Commun. Math. Phys. 140, 417–448 (1991)

Communications in Mathematical Physics

Hydrodynamic Limit for Attractive Particle Systems on $\mathbb{Z}^d$

Fraydoun Rezakhanlou*

Courant Institute of Mathematical Sciences, New York University, 251 Mercer St., New York, NY 10012, USA

Received August 3, 1990; in revised form November 15, 1990

Abstract. We study the hydrodynamic behavior of asymmetric simple exclusions and zero range processes in several dimensions. Under Euler scaling, a nonlinear conservation law is derived for the time evolution of the macroscopic particle density.

1. Notation and Summary

In this article, we study the hydrodynamic behavior of certain stochastic particle systems, such as simple exclusions and zero range processes. These systems consist of an infinite number of identical particles that move on a multidimensional lattice according to a Markovian law. Under Euler scaling, the microscopic particle density converges to a deterministic limit that is characterized as the solution of a nonlinear conservation law.

Before stating our main results we describe the simple exclusion model and the zero-range process is more detail.

Let E denote the space of configurations $\eta = (\eta(u): u \in \mathbb{Z}^d)$, where $\eta(u)$ is a nonnegative integer representing the occupation number of particles at site u. Let $(p(z): z \in \mathbb{Z}^d)$ be a probability transition function $\left(\text{i.e. } \sum_z p(z) = 1 \text{ and } p(z) \geqq 0\right)$ and $g: \mathbb{N} \to [0, \infty)$ be a bounded nondecreasing function with $0 = g(0) < g(1)$. The zero-range processes are defined as Markov processes with state space E and generator

$$\mathscr{L} f(\eta) = \sum_{u,v} p(v-u) g(\eta(u)) (f(\eta^{uv}) - f(\eta)),$$

* Research supported by NSF Grant No. DMS-8806731. Present address: Department of Mathematics, University of California at Berkeley, Berkeley, CA 94720, USA

where

$$\eta^{uv}(z)=\begin{cases}\eta(u)-1 & \text{if } z=u\\ \eta(v)+1 & \text{if } z=v\\ \eta(z) & \text{if } z\neq u,v\end{cases},$$

provided $\eta(u)\geqq 1$ and $u\neq v$; $\eta^{uv}\equiv\eta$ otherwise.

In the simple exclusion model, there is at most one particle per site (i.e. $\eta(u)=0$ or 1) and the generator is

$$\mathscr{L}f(\eta)=\sum_{u,v}p(v-u)\eta(u)(1-\eta(v))(f(\eta^{uv})-f(\eta)).$$

We refer the reader to [13] and [1] for the existence and construction of the above processes.

It is known that for any nonnegative ρ (in the simple exclusion case $\rho\in[0,1]$), there exists a unique translation invariant equilibrium measure ν^ρ with density ρ. More precisely, ν^ρ is a probability measure on E with the following properties:

$$\int \mathscr{L}f\,d\nu^\rho=0$$

for all cylinder functions f,

$$\int \eta(0)\nu^\rho(d\eta)=\rho,$$

and

$$\int \tau_u f\,d\nu^\rho=\int f\,d\nu^\rho$$

for all $u\in\mathbb{Z}^d$, where τ_u is the shift operator defined by

$$\tau_u f(\eta)=f(\tau_u\eta), \tag{1.1}$$

and

$$\tau_u\eta(v)=\eta(u+v)\quad u,v\in\mathbb{Z}^d. \tag{1.2}$$

(For the definition of ν^ρ, see the next section.)

Because of Euler scaling, we consider the speeded up generator $N\mathscr{L}$ for positive integers N, and denote the Markov process with generator $N\mathscr{L}$, by $(\eta_t^{(N)}(u):u\in\mathbb{Z}^d)$, and if there is no danger of confusion, simply by η_t. Associated with the particle configuration η_t, we define the Radon measure

$$\alpha^N(t,dx)=\frac{1}{N^d}\sum\delta_{u/N}(dx)\eta_t(u) \tag{1.3}$$

viewed as a random measure on $\mathbb{R}^d$. In other words, for any smooth function J with compact support

$$\int J(x)\alpha^N(t,dx)=\frac{1}{N^d}\sum_u J\left(\frac{u}{N}\right)\eta_t(u).$$

The object of this paper is to derive the hydrodynamic equation for the particle density $\alpha^N(t,dx)$, as N goes to infinity. Roughly speaking, we initially start with a distribution μ^N for η_0, whose *density profile* is some masurable function $\rho(\cdot)$, i.e.

$$\lim_{N\to\infty}\mu^N\left(\left\{\eta_0:\left|\frac{1}{N^d}\sum J\left(\frac{u}{N}\right)\eta_0(u)-\int J(x)\rho(x)dx\right|>\delta\right\}\right)=0$$

for any $\delta > 0$. We then expect that the distribution of η_t has a *density profile* $\rho(t, \cdot)$ which satisfies the conservation law

$$\frac{\partial \rho}{\partial t} + \sum_{i=1}^{d} \gamma_i \frac{\partial}{\partial x_i} h(\rho) = 0, \tag{1.4}$$

where $h(\rho) = \int g(\eta(0)) v^\rho(d\eta)$ for the zero range process and $h(\rho) = \rho(1-\rho)$ for the simple exclusion model, and

$$\gamma := (\gamma_1, \ldots, \gamma_d) = \sum_z z p(z). \tag{1.5}$$

When h is not linear, then Eq. (1.4) has no differentiable solution. So, it must be understood in the sense of distributions. Another feature of (1.4) is that its solutions are not determined uniquely by their initial data. Therefore, we need some criteria to pick the *relevant* solutions. These relevant solutions are normally called *entropy solutions* and they are characterized by the following criteria:

$$\frac{\partial}{\partial t} |\rho - c| + \sum_{i=1}^{d} \gamma_i \frac{\partial}{\partial x_i} q(\rho; c) \leqq 0 \tag{1.6}$$

for all constant $c \in \mathbb{R}$, where

$$q(\rho; c) = \operatorname{sign}(\rho - c)(h(\rho) - h(c)).$$

Here the inequalities (1.6) are interpreted in the sense of distributions, and we refer to them as the *entropy inequalities*. Kružkov's uniqueness theorem asserts that the entropy solution of Eq. (1.4) is unique, providing

$$\lim_{t \to 0} \int_{|x| \leqq k} |\rho(x, t) - \rho(x)| dx = 0 \tag{1.7}$$

for every constant k.

In order to prepare for the statements of our main results, we formulate several assumptions. These assumptions are of two types: on the initial distribution μ^N and on the transition probability function $p(z)$. We will assume throughout this paper:

Assumptions 1.1.
(a) $p(\cdot)$ is of finite range, i.e. $p(z) = 0$ if $|z| > r_0$ for some r_0.
(b) (For the zero range process) p is irreducible, i.e. $\sum_{n>0} p^{*n}(z) > 0$ for all $z \in \mathbb{Z}^d$, where p^{*n} denotes the n^{th} convolution of p.
(b′) (For the simple exclusion model) $p(z) + p(-z)$ is irreducible, i.e.

$$\sum_{n>0} (p^{*n}(z) + p^{*n}(-z)) > 0$$

for all $z \in \mathbb{Z}^d$.

See the next section for the motivation behind assumptions (b) and (b′).

Notation 1.2. Let μ^N be a sequence of probability measures on E, and let ρ be a bounded measurable function on $\mathbb{R}^d$. We then write $\mu^N \sim \rho$ if the following conditions hold:
(a) μ^N is a product measure,

(b) there exists a sequence $\rho_{u,N}$ such that

$$\mu^N(\eta(u)=k)=\nu^{\rho_{u,N}}(\eta(u)=k) \quad u\in\mathbb{Z}^d,\ k\in\mathbb{N},$$

$$\lim_{N\to\infty} \int_{|x|\leq k} |\rho_{[xN],N}-\rho(x)|\,dx=0$$

for every positive k($[xN]$ denotes the integer part of xN).

Note that if ρ is continuous, we may choose $\rho_{u,N}=\rho\left(\frac{u}{N}\right)$ for all $u\in\mathbb{Z}^d$.

We are now ready to state our main result. Let P^N denote the distribution of the process η_t with the initial distribution μ^N.

Theorem 1.3. *Suppose $\rho\in L^\infty(\mathbb{R}^d)$ and $\mu^N\sim\rho$. Then for every $t>0$, every smooth function J of compact support, and each $\delta>0$,*

$$\lim_{N\to\infty} P^N\left(\left|\frac{1}{N^d}\sum_u J\left(\frac{u}{N}\right)\eta_t(u)-\int J(x)\rho(t,x)dx\right|>\delta\right)=0,$$

where $\rho(t,x)$ is the unique solution of (1.4) satisfying (1.7) and the entropy inequalities (1.6).

Actually we will show the following stronger result: if Q^N is the law of $\alpha^N(t,dx)$ with respect to the probability measure P^N, then Q^N converges weakly to a probability measure Q that is concentrated on the single path $\rho(t,x)dx$ (see Theorem 5.1).

The key idea that will be used in the proof of the above theorem is the *monotonicity* or *attractiveness* property of the process η_t. That is, if certain inequalities initially hold between configurations, then they continue to hold at later times. Section 2 will be devoted to the precise definition of attractiveness, and some of its consequences that will be used frequently in the rest of the paper.

In Sect. 3, we will show that the inequalities (1.6) hold if we replace the macroscopic density $\rho(x,t)$ with the average density of particles in large microscopic blocks. We then need to verify that the microscopic particle densities of macroscopically close blocks do not fluctuate. This will be done in Sects. 5 and 6, using two different approaches.

Theorem 1.3 will be established in Sect. 5. The proof of Theorems 1.3 and 5.1 presented in Sect. 5 uses *DiPerna's uniqueness* theorem for *measure-valued solutions* (see Sect. 5 for the definition of measure-valued solution and see Lemma 5.3, for DiPerna's theorem).

In Sect. 6 we will give an alternative proof of Theorem 1.3 under the stringent assumption $p(1)+p(-1)=1$ (i.e. nearest neighbor jumps) that does not use DiPerna's theorem. The proof of DiPerna's theorem presented in [9] relies on the existence of the entropy solutions to (1.6). Our proof of Theorem 1.3 in Sect. 6 does not use any existence theory, therefore establishing the existence of entropy solutions using probabilitic arguments.

Section 7 is devoted to some of the implications and refinements of Theorem 1.3.

If $\rho(t,\cdot)$ is continuous at x, we expect that the distribution of $(\eta_t([Nx]+u):u\in\mathbb{Z}^d)$ converges weakly to the equilibrium measure $\nu^{\rho(t,x)}$. Results of this type have been recently proven by Benassi, Fouque, Saada and Vares [17] for the one dimensional zero-range processes with monotone initial densities. Some of the earlier references in this context are [2–5, 12, 13 Chap. VIII, and 14].

2. Monotonicity and Its Consequences

The Markov processes described in the previous section are *attractive* (or *monotone*). The primary purpose of this section is to present some of the consequences of *attractiveness* that will be used frequently in the succeeding sections.

We start with defining a class of Markov processes that includes simple exclusion and zero-range processes.

Let $b:\mathbb{N}\times\mathbb{N}\to[0,\infty)$ be a bounded function with the following properties:

$$\left.\begin{array}{ll}\text{(i)} & b(0,\cdot)\equiv 0,\\ \text{(ii)} & n\mapsto b(n,m) \text{ is a nondecreasing function for each } m,\\ \text{(iii)} & m\mapsto b(n,m) \text{ is a nonincreasing function for each } n.\end{array}\right\}\tag{2.1}$$

Given such b we define the process $(\eta_t^{(N)}(u):u\in\mathbb{Z}^d)$ as the unique Feller process with state space $E=\mathbb{N}^{\mathbb{Z}^d}$ (endowed with the product topology) and the infinitesimal generator $N\mathscr{L}$ where $\mathscr{L}$ acting on cylinder functions is defined by

$$\mathscr{L}f(\eta)=\sum_{u,v}p(v-u)b(\eta(u),\eta(v))(f(\eta^{uv})-f(\eta)).$$

Note that the factor N in front of $\mathscr{L}$ represents the effect of Euler scaling, since the process $(\eta_t^{(N)},t\geqq 0)$ in law is the same as $(\eta_{tN}^{(1)},t\geqq 0)$, where $\eta^{(1)}$ is the Markov process generated by $\mathscr{L}$. We find it more convenient to deal with $\eta_t^{(N)}$ instead of $\eta_{tN}^{(1)}$ in the succeeding sections. But this may appear a bit confusing in this section since N plays no role in the following discussions. When there is no danger of confusion, we drop the superscript N, and denote the process $\eta_t^{(N)}$ with η_t.

It is known that for any $\eta\in E$, there exists a unique probability measure P_η on the Skorohod space $D([0,+\infty);E)$ that solves the martingale problem associated to $\mathscr{L}$. Let S_t be the corresponding Markov semigroup. That is,

$$S_tf(\eta)=\int f(\eta_t^{(1)})dP_\eta$$

for f a bounded and continuous function on E, and we define μS_t by

$$\int f d(\mu S_t)=\int S_t f d\mu$$

for any probability measure μ on E.

An important problem concerning these processes is to characterize the set of invariant measures $\mathscr{I}$,

$$\mu\in\mathscr{I}\quad\text{if}\quad \mu S_t=\mu\quad\text{for all}\quad t\geqq 0.$$

Let τ_u, $u\in\mathbb{Z}^d$, be the shift operators acting on E by $(\tau_u\eta)(v)=\eta(u+v)$. They also act on functions by $(\tau_u f)(\eta)=f(\tau_u\eta)$, and on measures by

$$\int f d(\mu\tau_u)=\int \tau_u f d\mu$$

for any measurable function f on E. Let $\mathscr{S}$ denote the space of probability measures invariant under $(\tau_u;u\in\mathbb{Z}^d)$.

An important implication of the monotonicity assumptions 2.1 on b is the monotonicity or attractiveness of the process η_t. For this we first define the following partial order on $E; \eta \leqq \zeta$ if $\eta(u) \leqq \zeta(u)$ for all $u \in \mathbb{Z}^d$. This in turn defines an order between probability measures on $E: \mu_1 \leqq \mu_2$ if there exists a coupling measure $\tilde{\mu}$ on $E \times E$ such that $\tilde{\mu}(A \times E) = \mu_1(A)$, $\tilde{\mu}(E \times A) = \mu_2(A)$ and $\tilde{\mu}\{\eta, \zeta): \eta \leqq \zeta\} = 1$. The process η_t is attractive in the sense that $\mu_1 \leqq \mu_2$ implies $\mu_1 S_t \leqq \mu_2 S_t$ for all t. This is easily shown by constructing a coupled process (η_t, ζ_t) on the space $E \times E$ such that the evolutions of η_t and ζ_t are governed by $N\mathscr{L}$, and if $\eta_0 \leqq \zeta_0$ then $\eta_t \leqq \zeta_t$ for all t. The generator of (η_t, ζ_t) is given by $N\tilde{\mathscr{L}}$, where

$$\begin{aligned}\tilde{\mathscr{L}} f(\eta, \zeta) = &\sum_{u,v} p(v-u)(b(\eta(u), \eta(v)) \wedge b(\zeta(u), \zeta(v)))(f(\eta^{uv}, \zeta^{uv}) - f(\eta, \zeta)) \\ &+ \sum_{u,v} p(v-u)(b(\eta(u), \eta(v)) \\ &- b(\eta(u), \eta(v)) \wedge b(\zeta(u), \zeta(v)))(f(\eta^{uv}, \zeta) - f(\eta, \zeta)) \\ &+ \sum_{u,v} p(v-u)(b(\zeta(u), \zeta(v)) \\ &- b(\eta(u), \eta(v)) \wedge b(\zeta(u), \zeta(v)))(f(\eta, \zeta^{uv}) - f(\eta, \zeta)).\end{aligned}$$

Let $\tilde{\mathscr{I}}$ denote the space of invariant measures of $\tilde{\mathscr{L}}$ and let $\tilde{\mathscr{S}}$ denote the space of translation invariant measures on $E \times E$.

Note that if we choose $b(n,m) = g(n)$, we obtain the zero-range processes, and if we choose

$$b(n,m) = \begin{cases} 1 & \text{if} \quad n = 1, m = 0, \\ 0 & \text{otherwise}, \end{cases}$$

and restricting the process to $\{0,1\}^{\mathbb{Z}^d}$, we then obtain the simple exclusion model.

Both of the above examples have the following properties that would be essential for our arguments.

Properties 2.1.

(a) For each density $\rho \in [0, \infty)$ ($\rho \in [0,1]$ in the simple exclusion case), there exists a unique product measure ν^ρ in $\mathscr{I} \cap \mathscr{S}$ such that $\int \eta(0) \nu^\rho(d\eta) = \rho$.

(b) For $\mu \in \mathscr{I} \cap \mathscr{S}$ with $\int \eta(0) \mu(d\eta) < \infty$, there exists a probability measure β on $[0, \infty)$ such that

$$\mu = \int \nu^\rho \beta(d\rho).$$

(c) For every $\tilde{\mu} \in \tilde{\mathscr{I}} \cap \tilde{\mathscr{S}}$ we have

$$\tilde{\mu}\{(\eta, \zeta): \eta \leqq \zeta \text{ or } \zeta \leqq \eta\} = 1.$$

In the simple exclusion model the measure ν^ρ is characterized by the relation $\nu^\rho(\eta(u) = 1) = \rho$.

To construct ν^ρ for the zero-range process, we first define the following probability measure Θ_λ on $\mathbb{N}$, for each $\lambda \in [0, \sup_k g(k))$:

$$\Theta_\lambda(n) = \begin{cases} \dfrac{1}{Z(\lambda)} \dfrac{\lambda^n}{g(1) \cdots g(n)} & \text{if} \quad n \neq 0, \\ \dfrac{1}{Z(\lambda)} & \text{if} \quad n = 0, \end{cases}$$

where $Z(\lambda)$ is the normalizing factor. Set

$$\psi(\lambda) = \sum_{n=1}^{\infty} n\Theta_\lambda(n).$$

Then $\psi:[0, \sup_k g(k)) \to [0, \infty)$ is strictly increasing, and $\lim_{\lambda \to \sup_k g(k)} \psi(\lambda) = +\infty$. The inverse of ψ is denoted by $\lambda(\cdot)$ and let

$$\Theta^\rho = \Theta_{\lambda(\rho)}.$$

The probability measure ν^ρ is obtained by taking the product of Θ^ρ's, i.e. $\nu^\rho(d\eta) = \prod_{u \in \mathbb{Z}^d} \Theta^\rho(d\eta(n))$, so that

$$\nu^\rho(\eta(u) = k) = \Theta^\rho(k).$$

The expectation with respect to ν^ρ will be denoted by $\langle \cdot \rangle_\rho$. We certainly have

$$\langle \eta(0) \rangle_\rho = \rho,$$

$$\langle g(\eta(0)) \rangle_\rho = \lambda(\rho).$$

Properties 2.1 were shown by Ligget [13], for the simple exclusion model and by Andjel [1] for zero-range processes.

There is another class of examples for which Properties 2.1 hold. This class was introduced by Cocozza [6], and it includes the class of zero-range processes. We refer to this class as *Processus des misanthropes.* It is characterized by further assumptions on b:

$$b(n,m) > 0 \quad n \geqq 1$$

$$\frac{b(n,m)}{b(m+1,n+1)} = \frac{b(n,0)(b(1,m)}{b(m+1,0)b(1,n-1)}, \quad n \geqq 1,\ m \geqq 0,$$

$$b(n,m) - b(m,n) = b(n,0) - b(m,0) \quad n,m \geqq 0.$$

The measure ν^ρ is a product measure characterized by relations

$$\frac{\nu^\rho(\eta(u) = n+1)}{\nu^\rho(\eta(u) = n)} = \frac{\nu^\rho(\eta(u) = 1)}{\nu^\rho(\eta(u) = 0)} \frac{b(1,n)}{b(n+1,0)}$$

for $n \geqq 1$.

We end this section with the following definition:

$$h(\rho) := \langle b(\eta(0), \eta(z)) \rangle_\rho, \quad z \neq 0. \tag{2.2}$$

Since ν^ρ is a product measure, the right-hand side of (2.2) is independent of z. Indeed $h(\rho) = \rho(1-\rho)$ in the simple exclusion case, and $h(\rho) = \lambda(\rho)$ in the zero-range case.

3. Entropy Inequalities in Microscopic Form

Motivated by the work of Guo, Papanicolaou and Varadhan [10], we introduce an intermediate space scaling into our problem. While macroscopic regions have size of order N, in microscopic space scaling, we will take averages over microscopic regions of size l, where l increases to infinity only after N has already tended to infinity. In this section we will prove the entropy inequalities (1.6) with $\rho(t,x)$ replaced by the average density of particles in large microscopic blocks.

Let T_l be the cube of length $2l+1$ in $\mathbb{Z}^d$, centered at the origin:

$$T_l = \{u \in \mathbb{Z}^d : u = (u_1 \cdots u_d) \text{ with } |u_i| \leqq l \text{ for } 1 \leqq i \leqq d\},$$

and let $T_l(z) = T_l + z$, $z \in \mathbb{Z}^d$. Define

$$M_T(\eta) = \frac{1}{|T|} \sum_{u \in T} \eta(u)$$

for any $T \subseteqq \mathbb{Z}^d$, where $|T|$ denotes the total number of sites in T. We now state and prove the following version of the entropy inequalities:

Let $J(t,x)$ be a smooth test function with compact support in $(0,\infty) \times \mathbb{R}^d$. Let μ^N be a sequence of probability measures. For each N, P^N will denote the law of the process η_t with the infinitesimal generator $N\mathscr{L}$ and with the initial distribution μ^N.

Theorem 3.1. *Suppose that there exists a positive density ρ_0 such that $\mu^N \leqq \nu^{\rho_0}$ for all N. If $J \geqq 0$, $\varepsilon > 0$ and c is any constant, then*

$$\lim_{l\to\infty} \liminf_{N\to\infty} P^N \left\{ \int_0^\infty \frac{1}{N^d} \sum_u \frac{\partial J}{\partial s}\left(s, \frac{u}{N}\right) |M_{T_l(u)}(\eta_s) - c|\, ds + \int_0^\infty \frac{1}{N^d} \sum_u \gamma \cdot \nabla J\left(s, \frac{u}{N}\right) q(M_{T_l(u)}(\eta_s) : c)\, ds \geqq -\varepsilon \right\} = 1.$$

Proof. Step 1. The constant "c" can be obtained as the density of a Markov process that is generated by $N\mathscr{L}$ and distributed initially as ν^c. With this in mind, we consider the coupled process (η_t, ζ_t) with the generator $N\tilde{\mathscr{L}}$ and initial distribution $\mu^N \times \nu^c$. We denote the law of this process on $D([0,\infty) \times E^2)$ by $\tilde{P}^N$.

Set $A(s,\eta,\zeta) = \frac{1}{N^d} \sum_u J\left(s, \frac{u}{N}\right) |\eta(u) - \zeta(u)|$. Since $J(\cdot, u/N)$ has a bounded support in the open interval $(0,\infty)$, we have

$$B_t = \int_0^t \left[\frac{\partial}{\partial s} A(s,\eta_s,\zeta_s) + N\tilde{\mathscr{L}} A(s,\eta_s,\zeta_s) \right] ds$$

is a martingle for large t, and $B_t^2 - \langle B \rangle_t$ is also a martingle, where

$$\langle B \rangle_t = \int_0^t (N\tilde{\mathscr{L}} A^2(s) - 2NA(s)\tilde{\mathscr{L}} A(s))\, ds.$$

In order to compute B_t and $\langle B \rangle_t$, we start with

$$\begin{aligned}\tilde{\mathscr{L}} |\eta(u) - \zeta(u)| = \sum_v [&p(v-u)(b(\zeta(u),\zeta(v)) - b(\eta(u),\eta(v))) \\ &- p(u-v)(b(\zeta(v),\zeta(u)) - b(\eta(v),\eta(u)))] \cdot F_{u,v}(\eta,\zeta) \\ &+ \sum_v [p(u-v)(b(\zeta(v),\zeta(u)) - b(\eta(v),\eta(u)))\end{aligned}$$

$$-p(v-u)(b(\zeta(u),\zeta(v))-b(\eta(u),\eta(v))]\cdot F_{u,v}(\zeta,\eta)$$
$$-\sum_v [p(v-u)|b(\zeta(u),\zeta(v))-b(\eta(u),\eta(v))|$$
$$+p(u-v)|b(\zeta(v),\zeta(u))-b(\eta(v),\eta(u))|]\cdot G_{u,v}(\eta,\zeta), \quad (3.2)$$

where

$$F_{u,v}(\eta,\zeta)=\begin{cases}1 & \text{if } \eta(u)\geqq\zeta(u) \text{ and } \eta(v)\geqq\zeta(v)\\ 0 & \text{otherwise,}\end{cases}$$

and $G_{u,v}(\eta,\zeta)=1-F_{u,v}(\eta,\zeta)-F_{u,v}(\zeta,\eta)$. Formula (3.2) follows from a straightforward computation on $\mathscr{L}$ using monotonicity properties of b. Note that in (3.2) the third sum in the right-hand side is nonpositive, then if $J\geqq 0$,

$$\begin{aligned}B_t\leqq &\int_0^t \frac{1}{N^d}\sum_u \frac{\partial J}{\partial s}\left(s,\frac{u}{N}\right)|\eta_s(u)-\zeta_s(u)|\,ds\\
&+\int_0^t \frac{N}{N^d}\sum_{u,v} p(v-u)J\left(s,\frac{u}{N}\right)(b(\zeta_s(u),\zeta_s(v))-b(\eta_s(u),\eta_s(v)))F_{u,v}(\eta_s,\zeta_s)ds\\
&-\int_0^t \frac{N}{N^d}\sum_{u,v} p(u-v)J\left(s,\frac{u}{N}\right)(b(\zeta_s(v),\zeta_s(u))-b(\eta_s(v),\eta_s(u)))F_{u,v}(\eta_s,\zeta_s)ds\\
&+\int_0^t \frac{N}{N^d}\sum_{u,v} p(u-v)J\left(s,\frac{u}{N}\right)(b(\zeta_s(v),\zeta_s(u))-b(\eta_s(v),\eta_s(u)))F_{u,v}(\zeta_s,\eta_s)ds\\
&-\int_0^t \frac{N}{N^d}\sum_{u,v} p(v-u)J\left(s,\frac{u}{N}\right)(b(\zeta_s(u),\zeta_s(v))-b(\eta_s(u),\eta_s(v)))F_{u,v}(\zeta_s,\eta_s)ds.\end{aligned}$$

In the third and fourth term we interchange u with v and then we add up the second with third, and the fourth with fifth terms,

$$\begin{aligned}B_t\leqq &\int_0^t \frac{1}{N^d}\sum_u \frac{\partial J}{\partial S}\left(s,\frac{u}{N}\right)|\eta_s(u)-\zeta_s(u)|\,ds\\
&+\int_0^t \frac{N}{N^d}\sum_{u,v} p(v-u)\left(J\left(s,\frac{u}{N}\right)\right.\\
&\left.-J\left(s,\frac{v}{N}\right)\right)(b(\zeta_s(u),\zeta_s(v))-b(\eta_s(u),\eta_s(v)))F_{u,v}(\eta_s,\zeta_s)ds\\
&+\int_0^t \frac{N}{N^d}\sum_{u,v} p(v-u)\left(J\left(s,\frac{u}{N}\right)\right.\\
&\left.-J\left(s,\frac{v}{N}\right)\right)(b(\eta_s(u),\eta_s(v))-b(\zeta_s(u),\zeta_s(v)))F_{u,v}(\zeta_s,\eta_s)ds. \quad (3.3)\end{aligned}$$

Since p is of finite range, we have

$$p(v-u)\left(J\left(s,\frac{u}{N}\right)-J\left(s,\frac{v}{N}\right)\right)=\frac{1}{N}p(v-u)(u-v)\cdot\nabla J\left(s,\frac{u}{N}\right)+O\left(\frac{1}{N^2}\right).$$

Define

$$H(\eta,\zeta)=\sum_z p(z)F_{0,z}(\eta,\zeta)(b(\eta(0),\eta(z))-b(\zeta(0),\zeta(z)))z$$
$$+\sum_z p(z)F_{0,z}(\zeta,\eta)(b(\zeta(0),\zeta(z))-b(\eta(0),\eta(z)))z.$$

(Here, we use vector notation: $z\in\mathbb{Z}^d$ and $H(\eta,\zeta)\in\mathbb{R}^d$.)

Now (3.3) can be written as

$$B_t\leqq\int_0^t\frac{1}{N^2}\sum_u\frac{\partial J}{\partial s}\left(s,\frac{u}{N}\right)|\eta_s(u)-\zeta_s(u)|\,ds$$
$$+\int_0^t\frac{1}{N^d}\sum_u\nabla J\left(s,\frac{u}{N}\right)\cdot\tau_u H(\eta_s,\zeta_s)ds+O\left(\frac{1}{N}\right). \tag{3.4}$$

In Lemma 3.2 we will show

$$\tilde{E}^N\langle B\rangle_t=O\left(\frac{1}{N^d}\right), \tag{3.5}$$

where $\tilde{E}^N$ means the expectation with respect to $\tilde{P}^N$. Since $B_t^2-\langle B\rangle_t$ is a martingale, (3.5) implies $\tilde{E}^N\sup_{0\leqq s\leqq t}B_s^2=O(1/N^d)$. From this and (3.4) we conclude

$$\lim_{N\to\infty}\tilde{P}^N\left\{\int_0^t\frac{1}{N^d}\sum_u\frac{\partial J}{\partial s}\left(s,\frac{u}{N}\right)|\eta_s(u)-\zeta_s(u)|\,ds\right.$$
$$\left.+\int_0^t\frac{1}{N^d}\sum_u\nabla J\left(s,\frac{u}{N}\right)\cdot\tau_u H(\eta_s,\zeta_s)ds\geqq-\varepsilon\right\}=1. \tag{3.6}$$

Step 2. Let l be a fixed positive integer. Since J is smooth, in (3.6) we can replace $|\eta_s(u)-\zeta_s(u)|$ and $\tau_u H(\eta_s,\zeta_s)$ with their space averages over sites in a box of side length $2l+1$ and center u. Therefore to go from (3.6) to (3.1), it suffices to show that for any $k>0$,

$$\lim_{l\to\infty}\limsup_{N\to\infty}\tilde{E}^N\left[\int_0^t\frac{1}{|T_{kN}|}\sum_{u\in T_{kN}}\left|\frac{1}{|T_l(u)|}\sum_{z\in T_l(u)}|\eta_s(z)-\zeta_s(z)|-|M_{T_l(u)}(\eta_s)-c|\right|ds\right]=0 \tag{3.7}$$

$$\lim_{l\to\infty}\limsup_{N\to\infty}\tilde{E}^N\left[\int_0^t\frac{1}{|T_{kN}|}\sum_{u\in T_{kN}}\left|\frac{1}{|T_l(u)|}\tau_u H(\eta_s,\zeta_s)-q(M_{T_l(u)}(\eta_s);c)\gamma\right|ds\right]=0. \tag{3.8}$$

We only show (3.8) because (3.7) can be treated in the same way. Remember that (η_t,ζ_t) is generated by $N\tilde{\mathscr{L}}$, and initially distributed as $\mu^N\times\nu^c$. Let $\tilde{S}_t$ be the semigroup associated to $\tilde{\mathscr{L}}$. Let $\tilde{\mu}_t^N=\frac{1}{tN}\int_0^{tN}\frac{1}{|T_{kN}|}\sum_{u\in T_{kN}}(\mu^N\times\nu^c)\tau_u\tilde{S}_s ds$. Then (3.8) can be written as

$$\lim_{l\to\infty}\limsup_{N\to\infty}\int\left|\frac{1}{|T_l|}\sum_{u\in T_l}\tau_u H(\eta,\zeta)-q(M_{T_l}(\eta);c)\gamma\right|\tilde{\mu}_t^N(d\eta,d\zeta)=0. \tag{3.9}$$

Since ν^c is an invariant measure of ζ_t, the ζ-marginal of $\tilde{\mu}_t^N$ is always ν^c. On the other hand $\mu^N\leqq\nu^{\rho_0}$, therefore the η-marginal of $\tilde{\mu}_t^N$ is always (stochastically)

less than ν^{ρ_0}. Thus, the sequence $\{\tilde{\mu}_t^N\}$ is tight. Let $\mathscr{A}$ be the collection of all limit points of $\{\tilde{\mu}_t^N\}$. From the way we defined $\tilde{\mu}_t^N$, we clearly have
(a) Marginals of any μ in $\mathscr{A}$ are less than $\nu^{c+\rho_0}$
(b) $\mathscr{A} \subseteq \tilde{\mathscr{I}} \cap \tilde{\mathscr{S}}$.
(c) ζ-marginal of any μ on $\mathscr{A}$ is ν^c.
Thus, (3.9) would be implied by

$$\lim_{l\to\infty} \sup_{\mu\in\mathscr{A}} \int \left| \frac{1}{|T_l|} \sum_{u\in T_l} \tau_u H(\eta,\zeta) - q(M_{T_l}(\eta); M_{T_l}(\zeta))\gamma \right| \mu(d\eta, d\zeta) = 0, \tag{3.10}$$

where we have used the following conclusion of Ergodic Theorem

$$\lim_{l\to\infty} \nu^c(|M_{T_l} - c| > \delta) = 0 \tag{3.11}$$

to replace c with $M_{T_l}(\zeta)$ in (3.9).

Properties 2.1(c) guarantees

$$\mu\{(\eta,\zeta): \eta \leqq \zeta \text{ or } \zeta \leqq \eta\} = 1$$

for all $\mu\in\mathscr{A}$. Now we break the integral in (3.10) into two pieces, one over configurations (η,ζ) with $\eta\leqq\zeta$, and the other over configurations with $\zeta\leqq\eta$. We now write one of these two integrals,

$$\begin{aligned}
&\int_{\eta\leqq\zeta} \left| \frac{1}{|T_l|} \sum_{u\in T_l} \tau_u H(\eta,\zeta) - q(M_{T_l}(\eta); M_{T_l}(\zeta))\gamma \right| \mu(d\eta, d\zeta) \\
&= \int_{\eta\leqq\zeta} \left| \frac{1}{|T_l|} \sum_{u\in T_l} \sum_z p(z)(b(\zeta(u),\zeta(u+z)) - b(\eta(u),\eta(u+z)))z \right. \\
&\qquad \left. - (h(M_{T_l}(\zeta)) - h(M_{T_l}(\eta)))\gamma \right| \mu(d\eta, d\zeta).
\end{aligned}$$

Thus, it suffices to show

$$\lim_{l\to\infty} \sup_{\substack{\beta\in\mathscr{I}\cap\mathscr{S} \\ \beta\leqq\nu^{\rho_0}}} \int \left| \frac{1}{|T_l|} \sum_{u\in T_l} \sum_z p(z) b(\eta(u),\eta(u+z))z - h(M_{T_l}(\eta))\gamma \right| \beta(d\eta) = 0. \tag{3.12}$$

Using Properties 2.1(b), any β in $\mathscr{I}\cap\mathscr{S}$ of finite expectation can be written as

$$\beta = \int \nu^\rho \alpha(d\rho)$$

for some probability measure α. Since $\beta\leqq\nu^{\rho_0}$, α is concentrated on $[0,\rho_0]$. Therefore we only need to show (3.1) with β replaced by ν^ρ and with supremum over $0\leqq\rho\leqq\rho_0$.

Since every ν^ρ is a product measure and translation invariant, we have

$$\left\langle \sum_z p(z) b(\eta(0),\eta(z))z \right\rangle_\rho = h(\rho)\gamma.$$

Thus, by Ergodic Theorem,

$$\lim_{l\to\infty} \int \left| \frac{1}{|T_l|} \sum_{u\in T_l} \sum_z p(z) b(\eta(u),\eta(u+z))z - h(\rho)\gamma \right| \nu^\rho(d\eta) = 0. \tag{3.13}$$

Since h is bounded and using (3.11), we have

$$\lim_{l\to\infty} \int |h(M_{T_l}(\eta)) - h(\rho)| \nu^\rho(d\eta) = 0, \tag{3.14}$$

for any $\rho \geqq 0$. Convergence in both of (3.13) and (3.14) is uniform in ρ. (To see this for the first one, take the second moment of the integrand. For the second one, first truncate η and then do the same thing.) This completes the proof of (3.12). □

Lemma 3.2. $\tilde{E}^N \langle B \rangle_t = O\left(\frac{1}{N^d}\right)$.

Proof. Recall $A(s,\eta,\zeta) = \frac{1}{N^d} \sum_u J\left(s, \frac{u}{N}\right) |\eta(u) - \zeta(u)|$, and

$$\langle B \rangle_t = \int_0^t (N\tilde{\mathscr{L}} A^2(s) - 2NA(s)\tilde{\mathscr{L}} A(s))ds.$$

A straightforward computation yields

$$\begin{aligned} &N\tilde{\mathscr{L}}A^2 - 2NA\tilde{\mathscr{L}}A \\ &\quad = N\sum_{u,v} p(v-u)(b(\eta(u), n(v)) \wedge b(\zeta(u),\zeta(v)))(A(s,\eta^{uv},\zeta^{uv}) - A(s,\eta,\zeta))^2 \\ &\qquad + N\sum_{u,v} p(v-u)(b(\eta(u),\eta(v)) - b(\eta(u),\eta(v)) \wedge b(\zeta(u),\zeta(v)))(A(s,\eta^{uv},\zeta) - A(s,\eta,\zeta))^2 \\ &\qquad + N\sum_{u,v} p(v-u)(b(\zeta(u),\zeta(v)) - b(\eta(u),\eta(v)) \wedge b(\zeta(u),\zeta(v)))(A(s,\eta,\zeta^{uv}) - A(s,\eta,\zeta))^2. \end{aligned}$$

From the definition of A it follows that the first term is zero. We now focus on the remaining terms.

Suppose $b(\eta(u),\eta(v)) > b(\zeta(u),\zeta(v))$ and $\eta(u) \geqq 1$. Then

$$\begin{aligned} A(s,\eta^{uv},\zeta) - A(s,\eta,\zeta) &= \frac{1}{N^d} J\left(s,\frac{u}{N}\right)(|\eta(u) - 1 - \zeta(u)| - |\eta(u) - \zeta(u)|) \\ &\quad + \frac{1}{N^d} J\left(s,\frac{v}{N}\right)(|\eta(v) + 1 - \zeta(v)| - |\eta(v) - \zeta(v)|) \\ &= \frac{1}{N^d}\left(J\left(s,\frac{u}{N}\right) - J\left(s,\frac{v}{N}\right)\right) F_{u,v}(\zeta,\eta) \\ &\quad + \frac{1}{N^d}\left(J\left(s,\frac{v}{N}\right) - J\left(s,\frac{u}{N}\right)\right) F_{u,v}(\eta,\zeta) \\ &\quad - \frac{1}{N^d}\left(J\left(s,\frac{u}{N}\right) + J\left(s,\frac{v}{N}\right)\right) G_{u,v}(\eta,\zeta), \end{aligned}$$

where F, G are as in the proof of Theorem 3.1. Similarly, we can treat the case

$b(\zeta(u),\zeta(v)) > b(\eta(u),\eta(v))$. Therefore, for some k

$$N\tilde{\mathscr{L}}A^2 - 2NA\tilde{\mathscr{L}}A \leqq O\left(\frac{1}{N^{d+1}}\right) + \frac{8N\|J\|_\infty}{N^{2d}} \sum_{u,v\in T_{kN}} p(v-u)|b(\eta(u),\eta(v)) - b(\zeta(u),\zeta(v))|G_{u,v}(\eta,\zeta)$$

(because J has compact support).

The proof is complete if we can show:

Lemma 3.3. *For every positive* k,

$$\tilde{E}^N \int_0^t \frac{N}{N^d} \sum_{u,v\in T_{kN}} p(v-u)|b(\eta_s(u),\eta_s(v)) - b(\zeta_s(u),\zeta_s(v))|G_{u,v}(\eta_s,\zeta_s)ds = O(1). \quad (3.15)$$

Proof. Take $f(\eta,\zeta) = \frac{1}{N^d}\sum_{u\in T_{kN}} |\eta(u)-\zeta(u)|$. Then

$$\tilde{E}^N f(\eta_0,\zeta_0) - \tilde{E}^N f(\eta_t,\zeta_t) = -\tilde{E}\int_0^t N\tilde{\mathscr{L}} f(\eta_s,\zeta_s)ds.$$

The left-hand side is uniformly bounded in N, because $\mu^N \leqq \nu^{\rho_0}$ and η is integrable with respect to ν^{ρ_0}. The right-hand side is almost equal to the left-hand side of (3.15), except some error coming from the terms corresponding to the sites on the boundary of T_{kN}. The number of sites on the boundary of T_{kN} is of order N^{d-1} and this multiplied by N/N^d is uniformly bounded. □

4. Tightness

In this section we will prove a preliminary lemma that will be used in Sects. 5 and 6.

Let μ^N and P^N be as in Theorem 3.1, where P^N is viewed as a probability measure on the Skorohod space $D([0,\infty),E)$. Let $\mathscr{M}(\mathbb{R}^d)$ denote the space of Radon measures on $\mathbb{R}^d$, endowed with the topology of vague convergence. For each trajectory $(\eta_t : t\in[0,\infty))$ in $D([0,\infty),E)$, we define

$$\alpha^N(t,dx) = \frac{1}{N^d}\sum_u \delta_{u/N}(dx)\eta_t(u).$$

The law of α^N with respect to P^N, will give us a probability measure Q^N on the Skorohod space $D([0,\infty);\mathscr{M}(\mathbb{R}^d))$.

Lemma 4.1. *Suppose that for some ρ_0, $\mu^N \leqq \nu^{\rho_0}$ for all N. Then the sequence $\{Q^N\}$ is tight. Moreover if Q is any limit point of $\{Q^N\}$, then for almost all $\alpha(\cdot,\cdot)$ with respect to Q, $\alpha(t,\cdot)$ is weakly continuous in t.*

Proof. For the tightness of $\{Q^N\}$, and the continuity of $\alpha(t,\cdot)$, it suffices to show

$$\lim_{\delta\to\infty}\limsup_{N\to\infty} E^N \sup_{|t-s|<\delta} |\int J(x)\alpha^N(t,dx) - \int J(x)\alpha^N(s,dx)| = 0 \quad (4.1)$$

for any test function $J \in C_0^\infty(\mathbb{R}^d)$. Let $f(\eta) = \frac{1}{N^d}\sum J\left(\frac{u}{N}\right)\eta(u)$. Then

$$f(\eta_t) = f(\eta_0) + \int_0^t N\mathscr{L}f(\eta_s)ds + W_t,$$

where W_t is a martingale. On the other hand

$$E^N W_t^2 = E^N \int_0^t (N\mathscr{L}f^2 - 2Nf\mathscr{L}f)(\eta_s)ds.$$

A direct computation shows that the right-hand side is of order $O(1/N^d)$. Therefore $E^N \sup_{0 \leqq s \leqq t} W_s^2 = O\left(\frac{1}{N^d}\right)$. Thus

$$\frac{1}{N^d}\sum_u J\left(\frac{u}{N}\right)\eta_t(u) = \frac{1}{N^d}\sum_u J\left(\frac{u}{N}\right)\eta_s(u) + \frac{1}{N^d}\sum_{u,v}\int_s^t p(v-u)b(\eta_\theta(u),\eta_\theta(v))(u-v)\cdot\nabla J\left(\frac{u}{N}\right)d\theta + r_N + O\left(\frac{1}{N^d}\right), \tag{4.2}$$

where the second term is obtained after calculating $N\mathscr{L}f(\eta_s)$, summing by parts, and replacing $J\left(\frac{u}{N}\right) - J\left(\frac{v}{N}\right)$ with $\frac{1}{N}(u-v)\cdot\nabla J\left(\frac{u}{N}\right)$. r_N is the error coming from such a replacement, and since p is of finite range, the error r_N is uniformly of order $O\left(\frac{1}{N}\right)$. Clearly (4.2) implies (4.1). □

5. Hydrodynamic Limit and Measure Valued Solutions

In this section we derive the hydrodynamic equation by showing that the averages M_{T_l} in Theorem 3.1 will coincide with the macroscopic densities (as l tends to infinity), if the initial distribution μ^N satisfies certain conditions. Let Q^N be as in Sect. 4.

Theorem 5.1. *Suppose $\mu^N \sim \rho$ for some $\rho \in L^\infty(\mathbb{R}^d)$. Then the sequence $\{Q^N\}$ converges weakly to Q, where Q is concentrated on the single path $\alpha(t,dx) = \rho(t,x)dx$ that satisfies*

$$\int_0^\infty \int_{\mathbb{R}^d} \frac{\partial J}{\partial t}(t,x)|\rho(t,x) - c|dxdt + \int_0^\infty \int_{\mathbb{R}^d} \gamma\cdot\nabla J(t,x)q(\rho(t,x);c)dxdt \geqq 0 \tag{5.1}$$

and

$$\lim_{t\to\infty} \int_{|x|\leqq k} |\rho(t,x) - \rho(x)|dx = 0 \tag{5.2}$$

for each positive k, every $c \in \mathbb{R}$, and any nonnegative function $J \in C_0^\infty((0,\infty)\times\mathbb{R}^d)$.

The first step is to prove the above theorem when the initial density is integrable.

Lemma 5.2. *The conclusions of Theorem 5.1 hold if* $\rho \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. *Moreover*

$$\lim_{t \to 0} \int |\rho(t,x) - \rho(x)| dx = 0. \tag{5.3}$$

In order to prove Lemma 5.2, we appeal to more machinery from P.D.E. First we consider a more flexible notion of solutions, measure-valued solutions. We will also consider measure-valued solutions that satisfy the entropy inequalities (they will be called mve solutions in this paper).

Definition 5.3. Let $\pi(t,x;d\lambda)$ be a measurable map from $[0,\infty) \times \mathbb{R}^d$ into the space of probability measures over some bounded interval $[0,\rho_0]$. Then we say π is a mve solution if

$$\int_0^\infty \int_{\mathbb{R}^d} \left\{ \frac{\partial J}{\partial t}(t,x) \int_0^{\rho_0} \pi(t,x;d\lambda)|\lambda - c| + \gamma \cdot \nabla J(t,x) \int_0^{\rho_0} \pi(t,x;d\lambda) q(\lambda;c) \right\} dx\, dt \geqq 0$$

for any nonnegative test function $J \in C_0((0,\infty) \times \mathbb{R}^d)$ and all $c \in \mathbb{R}$. Any distributional entropy solution is also a mve solution. To see this choose $\pi(t,x;d\lambda) = \delta_{\rho(t,x)}(d\lambda)$, where $\rho(t,x)$ is a distributional entropy solution (i.e. satisfying (5.1)). The converse is also true if, in some sense, π does not oscillate for small t.

Lemma 5.4. *Suppose* π *is a* mve *solution satisfying the following conditions*:

(1) $$\sup_t \int_{\mathbb{R}^d} \left\{ \int_0^{\rho_0} \pi(t,x;d\lambda)|\lambda| \right\} dx < \infty,$$

(2) $$\lim_{t \to 0} \int_{\mathbb{R}^d} \left\{ \int_0^{\rho_0} \pi(t,x;d\lambda)|\lambda - \rho(x)| \right\} dx = 0$$

for some $\rho \in L^1 \cap L^\infty$. *Then* $\pi(t,x;d\lambda) = \delta_{\rho(t,x)}(d\lambda)$, *where* $\rho(\cdot,\cdot)$ *is the unique entropy solution satisfying* (5.1) *with initial condition* $\rho(0,x) = \rho(x)$.

We refer the reader to [9] for an excellent account on mve solutions and for the proof of the above lemma.

Let X denote the space of measurable maps from $[0,\infty)$ into $\mathcal{M}(\mathbb{R}^d \times \mathbb{R}^+)$ (the space of Radon measures endowed with the vague topology). As before η_t is the Markov process generated by $N\mathcal{L}$, with initial distribution μ^N. We assume $\mu^N \leqq \nu^{\rho_0}$ for some ρ_0.

Associated with a configuration η_t, we define the *Young measure* $\pi^{N,l}$ by

$$\int F(x,\lambda)\pi^{N,l}(t,dx;d\lambda) = \frac{1}{N^d}\sum_u F\left(\frac{u}{N}, M_{T_l(u)}(\eta_t)\right)$$

and the measure α^N by

$$\int J(x)\alpha^N(t,dx) = \frac{1}{N^d}\sum_u J\left(\frac{u}{N}\right)\eta_t(u)$$

for any $F \in C_0(\mathbb{R}^d \times \mathbb{R}^+)$. (See also [15] where the Young measures in the above form were introduced.) Note that $\pi^{N,l}$ is related to α^N by the formula

$$\int J(x)\lambda\pi^{N,l}(t;dx,d\lambda) \approx \int J(x)\alpha^N(t,dx), \quad J \in C_0(\mathbb{R}^d), \tag{5.4}$$

where the error is uniformly small in N and l.

The map $\eta_t \mapsto (\pi^{N,l}, \alpha^N)$ induces a probability measure $R^{N,l}$ on the space X. It is not hard to show that the sequence $\{R^{N,l}\}$ is tight as a sequence of probability measures on the space X. So we can take limit points of $\{R^{N,l}\}$ as N goes to infinity. If $\{R^l\}$ is a sequence of these limit points, we can further take their limit points as l tends to infinity. Let R be any limit point of $\{R^l\}$. Our main steps towards the proof of Lemma 5.2, are the following lemmas.

Lemma 5.5. *For almost all (π, α) with respect to R, we have*

(a) *For some measurable functions $\rho(t,x)$ and $\pi(t,x;d\lambda)$, we have*

$$\alpha(t,dx) = \rho(t,x)dx, \tag{5.5}$$

$$\pi(t,dx;d\lambda) = \pi(t,x;d\lambda)dx, \tag{5.6}$$

$$\int \lambda \pi(t,x;d\lambda) = \rho(t,x), \tag{5.7}$$

(b)
$$\pi(t,x;\mathbb{R} - [0,\rho_0]) = 0,$$

(c) *π is a* mve *solution.*

Lemma 5.6. *Suppose $\mu^N \sim \rho$, where ρ is a Lipschitz continuous function, with compact support. Then for almost all (π, α) with respect to R, we have*

(a)
$$\lim_{t \to 0} \int_{\mathbb{R}^d} \left\{ \int_0^{\rho_0} \pi(t,x;d\lambda)|\lambda - \rho(x)| \right\} dx = 0,$$

(b)
$$\pi(t,x;d\lambda) = \delta_{\rho(t,x)}(d\lambda).$$

Before proving these two lemmas, we first show how they imply Lemma 5.2.

Proof of Lemma 5.2.

Step 1. By Lemma 4.1, the sequence $\{Q^N\}$ is tight, and Q lives on $\alpha(t,dx)$ with α being weakly continuous in t. If we choose an initial density $\rho(\cdot)$ a smooth function with compact support, it certainly satisfies the assumptions of Lemmas 5.6. A combination of Lemma 5.5(c) and Lemma 5.6(b), proves (5.1), and Lemma 5.6(a) implies (5.2).

Step 2. Suppose $\rho(\cdot) \in L^\infty(\mathbb{R}^d)$ is any integrable function. Then, we may pick a sequence of $\{\rho_\varepsilon(\cdot)\} \subseteqq C_0^\infty(\mathbb{R}^d)$ such that

$$\int |\rho_\varepsilon(x) - \rho(x)| dx < \varepsilon$$

for any $\varepsilon > 0$. For each ε, let $\mu^{N,\varepsilon}$ be defined as

$$\mu^{N,\varepsilon}(d\eta) = \prod_u \Theta^{\rho_\varepsilon(u/N)}(d\eta(u)).$$

Recall that $\mu^N \sim \rho$. We also have $\mu^{N,\varepsilon} \sim \rho_\varepsilon$. For each ε, we construct a coupling $\tilde{\mu}^{N,\varepsilon}$ of μ^N and $\mu^{N,\varepsilon}$ such that

$$\eta(u) \leqq \zeta(u) \quad \text{if} \quad \rho_{u,N} \leqq \rho_\varepsilon\left(\frac{u}{N}\right), \quad \text{and} \quad \eta(u) \geqq \zeta(u) \quad \text{if} \quad \rho_{u,N} \geqq \rho_\varepsilon\left(\frac{u}{N}\right), \tag{5.8}$$

for all $u \in \mathbb{Z}^d$ and with probability one with respect to $\tilde{\mu}^{N,\varepsilon}$. Let $\tilde{P}^{N,\varepsilon}$ be the law of the process (η_t, ζ_t) generated by $N\tilde{\mathscr{L}}$, with initial distribution $\tilde{\mu}^{N,\varepsilon}$. Define

$$F^N(\eta,\zeta) = \frac{1}{N^d}\sum_u |\eta(u) - \zeta(u)|.$$

We first show $F^N(\eta,\zeta)$ is finite with probability one with respect to $\tilde{P}^{N,\varepsilon}$. In fact

$$\tilde{E}^{N,\varepsilon}F(\eta_0,\zeta_0) = \frac{1}{N^d}\sum_u \left|\rho_{u,N} - \rho_\varepsilon\left(\frac{u}{N}\right)\right|$$
$$= \int |\rho(x) - \rho_\varepsilon(x)|\,dx + r_N(\varepsilon), \tag{5.9}$$

where the first equality follows from (5.8), the second equality follows from $\mu^N \sim \rho$, and $r_N(\varepsilon)$ is an error that goes to zero as N tends to infinity. On the other hand

$$\tilde{E}^{N,\varepsilon}F(\eta_t,\zeta_t) = \tilde{E}^{N,\varepsilon}F(\eta_0,\zeta_0) + \int_0^t \left(\frac{1}{N^d}\sum_u N\tilde{E}^{N,\varepsilon}\tilde{\mathscr{L}}|\eta_s - \zeta_s|\right)ds$$
$$\leqq \tilde{E}^{N,\varepsilon}F(\eta_0,\zeta_0) \tag{5.10}$$

because if we add up the right-hand side of (3.2) over u, the first two sums cancel, and the third sum is always negative. Therefore, by (5.9)

$$\tilde{E}^{N,\varepsilon}\frac{1}{N^d}\sum_u |\eta_t(u) - \zeta_t(u)| \leqq \varepsilon + r_N(\varepsilon). \tag{5.11}$$

Step 3. Let $\tilde{Q}^{N,\varepsilon}$ be the law of the pair $(\alpha^N, \alpha^{N,\varepsilon})$ with respect to $\tilde{P}^{N,\varepsilon}$, where

$$\alpha^{N,\varepsilon}(t,dx) = \frac{1}{N^d}\sum_u \delta_{u/N}(dx)\zeta_t(u).$$

Let $\tilde{Q}^\varepsilon$ be any limit point of $\{\tilde{Q}^{N,\varepsilon}\}$, as N goes to infinity. Then for almost all pairs $(\alpha, \alpha^\varepsilon)$ with respect to $\tilde{Q}^\varepsilon$,

$$\alpha(t,dx) = \rho(t,x)dx,$$
$$\alpha^\varepsilon(t,dx) = \rho_\varepsilon(t,x)dx,$$

where $\rho_\varepsilon(t,x)$, by Step 1, satisfies (5.1) and (5.3). On the other hand, using (5.11), we have

$$E^{\tilde{Q}^\varepsilon}\int |\rho(t,x) - \rho_\varepsilon(t,x)|\,dx \leqq 2\varepsilon$$

for any $t \geqq 0$. This is because the functional $\alpha \mapsto \|\alpha\|$ is lower semicontinuous with respect to the vague convergence ($\|\cdot\|$ denotes the total variation). The α-marginal of $\tilde{Q}^\varepsilon$ is Q, therefore

$$E^Q\int |\rho(t,x) - \rho_\varepsilon(t,x)|\,dx \leqq 2\varepsilon,$$

because $\rho_\varepsilon(t,x)$ is uniquely determined by (5.1), (5.3) with ρ replaced with ρ_ε. According to Theorem 1 of [11], $\rho_\varepsilon(\cdot,\cdot)$ converges in L^1 sense, as $\varepsilon \to 0$, to the unique solution satisfying (5.1) and (5.3) with initial condition $\rho(\cdot)$. Thus, Q is concentrated on that unique limit, and this completes the proof. □

Proof of Lemma 5.5. Let f be a bounded function. Then for any positive $J \in C_0(\mathbb{R}^d)$, we have

$$\int J(x) f(\lambda) \pi^{N,l}(t, dx; d\lambda) \leqq \|f\|_\infty \left(\frac{1}{N^d} \sum_u J\left(\frac{u}{N}\right)\right).$$

This implies (5.6) and hence (5.5). Relation (5.7) follows from (5.3). To prove part (b), it suffices to establish

$$\lim_{l \to \infty} \limsup_{N \to \infty} \frac{1}{N^d} \sum_u J\left(\frac{u}{N}\right) P^N(M_{T_l(u)}(\eta_t) > \rho_1) = 0, \tag{5.12}$$

for any $\rho_1 > \rho_0$, and any test function $J \in C_0(\mathbb{R}^d)$. This can be shown by first coupling η_t with ζ_t, where ζ_t is initially distributed as ν^{ρ_0} and $\eta_t \leqq \zeta_t$, then replacing η with ζ in (5.12).

Finally, part (c) is nothing but a restatement of Theorem 3.1.

Proof of Lemma 5.6. Part (b) follows from part (a), using Lemmas 5.4 and 5.5, and

$$\begin{aligned} E^N \int_{\mathbb{R}^d} \int_0^\infty \lambda \pi^{N,\lambda}(t, dx, d\lambda) &= E^N \frac{1}{N^d} \sum_u \eta_t(u) \\ &= E^N \frac{1}{N^d} \sum_u \eta_0(u) \\ &= \int_{\mathbb{R}^d} \rho(x) dx < \infty, \end{aligned}$$

where the second identity is because the total number of particles is conserved, and the last identity follows from $\mu^N \sim \rho$.

We now turn to the proof of (a). It is enough to show

$$\lim_{t \to 0} \limsup_{l \to \infty} \limsup_{N \to \infty} E^N \frac{1}{N^d} \sum_u \left| M_{T_l(u)}(\eta_t) - \rho\left(\frac{u}{N}\right)\right| = 0. \tag{5.13}$$

The proof of this is carried out in several steps:

Step 1. For every $u \in \mathbb{Z}^d$, let ζ_t^u denote the Markov process generated by $N\mathscr{L}$ and initially distributed according to $\nu^{\rho(u/N)}$. For any pair $u, v \in \mathbb{Z}^d$, we construct a coupling measure $\mu_{u,v}^N$ on E^3

$$E^3 = \{(\eta, \zeta^u, \zeta^v) : \eta, \zeta^u, \zeta^v \in \mathbb{N}^{\mathbb{Z}^d}\}$$

with the following properties

$$\begin{aligned} &\eta\text{-marginal of } \mu_{u,v}^N \text{ is } \mu^N, \\ &\zeta^u\text{-marginal of } \mu_{u,v}^N \text{ is } \nu^{\rho(u/N)}, \\ &\zeta^v\text{-marginal of } \mu_{u,v}^N \text{ is } \nu^{\rho(v/N)}, \\ &\mu_{u,v}^N(\zeta^u \leqq \zeta^v \text{ or } \zeta^v \leqq \zeta^u) = 1, \quad \text{and} \\ &\mu_{u,v}^N(\zeta^u(u) = \eta(u) \quad \text{and} \quad \zeta^v(v) = \eta(v)) = 1. \end{aligned} \tag{5.14}$$

Such coupling can be constructed in the following way: for each $w \in \mathbb{Z}^d$, construct a coupling $\Theta^{w,u,v}$ of $\Theta^{\rho(w/N)}$, $\Theta^{\rho(u/N)}$ and $\Theta^{\rho(v/N)}$ with the above properties at site w, and then take the product of $\Theta^{w,u,v}$ over $w \in \mathbb{Z}^d$.

Next we couple the processes $\eta_t, \zeta_t^u, \zeta_t^v$ such that $(\eta_0, \zeta_0^u, \zeta_0^v)$ distributed according to $\mu_{u,v}^N$, and every two component of $(\eta_t, \zeta_t^u, \zeta_t^v)$ is generated by $N\tilde{\mathscr{L}}$. The generator of this three-process coupling is defined in a manner analogous to the definition of $\tilde{\mathscr{L}}$ (the particles of three coordinates at each site jump together as much as possible), and we omit its formal definition. The law of this coupling will be denoted by $P_{u,v}^N$ and expectations with respect to $P_{u,v}^N$ will be denoted by $E_{u,v}^N$. We also write E_u^N for the expectations involving coupled process (η_t, ζ_t^u).

We are now ready to rewrite (5.13) as

$$\lim_{t\to 0}\limsup_{l\to\infty}\limsup_{N\to\infty}\frac{1}{N^d}\sum_u E_u^N|M_{T_l(u)}(\eta_t)-M_{T_l(u)}(\zeta_t^u)|=0, \tag{5.15}$$

because by the Ergodic Theorem, $M_{T_l(u)}(\zeta_t^u)$ can be replaced with $\rho(u/N)$. Note that such replacement is uniform in u because ρ has compact support, and the convergence in (3.11) is uniform in bounded c-intervals.

Step 2. We would like to replace $\zeta_t^u(v)$ with $\zeta_t^v(v)$ for every $v\in T_l(u)$, and for this we need to show

$$\lim_{t\to 0}\limsup_{l\to\infty}\limsup_{N\to\infty}\frac{1}{N^d}\sum_u\frac{1}{|T_l|}\sum_{v\in T_l(u)}E_{u,v}^N|\zeta_t^u(v)-\zeta_t^v(v)|=0. \tag{5.16}$$

Without loss of generality, we may assume $\rho\left(\frac{u}{N}\right)\leqq\rho\left(\frac{v}{N}\right)$. This in turn implies $\zeta_0^u\leqq\zeta_0^v$ almost surely and therefore $\zeta_t^u\leqq\zeta_t^v$ for all $t\geqq 0$, because (ζ_t^u,ζ_t^v) is generated by $N\tilde{\mathscr{L}}$. Hence

$$\begin{aligned}E_{u,v}^N|\zeta_t^u(v)-\zeta_t^v(v)| &= E_{u,v}^N(\zeta_t^v(v)-\zeta_t^u(v))\\ &= E_{u,v}^N(\zeta_0^v(v)-\zeta_0^u(v))\\ &= \rho\left(\frac{v}{N}\right)-\rho\left(\frac{u}{N}\right),\end{aligned} \tag{5.17}$$

because ζ_t^v and ζ_t^u are separately at equilibrium. Now (5.16) is clear because ρ is uniformly Lipschitz continuous and it has a bounded support.

Step 3. Because of (5.16), we only need to check

$$\lim_{t\to 0}\limsup_{N\to\infty}\frac{1}{N^d}\sum_v E_v^N|\eta_t(v)-\zeta_t^v(v)|=0.$$

First we start with the basic identity

$$E_v^N|\eta_t(v)-\zeta_t^v(v)|=E_v^N|\eta_0(v)-\zeta_0^v(v)|+\int_0^t N\tilde{\mathscr{L}}(|\eta_s(v)-\zeta_s^v(v)|)ds.$$

By our assumption (5.14), the first term on the right-hand side vanishes. For the second term, we can use (3.2) to write

$$\begin{aligned}\tilde{\mathscr{L}}|\eta(v)-\zeta^v(v)|\leqq\sum_w[&p(w-v)(b(\zeta^v(v),\zeta^v(w))-b(\eta(v),\eta(w)))\\ &-p(v-w)(b(\zeta^v(w),\zeta^v(v))-b(\eta(w),\eta(v)))]\cdot F_{v,w}(\eta,\zeta^v)\end{aligned}$$

$$+\sum_{w}[p(v-w)(b(\zeta^v(w),\zeta^v(v))-b(\eta(w),\eta(v)))$$
$$-p(w-v)(b(\zeta^v(v),\zeta^v(w))-b(\eta(v),\eta(w)))]\cdot F_{v,w}(\zeta^v,\eta),$$

where

$$F_{v,w}(\eta,\zeta)=\begin{cases}1 & \text{if} \quad \eta(v)\geqq\zeta(v) \quad \text{and} \quad \eta(w)\geqq\zeta(w)\\ 0 & \text{otherwise}\end{cases}$$

Therefore

$$\frac{1}{N^d}\sum_{v}E_v^N|\eta_t(v)-\zeta_t^v(v)|$$
$$\leqq\frac{N}{N^d}\int_0^t\left\{\sum_{v,w}E_{v,w}^N[p(w-v)(b(\zeta_s^v(v),\zeta_s^v(w))-b(\eta_s(v),\eta_s(w)))F_{v,w}(\eta_s,\zeta_s^v)]\right.$$
$$-\sum_{v,w}E_{v,w}^N[p(v-w)(b(\zeta_s^v(w),\zeta_s^v(v))-b(\eta_s(w),\eta_s(v)))F_{v,w}(\eta_s,\zeta_s^v)]$$
$$+\sum_{v,w}E_{v,w}^N[p(v-w)(b(\zeta_s^v(w),\zeta_s^v(v))-b(\eta_s(w),\eta_s(v)))F_{v,w}(\zeta_s^v,\eta_s)]$$
$$\left.-\sum_{v,w}E_{v,w}^N[p(w-v)(b(\zeta_s^v(v),\zeta_s^v(w))-b(\eta_s(v),\eta_s(w)))F_{v,w}(\zeta_s^v,\eta_s)]\,ds\right\}$$
$$=I+II+III+IV.$$

Our goal is to show the left-hand side goes to zero. For this we only prove $I+II$ goes to zero, because $III+IV$ can be treated in the same way.

First we exchange v with w in II to obtain

$$I+II=\frac{N}{N^d}\int_0^t\left\{\sum_{v,w}E_{v,w}^N[p(w-v)(b(\zeta_s^v(v),\zeta_s^v(w))-b(\eta_s(v),\eta_s(w)))F_{v,w}(\eta_s,\zeta_s^v)]\right.$$
$$\left.-\sum_{v,w}E_{v,w}^N[p(w-v)(b(\zeta_s^w(v),\zeta_s^w(w))-b(\eta_s(v),\eta_s(w)))F_{v,w}(\eta_s,\zeta_s^w)]\right\}ds.$$

Further, we use the estimates

$$|F_{v,w}(\eta,\zeta^v)-F_{v,w}(\eta,\zeta^w)|\leqq|\zeta^v(v)-\zeta^w(v)|+|\zeta^v(w)-\zeta^w(w)|,$$

and

$$|b(\zeta^v(v),\zeta^v(w))-b(\zeta^w(v),\zeta^w(w))|\leqq\|b\|_\infty(|\zeta^v(v)-\zeta^w(v)|+|\zeta^v(w)-\zeta^w(w)|)$$

to write

$$I+II\leqq\frac{N}{N^d}\int_0^t\left\{4\|b\|_\infty\sum_{v,w}E_{v,w}^{N}p(u-w)(|\zeta_s^v(v)-\zeta_s^w(v)|+|\zeta_s^v(w)-\zeta_s^w(w)|)\right\}ds$$
$$=8\|b\|_\infty\frac{Nt}{N^d}\sum_{v,w}p(v-w)\left|\rho\left(\frac{v}{N}\right)-\rho\left(\frac{w}{N}\right)\right|,$$

where for the last equality we have used (5.17).

Finally since p is of finite range, and ρ is uniformly Lipschitz continuous

$$\limsup_{N\to\infty}(I+II) \leqq c_0 t$$

for some c_0, and this goes to zero as t tends to zero. □

Proof of Theorem 5.1. For any positive integer k, let ρ_k be an integrable function such that $\rho_k(x)=\rho(x)$ for $|x|\leqq k$. We also choose a probability measure $\mu^{N,k}$ and a coupling $\tilde{\mu}^{N,k}$, with the following properties:

(a) $\mu^{N,k} \sim \rho_k$
(b) η-marginal of $\tilde{\mu}^{N,k} = \mu^N$, ζ-marginal of $\tilde{\mu}^{N,k} = \mu^{N,k}$, and
(c) $\tilde{\mu}^{N,k}\{(\eta,\zeta):\eta(u)=\zeta(u) \text{ for all } |u|\leqq kN\}=1$.

Let (η_t,ζ_t) be the process generated by $N\tilde{\mathscr{L}}$, and initially distributed as $\tilde{\mu}^{N,k}$. According to Lemma 5.7 there exists a constant c_0 such that, $\eta_t=\zeta_t$ on $[-kN+c_0tN, kN-c_0tN]$ with probabilities close to 1.

Let $\rho_k(t,x)$ be the unique entropy solution of the hydrodynamic solution with $\rho_k(0,x)=\rho_k(x)$. If Q is as in Theorem 5.1, we then have $Q(\rho(t,x)=\rho_k(t,x)$ for $|x|\leqq k-c_0t)=1$. On the other hand $\rho_k(t,x)$ converges to the unique solution of the hydrodynamic equation satisfying (5.1) and (5.2), and this completes the proof. □

Lemma 5.7. *Suppose*

$$\tilde{P}^N(\eta_0(u)=\zeta_0(u) \text{ for } |u|\leqq kN)=1,$$

then

$$\lim_{N\to\infty}\tilde{P}^N(\eta_t(u)=\zeta_t(u) \text{ for } |u|\leqq (k-c_0t)^+N)=1, \tag{5.18}$$

where c_0 is a constant independent of N, k and t.

Proof. We would like to estimate how far η_t stays equal to ζ_t. For this we label η and ζ particles with superscript indices. For each $q\in\mathbb{N}$, η_t^q (respectively ζ_t^q) is the position of the q^{th} particle at time t. In particular

$$\{\#q:\eta_t^q=u\}=\eta_t(u)$$

with a similar relation between ζ^q and ζ. The labels q are chosen so that

$$\eta_0^q=\zeta_0^q \quad \text{if} \quad |\eta_0^q| \text{ or } |\zeta_0^q|\leqq kN$$

with probability one with respect to $\tilde{P}^N$. Suppose $\omega_N(t)$ is a suitable random variable such that

$$|\eta_t^q|, |\zeta_t^q| > \omega_N(t) \quad \text{if} \quad |\eta_0^q| \text{ or } |\zeta_0^q| > kN.$$

Then, from the way the coupled process is defined, we certainly have

$$\eta_t(u)=\zeta_t(u) \quad \text{if} \quad |u|\leqq \omega_N(t).$$

Thus, our task is to show that there exists a constant c_0 such that

$$\lim_{N\to\infty}\tilde{P}^N(\omega_N(t)\leqq (k-c_0t)N)=0. \tag{5.19}$$

We can view η_t^q and ζ_t^q as continuous time random walks in random media, whose holding times have rates bounded by $\|b(\cdot,\cdot)\|_\infty$ and whose transition probabilities are $p(v-u)$. Therefore we can couple (η_t^q, ζ_t^q) with a continuous random walk y_t^q with the following properties.

(1) The jump rate of y^q is $2\|b(\cdot,\cdot)\|_\infty$,
(2) $y_t^q - y_{t-}^q$ is either 0 or r_0,
(3) $|\eta_t^q - \eta_0^q|$ and $|\zeta_t^q - \zeta_0^q| \leqq y_{tN}^q$,

where r_0 is a bound on the range of p. It is not hard to see

$$\log Ee^{\lambda y_t^q} = 2t\|b\|_\infty (e^{\lambda r_0} - 1),$$

for every $\lambda \in \mathbb{R}$. This will guarantee the existence of a constant a_0 such that for every $a > a_0$,

$$P\left(\left|\frac{y_t^q}{t}\right| > a\right) \leqq e^{-ta}$$

(use Chebyshev's inequality). Thus

$$\tilde{P}^N(|\eta_t^q - \eta_0^q| > tNa) \leqq e^{-tNa},$$
$$\tilde{P}^N(|\zeta_t^q - \zeta_0^q| > tNa) \leqq e^{-tNa}.$$

Set $c_0 = a_0 + 1$. Then

$$\begin{aligned} P(|\eta_t^q| \leqq (k - c_0 t)N, |\eta_0^q| \geqq nN) &\leqq P(|\eta_t^q - \eta_0^q| \geqq (n - k + c_0 t)N) \\ &\leqq \exp(-(n-k+c_0 t)N). \end{aligned} \tag{5.20}$$

Define

$$A_n = \{\#q : nN < |\eta_0^q| \leqq (n+1)N\}.$$

Then, by (5.20)

$$\tilde{P}^N(|\eta_t^q| \leqq (k - c_0 t)N, |\eta_0^q| > kN) \leqq \tilde{E}^N \sum_{n=k}^{\infty} A_n \exp(-(n - k + c_0 t)N). \tag{5.21}$$

Since the initial distribution μ^N is less than ν^{ρ_0} with $\rho_0 = \|\rho\|_\infty$, we can easily estimate

$$\tilde{E}^N A_n \leqq \rho_0 (nN)^d.$$

Thus

$$\lim_{N\to\infty} \tilde{E}^N \sum_{n=k}^{\infty} A_n \exp(-(n - k + c_0 t)N) = 0.$$

Therefore the left-hand side of (5.21) converges to zero. (5.21) also holds if we replace η with ζ, and this will complete the proof of (5.19). □

6. Hydrodynamic Limit and Two Block Estimates

In this section we give a second proof of Theorem 5.1 under the following assumptions.

Assumption 6.1. $d=1$ and $p(1)+p(-1)=1$.

In words, particles move on the one-dimensional lattice $\mathbb{Z}$, and they only jump to nearest neighbor sites.

The following lemma is the crux of our approach in this section.

Lemma 6.2. *Under Assumption 6.1, and supposing $\mu^N \sim \rho$ for some $\rho \in L^\infty(\mathbb{R})$, we have*

$$\lim_{\varepsilon\to 0} \limsup_{l\to\infty} \limsup_{N\to\infty} \frac{1}{N} \sum_{u\in T_{kN}} P^N(|M_{T_l(u)}(\eta_t) - M_{T_{N\varepsilon}(u)}(\eta_t)| > \delta) = 0 \tag{6.1}$$

for every $k, t, \delta > 0$.

Before proving the above lemma, we discuss how a combination of Theorem 3.1 and Lemma 6.2 will imply the validity of the hydrodynamic limit. According to Theorem 3.1, the entropy inequalities hold for quantities involving averages over microscopically large blocks. For the hydrodynamic limit, however, we need the entropy inequalities with averages over macroscopically small blocks. The role of Lemma 6.2 is to fill the gap by showing that these two averages are close.

Our strategy for the proof of Lemma 6.2 is as follows: we first prove (6.1) for a class of functions ρ, that includes functions of bounded variation. We then extend the result to the class of bounded measurable functions.

We first start with some definitions.

Definition 6.3. *Let c be any positive value. We then call c a finite cross value of $\rho(\cdot)$ on an interval (x,y), if there exists a finite sequence $x=x_0<x_1<\dots<x_{r-1}<x_r=y$ such that for any $0\leqq i\leqq r-1$ either $\rho\leqq c$ or $\rho\geqq c$ on the interval $[x_i,x_{i+1}]$.*

Definition 6.4. *We say a function $\rho\in L^\infty(\mathbb{R})$ is admissible on an interval (x,y), if there exists a countable dense sequence $\{c_n\}$ such that every c_n is a finite cross value of ρ on (x,y).*

For any two configurations η,ζ, we let $\mathcal{N}(\eta,\zeta)$ denote the number of changes of sign of the sequence $(\eta(u)-\zeta(u):u\in\mathbb{Z})$. In other words, if $\mathcal{N}(\eta,\zeta)\leqq n+2$, then there exists a finite sequence $u_1<u_2<\dots<u_n$ such that on each interval $[u_i,u_{i+1}]\cap\mathbb{Z}$, and $(-\infty,u_1)\cap\mathbb{Z}$, $[u_n,+\infty)\cap\mathbb{Z}$, either $\eta\leqq\zeta$ or $\zeta\leqq\eta$.

For any bounded continuous density $\rho(\cdot)$, define μ^N by

$$\mu^N(d\eta) = \prod_{u\in\mathbb{Z}} \Theta^{\rho(u/N)}(d\eta(u)). \tag{6.2}$$

For any constant $c\in[0,\infty)$, let $\tilde{\mu}^{N,c}$ denote a coupling of μ^N and ν^c for which the following relations hold:

$$\eta(u)\leqq\zeta(u) \quad \text{if} \quad \rho\left(\frac{u}{N}\right)\leqq c,$$

and

$$\eta(u)\geqq\zeta(u) \quad \text{if} \quad \rho\left(\frac{u}{N}\right)\geqq c.$$

Let $\tilde{P}^{N,c}$ denote the law of the coupled process (η_t,ζ_t) generated by $N\tilde{\mathcal{L}}$ and initially distributed as $\tilde{\mu}^{N,c}$. If c is a finite cross value of ρ, we then have

$$\tilde{\mu}^{N,c}(\mathcal{N}(\eta,\zeta)\leqq r)=1$$

for some bound r independent of N. Next we show that $\mathcal{N}(\eta_t, \zeta_t)$ does not increase in t:

Lemma 6.5. $\tilde{P}^{N,c}(\mathcal{N}(\eta_t, \zeta_t) \leqq \mathcal{N}(\eta_0, \zeta_0)) = 1$.

Proof. Suppose that our process (η_t, ζ_t) starts from a configuration (η_0, ζ_0) for which $\mathcal{N}(\eta_0, \zeta_0) = r_0 + 2$. Therefore, there exists a sequence of sites $-\infty = u_0 < u_1 < \cdots < u_{r_0} < u_{r_0+1} = +\infty$ such that $\eta_0 - \zeta_0$ is of a definite sign over every block $[u_i, u_{i+1}] \cap \mathbb{Z}$. We call a block positive (respectively negative) if $\eta_0 - \zeta_0$ is positive (respectively negative) on that block. When we start the evolution, these blocks either shrink, or expand, or stay unchanged in length (while keeping their sign unchanged), but no new block will be created. This follows from the way the coupling process is constructed. Of course it is possible that a block shrinks to zero, thus $\mathcal{N}(\eta_t, \zeta_t)$ could decrease.

One can make the above argument rigorous by showing $\tilde{\mathscr{L}}F(\eta, \zeta) \leqq 0$, where $F(\eta, \zeta) = 1(\mathcal{N}(\eta, \zeta) > r_0 + 2)$. □

Lemma 6.6. *The conclusion of Theorem 6.2 holds if μ^N is defined as (6.2), and $\rho(\cdot)$ is continuous and admissible on* $\mathbb{R}$.

Proof. Let $\{c_n\}$ be a countable dense subset of $[0, \infty)$ such that every c_n is a finite-cross value of ρ. For each n, there exists a constant r_n such that $\mu^{N,c_n}(\mathcal{N}(\eta, \zeta) \leqq r_n) = 1$ uniformaly in N. By Lemma 6.5, we have

$$\tilde{P}^{N,c_n}(\mathcal{N}(\eta_t, \zeta_t) \leqq r_n) = 1. \tag{6.3}$$

In particular, for every positive k, there exists some $\tilde{r}_n(k)$ such that

$$\frac{1}{N}\{\#i{:}i \in [-kN, kN] \text{ such that } \eta_t \nleqq \zeta_t \text{ or } \zeta_t \nleqq \eta_t \text{ over } [i - N\varepsilon, i + N\varepsilon]\} \leqq \tilde{r}_n(k)\varepsilon, \tag{6.4}$$

with probability one with respect to $\tilde{P}^{N,c_n}$. This is because, by (6.3), there are at most $[2N\varepsilon]\cdot r_n$ intervals of the form $[i - N\varepsilon, i + N\varepsilon]$ that fail to satisfy $\eta_t \leqq \zeta_t$ or $\zeta_t \leqq \eta_t$, providing i is not close to $\pm kN$. Taking into account such i's, we may choose $\tilde{r}_n(k) = 2(3r_n + 4)k$.

The inequality (6.4) implies that if ε is small then for most intervals $[i - N\varepsilon, i + N\varepsilon]$, we either have $\eta_t \leqq \zeta_t$ or $\zeta_t \leqq \eta_t$. In every such interval, we take the averages $M_{T_{N\varepsilon}(i)}(\eta_t)$, $M_{T_{N\varepsilon}(i)}(\zeta_t)$ and $M_{T_l(i)}(\eta_t)$, $M_{T_l(i)}(\zeta_t)$. The same relations hold:

$$M_{T_l(i)}(\eta_t) \leqq M_{T_l(i)}(\zeta_t) \quad \text{and} \quad M_{T_{N\varepsilon}(i)}(\eta_t) \leqq M_{T_{N\varepsilon}(i)}(\zeta_t)$$

or

$$M_{T_l(i)}(\eta_t) \geqq M_{T_l(i)}(\zeta_t) \quad \text{and} \quad M_{T_{N\varepsilon}(i)}(\eta_t) \geqq M_{T_{N\varepsilon}(i)}(\zeta_t),$$

where $l \leqq N\varepsilon$. Since ζ_t is initially distributed according to the equilibrium measure ν^{c_n}, we can replace both $M_{T_l(i)}(\zeta_t)$ and $M_{T_{N\varepsilon}(i)}(\zeta_t)$ with c_n, providing N and l are sufficiently large. Thus, for any positive δ_n

$$\lim_{\varepsilon \to 0} \liminf_{l \to \infty} \liminf_{N \to \infty} \frac{1}{N} \sum_{i \in T_{kN}} P^N(M_{T_l(i)}(\eta_t), M_{T_{N\varepsilon}(i)}(\eta_t) \leqq c_n + \delta_n; \quad \text{or}$$

$$M_{T_l(i)}(\eta_t), M_{T_{N\varepsilon}(i)}(\eta_t) \geqq c_n - \delta_n) = 1.$$

Fix $\delta > 0$. In any bounded interval $[0, A]$, we can find a finite set of finite-cross values $c_{n_1}, c_{n_2}, \ldots, c_{n_q}$, and suitable $\delta_{n_1}, \ldots, \delta_{n_q}$ such that for any pair $x, y \in [0, A]$, if either $x, y \leqq c_{n_i} + \delta_{n_i}$ or $x, y \geqq c_{n_i} + \delta_{n_i}$ for $i = 1, \ldots, q$; then $|x - y| \leqq \delta$. This is certainly possible, because $\{c_n\}$ is dense. Thus, for any positive A we have

$$\lim_{\varepsilon\to 0}\liminf_{l\to\infty}\liminf_{N\to\infty}\frac{1}{N}\sum_{i\in T_{kN}} P^N(|M_{T_l(i)} - M_{T_{N\varepsilon}(i)}| > \delta, \text{or } M_{T_l(i)} > A, \text{or } M_{T_{N\varepsilon}(i)} > A) = 1. \tag{6.5}$$

Since $\rho(\cdot)$ is bounded, we

$$\lim_{\varepsilon\to\infty}\liminf_{l\to\infty}\liminf_{N\to\infty}\frac{1}{N}\sum_{i\in T_{kN}} P^N(M_{T_l(i)} > A \text{ or } M_{T_{N\varepsilon}(i)} > A) = 0,$$

providing $A > \|\rho\|_\infty$. This and (6.5) imply (6.1). □

Proof of Lemma 6.2. So for we have shown (6.1) for admissible ρ's. Using the coupling introduced in the proof of Lemma 5.2 (Step 2) we can extended the result to any $\rho \in L^1$. Using the coupling introduced in the proof of Theorem 5.2, we can extend the result to any $\rho \in L^\infty$. □

Remark 6.7. It is not hard to see that our results in the previous section will imply (6.1), but only if we integrate the sum in the left-hand side over a finite t-interval. This is because our probability measures $R^{N,l}$ in the previous section only converge weakly as probability measures on the space X.

We end this section by giving a proof of (5.1), under Assumption 6.1, using Lemma 6.2:

By Lemma 6.2, we can replace $M_{T_l(u)}$ with $M_{T_{N\varepsilon}(u)}$ is (3.1). Thus

$$\begin{aligned}\lim_{\varepsilon\to 0}\liminf_{N\to\infty} P^N\Bigg\{&\int_0^\infty \frac{1}{N}\sum_u \frac{\partial J}{\partial s}\left(s, \frac{u}{N}\right)|M_{T_{N\varepsilon}(u)}(\eta_s) - c|\,ds \\ &+ \int_0^\infty \frac{1}{N}\sum_u \gamma\frac{\partial J}{\partial x}\left(s, \frac{u}{N}\right) q(M_{T_{N\varepsilon}(u)}(\eta_s); c)\,ds \geqq -\varepsilon\Bigg\} = 1.\end{aligned} \tag{6.6}$$

From the definition of α^N, we have

$$M_{T_{N\varepsilon}(u)}(\eta_t) = \frac{1}{2\varepsilon}\alpha^N\left(t, \left[\frac{u}{N} - \varepsilon, \frac{u}{N} + \varepsilon\right]\right).$$

We then consider the functionals

$$\begin{aligned}V_c(N, \varepsilon; \alpha) = &\int_0^\infty \frac{1}{N}\sum_u \frac{\partial J}{\partial s}\left(s, \frac{u}{N}\right)\left|\frac{\alpha(t, [u/N - \varepsilon, u/N + \varepsilon])}{2\varepsilon} - c\right| ds \\ &+ \int_0^\infty \frac{1}{N}\sum_u \gamma\frac{\partial J}{\partial x}\left(s, \frac{u}{N}\right) q\left(\frac{\alpha(t, [u/N - \varepsilon, u/N + \varepsilon]}{2\varepsilon}; c\right) ds\end{aligned}$$

and

$$\begin{aligned}V_c(\varepsilon; \alpha) = &\int_0^\infty \int \frac{\partial J}{\partial s}(s, x)\left|\frac{\alpha(t, [x - \varepsilon, x + \varepsilon])}{2\varepsilon} - c\right| dx\,ds \\ &+ \int_0^\infty \int \gamma\frac{\partial J}{\partial x}(s, x) q\left(\frac{\alpha(t, [x - \varepsilon, x + \varepsilon])}{2\varepsilon}, c\right) dx\,ds.\end{aligned}$$

We now restate (6.6) as

$$\lim_{\varepsilon\to 0}\liminf_{N\to\infty} Q^N\{\alpha: V_c(N,\varepsilon;\alpha)\geqq -\varepsilon\}=1.$$

Since $V_c(N,\varepsilon;\alpha)$ converges to $V_c(\varepsilon;\alpha)$, we can use Fatou's lemma to deduce

$$\lim_{\varepsilon\to 0} Q\{\alpha: V_c(\varepsilon;\alpha)\geqq -\varepsilon\}=1$$

for any limit point Q of Q^N. Finally we can use (5.5) and pass to limit $\varepsilon\to 0$ and this completes the proof of (5.1). □

7. Local Equilibrium

For any cylinder function f (depending only on $(\eta(u):|u|\leqq r)$ for some fixed r), let

$$\hat{f}(\rho)=\int f d\nu^\rho.$$

In this section we will establish the following theorems:

Theorem 7.1. *Under the assumptions of Theorem 1.3, we have*

$$\lim_{N\to\infty} E^N \frac{1}{N^d}\sum_u J\left(\frac{u}{N}\right)\tau_u f(\eta_t)=\int J(x)\hat{f}(\rho(t,x))dx \tag{7.1}$$

for every $J\in C_0(\mathbb{R}^d)$, any cylinder function f, and each t.

Theorem 7.2. *Under the assumptions of Lemma 6.2, we have*

$$\lim_{N\to\infty} E^N\left|\frac{1}{N}\sum_u J\left(\frac{u}{N}\right)\tau_u f(\eta_t)-\int J(x)\hat{f}(\rho(t,x))dx\right|=0. \tag{7.2}$$

Theorem 7.3. *Suppose $d=1$, $\mu^N\sim\rho$ with $\rho_{u,N}=\rho\left(\frac{u}{N}\right)$, and that ρ is locally of bounded variation. Then for any cylinder function f, each $k>0$ and every t*

$$\lim_{N\to\infty}\int_{|x|\leqq k}|E^N\tau_{[Nx]}f(\eta_t)-\hat{f}(\rho(t,x))|dx=0. \tag{7.3}$$

We start with the following lemma which is essential for the proof of Theorem 7.2.

Lemma 7.4. *Under the assumptions of Theorem 1.3,*

$$\lim_{l\to\infty}\limsup_{N\to\infty}\sup_{s_0\leqq t\leqq t_0} E^N\left|\frac{1}{N^d}\sum_u J\left(\frac{u}{N}\right)(\tau_u f(\eta_t)-\hat{f}(M_{T_l(u)}(\eta_t))\right|=0 \tag{7.4}$$

for every positive t_0 and s_0.

Proof. Let S_t be the semigroup associated to $\mathscr{L}$. For any positive k, set

$$\mu_t^{N,k}=\frac{1}{|T_{kN}|}\sum_{u\in T_{kN}}\mu^N\tau_u S_{tN}.$$

Since J is continuous and it has compact support, (7.4) would follow if we can show

$$\lim_{l\to\infty}\limsup_{N\to\infty}\sup_{s_0\leqq t\leqq t_0}\int\left|\frac{1}{T_l}\sum_{u\in T_l}\tau_u f(\eta)-\hat{f}(M_{T_l}(\eta))\right|\mu_t^{N,k}(d\eta)=0, \tag{7.5}$$

for every k.

The sequence $\{\mu_t^{N,k}\}$ is tight because $\mu_t^{N,k}\leqq\nu^{\rho_0}$ for all N. Let $\mathscr{A}$ denote the space of the limit points of $\{\mu_t^{N,k}:s_0\leqq t\leqq t_0\}$ as $N\to\infty$. Then for (7.5), it suffices to show

$$\lim_{l\to\infty}\sup_{\mu\in\mathscr{A}}\int\left|\frac{1}{|T_l|}\sum_{u\in T_l}\tau_u f(\eta)-\hat{f}(M_{T_l}(\eta))\right|\mu(d\eta)=0. \tag{7.6}$$

We certainly have $\mu\leqq\nu^{\rho_0}$ for all $\mu\in\mathscr{A}$, and we will show in Lemma 7.5 that $\mathscr{A}\subseteqq\mathscr{I}\cap\mathscr{S}$. We then can repeat the proof of Theorem 3.1 after (3.12) to conclude (7.6). □

The proof of the following lemma is very similar to the proof of Theorem 3.9 in [13] Ch. VIII, or Proposition 5.1 in [1].

Lemma 7.5. $\mathscr{A}\subseteqq\mathscr{I}\cap\mathscr{S}$.

Proof. Let $\mu\in\mathscr{A}$. We certainly have $\mu\in\mathscr{S}$. According to Lemma 3.6 of [13] Ch. VIII, μ is also in $\mathscr{I}$ if we can show that for every equilibrium measure ν^c, there exists a coupling $\tilde{\mu}^c$ of μ and ν^c such that

$$\tilde{\mu}^c\{(\eta,\zeta):\eta\leqq\zeta \text{ or } \zeta\leqq\eta\}=1. \tag{7.7}$$

We choose $\tilde{\mu}^c$ to be any limit point of the set $\{\tilde{\mu}_{t,c}^{N,k}:s_0\leqq t\leqq t_0\}$ as $N\to\infty$, where

$$\tilde{\mu}_{t,c}^N:=\frac{1}{|T_{kN}|}\sum_{u\in T_{kN}}(\mu^N\times\nu^c)\tau_u\tilde{S}_{tN}.$$

Here $\tilde{S}$ denotes the semigroup associated to $\tilde{\mathscr{L}}$. The rest of the proof is devoted to showing (7.7) holds and it will be carried out in several steps.

Step 1. The object of this step is to show

$$\tilde{\mu}^c\{(\eta,\zeta):\eta(u)>\zeta(u)=0,\ \eta(v)<\zeta(v)\}=0$$

for all $u,v\in Z^d$ with $p(v-u)>0$. For this it suffices to show

$$\lim_{N\to\infty}f_N(t)=0 \tag{7.8}$$

uniformly in $t\in[s_0,t_0]$, where

$$f_N(t)=\frac{1}{|T_{kN}|}\sum_{v,w\in T_{kN}(u)}\tilde{E}^N[p(w-v)|b(\zeta_t(v),\zeta_t(w))-b(\eta_t(v),\eta_t(w))|G_{v,w}(\eta_t,\zeta_t)].$$

Here, as before (η_t,ζ_t) is the process generated by $N\tilde{\mathscr{L}}$ and initially distributed as $\mu^N\times\nu^c$. Set

$$g_N(t)=\frac{1}{|T_{kN}|}\sum_{v\in T_{kN}(u)}\tilde{E}^N|\eta_t(v)-\zeta_t(v)|.$$

Using (3.2), we can write

$$g_N(t)-g_N(0)=-N\int_0^t f_N(s)ds+\int_0^t a_N(s)ds, \tag{7.9}$$

where $\int_0^t a_N(s)ds$ represents the errors coming from terms corresponding to the boundary sites of $T_{kN}(u)$, and $a_N(s)$ is uniformly bounded in N and s (see the proof of Lemma 3.3).

By applying $N\tilde{\mathscr{L}}$ once more, it is not hard to see

$$\sup_s \frac{d}{ds} f_N(s)=O(N). \tag{7.10}$$

Next we write

$$\frac{1}{N(t-s)}(g_N(t)-g_N(s))=-\frac{1}{t-s}\int_s^t f_N(\theta)d\theta+\frac{1}{N(t-s)}\int_s^t a_N(\theta)d\theta \tag{7.11}$$

and we choose $s=t-\delta/N$ for some $\delta>0$. On the other hand

$$\frac{1}{t-s}\int_s^t f_N(\theta)d\theta=f_N(t)+\frac{1}{2}\frac{d}{ds}f_N(s^*)(t-s)$$

for some $s^*\in[s,t]$. So by (7.10)

$$\lim_{\delta\to 0}\limsup_{N\to\infty}\left|\frac{1}{t-s}\int_s^t f_N(\theta)d\theta-f_N(t)\right|=0. \tag{7.12}$$

Therefore, (7.8) follows from (7.11), if we can show that the sequence $\{g_N(t)\}$ is equicontinuous. This will be shown in Lemma 7.6.

Step 2. Next we prove

$$\mu^c\{(\eta,\zeta):\eta(u)>\zeta(u),\eta(v)<\zeta(v)\}=0 \tag{7.13}$$

for all $u,v\in\mathbb{Z}^d$ with $p(v-u)>0$. Let $I_m(\eta,\zeta)$ denote the indicator function of the set

$$\{(\eta,\zeta):\eta(u)>\zeta(u)=m,\ \eta(v)<\zeta(v)\}.$$

We then prove $\int I_m(\eta,\zeta)\mu^c(d\eta,d\zeta)=0$, by showing

$$\lim_{N\to\infty}\int I_m d\tilde{\mu}_t^{N,c}=0$$

uniformly in $t\in[s_0,t_0]$. This will be proved by induction on m. For $m=0$, we proved it in Step 1. Suppose it is true for $m-1$. By Semigroup Theory

$$\int I_{m-1}d\tilde{\mu}_t^{N,c}-\int I_{m-1}d\tilde{\mu}_s^{N,c}=N\int_s^t\left(\int\tilde{\mathscr{L}}I_{m-1}d\tilde{\mu}_\theta^{N,c}\right)d\theta. \tag{7.14}$$

A simple computation shows

$$\hat{f}_N(t):=\int(\tilde{\mathscr{L}}I_{m-1})(1-I_{m-1})d\tilde{\mu}_t^{N,c}\geqq\int b(m,\zeta(v))I_m d\tilde{\mu}_t^{N,c}.$$

Therefore

$$\hat{f}_N(t)\geqq b(m,n)\int_{\zeta(v)\leqq n} I_m d\tilde{\mu}_t^{N,c}$$

for every positive integer n. Thus, our task is to show

$$\lim_{N\to\infty} \hat{f}_N(t) = 0. \tag{7.15}$$

Set $\hat{g}_N(t) = \int I_{m-1} d\tilde{\mu}_t^{N,c}$. Hence (7.14) can be written as

$$\hat{g}_N(t) - \hat{g}_N(s) = N\int_s^t \hat{f}_N(\theta)d\theta + N\int_s^t \hat{a}_N(\theta)d\theta, \tag{7.16}$$

where

$$\hat{a}_N(\theta) = \int (\tilde{\mathscr{L}} I_{m-1}) I_{m-1} d\tilde{\mu}_\theta^{N,c}.$$

It is not hard to see

$$|\hat{a}_N(\theta)| \leqq c_0 \int I_{m-1} d\tilde{\mu}_\theta^{N,c},$$

for some constant c_0, therefore by induction hypothesis $\lim_{N\to\infty} \hat{a}_N(\theta) = 0$. Next we choose $s = t - \delta/N$ for some positive δ, and we write

$$\frac{1}{N(t-s)}(\hat{g}_N(t) - \hat{g}_N(s)) = \frac{1}{t-s}\int_s^t \hat{f}_N(\theta)d\theta + \frac{1}{t-s}\int_s^t \hat{a}_N(\theta)d\theta.$$

By induction hypothesis, the left-hand side tends to zero, as $N\to\infty$. The second term on the right-hand side also converges to zero. On the other hand, since $\sup_s \frac{d}{ds}\hat{f}_N(s) = O(N)$, we have (7.12) with f_N replaced with $\hat{f}_N$. Thus we can let $\delta \to 0$, in order to complete the proof of (7.15).

Final Step. So far we have shown (7.13) for u, v with $p(v-u) > 0$ (reviewing the proof reveals that for the simple exclusion model we only need $p(v-u) + p(u-v) > 0$). Let u, v be two arbitrary sites. By irreducibility of p, there exists a sequence $u_0 = u$, $u_1, \ldots, u_n = v$ such that $p(u_{i+1} - u_i) > 0$ for $i = 0, \ldots, n-1$. To prove (7.13) for u, v we use induction on n. We omit the rest of the proof which is almost a repetition of Step 2 (see also the proof of Lemma 4.7 of [1]). □

Lemma 7.6. *The sequence $\{g_N(t)\}$ is equicontinuous.*

Proof. We prove the equicontinuity by showing that any subsequence of $\{g_N(t)\}$ has a uniformly convergent subsequence. Since $\left\{\int_0^t a_N(s)ds\right\}$ is equicontinuous, it is enough to check the equicontinuity of

$$\tilde{g}_N(t) := g_N(t) - g_N(0) - \int_0^t a_N(s)ds.$$

Because of (7.9), each $\tilde{g}_N$ is nonincreasing in t. Therefore, by Helley's selection theorem, we can always pick a subsequence of $\{\tilde{g}_N\}$ that is pointwise convergent, and the convergence would be uniform, if we can show that the limiting function is continuous. For this, it suffices to check that any limit of the sequence $\{g_N\}$ is continuous in t.

It follows from (3.7) that

$$\lim_{l\to\infty} \limsup_{N\to\infty} \left[\int_0^t g_N(s)ds - \int_0^t \frac{1}{|T_{kN}|} \sum_{v\in T_{kN}(u)} E^N |M_{T_l(v)}(\eta_s) - c| ds \right] = 0,$$

and then by Lemma 5.6(b), we can replace $M_{T_l(v)}(\eta_s)$ with $\rho(s, v/N)$. Therefore

$$\lim_{N\to\infty}\int_0^t g_N(s)ds = \frac{1}{(2k)^d}\int_{|x|\leq k}\int_0^t |\rho(s,x)-c|\,ds\,dx.$$

Thus if a subsequence of g_n converges to g_∞, then

$$g_\infty(t) = \frac{1}{(2k)^d}\int_{|x|\leq k} |\rho(t,x)-c|\,dx.$$

Since any solution of the hydrodynamic equation (5.1) and (5.2) is L^1_{loc}-continuous in t $\left(\text{i.e. } \lim_{t\to s}\int_{|x|\leq k}|\rho(t,x)-\rho(s,x)|dx = 0\right)$, the limiting function g_∞ is also continuous and this completes the proof. (The L^1-continuity of solutions is mentioned in [7] under the assumption $\rho(\cdot)\in L^1$. If ρ is merely bounded, we can cut it off outside a bounded set to get an integrable function, and use Kružkov's comparison theorem (Theorem 1 in [11]) in order to obtain L^1_{loc}-continuity). □

Corollary 7.7. The sequence $A_N(t) = \frac{1}{|T_{kN}|}\sum_{u\in T_{kN}} E^N|M_{T_l(u)}(\eta_t) - c|$ *is equicontinuous* in t.

Proof. It follows from the proof of Lemma 7.4 that $\sup_{s_0\leq t\leq t_0}|g_N(t) - A_N(t)|\to 0$ for every $s_0, t_0 > 0$, where g_N is as in Lemma 7.5. By Lemma 7.6 the sequence g_N is equicontinuous, and this completes the proof. □

We are now ready to prove our theorems:

Proof of Theorem 7.2. In view of Lemma 7.4, we only need to verify

$$\lim_{l\to\infty}\limsup_{N\to\infty} E^N\left|\frac{1}{N}\sum_u J\left(\frac{u}{N}\right)\hat{f}(M_{T_l(u)}(\eta_t)) - \int J(x)\hat{f}(\rho(t,x))dx\right| = 0.$$

By Lemma 6.2 we can replace $\hat{f}(M_{T_l(u)}(\eta_t))$ with $\hat{f}(M_{T_{N\varepsilon}(u)}(\eta_t))$. Moreover

$$\lim_{\varepsilon\to\infty}\limsup_{N\to\infty} E^N\left|\frac{1}{N}\sum J\left(\frac{u}{N}\right)\hat{f}(M_{T_{N\varepsilon}(u)}(\eta_t)) - \int J(x)\hat{f}(\rho(t,x))dx\right| = 0$$

that follows the discussion at the end of Sect. 6. □

Proof of Theorem 7.1. Step 1. In view of Lemma 7.4, we only need to show

$$\lim_{l\to\infty}\limsup_{N\to\infty} E^N\left[\frac{1}{N^d}\sum_u J\left(\frac{u}{N}\right)\hat{f}(M_{T_l(u)}(\eta_t))\right] = \int J(x)\hat{f}(\rho(t,x))dx. \tag{7.17}$$

Set

$$F_{N,l}(t) = E^N\left[\frac{1}{N^d}\sum_u J\left(\frac{u}{N}\right)\hat{f}(M_{T_l}(\eta_t))\right]. \tag{7.18}$$

It follows the proof of Lemma 5.5(*b*) and Theorem 5.1

$$\lim_{l\to\infty}\limsup_{N\to\infty}\int_0^t F_{N,l}(s)ds = \int_0^t\int J(x)\hat{f}(\rho(s,x))dx\,ds.$$

Note that the integration in time is necessary, because our probability measures $R^{N,l}$ and R^l they only converge as probability measures on the space X. This technical problem can be taken care of if we can show that the sequence $\{F_{N,l}\}$ is equicontinuous. Then we can drop the time-integral and recover (7.17).

Step 2. Our goal is to show $\{F_{N,l}\}$ is equicontinuous in the following sense:

$$\lim_{\delta\to\infty}\limsup_{l\to\infty}\limsup_{N\to\infty}\sup_{\substack{|t-s|<\delta \\ s_0\leq t,s\leq t_0}}|F_{N,l}(t)-F_{N,l}(s)|=0 \tag{7.19}$$

for every $s_0, t_0 > 0$.

Let $c_0 > \|\rho\|_\infty$. Then by coupling η-process with a ζ-process that starts from v^{c_0}, we can show

$$\lim_{l\to\infty}\limsup_{N\to\infty}\sup_t E^N\left|\frac{1}{N^d}\sum_u J\left(\frac{u}{N}\right)\hat{f}(M_{T_l(u)}(\eta_t))1(M_{T_l(u)}(\eta_t)>c_0)\right|=0.$$

Thus we may assume $\hat{f}$ has compact support, without loss of generality. Let $\mathscr{F}$ denote the space of all pairs $(J,\hat{f})$ with $\hat{f}:[0,c_0]\to\mathbb{R}$, for which (7.19) holds. $\mathscr{F}$ is certainly a linear space, and it is closed with respect to the uniform topology. That is, if $(J_n,\hat{f}_n)\in\mathscr{F}$ and $\|J_n-J\|_\infty\to 0$, $\|\hat{f}_n-\hat{f}\|_\infty\to 0$ then $(J,\hat{f})\in\mathscr{F}$.

We would like to show $\mathscr{F}\supseteqq C_0(\mathbb{R}^d)\times C_b$, and for this we only need to check $(J^{a,b},f^c)\in\mathscr{F}$ for every c and k, where $J^{a,b}(x)=1_{[a,b]}$ and $f^c(\rho)=|\rho-c|$. This is because every piecewise linear continuous f can be written as a linear combination of f^c's and constants, and every piecewise constant J can be written as a linear combination of $J^{a,b}$'s. Finally, $(J^{a,b},f^c)\in\mathscr{F}$ because this is exactly the content of Corollary 7.8. □

Proof of Theorem 7.3. Step 1. From the proof of Theorem 5.1, in particular formula (5.18), we conclude that we only need to prove (7.3) for initial densities ρ that are of bounded variations.

Step 2. (the arguments of this step are taken from [16]).

Suppose ρ is of bounded variation. Then there exists a coupling $\tilde{\mu}^N$ of μ^N and $\mu^N\tau_1$ *such that*

$$\sup_N\int\sum_u|\eta(u)-\zeta(u)|\tilde{\mu}^N(d\eta,d\zeta)<\infty.$$

The proof of (5.10) implies

$$\sup_N\tilde{E}^N\sum_u|\eta_t(u)-\zeta_t(u)|<\infty,$$

where $\tilde{E}^N$ denotes the expectation with respect to the process generated by $N\tilde{\mathscr{L}}$ and initially started from $\tilde{\mu}^N$. As a result

$$\sum_u|E^N\tau_u f(\eta_t)-E^N\tau_{u+1}f(\eta_t)|<\infty. \tag{7.20}$$

Final Step. Set

$$\phi_N(x)=E^N\tau_{[Nx]}f(\eta_t).$$

Let $\operatorname{Var}(\phi_N)$ denote the total variation of ϕ_N. Then (7.20) means $\sup_N\operatorname{Var}(\phi_N)<\infty$.

The sequence $\{\phi_N\}$ has the following properties:

(a) $\sup_N \|\phi_N\|_\infty < \infty$,

(b) Every subsequence of $\{\phi_N\}$ contains a further subsequence that converges everywhere,

(c) $\lim_{N\to\infty} \int J(x)\phi_N(x)dx = \int J(x)\hat{f}(\rho(t,x))dx$,

where (b) follows from Helley's selection theorem, and (c) follows from Theorem 7.1. It is not hard to see that (a), (b) and (c) imply (7.3). □

Acknowledgements. I am very grateful to H. T. Yau for suggesting the use of DiPerna's uniqueness theorem for the proof of Lemma 5.6. I also wish to thank S. R. S. Varadhan for an essential hint concerning the proof of Lemma 5.7. Thanks are also given to CIMS for its hospitality and support.

References

1. Andjel, E. D.: Invariant measures for the zero range process. Ann. Prob. **10**, 525–547 (1982)
2. Andjel, E. D., Kipnis, C.: Derivation of the hydrodynamical equation for the zero range interaction process. Ann. Prob. **12**, 325–334 (1984)
3. Andjel, E. D., Vares, M. E.: Hydrodynamic equation for attractive particle systems on $\mathbb{Z}$. J. Stat. Phys. **47**, 265–288 (1987)
4. Benassi, A., Fouque, J. P.: Hydrodynamical limit for the asymmetric simple exclusion process. Ann. Prob. **15**, 546–560 (1987)
5. Benassi, A., Fouque, J. P.: Hydrodynamical limit for the asymmetric zero range process. Ann. Inst. Henri Poincaré 189–200 (1988)
6. Cocozza, C. T.: Processus des misanthropes. Z. Wahrs. Verw. Gebiete **70**, 509–523 (1985)
7. Crandall, M.: The semigroup approach to first-order quasilinear equations in several space variables. Israel J. Math **12**, 108–132 (1972)
8. DeMasi, A., Ianiro, N., Pellegrinotti, A., Presutti, E.: A survey of the hydrodynamical behavior of many particle systems. In: Studies in Statistical Mechanics, Vol **11**, Non Equilibrium Phenomena II, from Stochastics to Hydrodynamics. Amsterdam: North-Holland (1984)
9. DiPerna, R. J.: Measure-valued solutions to conservation laws. Arch. Rational Mech. Anal. **88**, 223–270 (1985)
10. Guo, M. Z., Papanicolaou, G. C., Varadhan, S. R. S.: Nonlinear diffusion limit for a system with nearest neighbor interactions. Commun. Math. Phys. **118**, 31–59 (1988)
11. Kružkov, S. N.: First order quasilinear equations in several independent variables. Math. USSR-Sb. **10**, 217–243 (1970)
12. Landim, C.: Hydrodynamical equation for attractive particle systems in $\mathbb{Z}^d$, preprint
13. Liggett, T. M.: Interacting particle systems. Berlin, Heidelberg, New York: Springer 1985
14. Rost, H.: Non-equilibrium behavior of a many particle process: density profile and local equilibria. Z. Wahrs. Verw. Gebiete **58**, 41–43 (1981)
15. Varadhan, S. R. S.: Scaling limits for interacting diffusions, preprint
16. Wick, W. D.: Entropy arguments in the study of the hydrodynamical limit. CARR reports in Mathematical Physics 3/88, May 1988
17. Benassi, A., Fouque, J. P., Saada, E., Vares, M. E.: Asymmetric attractive particle systems on Z: Hydrodynamical limit for monotone initial profiles, preprint

Communicated by J. L. Lebowitz