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Geometry of Virasoro Constraints in Nonperturbative 2-*d* Quantum Gravity

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Abstract. The string equation and the Virasoro constraints for arbitrary hermitian multimatrix models are derived using the Lie–Bäcklund symmetries of the generalised KdV equations. From this point of view the origin of the string equation and the meaning of the Virasoro constraints are explained. Some speculation about the appearance of extra constraints, which we suspect to be the W-constraints, is also given.

1. Introduction

The mystery of integrable systems continually surprises us. Especially, the 2-*d* integrable systems are well studied and provide us with rich structures in both physics and mathematics. Also in many aspects of string Theory and conformal field theory we have seen the relevance of certain integrable systems.

On the other hand, the quantum theory of gravity has been a long standing unsolved problem. But, for the theories based on 2-*d* random surfaces like string theory the situation is somewhat better. In particular there has been some recent success in formulating a nonperturbative 2-*d* quantum gravity, using the dynamically triangulated random lattice model based on the matrix models in the double scaling limit, i.e. tuning the coupling constant to get higher topology contributions, while taking the large-N and continuum limit [1–3].

In these models perturbation theory is used with respect to the multicritical points determined by the so-called "string equation." In the one-matrix model case the general string equation even away from the critical points can be neatly expressed in the Korteweg–de Vries (KdV) hierarchy formulation [3–5]. Thus again we see the intricate appearance of 2-*d* integrable systems. In this paper we will show how one can recover the full structure of the one-matrix model starting from

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the KdV equation, using some symmetry transformation properties. Some of the preliminary results were presented before [6] and here we shall report the rest, including the generalization to the multimatrix model cases.

In the two-matrix model case we start with the Boussinesq equation, which generates the 3rd generalized KdV hierarchy again by the symmetry arguments. Douglas conjectured that the M-matrix model approach to nonperturbative 2-d gravity will be described by the $(M+1)^{\text{th}}$ generalized KdV hierarchy [7]. For other multimatrix models we first generalize the KdV equation and then generate the hierarchy by the symmetries; then we get the Virasoro constraints up to some coefficients, which can be determined once we know the explicit form of the recursion operator. Though the recursion operator is not known yet, we have the generic structure of the Virasoro constraints. We believe this can be generalized even to the $c = 1$ model [8] using the KP equation.

One of the lessons we have learned from the success of QFT's based on the Lagrangian formalism is that the investigation of symmetries leads to an understanding of the various deep structures of the theories. In QFT a symmetry is usually defined by a transformation which leaves the action invariant. But sometimes we also use the invariance of the equation of motion under some transformation, for example, the charge conjugation symmetry for the Dirac equation. But still the action itself is also invariant.

For integrable systems based on differential equations we need to use a different notion of symmetries. We define a symmetry of a differential equation as a group of transformations which leave the differential equation invariant, therefore mapping one solution to another solution[1]. Thus such symmetries can be understood as symmetries in the space of solutions to a given differential equation. Such a symmetry argument can be generalized to a set of differential equations.

In ref. [6] we used these symmetries in nonperturbative 2-d quantum gravity with respect to the KdV hierarchy and showed that the Virasoro constraints derived in refs. [10 and 11] (where they are derived from the loop equation [12–14]) are due to non-isospectral symmetries, an example of Lie–Bäcklund symmetries for KdV equation. The isospectral symmetries, i.e. a hierarchy of symmetries generated from the Lie-point symmetry, are related to the integrability in the sense that they are related to the Hamiltonian structure. Note that the symmetries of the KdV equation can be interpreted as symmetries of 2-d quantum gravity because certain solutions of the KdV equation provide the partition function of 2-d gravity.

From this point of view the meaning of the Virasoro constraints is clear. The string equation and the Virasoro constraints are nothing but the vanishing Lie–Bäcklund evolution equations, whose time variables can be interpreted as the moduli of some auxiliary infinite genus Riemann surfaces in the $(M+1)^{\text{th}}$ Grassmannian, i.e. $\text{Gr}^{(M+1)}$. Thus the spectral space of all the multimatrix models can be imbedded into the infinite Grassmannian [15–16]. Note that in fact points of $\text{Gr}^{(M+1)}$ correspond to solutions of the $(M+1)^{\text{th}}$ KdV hierarchy [17]; therefore we can imbed the space of solutions of 2-d nonperturbative quantum gravity into the Grassmannian. Naturally, the next question is whether there is any field

[1] Note that when we apply this to QFT, the action is not necessarily invariant under the transformation which leaves the relevant equation invariant. For example, see [9]

theoretical explanation of such vanishing evolution equations. We shall leave this as a subject for future study.

This paper is organized as follows. In Sect. 2 we shall review the main aspects of the Lie–Bäcklund symmetries. In Sect. 3 the one-matrix model using the KdV equation and in Sect. 4 the two-matrix model using the Boussinesq equation are described. Then in Sect. 5 we have generalized everything for the multimatrix models. Finally, in Sect. 6 some speculations concerning the W-constraints and the loop operators for the multimatrix model are given. Further discussions are also given.

2. Lie–Bäcklund Symmetries

Here we shall review some relevant structures of the Lie-point and theLie–Bäcklund symmetries; more details can be found in refs. [18–19]. Note that the Lie–Bäcklund transformation is not necessarily the same as the usual Bäcklund transformation.

Let us consider a system of partial differential equations

$$\mathscr{P}_n[t_\mu, w_a, \partial_\mu w_a, \partial_\mu \partial_\nu w_a, \ldots] = 0, \tag{2.1}$$

where $\mu, \nu = 1, 2, \ldots, N$, $a = 1, 2, \ldots, Z$, and $w_a = w_a(t_\mu), \partial_\mu$ denotes the partial differentiation with respect to t_μ. $n = 1, 2, \ldots, M, M$ is the number of equations. Let $Y^I \equiv \{t_\mu, w_a, \partial_\mu w_a, \partial_\mu \partial_\nu w_a, \ldots\}$ be a generalized coordinate of an infinite dimensional space so that we can simply write Eq. (2.1) as $\mathscr{P}_n[Y^I] = 0$.

Then the Lie–Bäcklund transformation is

$$\tilde{Y}^I = \tilde{Y}^I(Y^J; \varepsilon), \tag{2.2}$$

which is an extension to the derivatives of

$$\begin{aligned} \tilde{t}_\mu &= \tilde{t}_\mu(t_\nu, w_b, \partial_\nu w_b, \partial_\nu \partial_\lambda w_b, \ldots; \varepsilon), \\ \tilde{w}_a &= \tilde{w}_a(t_\nu, w_b, \partial_\nu w_b, \partial_\nu \partial_\lambda w_b, \ldots; \varepsilon), \end{aligned} \tag{2.3}$$

where ε is some deformation parameter such that at $\varepsilon = 0$, $\tilde{t}_\mu = t_\mu$, etc.. We can introduce such an ε if the transformations form a one-parameter transformation group.

Then the Lie–Bäcklund symmetry is defined by the statement:

$$\text{If } \mathscr{P}_n[Y^I] = 0, \quad \text{then} \quad \mathscr{P}_n[\tilde{Y}^I] = 0.$$

This is equivalent to saying that the Lie–Bäcklund transformation $\tilde{Y}^I$ is a symmetry of $\mathscr{P}_n[Y^I] = 0$ if and only if for $\mathscr{P}_n[Y^I] = 0$,

$$\mathbf{X}\mathscr{P}_n[Y^I] = 0, \tag{2.4}$$

where the Lie–Bäcklund operator $\mathbf{X}$ is

$$\mathbf{X} = \eta_\mu \partial_\mu + \xi_a \frac{\partial}{\partial w_a} + (\mathscr{D}_\mu \xi_a - (\partial_\nu w_a)\mathscr{D}_\mu \eta_\nu) \frac{\partial}{\partial(\partial_\mu w_a)} + \cdots \tag{2.5}$$

and

$$\mathscr{D}_\mu = \partial_\mu + \partial_\mu w_a \frac{\partial}{\partial w_a} + \partial_\mu \partial_\nu w_a \frac{\partial}{\partial(\partial_\nu w_a)} + \cdots. \tag{2.6}$$

The first differentiation ∂_μ in $\mathscr{D}_\mu$ acts with respect to t_μ in the generalized coordinate Y^I. Equation (2.4) is equivalent to the condition $\frac{d}{d\varepsilon}\mathscr{P}_n[\tilde{Y}]|_{\varepsilon=0}=0$. From now on we will call Eq. (2.4) symmetry condition. Sometimes to make sure the dependence of η and ξ for the Lie–Bäcklund operator we shall explicitly write as $\mathbf{X}_{\eta,\xi}$.

Note that Eq. (2.3) implies that η_μ and ξ_a are functions of all the variables,

$$\eta_\mu = \eta_\mu(t_\nu, w_b, \partial_\nu w_b, \partial_\nu \partial_\lambda w_b, \ldots),$$
$$\xi_a = \xi_a(t_\nu, w_b, \partial_\nu w_b, \partial_\nu \partial_\lambda w_b, \ldots), \tag{2.7}$$

and $\eta_\mu = 0$ can be chosen without loss of generality if $\mathscr{P}_n$ does not depend on t_μ explicitly. This will be the case for all the equations to be studied in this paper.

Thus Eq. (2.3) is a transformation between two solutions of $\mathscr{P}_n = 0$ and that we have an evolution equation for w_a as

$$\left.\frac{d\tilde{w}_a}{d\varepsilon}\right|_{\varepsilon=0} = \xi_a. \tag{2.8}$$

In general, as ε changes $(\tilde{t}_\mu, \tilde{w}_a)$ moves along the hyperspace of the solutions of $\mathscr{P}_n = 0$ in the space with a coordinate provided by the generic (t_μ, w_a). It turns out that the evolution equation, Eq. (2.8), has a very significant meaning. We shall come back to this point again later for our specific examples.

Once we have a symmetry, we can sometimes generate an infinite hierarchy of symmetries. Rewrite the symmetry condition Eq. (2.4) as an equation of operator $\mathscr{Q}_a$ acting on ξ_a such that

$$\mathbf{X}_\xi \mathscr{P} = \mathscr{Q}_a \xi_a = 0, \tag{2.9}$$

where we set $\eta = 0$ for convenience and $\mathscr{Q}_a$ can be properly derived from $\mathbf{X}_\xi \mathscr{P}$. If there is any operator $\mathscr{R}$ that commutes with $\mathscr{Q}$, i.e. $[\mathscr{R}, \mathscr{Q}_a]\xi_a = 0$, then $\mathscr{R}^n \xi_a$ is also a symmetry for any positive integer n. Such an operator $\mathscr{R}$ is called a *recursion operator.*

Now we shall define a special case of Lie–Bäcklund symmetry, which we call the Lie-point symmetry. The Lie-point symmetry is generated by the Lie-point transformation as

$$\tilde{Y}^I = \tilde{Y}^I(Y^J; \varepsilon), \tag{2.10}$$

which is an extension to the derivatives of

$$\tilde{t}_\mu = \tilde{t}_\mu(t_\nu, w_b; \varepsilon),$$
$$\tilde{w}_a = \tilde{w}_a(t_\nu, w_b; \varepsilon). \tag{2.11}$$

Note that Eq. (2.11) implies now that

$$\eta_\mu = \eta_\mu(t_\nu, w_b), \quad \xi_a = \xi_a(t_\nu, w_b). \tag{2.12}$$

In fact the Lie–Bäcklund symmetry has been discovered as a generalization of the Lie-point symmetry because somehow the Lie-point transformation does not generate any symmetry which involves t_μ for $\eta_\nu = 0$. This is due to the fact that from the evolution equation, Eq. (2.8), ξ_a will always contain $\partial_\nu w$ if $\tilde{t}_\mu$ depends on ε explicitly, but by definition ξ_α cannot contain $\partial_\nu w$ for the Lie-point transformation so that $\tilde{t}_\mu$ cannot depend on ε explicitly if $\eta_\nu = 0$. In many physically interesting cases $\eta_\nu = 0$ and we still have the space-time symmetries. Thus we need to generalize to accommodate such cases.

Indeed such a symmetry exists and even in some cases the transformations form a transformation group. This is the Lie–Bäcklund symmetry discovered by Ibragimov and Anderson [20]. For our purpose here we will restrict ourselves to the case when they form a one-parameter group. For more general Lie–Bäcklund transformations, see refs. [18, 19]. When we wrote down the Lie–Bäcklund operator **X**, Eq. (2.5), we already used this group property.

In many cases even though Eq. (2.3) is of the form of Eq. (2.11) up to the first order of ε, Eq. (2.4) leads to a Lie–Bäcklund symmetry completely different from the Lie-point symmetry due to Eq. (2.7).

3. KdV Equation: One-Matrix Model

3.1. Loop Operator. Let us start with the loop operator which connects the matrix model and the KdV hierarchy in the following sense. Surprisingly, the Schrödinger operator which defines the one dimensional quantum mechanical system,

$$\mathscr{H} = -D^2 + u(x), \quad D \equiv \frac{\partial}{\partial x}, \tag{3.1.1}$$

appears in many important cases related to Conformal Field Theories. Once again we are led to use this operator to study some of the very important structures of the nonperturbative 2-d quantum gravity, which recovers some results of the Liouville quantum gravity at the planar limit [21–23].

Based on the KdV hierarchy formulation of the one-matrix model approach, one can identify the expectation value of the loop operator ω_l of length l with the partition function of Hamiltonian $\mathscr{H}$ using l as an inverse temperature [4], i.e.

$$\langle \omega_l \rangle = \mathrm{tr}_x \, e^{-l\mathscr{H}}, \tag{3.1.2}$$

where tr_x denotes the trace taken only up to the Fermi level, which is related to the renormalized cosmological constant x.

This can be expanded for small l by the heat kernel method as

$$\langle \omega_l \rangle \sim \frac{I_0}{2\sqrt{\pi l}} + \sum_{k=1}^{\infty} \frac{(-1)^k l^{k-1/2} I_k}{2\sqrt{\pi}(2k-1)!!}, \quad I_k = \int_x^{\infty} d\tilde{x} \frac{\delta H_k}{\delta u}, \tag{3.1.3}$$

where $H_k = \int_{-\infty}^{\infty} d\tilde{x} h_k$, $(-1)!! \equiv 1$, and h_k's are polynomials of u and its derivatives. For example, $h_0 = u, I_0 = 1$. With $L \equiv -\frac{1}{2}D^3 + uD + Du$, the H_k's satisfy the following recursion relation:

$$D \frac{\delta H_k}{\delta u} = L \frac{\delta H_{k-1}}{\delta u}, \quad k \geqq 1. \tag{3.1.4}$$

Note that $\langle \omega_l \rangle$ now depends on the eigenvalues of $\mathscr{H}$ as well as the renormalized cosmological constant x. As $l \to 0$, $\langle \omega_l \rangle \to \infty$, i.e. in fact ω_l is a unnormalized loop operator. Thus, by regularizing the leading divergence term, we can define a renormalized loop operator, which is used to derive the continuum loop equation in ref. [23].

Under Laplace transform

$$\begin{aligned}\langle \omega(\zeta) \rangle &= \int_0^\infty dl e^{-\zeta l} \operatorname{tr} e^{-l\mathscr{H}} \\ &= \operatorname{tr} \frac{1}{\mathscr{H} + \zeta} \\ &= \frac{1}{2\sqrt{\zeta}} + \int_x^\infty d\tilde{x} \sum_{k=1}^\infty \frac{R_k[u(\tilde{x})]}{\zeta^{k+1/2}},\end{aligned} \tag{3.1.5}$$

where R_k's are the Gelfand–Dickii polynomials [24] and $R_k = (-2)^{-k}\delta H_k/\delta u$. Then the string equation for the general massive model interpolating between multicritical points is given by [4]

$$-x = \sum_{k=1}^\infty (-2)^k(2k+1)t_{2k+1}R_k[u], \tag{3.1.6}$$

where the m^{th} multicritical point is determined by $t_{2k+1} = 0$ $(k \neq m)$ and choosing t_{2m+1} such that $(-2)^m(2m+1)t_{2m+1}R_m = -u^{2m-2} + \cdots$. We can notice that from Eq. (3.1.4)

$$\frac{\partial u}{\partial t_{2k+1}} = V_k u \equiv D\frac{\delta H_{k+1}}{\delta u} = (-2)^{k+1}DR_{k+1}, \quad k \geqq 0, \tag{3.1.7}$$

form the KdV hierarchy.

Now from Eq. (3.1.7) $k = 0$ implies $\left(\frac{\partial}{\partial t_1} - \frac{\partial}{\partial x}\right)u = 0$, which can be interpreted as a chirality condition in the (x, t_1) plane. Thus from now on we can safely identify $t_1 = x$, trading with this equation.

3.2. KdV *Hierarchy and the Symmetries.* The KdV hierarchy has a well known set of isospectral symmetries which are related to the integrability of the system. Let $\mathscr{U} = \{u(x, \vec{t})\}$ be the space of all real differentiable functions u deformed from $u(x)$ in Eq. (3.1.1) by extra variables $\vec{t}$ such that the eigenvalues of the Schrödinger operator, Eq. (3.1.1), are the same, i.e. isospectral.

Let us see how this can happen [25]. Let $\mathscr{H}(0) \equiv \mathscr{H}[u(x)]$ and $\mathscr{H}(t_{2k+1}) \equiv \mathscr{H}[u(x, t_{2k+1})]$, $k \geqq 0$, such that there exists a unitary transformation, which leaves the eigenvalues of $\mathscr{H}$ unchanged,

$$U^\dagger(t_{2k+1})\mathscr{H}(t_{2k+1})U(t_{2k+1}) = \mathscr{H}(0), \tag{3.2.1}$$

where $U(t_{2k+1})U^\dagger(t_{2k+1}) = U^\dagger(t_{2k+1})U(t_{2k+1}) = 1$. Taking a derivative with respect to t_{2k+1}, we get

$$\frac{\partial \mathscr{H}(t_{2k+1})}{\partial t_{2k+1}} = [P(t_{2k+1}), \mathscr{H}(t_{2k+1})], \tag{3.2.2}$$

where

$$P(t_{2k+1}) \equiv \frac{\partial U(t_{2k+1})}{\partial t_{2k+1}} U^{\dagger}(t_{2k+1}).$$

If there is any solution of this equation, then $\mathscr{U}$ is nontrivial and we have isospectral symmetries.

Indeed $P(t_{2k+1})$ can be found and particularly for the KdV hierarchy it is known as

$$P(t_{2k+1}) = [\mathscr{H}(t_{2k+1})]_{+}^{(2k+1)/2}, \tag{3.2.3}$$

where the subscript "+" denotes to take only nonnegative powers of differentiation, i.e. differential operator part, and P and $\mathscr{H}$ are known as the Lax pairs of the KdV hierarchy. Thus indeed $\mathscr{U}$ is not trivial.

The integrability can be shown by defining a Poisson structure on $\mathscr{U}$ such that the H_k's in Eq. (3.1.3) commute each other under such a Poisson bracket. V_k's in Eq. (3.1.7) form commuting vector fields on $\mathscr{U}$, which generate Hamiltonian vector flows and the parameters are denoted by t_{2k+1}.[2]

Note that $\langle \omega_l \rangle$ is a solution of the loop equation and is a function of u and its derivatives, while u is a solution of the KdV hierarchy. Thus the symmetry of the KdV hierarchy still survives as a symmetry of 2-d quantum gravity in the sense that the solution of the KdV equation provides a solution of the loop equation. In particular, for a given u we always have a partition function of 2-d quantum gravity in terms of the τ-function. In this sense we have the isospectral symmetries, $u \to \tilde{u} = u + \varepsilon \frac{\partial u}{\partial t_k}$, where u and $\tilde{u}$ are both solutions of the KdV equation.

We can also find other symmetries which change the eigenvalues but still solves the same KdV hierarchy. Such symmetries are known as nonisospectral symmetries and are first found in the context of the KdV equation [27]. Similar symmetries are also found in the Kadomtsev–Petviashvili (KP) equation [28] and in KP hierarchy [29].

First, let us recapture the case of the KdV equation [27]. From Eq. (3.1.7) we get the usual KdV equation

$$\dot{u} = 3uu' - \tfrac{1}{2}u''', \tag{3.2.4}$$

where $\dot{} \equiv \partial/\partial t, ' \equiv D$. By introducing $w' = u$ to get a nontrivial Lie-point symmetry we can rewrite the above as

$$\dot{w} - \tfrac{3}{2}w'^2 + \tfrac{1}{2}w''' = 0. \tag{3.2.5}$$

Whenever we get a solution of Eq. (3.2.5), by taking a derivative with respect to x we always get a solution of Eq. (3.2.4). In this sense we can always find a symmetry of Eq. (3.2.4) from a symmetry of Eq. (3.2.5).

The Lie–Bäcklund tangent operator is now

$$\mathbf{X}_f = f\frac{\partial}{\partial w} + \mathscr{D}_t f\frac{\partial}{\partial \dot{w}} + \mathscr{D}_x f\frac{\partial}{\partial w'} + \cdots, \tag{3.2.6}$$

[2] The integrability from the lattice model point of view has been studied in refs. [26]

where $f = f(x, t, w, w', \ldots)$ and for the notational convenience we use f in place of ξ in Eq. (2.5). Then $\mathbf{X}_f$ acting on Eq. (3.2.5) gives a symmetry condition, Eq. (2.9),

$$\mathcal{Q}f \equiv (\mathcal{D}_t - 3w'\mathcal{D}_x + \tfrac{1}{2}\mathcal{D}_x^3)f = 0. \tag{3.2.7}$$

The solutions of this equation determined the symmetries of the KdV equation.

Note that a hierarchy of the isospectral symmetries can be generated by a simple translation of w without any coordinate change as

$$w \to \tilde{w} = w + \varepsilon \tag{3.2.8}$$

so that

$$f = \left.\frac{d\tilde{w}}{d\varepsilon}\right|_{\varepsilon=0} = 1. \tag{3.2.9}$$

This is in fact a Lie-point symmetry. One may understand now why we use Eq. (3.2.5) instead of Eq. (3.2.4) because the corresponding symmetry of Eq. (3.2.5) is trivially $u \to \tilde{u} = u$.

Following the general strategy discussed after Eq. (2.9) we now generate the higher order symmetries

$$\frac{\partial w}{\partial t_{2k+1}} = \mathcal{R}^{k+1}\cdot 1 = \mathcal{R}^{k-1}\frac{\partial w}{\partial t_3}, \quad k \geqq 0, \tag{3.2.10}$$

where $\mathcal{R} \equiv D^{-1}L$ is the recursion operator for the KdV equation such that $[\mathcal{Q}, \mathcal{R}]f = 0$ and $D^{-1} \equiv \int dx$. The symmetries of Eq. (3.2.4) can be derived using $\frac{\partial u}{\partial t_{2k+1}} = \frac{\partial w'}{\partial t_{2k+1}}$. Note that this is nothing but the KdV hierarchy. Thus we have observed that the existence of the KdV hierarchy is due to the existence of an infinite hierarchy of symmetries for the KdV equation, i.e. the isospectral symmetries. We want to direct the reader's attention to the fact that, though Eq. (3.2.9) is a Lie-point symmetry, the higher order symmetries Eq. (3.2.10) are Lie–Bäcklund symmetries. Note that each higher order symmetry tells us $w \to \tilde{w} = w + \varepsilon\frac{\partial w}{\partial t_{2k+1}}$ and $\tilde{w}$ is also a solution of the KdV equation.

Equation (3.2.10) does not exhaust all possible symmetries of Eq. (3.2.5). We can also solve Eq. (3.2.7) using a Galilean transformation for small ε,

$$\begin{aligned} x &\to \tilde{x} = x + at\varepsilon, \\ w &\to \tilde{w} = w + x\varepsilon + \mathcal{O}(\varepsilon^2), \end{aligned} \tag{3.2.11}$$

where a is some constant. Then we can compute the change as an evolution equation for w with respect to "time" ε as in Eq. (2.8), $f = \left.\frac{d\tilde{w}}{d\varepsilon}\right|_{\varepsilon=0}$, where $\tilde{w}(\varepsilon) = \tilde{w}(0) + \varepsilon\left.\frac{d\tilde{w}}{d\varepsilon}\right|_{\varepsilon=0}$. Claiming that f should satisfy Eq. (3.2.7) to be a symmetry of the KdV equation, we can determine $a = 3$ so that we have a Lie–Bäcklund symmetry[2]

$$f = 3tw' + x = 3tu + x. \tag{3.2.12}$$

Then we can define the corresponding symmetry as an evolution equation for w by introducing a time β_{-1}

$$\frac{\partial w}{\partial \beta_{-1}} = f = \left.\frac{d\tilde{w}}{d\varepsilon}\right|_{\varepsilon=0}. \tag{3.2.13}$$

In general, we get for the KdV equation a family of evolution equations which can be derived by applying the recursion operator from Eq. (3.2.13) [27],

$$\frac{\partial w}{\partial \beta_n} = (D^{-1}L)^{n+1} f, \quad n \geqq -1. \tag{3.2.14}$$

Note that, since these symmetries depend on x as well as t, there is no reason that the eigenvalues of $\mathscr{H}$ are preserved. In fact there is no unitary transformation of Eq. (3.2.1) in this case. This explains why we call them nonisospectral symmetries.

We now have two hierarchies of symmetries of the KdV equation, those parametrized by t_{2k+1} and β_n. Now let us look for the Lie–Bäcklund symmetries of the full KdV hierarchy. Here we shall identify such symmetries of the KdV hierarchy and show that they are related to nothing but the Virasoro constraints in refs. [10, 11]. Thus we can understand what is the symmetry principle underlying the Virasoro constraints.

Our strategy is again to begin with a simplest symmetry, then applying the recursion operator to get the whole symmetries. Since now in general all the t_{2k+1} will be nonzero, our initial Galilean type ansatz has to be more complicated. For the KdV hierarchy, Eq. (3.2.10), our generalized Galilean transformation is

$$\begin{aligned} &w \to \tilde{w} = w + t_1 \varepsilon + \mathcal{O}(\varepsilon^2), \quad t_1 \equiv x, \quad t_3 \equiv t, \\ &t_{2k+1} \to \tilde{t}_{2k+1} = t_{2k+1} + a_{k+1} t_{2(k+1)+1} \varepsilon + \mathcal{O}(\varepsilon^2), \quad k \geqq 0. \end{aligned} \tag{3.2.15}$$

Now a set of symmetry conditions for all the k's arises from the action of (3.2.6) on the KdV hierarchy:

$$\left(\mathscr{D}_x \mathscr{D}_{t_{2k+1}} + \tfrac{1}{2}\mathscr{D}_x^3 \mathscr{D}_{t_{2k-1}} - 2w' \mathscr{D}_x \mathscr{D}_{t_{2k-1}} - 2\frac{\partial w'}{\partial t_{2k-1}} \mathscr{D}_x - w'' \mathscr{D}_{t_{2k-1}} - \frac{\partial w}{\partial t_{2k-1}} \mathscr{D}_x^2\right) F = 0. \tag{3.2.16}$$

The simplest solution is a generalization of Eq. (3.2.12):

$$\begin{aligned} F = \left.\frac{d\tilde{w}}{d\varepsilon}\right|_{\varepsilon=0} &= \left.\frac{\partial \tilde{w}}{\partial \varepsilon}\right|_{\varepsilon=0} + \sum_{k=0}^{\infty} \left.\frac{\partial \tilde{t}_{2k+1}}{\partial \varepsilon} \frac{\partial \tilde{w}}{\partial \tilde{t}_{2k+1}}\right|_{\varepsilon=0} \\ &= t_1 + \sum_{k=1}^{\infty} a_k t_{2k+1} (D^{-1}L)^{k-1} u, \end{aligned} \tag{3.2.17}$$

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[3] Note that $f = 0$ does not produce the Painlevé equation, though there is a Bäcklund transformation between the KdV equation and the Painlevé equation. But our Lie–Bäcklund transformation here is not the Bäcklund transformation between the KdV equation and the Painlevé equation, which one can find, for example, in ref. [30]. Later we shall get the Painlevé equation from the evolution equation, but only after imposing the $m = 2$ multicritically condition, which sets $t = 0$

where $a_k = 2k + 1$. Exactly as before we now get a new hierarchy of symmetries as evolution equations

$$\frac{\partial w}{\partial \beta_n} = (D^{-1}L)^{n+1}F, \tag{3.2.18a}$$

or

$$\frac{\partial u}{\partial \beta_n} = (LD^{-1})^n LF. \tag{3.2.18b}$$

3.3. String Equation and Virasoro Constraints. We can make a remarkable observation. Amazingly the string equation, Eq. (3.1.6), turns out to be just the constraint

$$\frac{\partial w}{\partial \beta_{-1}} = 0: \quad \text{Massive String Equation.}$$

Note that $(LD^{-1})^{-1} = DL^{-1}$. Similarly, with $u = -2D^2 \ln \tau$, where τ is the τ-function [16], requiring $\partial u/\partial \beta_n = 0$, we get the Virasoro constraints

$$L_n \tau = 0, \quad n \geqq -1, \tag{3.3.1}$$

where

$$L_{-1} = \sum_{k=1}^{\infty} \left(k + \frac{1}{2}\right) t_{2k+1} \frac{\partial}{\partial t_{2k-1}} - \frac{1}{8} t_1^2,$$

$$L_0 = \sum_{k=0}^{\infty} \left(k + \frac{1}{2}\right) t_{2k+1} \frac{\partial}{\partial t_{2k+1}} + \frac{1}{16},$$

$$L_n = \sum_{k=0}^{\infty} \left(k + \frac{1}{2}\right) t_{2k+1} \frac{\partial}{\partial t_{2(k+n)+1}} - \frac{1}{2} \sum_{k=1}^{n} \frac{\partial^2}{\partial t_{2k-1} \partial t_{2(n-k)+1}}, \quad n \geqq 1. \tag{3.3.2}$$

In ref. [10] such a τ-function is identified as the square root of the partition function of 2-*d* quantum gravity. The relevance of τ-function in 2-*d* gravity is considered in ref. [31], too.

Thus we have found both a simple interpretation of the Virasoro constraints and a connection to the massive string equation: they both amount to saying that certain symmetry generating vector fields, i.e. the evolution equations, vanish.

Clearly from Eq. (3.2.18b) $L_{-1}\tau = 0$ implies $L_n \tau = 0$, $n \geqq 0$, because

$$\frac{\partial u}{\partial \beta_n} = (LD^{-1})^{n+1} \frac{\partial u}{\partial \beta_{-1}}, \tag{3.3.3}$$

which recovers the recursion relation in ref. [10] derived from the loop equation, which in turn is derived using the KdV hierarchy. Now the nonisospectral symmetry clarifies the existence of such a recursion relation, which is due to the recursion operator.

Note that the existence of Eq. (3.2.18a) implies the nonisospectral symmetries. Moreover the eigenvalues are now parametrized in terms of β_n's so that, as w, or equivalently u, changes along with β_n, in general the eigenvalues also vary. Since none of the nonisospectral symmetries leave the eigenvalues invariant, by observing the change of the eigenvalues we can measure how u moves in $\mathscr{U} = \{u\}$. In this sense we can regard β_n's as intrinsic coordinates of $\mathscr{U}$. The Virasoro constraints, or $\frac{\partial w}{\partial \beta_n} = 0$, are conditions for some hyperspace independent of β_n in $\mathscr{U}$ so that we can interpret these constraints as saying that w does not depend on the change of the eigenvalues of Schrödinger operator, Eq. (3.1.1). As a result, $\langle \omega_l \rangle$ does not depend on the change of the eigenvalues, either.

In the Krichever's algebraic geometrical construction of soliton solutions [17] a set of eigenvalues determines branch cuts of surface and identifies a hyperelliptic Riemann surface in $\mathrm{Gr}^{(2)}$. The change of eigenvalues corresponds to the change of the complex structures after imposing $SL(2, \mathbf{C})$ invariance on a complex plane where all the eigenvalues, which are real and positive definite, are imbedded. Since L_{-1}, L_0, L_1 form $SL(2, \mathbf{C})$, we can interpret $\beta_n, n \geqq 2$, as moduli of hyperelliptic Riemann surfaces[4] which in this case are infinite genus and the corresponding Virasoro constraints as the conditions for the independence of such moduli. Since $\mathscr{U}$ can be imbedded into $\mathrm{Gr}^{(2)}$, the interpretation of some intrinsic coordinates of $\mathscr{U}$ as moduli of the hyperelliptic Riemann surfaces is perfectly legitimate.

It would be very important to have some field theoretical understanding of this structure.

3.4. Symmetry Algebra. Note that V_k's in Eq. (3.1.7) form a commuting isospectral symmetry algebra and that, together with L_n's, form the following algebra:

$$\begin{aligned} &[L_n, L_m] = (n-m)L_{n+m}, \quad m, n \geqq -1, \\ &[V_p, L_n] = (p+\tfrac{1}{2})V_{p+n}, \quad p \geqq 0, \quad p+n \geqq 0, \\ &[V_0, L_{-1}] = -\tfrac{1}{4}t_0, \\ &[V_p, V_q] = 0, \quad p, q \geqq 0. \end{aligned} \tag{3.4.1}$$

These can be derived easily from the definitions Eqs. (3.1.7) (3.3.2), assuming the last commuting algebra. The proof for the last commuting algebra can be found in ref. [32].

We would like to call the reader's attention to the fact that the above algebra is not closed. Either without V_0 and L_{-1} or just without L_{-1} the above becomes a closed algebra. Thus there are four different closed subalgebras of (3.4.1), which include isospectral and/or nonisospectral symmetry algebra. While L_n's act on the τ-function and annihilate it, V_p's do not annihilate the τ-function.

4. Boussinesq Equation: Two Matrix Model

Naturally, we should expect that a similar analysis can be applied to other generalized KdV equations. Note that in fact we never need to start with the full hierarchy because the hierarchy itself is just due to the existence of a family of symmetries for the generalized KdV equation like as shown in the KdV equation case. We shall follow this strategy here too.

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[4] Note that these Riemann surfaces are not the two-dimensional random surfaces for the 2-*d* gravity. These are just some auxiliary spaces

4.1. Loop Operators. To construct the loop operators for the multimatrix model is not so simple as for the one-matrix model. We do not have any obvious eigenvalue problems to define the expectation values of the loop operators. Fortunately, even though we do not have a Hamiltonian operator which corresponds to any well known physical system like in the one-matrix model case, we shall have a well-defined formal eigenvalue problem. Let $\mathscr{H}_{M+1} = -D^{M+1} + uD^{M-1} + \cdots$ be a differentail operator such that

$$\mathscr{H}_{M+1}\psi = \lambda\psi, \tag{4.1.1}$$

where ψ is the Baker function. From the KP hierarchy's point of view, this is legitimate [16].

But the M-matrix model has M different loop operators so that naively computing the partition function of $\mathscr{H}_{M+1}$ does not count the expectation values of all loop operators in an obvious way. We need more structures, but there is no clear answer. Some speculation to construct the loop operators will be given in the final discussion.

4.2. Boussinesq Hierarchy and the Symmetries. Let us start with the Boussinesq equation

$$\ddot{u} + \tfrac{1}{2}u'''' - \tfrac{3}{2}(u^2)'' = 0, \tag{4.2.1}$$

which is a generalization of the KdV equation in the sense that

$$\partial_t^M u + D^M(\tfrac{1}{2}u'' - \tfrac{3}{2}u^2) = 0 \tag{4.2.2}$$

leads to the KdV equation for $M = 1$ and to the Boussinesq equation for $M = 2$. As before we prefer to use w such that $u = w'$ to get a nontrivial Lie-point symmetry. Then for w Eq. (4.2.1) becomes

$$\mathscr{E}_3[w] \equiv \ddot{w} + \tfrac{1}{2}w'''' - 3w'w'' = 0. \tag{4.2.3}$$

Somehow we could not find the recursion operator for Eq. (4.2.3) directly due to the second order time derivative. So we need a first order form. Indeed we can equivalently write Eq. (4.2.3) by introducing v as

$$\mathscr{P}_1[w, v] \equiv \dot{w} - v' = 0,$$

$$\mathscr{P}_2[w, v] = \dot{v} + \tfrac{1}{2}w'''' - \tfrac{3}{2}w'^2 = 0. \tag{4.2.4}$$

We now need symmetry conditions. Using the Lie–Bäcklund operator

$$\mathbf{X}_\xi = \xi_1\frac{\partial}{\partial w} + \mathscr{D}_t\xi_1\frac{\partial}{\partial \dot{w}} + \mathscr{D}_x\xi_1\frac{\partial}{\partial w'} + \cdots + \xi_2\frac{\partial}{\partial v} + \mathscr{D}_t\xi_2\frac{\partial}{\partial \dot{v}} + \mathscr{D}_x\xi_2\frac{\partial}{\partial v'} + \cdots,$$

we get

$$\mathbf{X}_\xi\mathscr{P}_1 = \mathscr{D}_t\xi_1 - \mathscr{D}_x\xi_2 = 0, \tag{4.2.5a}$$

$$\mathbf{X}_\xi\mathscr{P}_2 = \mathscr{D}_t\xi_2 + \tfrac{1}{2}\mathscr{D}_x^3\xi_1 - 3w'\mathscr{D}_x\xi_1 = 0, \tag{4.2.5b}$$

which can be rewritten in a matrix form

$$\hat{\mathscr{Q}}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} \equiv \begin{pmatrix}\mathscr{D}_t & -\mathscr{D}_x\\ \tfrac{1}{2}\mathscr{D}_x^3 - 3w'\mathscr{D}_x & \mathscr{D}_t\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} = 0. \tag{4.2.6}$$

Note that $\mathscr{D}_t\xi(Y^I) = \partial_t\xi(t,x),\ldots$ etc. so that in the (t,x) coordinate

$$\hat{\mathscr{Q}} = \begin{pmatrix} \partial_t & -D \\ \frac{1}{2}D^3 - 3w'D & \partial_t \end{pmatrix}. \tag{4.2.7}$$

For the practical computations we in fact use this representation.

The simplest solutions to Eq. (4.2.6) are three Lie-point symmetries

$$\xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} t \\ x \end{pmatrix}, \tag{4.2.8}$$

which can be used as seeds for the hierarchy of symmetries. The first two constant solutions are the seeds for the Boussinesq hierarchy and the last one is the simplest solution for the other Lie–Bäcklund symmetries.

Now we need a recursion operator. After some work we get

$$\hat{\mathscr{R}} = D^{-1}\hat{\mathscr{L}}, \tag{4.2.9}$$

where

$$\hat{\mathscr{L}} \equiv \begin{pmatrix} \frac{1}{2}v'D + Dv' & L_{(1)} \\ L_{(2)} & v'D + \frac{1}{2}Dv' \end{pmatrix}, \tag{4.2.10}$$

$$L_{(1)} \equiv -\tfrac{2}{3}D^3 + \tfrac{1}{2}(uD + Du),$$

$$L_{(2)} = \tfrac{1}{3}D^5 - \tfrac{5}{4}(uD^3 + D^3u) + \tfrac{3}{2}(u^2D + Du^2) + \tfrac{3}{4}(u''D + Du''), \tag{4.2.11}$$

and $[\hat{\mathscr{Q}}, \hat{\mathscr{R}}]\xi = 0$ so that we have a hierarchy of symmetries defined by $\hat{\mathscr{R}}^n\xi$ for any nonnegative integer n. Due to the existence of two constant ξ's we have two sets of hierarchies, which all together consist of the Boussinesq hierarchy. Thus we can introduce two sets of time parameters, or flow coordinates, for example, $t_{3n+1}, t_{3n+2}, n \geqq 0$, for the evolution equations.

Let $W \equiv \begin{pmatrix} w \\ v \end{pmatrix}$ and

$$\frac{\partial W}{\partial t_{3n+1}} \equiv \begin{pmatrix} \dfrac{\partial w}{\partial t_{3n+1}} \\ \dfrac{\partial v}{\partial t_{3n+1}} \end{pmatrix} = 2\hat{\mathscr{R}}^{n+1}\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad n \geqq 0, \tag{4.2.12}$$

$$\frac{\partial W}{\partial t_{3n+2}} = \begin{pmatrix} \dfrac{\partial w}{\partial t_{3n+2}} \\ \dfrac{\partial v}{\partial t_{3n+2}} \end{pmatrix} = \hat{\mathscr{R}}^{n+1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad n \geqq 0. \tag{4.2.13}$$

Note that the $n = 0$ cases lead to the identification $t_1 \equiv x, t_2 \equiv t$. Equation (4.2.13) for $n = 0$ is nothing but the original Boussinesq equation.

Now we have the Boussinesq hierarchy

$$\frac{\partial W'}{\partial t_{3n+1}} \equiv \begin{pmatrix} \dfrac{\partial u}{\partial t_{3n+1}} \\ \dfrac{\partial \dot{w}}{\partial t_{3n+1}} \end{pmatrix} = 2(\hat{\mathscr{L}}D^{-1})^n\hat{\mathscr{L}}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = (\hat{\mathscr{L}}D^{-1})^n \begin{pmatrix} \dfrac{\partial u}{\partial t_1} \\ \dfrac{\partial \dot{w}}{\partial t_1} \end{pmatrix}, \quad n \geqq 0, \tag{4.2.14}$$

$$\frac{\partial W'}{\partial t_{3n+2}} = \begin{pmatrix} \dfrac{\partial u}{\partial t_{3n+2}} \\ \dfrac{\partial \dot{w}}{\partial t_{3n+2}} \end{pmatrix} = (\hat{\mathscr{L}} D^{-1})^n \hat{\mathscr{L}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (\hat{\mathscr{L}} D^{-1})^n \begin{pmatrix} \dfrac{\partial u}{\partial t_2} \\ \dfrac{\partial \dot{w}}{\partial t_2} \end{pmatrix}, \quad n \geqq 0. \tag{4.2.15}$$

Like in the KdV equation case, the Boussinesq hierarchy is also recovered as a result of the Lie–Bäcklund symmetries of the Boussinesq equation.

For such Boussinesq hierarchy with the following generalized Galilean transformation we can generate the Lie–Bäcklund symmetries:

$$\begin{aligned} v \to \tilde{v} &= v + t_1 \varepsilon + \mathcal{O}(\varepsilon^2), \quad t_1 \equiv x, \\ w \to \tilde{w} &= w + t_2 \varepsilon + \mathcal{O}(\varepsilon^2), \quad t_2 \equiv t, \\ t_{3k+1} \to \tilde{t}_{3k+1} &= t_{3k+1} + a_{k+1} t_{3(k+1)+1} \varepsilon + \mathcal{O}(\varepsilon^2), \quad k \geqq 0, \\ t_{3k+2} \to \tilde{t}_{3k+2} &= t_{3k+2} + b_{k+1} t_{3(k+1)+2} \varepsilon + \mathcal{O}(\varepsilon^2), \quad k \geqq 0. \end{aligned} \tag{4.2.16}$$

Similar equations like Eq. (2.8) lead to the solutions of the symmetry conditions, which are Eq. (4.2.6) and its higher order generalizations arising from the action of the Lie–Bäcklund operator on the Boussinesq hierarchy, Eqs. (4.2.14), (4.2.15). Thus we have

$$\xi_1 = \left.\frac{d\tilde{w}}{d\varepsilon}\right|_{\varepsilon=0} = t_2 + \sum_{k=0}^{\infty} a_{k+1} t_{3(k+1)+1} \frac{\partial w}{\partial t_{3k+1}} + \sum_{k=0}^{\infty} b_{k+1} t_{3(k+1)+2} \frac{\partial w}{\partial t_{3k+2}}, \tag{4.2.17a}$$

$$\xi_2 = \left.\frac{d\tilde{v}}{d\varepsilon}\right|_{\varepsilon=0} = t_1 + \sum_{k=0}^{\infty} a_{k+1} t_{3(k+1)+1} \frac{\partial v}{\partial t_{3k+1}} + \sum_{k=0}^{\infty} b_{k+1} t_{3(k+1)+2} \frac{\partial v}{\partial t_{3k+2}}, \tag{4.2.17b}$$

where

$$a_k = \frac{3k+1}{2}, \quad b_k = \frac{3k+2}{2}. \tag{4.2.18}$$

4.3. String Equation and Virasoro Constraints. If we set $\xi_2 = 0$, we derive the massive string equation for the two-matrix model. $\xi_1 = 0$ is equivalent to this because of Eq. (4.2.6). Thus the massive string equation for the two-matrix model is

$$\frac{\partial W}{\partial \beta_{-1}} \equiv \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0.$$

Now we can recover the Ising model coupled to 2-*d* gravity easily as $m = 3$ multicritical point, i.e. $t_7 = -\frac{8}{189}$, others $= 0$, so that the string equation reduces to $x = \frac{4}{27} \frac{\partial v}{\partial t_4} = \frac{4}{27} \left(\frac{1}{3} u'''' - 3uu'' - \frac{3}{2} u'^2 + 2u^3 + \dot{w}^2 \right)$. At $t_2 = 0$, $\dot{w} = 0$ so that by rescaling $u \to \frac{3}{2} u$ we recover the results in ref. [33]. Other multicritical points can be also easily derived in a similar way from Eq. (4.2.17b), since we know the explicit form of the recursion operator.

As before the Lie–Bäcklund symmetries will generate other Virasoro constraints, but the string equation is good enough to determine the 2-*d* gravity partition function at given multicritical points modulo the boundary conditions.

We can get the other Virasoro constraints applying the recursion operator to the massive string equation $\frac{\partial W}{\partial \beta_{-1}}$ to have a hierarchy of evolution equations such that

$$\frac{\partial W}{\partial \beta_n} = \hat{\mathscr{R}}^{n+1} \frac{\partial W}{\partial \beta_{-1}}, \quad n \geqq -1. \tag{4.3.1}$$

Since $\frac{\partial W}{\partial \beta_{-1}} = 0$ is the string equation so that $\frac{\partial W}{\partial \beta_n} = 0$ follows from this due to the above recursion relation. With $w = -2D \ln \tau$, $\frac{\partial W}{\partial \beta_n} = 0$ lead to the Virasoro constraints such that

$$L_n \tau = 0, \quad n \geqq -1, \tag{4.3.2}$$

where

$$L_{-1} = \sum_{k=0}^{\infty} \left(k + \frac{4}{3}\right) t_{3k+4} \frac{\partial}{\partial t_{3k+1}} + \sum_{k=0}^{\infty} \left(k + \frac{5}{3}\right) t_{3k+5} \frac{\partial}{\partial t_{3k+2}} - \frac{1}{3} t_1 t_2,$$

$$L_0 = \sum_{k=0}^{\infty} \left(k + \frac{1}{3}\right) t_{3k+1} \frac{\partial}{\partial t_{3k+1}} + \sum_{k=0}^{\infty} \left(k + \frac{2}{3}\right) t_{3k+2} \frac{\partial}{\partial t_{3k+2}} + \frac{1}{9},$$

$$L_n = \sum_{k=0}^{\infty} \left(k + \frac{1}{3}\right) t_{3k+1} \frac{\partial}{\partial t_{3(k+n)+1}} + \sum_{k=0}^{\infty} \left(k + \frac{2}{3}\right) t_{3k+2} \frac{\partial}{\partial t_{3(k+n)+2}}$$
$$- \frac{1}{6} \sum_{k=0}^{n-1} \frac{\partial^2}{\partial t_{3k+1} \partial t_{3(n-k-1)+2}}, \quad n \geqq 1. \tag{4.3.3}$$

As before we can identify β_n as the moduli of some auxiliary infinite genus Riemann surfaces in $\mathrm{Gr}^{(3)}$ so that the Virasoro constraints are due to the independence of such moduli for w, or equivalently the τ-function.

4.4. Symmetry Algebra. It is straightforward to derive the symmetry algebra as following:

$$[L_n, L_m] = (n-m) L_{n+m}, \quad m, n \geqq -1,$$
$$[V_{3k+i}, L_n] = \left(k + \frac{i}{3}\right) V_{3(k+n)+i}, \quad k \geqq 0, \quad k+n \geqq 0,$$
$$[V_i, L_{-1}] = -\tfrac{1}{3} t_{3-i}, \quad i = 1, 2,$$
$$[V_{3k+i}, V_{3l+j}] = 0, \quad k, l \geqq 0, \quad i, j = 1, 2. \tag{4.4.1}$$

Again the algebra is not closed. Once we find out the W-constraints as a result of symmetries, if there are any, we can add them here. But it is not quite clear from our investigation that the W-constraints are related as Lie–Bäcklund symmetries. Rather, we suspect that they may appear as some Bäcklund symmetries between differential equations as speculated in the final discussion.

5. $(M+1)^{\text{th}}$ Generalized KdV Equation: M-Matrix Model

Since in principle the explicit computation can be done for each case, we shall describe the generic features only.

For the M-matrix model we may be tempted to take Eq. (4.2.2) as a $(M+1)^{\text{th}}$ generalized KdV equation and construct a system of partial differential equations with first order time derivative as following:

$$\begin{aligned} \dot{w}_1 &= w'_2 \\ \dot{w}_2 &= w'_3 \\ &\vdots \\ \dot{w}_M &= -\tfrac{1}{2}w'''_1 + \tfrac{3}{2}w'^2_1. \end{aligned} \tag{5.1}$$

Then applying the Lie–Bäcklund operator we get $\hat{\mathcal{O}}\xi = 0$, where $\hat{\mathcal{Q}}$ is simply

$$\hat{\mathcal{Q}} \equiv \begin{pmatrix} \partial_t & -D & 0 & \cdots & \cdot \\ 0 & \partial_t & -D & \cdots & \cdot \\ 0 & 0 & \partial_t & \cdots & \cdot \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \tfrac{1}{2}D^3 - 3w'_1 D & 0 & 0 & \cdots & \partial_t \end{pmatrix}, \tag{5.2}$$

But it is not clear how one can get such an equation from the Lax pair representation of the generalized KdV hierarchy, even though it is most likely to be true. Therefore, instead of using Eqs. (5.1), (5.2), we shall use the generalized KdV equation arising from the Lax pair representation.

Anyhow, even for the Lax pair representation we always have the $(M+1)^{\text{th}}$ generalized KdV equation as a system of M equations which involve only the first order time derivatives. For example, in the $M = 3$ case for $\mathcal{H}_4$ in Eq. (4.1.1) we have

$$\begin{aligned} \dot{v}_1 &\equiv \dot{u} = 2v'_2, \\ \dot{v}_2 &= v'_3, \\ \dot{v}_3 &= -v'''_2 - v'_1 v_2 - 2v_1 v'_2. \end{aligned} \tag{5.3}$$

Thus the generic feature is not much different from Eq. (5.1). Unfortunately, now $\hat{\mathcal{Q}}$ is not so simple that the recursion operator will be also complicated.

Again we shall have M Lie-point symmetries as M trivial constant solutions for the corresponding $\hat{\mathcal{Q}}\xi = 0$ so that we can generate a hierarchy of symmetries for the $(M+1)^{\text{th}}$ generalized KdV equation as

$$\frac{\partial W}{\partial t_{(M+1)n+i}} \equiv \frac{\partial}{\partial t_{(M+1)n+i}} \begin{pmatrix} w_1 \\ \vdots \\ w_M \end{pmatrix} = \hat{\mathcal{R}}^{n+1} \begin{pmatrix} 0 \\ \vdots \\ \xi_i = 1 \\ \vdots \\ 0 \end{pmatrix}, \tag{5.4}$$

where $\hat{\mathcal{R}} = D^{-1}\hat{\mathcal{L}}$ is the recursion operator, $\{w_i\}$ are generic solutions with $w'_1 = u$, and M sets of time parameters are chosen as $t_{(M+1)n+i}$, $n \geqq 0, M \geqq i \geqq 0$. Equation (5.4) can be regarded as $(M+1)^{\text{th}}$ generalized KdV hierarchy. At this moment we do not know any good systematic way to derive such a recursion operator, but we can always derive in a tedious way because we can guess the generic form for the given generalized KdV equation.

Now we can get the Lie–Bäcklund symmetries as solutions of $\hat{\mathscr{Q}}\xi = 0$ with the following generalized Galilean transformation,

$$
\begin{aligned}
&w_1 \to \tilde{w}_1 = w_1 + t_2\varepsilon + \mathcal{O}(\varepsilon^2), \quad t_2 \equiv t,\\
&w_2 \to \tilde{w}_2 = w_2 + t_1\varepsilon + \mathcal{O}(\varepsilon^2), \quad t_1 \equiv x,\\
&\vdots\\
&t_{(M+1)k+i} \to \tilde{t}_{(M+1)k+i} = t_{(M+1)k+i} + a_{k,i}t_{(M+1)(k+1)+i}\varepsilon + \mathcal{O}(\varepsilon^2), \quad k \geqq 0, \quad M \geqq i \geqq 1,
\end{aligned} \tag{5.5}
$$

then the Lie–Bäcklund evolution equations are

$$\xi_1 = \left.\frac{d\tilde{w}_1}{d\varepsilon}\right|_{\varepsilon=0} = t_2 + \sum_{k=0,i=1}^{k=\infty,i=M} a_{k,i}t_{(M+1)(k+1)+i}\frac{\partial w_1}{\partial t_{(M+1)k+i}}, \tag{4.2.17a}$$

$$\xi_2 = \left.\frac{d\tilde{w}_2}{d\varepsilon}\right|_{\varepsilon=0} = t_1 + \sum_{k=0,i=1}^{k=\infty,i=M} a_{k,i}t_{(M+1)(k+1)+i}\frac{\partial w_2}{\partial t_{(M+1)k+i}}, \tag{4.2.17b}$$

and $\dfrac{\partial W}{\partial \beta_n} = \hat{\mathscr{R}}^{n+1}\cdot\xi$, $n \geqq -1$. To determine $a_{k,i}$ from the corresponding symmetry conditions we need to know the explicit form of $\hat{\mathscr{L}}$, but at least we can expect $a_{k,i} \propto (M+1)k + i$. Again we should get the string equation by claiming the vanishing Lie–Bäcklund equation $\dfrac{\partial W}{\partial \beta_{-1}} = 0$ and the Virasoro constraints from $\dfrac{\partial W}{\partial \beta_n} = 0$, $n \geqq -1$, with $w_1 = -2D\ln\tau$ such that $L_n\tau = 0$. It is not so difficult to show that L_n's actually form a Virasoro algebra because we do not need to know the explicit form of $\hat{\mathscr{R}}$ to do this. We would not bother writing them here. As before the full algebra with the generalized KdV flows is not closed, but now it becomes closed at $M \to \infty$ limit because the coefficient of t_1t_2 term in L_{-1} is proportional to $\dfrac{1}{M}$. Thus now we understand the origin of Virasoro constraints in all the multimatrix models.

6. Discussions and Conclusions

We have shown so far that the derivation based on Lie–Bäcklund symmetries is a very powerful way to obtain the string equation and to understand the geometrical origin of the Virasoro constraints. The Lie–Bäcklund framework easily produces the KdV flows, which are the Hamiltonian vector flows, and hence essentially shows integrability. But we have also found in the same way another intrinsic geometrical structure based on the coordinates $\{\beta_n\}$ on the space $\mathscr{U} = \{u(x, \vec{t})\}$, which is responsible for the existence of the Virasoro structure. It would be interesting to see any relation between such a structure and the one given in ref. [31].

The geometrical interpretation of our results is therefore the following. Now we can interpret the Virasoro constraints, i.e. the vanishing of certain evolution generators, as independence of the moduli in the infinite Grassmannian viewed as a Universal Moduli Space [34, 16]. Note that the recursion operator $\hat{\mathscr{R}}$ interpolates different multicritical points of a given matrix model, which can be seen by comparing the string equations at each multicritical point. For example, see Eq. (3.2.17) for the one-matrix model. For the other multimatrix models, $\frac{\partial W}{\partial \beta_n} = 0$ leads to a similar observation. Thus in this sense it acts on $\mathrm{Gr}^{(M+1)}$ for the M-matrix model, using Krichever's construction [17, 16].

From this point of view we expect that there should be some action which changes the moduli of a given generic Riemann surface, interpolating $\mathrm{Gr}^{(m)}$ to $\mathrm{Gr}^{(n)}$. We expect that such an action is responsible for the possible extra constraints, which have been conjectured to be W-constraints [10, 11]. For example, from Eq. (4.2.2) one may suspect that there may be some relation between one-matrix models and two-matrix models. One can imagine a Bäcklund transformation which maps a solution of the KdV equation to a solution of the Boussinesq equation. Using such a Bäcklund transformation, in principle we can rewrite the Virasoro constraints of the one-matrix model in terms of the τ-function of the Boussinesq hierarchy. These may become new constraints for the two-matrix model. These new constraints would not look like Virasoro constraints. Thus we can suspect them to be W-constraints. There is some circumstantial evidence for this expectation. Some preliminary results say that, compared to the W-constraints proposed in ref. [11], some terms are indeed the same. The details will be discussed in a separate paper [35].

If we accept this circumstantial evidence, we can suspect that the W-constraints of multimatrix models may be related to the Virasoro constraints of lower multimatrix models so that for the M-matrix model we can have $(M-1)$ sets of extra constraints and no other constraints for the one-matrix model. This precisely accounts for the correct number of sets of constraints for a given multimatrix model.

Presumably in this way we can also construct the M loop operators for the M-matrix model. The expectation values of loop operators for the M-matrix model are the expectation values of loop operators including those in the lower matrix model written in terms of the τ-function of $(M+1)^{\text{th}}$ KdV hierarchy.

Perhaps the investigation of such a structure may be an important clue even for the study of nonperturbative aspects of critical string theory based on the (compactified) Universal Moduli Space of Friedan and Shenker [34].

Some evidence that the infinite Grassmannian may be a nice place to study the true nonperturbative structure is that for the subspace $\mathrm{Gr}^{(M+1)}$ the symmetry algebra is not closed, but becomes closed at $M \to \infty$ limit. Note that $\mathrm{Gr}^{(M+1)}$ can be imbedded into the infinite Grassmannian for any M. This suggests that there may be a larger closed symmetry algebra like the Virasoro–Kac–Moody algebra [36], which will count the true symmetry structure of the nonperturbative 2-d Quantum Gravity.

Another attempt to enlarge the symmetry is given in ref. [9], but the final answer is still elusive.

Note that our construction clearly shows why the $(M+1)^{\text{th}}$ generalized KdV hierarchy is obtained from the KP hierarchy by setting certain t's to zero, namely the restricting condition $t_{(M+1)k} = 0$ so that the remaining coordinates are

$t_{(M+1)k+i}$, $i = 1, 2, \ldots, M$ [15]. Thus this may be more evidence that the KP equation is an important place to further investigate, and may even provide the structure of the $c \geqq 1$ cases.

The appearance of a KdV hierarchy from the point of view of topological gravity [37–40] is still elusive, though Eq. (3.1.2) suggests that there may be an explanation why all such structures are related to topological gravity. Somehow the existence of the scaling operators should explain the existence of the KdV flows. We also need to understand what kind of topological property is responsible for the other Lie–Bäcklund symmetry generators related to the Virasoro constraints.

There are still many questions which need further study, but we believe that the Lie–Bäcklund symmetries for integrable models will lead to some clues to reveal the deep nonperturbative structures of critical string theory.

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