ on $\mathscr{D}$ given by $j_F(b) = b(F_k)$, where for a polynomial $b \in T(V)$ and elements a_k, $k \in I$, of a $*$-algebra $\mathscr{A}$ we use the notation $b(a_k)$ for the element in $\mathscr{A}$ obtained from b by replacing v_k by a_k and v_k^* by $(a_k)^*$. Clearly, the random variables on $T(V)$ can be identified with families F of operators on $\mathscr{D}$ together with a unit vector Ω in $\mathscr{D}$. The distribution of F is then the linear functional φ on $T(V)$ given by $\varphi(b) = \langle \Omega, j_F(b)\Omega \rangle$. A q-independent ($q \neq 0$), stationary increment process on the left q-*-bialgebra $T(V)$ can be identified with families $(F_t)_{t\geq 0}$, $F_t = (F_{t,k})_{k\in I}$, $F_{t,k} \in H(\mathcal{D})$, $\mathcal{D}$ a pre-Hilbert space, satisfying (we put $(F_{st})_k = F_{t,k} - F_{s,k}$)

$$\begin{aligned}(F_{st})_k(F_{s't'})_{k'} &= q(F_{s't'})_{k'}(F_{st})_k\,,\\ ((F_{st})_k)^*(F_{s't'})_{k'} &= q^{-1}(F_{s't'})_{k'}((F_{st})_k)^*\end{aligned} \tag{4.5}$$

for $0 \leq s \leq t \leq s' \leq t'$, $k, k' \in I$, with the additional property that there is a stationary product state for the increment process $j_{F(s,t)}$ on $T(V)$ determined by F_t and that the φ_t are even and $t \mapsto \varphi_t$ is weakly continuous. In the right case, (4.5) is replaced by

$$((F_{st})_k)^*(F_{s't'})_{k'} = q(F_{s't'})_{k'}((F_{st})_k)^*.$$

Let us call a process F_t with these properties a left (right) additive q-white noise on V. Notice that, if $|q| = 1$, any right additive q-white noise can be regarded as a left additive q-white noise, beause the two graduations are the same in this case. We have

4.1. Theorem. *Let there be given*

- *a family $(D_{(m)})_{m\in\Lambda}$ of pre-Hilbert spaces*
- *a family $(B_k)_{k\in I}$, $B_k \in H(D)$, $D = \bigoplus_{m\in\Lambda} D_{(m)}$, such that B_k maps $D_{(m)}$ to $D_{(m+1)}$*
- *two families $(\xi_k)_{k\in I}$ and $(\zeta_k)_{k\in I}$ of vectors in $D_{(1)}$ and $D_{(-1)}$ respectively*
- *a family $(h_k)_{k\in I}$ of complex numbers with $h_k = 0$ unless $q = 1$.*

Then $(\check{B}_t)_{t\geq 0}$ with

$$\check{B}_{t,k} = \int J_{0\tau}(\mathrm{d}A_\tau^*(\xi_k) + \mathrm{d}\Lambda_\tau(B_k) + \mathrm{d}A_\tau(\zeta_k) + h_k\mathrm{d}\tau)\,, \tag{4.6}$$

in the vacuum state, is a left additive q-white noise. Here $J_{0t} = J_{0t}(\varrho_\Gamma(y))$ with $\varrho_\Gamma(y)$ the linear operator on D which is equal to multiplication by q^{-m} on $D_{(m)}$. In the right case, the solution of the quantum stochastic integral equation

$$L_{t,k} = \int_0^t L_{\tau,k}\mathrm{d}\Lambda_\tau(\varrho_\Gamma(y) - \mathrm{id}) + A_t^*(\xi_k) + \Lambda_t(B_k) + A_t(\zeta_k) + h_k t$$

exists on $\mathcal{D}$, and, in the vacuum state, the process $(B_t)_{t\geq 0}$ with

$$B_{t,k} = L_{t,k}J_{t,\infty}$$

is a right additive q-white noise. Conversely, any left (right) additive q-white noise is of the type $(\check{B}_t)_{t\geq 0}((B_t)_{t\geq 0})$ above. More precisely, let ψ be the generator of the process with associated triplet (D, ϱ, η). Then one can choose

- *$D_{(m)} = \eta(T(V)_{(m)})$*
- *$B_k = \varrho(v_k)$*
- *$\xi_k = \eta(v_k)$ and $\zeta_k = \eta(v_k^*)$*
- *$h_k = \psi(v_k)$.*

Proof. This is a direct consequence of Theorem 3.5. □

Formula (3.8) for the right case becomes

$$\begin{aligned}B_{t,k} = J_{0,\infty}\int_0^t &(J_{0\tau})^{-1}(\mathrm{d}A_\tau^*(\eta_\Gamma(y^{-1}v_k))\\ &+ \mathrm{d}\Lambda_\tau(\varrho_\Gamma(y^{-1}v_k)) + \mathrm{d}A_\tau(\eta(v_k^*)) + \psi(v_k)\mathrm{d}\tau)\,.\end{aligned} \tag{4.7}$$

In comparing (4.6) and (4.7) with (3.5) and (3.8), one sees that the integral equations for a general q-white noise involve integrals against additive q-white noise.

5. Unitary q-White Noise

For $d \in \mathbb{N}$ let $M_{\mathbb{C}}\langle d\rangle$ be the free $*$-algebra generated by d^2 indeterminates x_{kl}, $k, l = 1, \ldots, d$. Then $M_{\mathbb{C}}\langle d\rangle$ is the free algebra generated by $2d^2$ indeterminates x_{kl} and x_{kl}^*, $k, l = 1, \ldots, d$, with the involution given by $(x_{kl})^* = x_{kl}^*$. We define a $\mathbb{Z}_\nu$-graduation, $\nu \in \mathbb{N} \cup \{\infty\}$, on $M_{\mathbb{C}}\langle d\rangle$ by setting $d(x_{kl}) = k - l$ and $d(x_{kl}^*) = l - k$. If we set Λ and Γ equal to $\mathbb{Z}_\nu$ with the involution given by the taking of inverses then $M_{\mathbb{C}}\langle d\rangle$ becomes a Λ-Γ-graded $*$-algebra with both graduations equal to the one above. Let q be a complex number of modulus 1. W set $\Lambda = \Gamma = \mathbb{Z}_{\nu(q)}$; see the remark following Proposition 2.1. Since it is not necessary to distinguish between left and right q-$*$-bialgebra structures on $M_{\mathbb{C}}\langle d\rangle$, in this section, by a q-$*$-bialgebra (q-white noise) we always mean a left q-$*$-bialgebra (left q-white noise). We turn $M_{\mathbb{C}}\langle d\rangle$ into a q-$*$-bialgebra by setting

$$\Delta x_{kl} = \sum_{n=1}^{d} x_{kn} \oplus x_{nl}$$

and

$$\delta x_{kl} = \delta_{kl} \text{ (Kronecker delta)}$$

and extending Δ and δ to a $*$-algebra homomorphism

$$\Delta : M_{\mathbb{C}}\langle d\rangle \to M_{\mathbb{C}}\langle d\rangle \otimes_q M_{\mathbb{C}}\langle d\rangle$$

and

$$\delta : M_{\mathbb{C}}\langle d\rangle \to \mathbb{C}.$$

One checks that the ideal $\mathcal{J}$ in $M_{\mathbb{C}}\langle d\rangle$ generated by the elements

$$\sum_{n=1}^{d} x_{kn} x_{ln}^* - \delta_{kl}\mathbf{1}, \qquad \sum_{n=1}^{d} x_{nk}^* x_{nl} - \delta_{kl}\mathbf{1},$$

$k, l = 1, \ldots, d$, is a graded $*$-ideal and a coideal. We denote by $\mathcal{K}\langle d\rangle$ the Λ-Γ-graded $*$-algebra $M_{\mathbb{C}}\langle d\rangle/\mathcal{J}$. It inherits from $M_{\mathbb{C}}\langle d\rangle$ the structure of a q-$*$-bialgebra which may be regarded as a q-deformation of the so-called non-commutative coefficient algebra of the unitary group (the latter was introduced in [40]; see also [3, 17, 32]). The $*$-algebra $\mathcal{K}\langle d\rangle_\Gamma$ is obtained from $\mathcal{K}\langle d\rangle$ by adding generators y and y^{-1} with relations

$$\begin{aligned} yy^{-1} &= y^{-1}y = \mathbf{1}, \\ y^* &= y^{-1} \text{ (plus } y^{\nu(q)} = \mathbf{1} \text{ if } \nu(q) > \infty), \\ x_{kl}y &= q^{k-l} y x_{kl}. \end{aligned}$$

The comultiplication of the $*$-bialgebra $\mathcal{K}\langle d\rangle_\Gamma$ is given by

$$\Delta_\Gamma x_{kl} = \sum_{n=1}^{d} x_{kn} y^{n-l} \otimes x_{nl}.$$

If U is a unitary operator on $\mathbb{C}^d \otimes \mathscr{H}$, $\mathscr{H}$ a Hilbert space, then a $*$-algebra homomorphism j_U from $\mathscr{K}\langle d\rangle$ to $B(\mathscr{H})$ is uniquely determined by

$$j_U(x_{kl}) = U_{kl},$$

where we identified U with the corresponding $d \times d$-matrix $(U_{kl})_{k,l=1,\ldots,d}$ with $U_{kl} \in B(\mathscr{H})$. A q-white noise on $\mathscr{K}\langle d\rangle$ can be identified with a family $(U_t)_{t\geq 0}$ of unitary operators on a Hilbert space $\mathbb{C}^d \otimes \mathscr{H}$ satisfying (we put $U_{st} = U_s^{-1}U_t$)

$$(U_{st})_{kl}(U_{s't'})_{k'l'} = q^{(k-l)\,(k'-l')}(U_{s't'})_{k'l'}(U_{st})_{kl},$$
$$((U_{st})^*)_{lk}(U_{s't'})_{k'l'} = \bar{q}^{(k-l)\,(k'-l')}(U_{s't'})_{k'l'}((U_{st})^*)_{lk}$$

for $0 \leq s \leq t \leq s' \leq t'$, k, l, k', $l' = 1,\ldots,d$, with the additional properties that there is a stationary product state for the process $j_{U(s,t)}$ on $\mathscr{K}\langle d\rangle$ determined by U_t and that φ_t are even and $t \mapsto \varphi_t$ is weakly continuous. We call a process with these properties a unitary q-white noise (with d-dimensional initial space).

The next theorem shows that unitary q-white noise arises from special forms or ordinary unitary white noise by multiplication with a second quantization of multiplication by powers of q operator.

5.1. Theorem. *Let there be given*

- *a family $(H_{(m)})_{m\in\Lambda}$ of Hilbert spaces*
- *a unitary operator $\underline{B} = (B_{kl})_{kl}$ on $\mathbb{C}^d \otimes H$, $H = \bigotimes_{m\in\Lambda} H_{(m)}$, such that B_{kl} maps $H_{(m)}$ to $H_{(m+k-l)}$*
- *a $d \times d$-matrix $\underline{\xi} = (\xi_{kl})_{kl}$ with $\xi_{kl} \in H_{(k-l)}$*
- *a hermitian complex $d \times d$-matrix $\underline{h} = (h_{kl})_{kl}$ with $h_{kl} = 0$ unless $k - l = 0$.*

Denote by $\mathrm{d}I_t$ *the* $d \times d$*-matrix of differentials with*

$$(\mathrm{d}I_t)_{kl} = (J_{0t})\,(\varrho_\Gamma(y))^{k-l}\Bigg(\mathrm{d}A_t^*((\underline{B}\underline{\xi})_{kl}) + \mathrm{d}\Lambda_t(B_{kl} - \delta_{kl}\,\mathrm{id})$$
$$+\,\mathrm{d}A_t(\xi_{lk}) + \left(ih_{kl} - \frac{1}{2}\sum_{n=1}^{d}\langle \xi_{nk}, \xi_{nl}\rangle\right)\mathrm{d}t\Bigg),$$

where $\varrho_\Gamma(y)$ is the unitary operator on H which is equal to multiplication by $\bar{q}^m$ on $H_{(m)}$. Then the solution of the equation

$$U_t = \mathrm{id} + \int_0^t U_\tau\,\mathrm{d}I_\tau,$$

in the vacuum state, forms a unitary q-white noise $(U_t)_{t\geq 0}$. Conversely, any unitary q-white noise $(V_t)_{t\geq 0}$ is equivalent to one of the type $(U_t)_{t\geq 0}$ above. More precisely, let ψ be the generator of $(V_t)_{t\geq 0}$ with associated triplet (D,η,ϱ). Then one can choose

- *$H_{(m)}$ the completion of $\eta(\mathscr{K}\langle d\rangle_{(m)})$*
- *$B_{kl} = \varrho(x_{kl})$*
- *$\eta_{kl} = \eta(x_{lk}^*)$*
- $$h_{kl} = -i\left(\psi(x_{kl}) + \frac{1}{2}\sum_{n=1}^{d}\langle \xi_{nk}, \xi_{nl}\rangle\right).$$

Moreover,

$$\tilde{U}_t = U_t \mathscr{J}_t$$

with $(\mathscr{J}_t)_{kl} = \delta_{kl} J_{0t}(\varrho_\Gamma(y))^k$, *in the vacuum state, form an ordinary unitary white noise* $(\tilde{U}_t)_{t \geq 0}$.

Proof. This follows from an application of Theorem 3.5, and from

$$\Delta_\Gamma(x_{kl}y^l) = \sum_{n=1}^{\infty} (x_{kn}y^n) \otimes (x_{nl}y^l). \qquad \square$$

6. A Coalgebra Central Limit Theorem

This section serves as a preparation for Sect. 7. The following simple lemma will be crucial.

6.1. Lemma. *Let* φ_{nk}, $n \in \mathbb{N}$, $k = 1, \ldots, k_n$ $(k_n \in \mathbb{N})$, *be linear functionals on a coalgebra* $\mathscr{C}$ *satisfying*

(i) $\varphi_{n_1}, \ldots, \varphi_{nk_n}$ *commute for each* $n \in \mathbb{N}$ w.r.t. *convolution*

(ii)

$$\lim_{n\to\infty} \max_{1\leq k\leq k_n} |(\varphi_{nk} - \delta)(c)| = 0$$

for all $c \in C$.

(iii)

$$\sup_{n\in\mathbb{N}} \sum_{1\leq k\leq k_n} |(\varphi_{nk} - \delta)(c)| < \infty$$

for all $c \in \mathscr{C}$.

Suppose further that there exists a linear functional ψ *on* $\mathscr{C}$ *such that for all* $c \in \mathscr{C}$

$$\lim_{n\to\infty} \left(\sum_{1\leq k\leq k_n} (\varphi_{nk} - \delta) \right)(c) = \psi(c). \tag{6.1}$$

Then

$$\lim_{n\to\infty} \left(\prod_{1\leq k\leq k_n}^{*} \varphi_{nk} \right)(c) = (\exp_* \psi)(c) \tag{6.2}$$

for all $c \in \mathscr{C}$ *(where the product* $\prod^*$ *is the convolution product).*

Proof. Let (φ_{nk}) satisfy (i), (ii), (iii) and (6.1). For $c \in \mathscr{C}$ there is a finite-dimensional sub-coalgebra $\mathscr{D}_c$ of $\mathscr{C}$ containing c by the fundamental theorem on coalgebras. The linear operators $T_{nk} = (\mathrm{id} \otimes \varphi_{nk}) \circ \Delta$ on $\mathscr{C}$ leave $\mathscr{D}_c$ invariant. By (ii) and (iii)

$$\lim_{n\to\infty} \max_{1\leq k\leq k_n} \| T_{nk} \lceil \mathscr{D}_c - \mathrm{id} \| = 0$$

and

$$\sup_{n\in\mathbb{N}} \sum_{1\leq k\leq k_n} \| T_{nk} \lceil \mathscr{D}_c - \mathrm{id} \| < \infty .$$

Moreover, by (6.1)

$$\lim_{n\to\infty} \sum_{1\leq k\leq k_n} (T_{nk} \lceil \mathscr{D}_c - \mathrm{id}) = R \lceil \mathscr{D}_c$$

with $R = (\mathrm{id} \otimes \psi) \circ \delta$. Since by (i) the operators $T_{n1}, \ldots, T_{nk_n}$ commute for each $n \in \mathbb{N}$ we can use the same arguments as for complex numbers (see e.g. [12, p. 184]) to obtain

$$\lim_{n \to \infty} \prod_{1 \leq k \leq k_n} T_{nk} \lceil \mathscr{D}_c = \exp(R \lceil \mathscr{D}_c) . \tag{6.3}$$

If we apply the counit δ to both sides of (6.3) and evaluate at c this gives (6.2). □

As a consequence we have the following.

6.2. Proposition. *Let φ_n, $n \in \mathbb{N}$, be linear functionals on a coalgebra $\mathscr{C}$ such that*

$$\lim_{n \to \infty} n(\varphi_n - \delta)(c) = \psi(c)$$

for some linear functional ψ on $\mathscr{C}$ and for all $c \in \mathscr{C}$. Then

$$\lim_{n \to \infty} (\varphi_n)^{\star n}(c) = (\exp_\star \psi)(c)$$

for all $c \in \mathscr{C}$.

Proof. This is immediate from Lemma 6.1 if we put $k_n = n$ and $\varphi_{nk} = \varphi_n$. □

For an $\mathbb{N}$-graded coalgebra $\mathscr{C}$ and a complex number z we define the linear operator $s(z)$ on $\mathscr{C}$ by

$$s(z)c = z^{\varepsilon(c)} c ,$$

where $\varepsilon(c)$ denotes the degree of c. Then for linear functionals φ and ψ on $\mathscr{C}$

$$(\varphi \star \psi) \circ s(z) = (\varphi \circ s(z)) \star (\psi \circ s(z)) .$$

6.3. Theorem. *Let $\mathscr{C}$ be an $\mathbb{N}$-graded coalgebra and let $\kappa \in \mathbb{N}$. If a linear functional φ on $\mathscr{C}$ satisfies*
(i) $\varphi \lceil \mathscr{C}^{(l)} = 0$ *for* $0 < l < \kappa$,
(ii) $\varphi \lceil \mathscr{C}^{(0)} = \delta \lceil \mathscr{C}^{(0)}$,
then for all $c \in \mathscr{C}$,

$$\lim_{n \to \infty} (\varphi^{\star n} \circ s(n^{-\frac{1}{\kappa}}))(c) = (\exp_\star g_\varphi)(c) ,$$

where g_φ denotes the linear functional on $\mathscr{C}$ with

$$g_\varphi \lceil \mathscr{C}^{(l)} = 0 \quad \textit{for all} \quad l \neq k ,$$
$$g_\varphi \lceil \mathscr{C}^{(k)} = \varphi \lceil \mathscr{C}^{(\kappa)} .$$

Proof. This is a corollary to Proposition 6.2. For, if we put $\varphi_n = \varphi \circ s(n^{-\frac{1}{\kappa}})$ then for $c \in \mathscr{C}$ with $\varepsilon(c) \geq \kappa$

$$n(\varphi_n - \delta)(c) = n\varphi(c) = n \cdot n^{-\frac{\varepsilon(c)}{\kappa}} \varphi(c)$$

which is equal to $\varphi(c)$ if $\varepsilon(c) = \kappa$ and tends to 0 for $\varepsilon(c) > \kappa$. Moreover, by assumption $n(\varphi_n - \delta)(c) = 0$ for $\varepsilon(c) < \kappa$. □

For $\kappa = 1$ the above theorem is a law of large numbers and for $\kappa = 2$ it is a central limit theorem; cf. [15, 41] where the special case of a graded tensor algebra with "Boson and Fermion convolution" was treated.

7. A q-Central Limit Theorem

We now come to an application of Sects. 4 and 6. First we describe what will be the logarithm of the limiting functional of our central limit theorem. Let Q be a positive definite sesquilinear form on V which is also even, i.e. $Q(v,w) = 0$ unless $\mathrm{d}(v) = \mathrm{d}(w)$. We denote by g_Q the linear functional on $T(V)$ given by

$$g_Q(M) = \begin{cases} Q(v,w) & \text{if} \quad M = v^* w, v, w \in V \\ 0 & \text{otherwise} \end{cases} \tag{7.1}$$

for $M \in T(V)$ a monomial. Clearly, g_Q is even and conditionally positive. It follows from Theorem 2.7 that $\exp_\star g_Q$ is an even state. If $q \in \{\pm 1\}$ any sesquilinear form is even, but if $q \notin \{\pm 1\}$ we have $Q(v_k^*, v_l) = Q(v_k, v_l^*) = 0$ and Q is built up from the two positive definite matrices $\underline{\alpha} = (\alpha_{kl})_{k,l\in I}$, $\alpha_{kl} = Q(v_k^*, v_l)$ and $\underline{\beta} = (\beta_{kl})_{k,l\in I}$, $\beta_{kl} = Q(v_k, v_l^*)$. In this case, we write $g_Q = g_{\underline{\alpha},\underline{\beta}}$.

A linear functional φ on $T(V)$ is called centralized if $\varphi(v) = 0$ for all $v \in V$. (If $q \neq 1$ all even functionals are centralized.)

7.1. Theorem. *Let $q \in \mathbb{C}^*$ and let φ be an even, centralized state on $T(V)$. Then for both the left and the right q-$*$-bialgebra structure*

$$\lim_{n\to\infty} \varphi^{\star n}\left(b\left(\frac{v_k}{\sqrt{n}}\right)\right) = (\exp_\star g_Q)(b),$$

where Q is the even non-negative definite sesquilinear form with $Q(v,w) = \varphi(v^* w)$. If $q \notin \{\pm 1\}$ we have $g_Q = g_{\underline{\alpha},\underline{\beta}}$ with $\alpha_{kl} = \varphi(v_k v_l^*)$ and $\beta_{kl} = \varphi(v_k^* v_l)$.

Proof. Since φ is even we must have $\varphi(v_k^2) = \varphi((v_k^*)^2) = 0$ for all $k \in I$ in the case when $q \notin \{\pm 1\}$. But then everything follows from Theorem 6.3. $\square$

We also have a version of this central limit theorem for $T(V)_{\mathbb{N},0}$.

7.2. Theorem. *Let φ be a state on $T(V)_{\mathbb{N},0}$ such that*

$$\varphi(y) = \varphi(y^2) = 1. \tag{7.2}$$

Then

$$\lim_{n\to\infty} \varphi^{\star n}\left(b\left(\frac{v_k}{\sqrt{n}}, y\right)\right) = (\exp_\star g_{\underline{\alpha},0})(b),$$

for $b \in T(V)_{\mathbb{N},0}$, where $g_{\underline{\alpha},0}$ is again defined by (7.1) *and the matrix $\underline{\alpha}$ with $\alpha_{kl} = \varphi(v_k v_l^*)$ is positive definite.*

Proof. From (7.2) we obtain for $b \in T(V)_{\mathbb{N},0}$,

$$|\varphi((y - \mathbf{1})b)^2 \leq \varphi((y-\mathbf{1})^2)\varphi(b^* b) = 0$$

which gives

$$\varphi(y^{k_1} b y^{k_2}) = \varphi(b) \tag{7.3}$$

for all $k_1, k_2 \in \mathbb{N}$, $b \in T(V)$. If we set $\varepsilon(y) = 0$ we can extend the $\mathbb{N}$-graduation of $T(V)$ to an $\mathbb{N}$-graduation of $T(V)_{\mathbb{N},0}$. Then $T(V)_{\mathbb{N},0}$ is an $\mathbb{N}$-graded coalgebra, and we know from Theorem 6.3. That $\varphi^{\star n}\left(b\left(\frac{v_k}{\sqrt{n}}\right), y\right)$ converges to

$(\exp_\star g_\varphi)(b)$ for all $b \in T(V)_{\mathbb{N},0}$. But from (7.3) and from the relation $v_k y = 0$ in $T(V)_{\mathbb{N},0}$ we obtain

$$g_\varphi(y^{k_1} v_k y^{k_2} v_l y^{k_3}) = g_\varphi(y^{k_1} v_k^* y^{k_2} v_l y^{k_3}) = 0$$

for all $k_1, k_2, k_3 \in \mathbb{N}$ and $k, l \in I$. Finally,

$$g_\varphi(y^{k_1} v_k y^{k_2} v_l^* y^{k_3}) = g_\varphi(v_k y^{k_2} v_l^*)$$

which is 0 unless $k_2 = 0$. □

Using Theorem 4.1, we wish to identify our central limit $\exp_\star g_Q$ as the distribution of quantum Azéma noise and quantum q-Brownian motion. For simplicity, we assume that $\dim V_+ = 1$. Thus V is spanned by x and x^*, $x \in V_+ \backslash \{0\}$. We write $\mathbb{C}\langle x, x^* \rangle$ for $T(V)$. For $\alpha, \beta \in \mathbb{R}_+$ the linear functional $g_{\alpha,\beta}$ on $\mathbb{C}\langle x, x^* \rangle$ is even and conditionally positive. Non-diagonal covariance matrices Q are only admitted if $q \in \{\pm 1\}$. That is why we now restrict ourselves to diagonal Q. For simplicity, we also assume that $q \in \mathbb{R}\backslash\{0\}$ or $|q| = 1$ for the rest of the paper. For $\psi = g_{\alpha,\beta}$ the corresponding ψ_Γ is

$$\psi_\Gamma(M) = \begin{cases} q^{k_2}\alpha & \text{if } \ M = y^{k_1} x y^{k_2} x^* y^{k_3} \\ (\bar{q})^{-k_2}\beta & \text{if } \ M = y^{k_1} x^* y^{k_2} x k^{k_3} \\ 0 & \text{otherwise} \end{cases}$$

for M a monomial. If both α and β are non-zero we have $\dim D = 2$, and if $\alpha = 0$ or $\beta = 0$ but $\alpha + \beta \neq 0$ we have $\dim D = 1$. The 1-cocycle η_Γ is given by

$$\eta_\Gamma(M) = \begin{cases} q^{k_1} \begin{pmatrix} \sqrt{\alpha} \\ 0 \end{pmatrix} & \text{if } \ M = y^{k_1} x^* y^{k_2} \\ (\bar{q})^{-k_2} \begin{pmatrix} 0 \\ \sqrt{\beta} \end{pmatrix} & \text{if } \ M = y^{k_1} x y^{k_2} \\ 0 & \text{otherwise.} \end{cases}$$

For the representation ϱ_Γ we have $\varrho_\Gamma(x) = 0$ and

$$\varrho_\Gamma(y) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = q \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$
$$\varrho_\Gamma(y) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = (\bar{q})^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The equation for the corresponding q-white noise is

$$\check{B}_t = \int_0^t J_{0\tau}\left(\begin{pmatrix} q & 0 \\ 0 & (\bar{q})^{-1} \end{pmatrix}\right)\left(\mathrm{d}A_\tau^*\left(\begin{pmatrix} 0 \\ \sqrt{\beta} \end{pmatrix}\right) + \mathrm{d}A_\tau\left(\begin{pmatrix} \sqrt{\alpha} \\ 0 \end{pmatrix}\right)\right) \tag{7.4}$$

in the left case, and

$$B_t = L_t J_{t,\infty}\left(\begin{pmatrix} q & 0 \\ 0 & (\bar{q})^{-1} \end{pmatrix}\right),$$

$$L_t = \int_0^t L_\tau \mathrm{d}\Lambda_\tau\left(\begin{pmatrix} q-1 & 0 \\ 0 & (\bar{q})^{-1} - 1 \end{pmatrix}\right) + A_t^*\left(\begin{pmatrix} 0 \\ \sqrt{\beta} \end{pmatrix}\right) \tag{7.5}$$
$$+ A_t\left(\begin{pmatrix} \sqrt{\alpha} \\ 0 \end{pmatrix}\right)$$

in the right case. If $q = 0$ we have for $\psi = g_{\alpha,0}$ on $\mathbb{C}\langle x, x^*\rangle_{\mathbb{N},0}$ that $\dim D = 1$ and $\eta(M) = 0$ unless $M = x^* y^{k_2}$ and $\eta(x^* y^{k_2}) = \sqrt{\alpha}$. Moreover, $\varrho(x) = 0$ and $\varrho(y) = 0$. The equations for the corresponding white noise on $\mathbb{C}\langle x, x^*\rangle_{\mathbb{N},0}$ are

$$\check{B}_t = \sqrt{\alpha} \int_0^t J_{0\tau}(0) \mathrm{d}A_\tau \tag{7.6}$$

(left case) and

$$\begin{aligned} B_t &= L_t J_{t,\infty}(0), \\ L_t &= -\int_0^t L_\tau \mathrm{d}\Lambda_\tau + \sqrt{\alpha} A_t \end{aligned} \tag{7.7}$$

(right case). For $\alpha = 1$ and $\beta = 0$ we are back to Eqs. (1.1) and (1.2) of the introduction.

Before we proceed with the q-central limit theorem, we prove the following generalization of the canonical commutation ($q = 1$) and anti-commutation ($q = -1$) relations. These relations can be proved using quantum Ito's formula (for real q and $\alpha = 1$, $\beta = 0$, see [28, 24]). We here show that the relations follow from the $*$-bialgebra structure of $\mathbb{C}\langle x, x^*\rangle_{\Gamma,q}$.

7.3. Theorem. *The process $\check{B}_t$ given by* (7.4) *or* (7.6) *satisfies the commutation relations*

$$\check{B}_r \check{B}_t^* - q\check{B}_t^* \check{B}_t = (\alpha - q\beta) \int_0^t J_{0\tau}(J_{0\tau})^* \mathrm{d}\tau + (1 - q\bar{q}^{-1}) \int_0^t \check{B}_\tau \mathrm{d}\check{B}_\tau^* .$$

Proof. Using $yy^* = y^* y$, a short computation shows that

$$\begin{aligned} \Delta_\Gamma(xx^* - qx^* x) = (xx^* - qx^* x) \otimes \mathbf{1} + (yy^*) \otimes (xx^* - qx^* x) \\ + (\bar{q} - q)(y^* x) \otimes x^* . \end{aligned}$$

Moreover, $\eta(xx^* - qx^* x) = 0$, $\varrho(xx^* - qx^* x) = 0$ and $g_{\alpha,\beta}(xx^* - qx^* x) = \alpha - q\beta$, and $\varrho(x) = 0$, $\psi(x) = 0$, $\eta(x^*) = \begin{pmatrix} \sqrt{\alpha} \\ 0 \end{pmatrix}$ and $\eta(x) = \begin{pmatrix} 0 \\ \sqrt{\beta} \end{pmatrix}$ which proves the theorem. $\square$

Denote by Ω the vacuum state in $\mathscr{F}_H$. We have the

7.4. q-Central Limit Theorem. (a) *Let φ be an even, centralized state on $\mathbb{C}\langle x, x^*\rangle$. Then, with respect to the left q-$*$-bialgebra structure on $\mathbb{C}\langle x, x^*\rangle$,*

$$\lim_{n\to\infty} \varphi^{\star n}\left(b\left(\frac{v_k}{\sqrt{n}}\right)\right) = \langle \Omega, b(\check{L}_1)\Omega\rangle ,$$

where $\check{L}_1 = \check{B}_1$ is given by (7.4) *with $\beta = \varphi(x^* x)$ and $\alpha = \varphi(xx^*)$. With respect to the right q-$*$-bialgebra structure,*

$$\lim_{n\to\infty} \varphi^{\star n}\left(b\left(\frac{v_k}{\sqrt{n}}\right)\right) = \langle \Omega, b(L_1)\Omega\rangle ,$$

where L_1 is given by the solution of (7.5) *with $\beta = \varphi(x^* x)$ and $\alpha = \varphi(xx^*)$.*

(b) *Let φ be a centralized state on $\mathbb{C}\langle x, x^*\rangle_{\mathbb{N},0}$ satisfying* (7.2). *Then, in the left case,*

$$\lim_{n\to\infty} \varphi^{\star n}\left(b\left(\frac{v_k}{\sqrt{n}}, y\right)\right) = \langle \Omega, b(\check{L}_1, J_{01}(0))\Omega\rangle,$$

where $\check{L}_1$ is given by (7.6) *with $\alpha = \varphi(xx^*)$. In the right case,*

$$\lim_{n\to\infty} \varphi^{\star n}\left(b\left(\frac{v_k}{\sqrt{n}}, y\right)\right) = \langle \Omega, b(L_1, J_{01}(0))\Omega\rangle,$$

where L_1 is given by the solution of (7.7) *with $\alpha = \varphi(xx^*)$.*

Proof. Application of Theorems 7.1, 7.2, of the formulae (7.4)–(7.7) and of the fact that $\langle \Omega, b(B_1)\Omega\rangle = \langle \Omega, b(L_1)\Omega\rangle$. $\square$

The convolution product of states on $\mathbb{C}\langle x, x^*\rangle$ generalizes the convolution product of moment functionals of probability measures on $\mathbb{R}^2$ (see [33]), and the moment functional of the distribution of a sum of q-independent quantum random variables is the convolution product of the moment functionals of the distributions of the summands (see the beginning of Sect. 2). In this sense, quantum Azéma noise (L_t, L_t^*) and the q-interpolations $(\check{L}_t, \check{L}_t^*)$ appear as central limits of sums of q-independent, identically distributed quantum random variables.

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Note added in proof. Remark on Theorem 7.4. The moments are the same in the left and in the right case. However, the multi-time correlations of the processes differ. Theorems 7.1 and 7.2 yield q-analogues of the Donsker invariance principle and for these analogues the left and the right case must be distinguished.