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An Algebraic Geometry Study of the *b–c* System with Arbitrary Twist Fields and Arbitrary Statistics

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Abstract. We present an analysis of the general b–c system (including the β–γ system) on a compact Riemann surface of arbitrary genus $g \geqq 0$ by postulating that its correlation functions should only have the singularities imposed by the operator product expansion (OPE) of the system. Studying a very (in fact optimally) general form of the b–c system, we prove rigorously that the standard practice of eliminating zero modes, and even the standard lagrangian, follow from the analyticity structure dictated by the OPE alone. We extend the analysis to consider the most general case of the presence of twist (e.g. spin) fields. We then determine all the possible correlation functions of the b–c system, with statistics unspecified, compatible with the OPE. On imposing Fermi and Bose statistics, we obtain the correlation functions of the fermionic b–c and β–γ systems, respectively.

1. Introduction

We consider a system consisting of a pair of quantum fields b, c on a compact, connected Riemann surface M of genus $g \geqq 0$. Our aim is to study the *correlation functions*, written symbolically as

$$C(m,n) \equiv \langle b(Q_1) \dots b(Q_m) c(P_1) \dots c(P_n) \rangle , \tag{1.1}$$

where the Q's and P's are arbitrary (but distinct) points on M, by postulating the *operator product expansion* (OPE)

$$b(Q)c(P) \sim \frac{I}{Q-P} . \tag{1.2}$$

Here $Q, P \in M$ while I, on the right-hand side of (1.2), is the *identity operator*. Equation (1.2) is to be understood as holding inside a correlation function $C(m,n)$. An immediate consequence of (1.2) is that a general correlation function $C(m,n)$ has a pole when the arguments of a b field and a c field coincide. We shall study the consequences of the physical assumption that the $C(m, n)$ have no poles other than those implied by the OPE (1.2). This can be viewed as a kind of *principle of maximal analyticity*.

The full specification of the system requires precise hypotheses on the fields b, c. These will be stated in Sect. 2, but let us note that they will be such as to include both the usual fermionic spin $(1-J), J$ b–c system [1], as well as its bosonic counterpart, known as the β–γ system [1]. In fact, we shall not specify any statistics in advance. An important feature is that, since we start from the OPE and not from a lagrangian or equation of motion, we have no definition of a "*zero mode*" from the present point of view. Hence we have no means of identifying "zero modes" and no reason to eliminate them as is conventional [2–10]. One of our aims is to study whether the elimination of "zero modes" is required by the OPE and maximal analyticity alone.

The fermionic b–c system, as well as the β–γ system, were discussed over the complex plane in the classic paper of Friedan, Martinec, and Shenker [1]. Subsequently, a lot of interesting work has been done on the fermionic b–c system over a compact Riemann surface. A partial list of the techniques that have been used would be *bosonization*, both *chiral* [2] and *nonchiral* with holomorphic factorization [3], the *grassmannian* approach [4], *multiplicative Ward identities* [5], the *stress tensor* method [6, 7], *Krichever-Novikov expansions* [8], operatorial *Baker-Akhiezer* method [9]. There has also been considerable work on the β–γ system [6, 7, 10], mostly based on the bosonization formulae of [1].

In the present work we show that there is a very simple physical viewpoint for understanding this system, which is obscured by the field theoretic machinery used in earlier discussions [2–10], viz. the principle that the OPE (1.2) should determine the singularity structure of the correlation functions. This viewpoint has the advantage that it can be immediately translated into precise mathematics. In addition, one can raise and answer questions which are important for understanding the system, but cannot even be formulated by previous methods. In a recent series of papers [11–14] we have shown that this viewpoint leads to a satisfactory analysis of the fermionic b–c system with zero modes excluded. We showed that the principle of the singularities being determined by the OPE leads rigorously to the result that $C(m, n) = 0$ for $m \neq n$, which is consistent with the standard argument of *charge conservation*, and that the $2n$-point function $C(n, n)$ is a determinant of two point functions, i.e. that Wick's theorem holds. The proofs of these results in [11–13] actually hold for all genera $g \geqq 0$. (They even hold if the ground field $\mathbb{C}$ is replaced by any algebraically closed field.) We then showed [11, 13, 14] how to obtain an explicit expression for the $2n$-point function of the fermionic b–c system for $g \geqq 2$. This led to a new, purely algebraic geometry, proof of *Fay's identity* [15]. In Appendix A and Appendix B of the present paper we develop the methods necessary to write down the $2n$-point function of the fermionic b–c system for $g = 0$ and 1, respectively. As a result, we get an identity due to Cauchy [16] for the case of $g = 0$ and an identity due to Frobenius [17] for the case $g = 1$.

In Sects. 2 and 3 we discuss only the two point function $C(1, 1)$ and so the discussion there does not involve the question of statistics, which we come to in Sect. 4. In Sect. 2 we start with a generalized b–c system, which encompasses the usual spin $(1-J), J$ system, and allows the widest possibilities for the appearance of "zero modes" (in the language of [1–10]). Theorem 2.2 states that the criterion that the system should have a nonvanishing two point function, with singularity structure determined by the OPE, uniquely picks out the (twisted) spin $\frac{1}{2}$ b–c

system with zero modes excluded which was discussed by us in [11–13]. This has important implications which are discussed immediately after the statement of Theorem 2.2. The rest of Sect. 2 is devoted to the proof. In Sect. 3 we discuss the situation when there are a number of distinguished points on the Riemann surface around which the b and c fields have a specified, rational monodromy. We consider the most general problem of this kind from the same viewpoint as before, viz. that the singularities of the two point function should be determined by the OPE and the additional data. A complete solution is obtained in Theorem 3.2 by showing how the problem can be reduced to the problem considered in Sect. 2. In Sect. 4 we convert the conditions imposed by the OPE on the correlation functions into precisely stated postulates and determine all the possible $2n$-point functions compatible with these conditions. Our postulates contain the constraints implied by the OPE only in a very weak form. As a result, although the imposition of Fermi or Bose statistics leads to the expected results for the fermionic b–c and β–γ systems, these two cases do not exhaust all possibilities. Throughout this paper we demonstrate, as in our earlier papers [11–14], the advantages of an abstract analysis, using algebraic geometry, over detailed computations involving theta functions and prime forms [2–10]. Nothing is lost in the process of abstraction and we can finally write down the results explicitly. The requisite machinery was developed in our earlier papers [11, 13, 14] for $g \geqq 2$. We complete this in Appendix A and Appendix B of the present paper by taking up explicitly the cases $g=0$ and $g=1$, respectively.

Our notations are standard and we refer to Griffiths and Harris [18] and Hartshorne [19] for basic definitions and results in algebraic geometry. We denote by $\mathrm{Pic}(M)$ the *Picard group* of holomorphic line bundles on M, while $\mathrm{Pic}^d(M)$ is the subset of *degree* (or *Chern class*) $d \in \mathbb{Z}$. For $\gamma \in \mathrm{Pic}(M)$, $\deg(\gamma)$ denotes its degree. We denote by Θ the *canonical theta divisor*, i.e. the subset of $\mathrm{Pic}^{g-1}(M)$ of line bundles with at least one nonzero holomorphic section, and by K the holomorphic cotangent bundle of M. If D is a divisor of a compact, connected complex manifold X, then $\mathcal{O}(D)$ denotes the holomorphic line bundle canonically associated to D. We denote by $\mathcal{O}_X$ the structure sheaf of X, by $H^i(X, \mathcal{F})$ the i^{th} cohomology group of X with coefficients in the sheaf $\mathcal{F}$ and by $h^i(X, \mathcal{F})$ its dimension.

This paper has been written for mathematicians, interested in seeing how algebraic geometry can be applied in physics, as well as for physicists, interested in conformal field theory, but unfamiliar with the techniques we use. In order to enhance readability, we have tried to avoid burdening the text with excessive details of a routine nature in the proofs, while giving all the steps as well as references to standard textbooks for all the techniques used. The results are, however, discussed extensively and the proofs can be omitted by those interested only in the results.

2. The Two Point Function, Zero Modes and Algebraic Geometry

We shall consider a *generalised b–c system*, which we define to consist of a pair of quantum fields b, c on a compact, connected Riemann surface M of genus $g \geqq 0$. We suppose that c (respectively b) is an "operator-valued section" of a holomorphic line bundle α (respectively β) of degree p (respectively q), where $p+q=2g-2$ (in fact the condition $p+q \leqq 2g-2$ suffices). By specifying only the Chern class of the holomorphic line bundles α and β, note that in each case we have the freedom to choose any element out of a g-dimensional compact complex manifold. We shall have in mind that the fields b and c obey the OPE (1.2), but since quantum fields are mathematically complicated objects to deal with, we shall deal directly only with the correlation functions (1.1). The basic correlation function is the *two point function* $\langle b(Q)c(P)\rangle$. Our aim in this section is to investigate the conditions for our system to have a nonvanishing two point function, which will be our criterion for the system to be nontrivial.

The following definition is motivated by the OPE (1.2) and the criterion of maximal analyticity:

Definition 2.1. Let $p_i: M \times M \to M$ $(i=1,2)$ be the canonical projections onto the first and second factors of $M \times M$. The generalised b–c system will be said to have a *two point function* (not necessarily unique), denoted $\langle b(Q)c(P)\rangle$, if there is at least one meromorphic section (not identically zero) of $p_1^*(\beta)\otimes p_2^*(\alpha)$ whose only singularity is a simple pole along the diagonal Δ of $M \times M$ (i.e. for $Q=P$).

The principal result of this section is the following theorem:

Theorem 2.2. *Necessary and sufficient conditions for the generalised b–c system to have a two point function in the sense of Definition 2.1 are that* $p=q=g-1$, $\beta=K\otimes\alpha^{-1}$, *and* α *(hence also* β*) is in the complement of the canonical theta divisor* Θ *in* $\mathrm{Pic}^{g-1}(M)$. *Under these conditions the two point function is also unique.*

Before discussing the proof, let us consider the significance of Theorem 2.2. This theorem is the key to understanding the structure of the b–c system. It says that, apart from the question of statistics which is relevant only for higher point functions, there is really only one b–c system, viz. the twisted spin $\frac{1}{2}$ b–c system with zero modes (viz. holomorphic sections of α) excluded, which was discussed by us earlier [11–13]. Thus if we want to discuss e.g. the spin $(1-J), J$ b–c system, in which case $p=2J(g-1)$ and $q=2(1-J)(g-1)$, Theorem 2.2 tells us that *we must introduce new singularities not contained in the* OPE (1.2). If we restrict the new singularities to poles, then Theorem 2.2 tells us precisely how to introduce them, viz. in such a way as to obtain a new system satisfying Theorem 2.2 (in this connection, see [14]). In order to do this we find that if we introduce points on M at which e.g. the b field has poles, then the c field must have zeros at those points, and vice versa. Thus, in a sense, Theorem 2.2 is telling us that b and c are *conjugate fields* even though we are not dealing directly with the fields. Moreover, while the generalised b–c system we started with has no action principle, Theorem 2.2 leads us to a system with the action

$$\int_M b\bar{\partial}c,$$

where $\bar{\partial}$ is the usual $\bar{\partial}$-operator on M. Of course Theorem 2.2 does not insist on the use of poles to eliminate zero modes and it will be clear from our discussion in Sect. 3 that *branch point singularities*, introduced through *twist fields*, could also be used. Theorem 2.2 tells us precisely how this can be achieved. We shall also see in Sect. 3 how the generalized b–c system arises even when we start with the usual b–c system.

We shall prove Theorem 2.2 using methods of algebraic geometry, particularly sheaf cohomology. As a first step we shall have to convert the question of the existence of a *meromorphic* section, viz. the two point function, into one concerning *holomorphic* sections. One concept that figures throughout our work is the concept of the *prime form*, which we now define.

Definition 2.3. Let Δ be the diagonal of $M \times M$ and $\mathcal{O}(\Delta)$ the holomorphic line bundle on $M \times M$ associated to the divisor Δ. The *prime form* $E(Q,P)$ is the holomorphic section of $\mathcal{O}(\Delta)$ with divisor Δ.

In fact $\mathcal{O}(\Delta)$ has a one dimensional space of holomorphic sections for $g \geqq 1$, as shown in Proposition B.1 of Appendix B for $g=1$ and in Proposition 5.3 of [11] for $g \geqq 2$. In Proposition 6.3 of [11] we have discussed how this definition of the prime form is related to the standard one [15, 20] for $g \geqq 2$. In Theorems A.2 and B.4 we discuss the case of $g=0$ and $g=1$, respectively. We shall, therefore, use the symbol $E(Q,P)$ for all genera $g \geqq 0$.

The diagonal Δ of $M \times M$ is a closed subscheme and hence [19] we have the canonical short exact sequence

$$0 \to \mathcal{O}(-\Delta) \to \mathcal{O} \to \mathcal{O}|\Delta \to 0 \tag{2.1}$$

between the structure sheaf $\mathcal{O}$ of $M \times M$, the ideal sheaf $\mathcal{O}(-\Delta)$ of Δ, and the quotient sheaf $\mathcal{O}|\Delta$, which is the structure sheaf of Δ. Tensoring (2.1) by $\mathcal{O}(\Delta)$ (exactness is preserved) we get another useful form of (2.1):

$$0 \to \mathcal{O} \to \mathcal{O}(\Delta) \to K_\Delta^{-1} \to 0, \tag{2.2}$$

where K_Δ is the holomorphic cotangent bundle of Δ (which is isomorphic to M) and we have used Lemma 5.1 of [11], which holds for all $g \geqq 0$.

Let $\mathcal{F}_{\alpha\beta} \equiv p_1^*(\beta) \otimes p_2^*(\alpha)$ and $\mathcal{M}_{\alpha\beta} \equiv \mathcal{F}_{\alpha\beta} \otimes \mathcal{O}(\Delta)$. Tensoring the exact sequence (2.1) by $\mathcal{M}_{\alpha\beta}$ we get the exact sequence

$$0 \to \mathcal{F}_{\alpha\beta} \to \mathcal{M}_{\alpha\beta} \to \mathcal{M}_{\alpha\beta}|\Delta \to 0. \tag{2.3}$$

Since the global section functor $H^0(M \times M, \cdot)$ is left exact [19] we get:

$$0 \to H^0(M \times M, \mathcal{F}_{\alpha\beta}) \xrightarrow{i} H^0(M \times M, \mathcal{M}_{\alpha\beta}). \tag{2.4}$$

The implication of the exact sequence (2.4) is that, while every two point function in the sense of Definition 2.1 gives rise to a holomorphic section of $\mathcal{M}_{\alpha\beta}$ by multiplying it by the prime form, the converse is not necessarily true. If we have a holomorphic section of $\mathcal{M}_{\alpha\beta}$ and divide it by the prime form, we may merely get a holomorphic section of $\mathcal{F}_{\alpha\beta}$. We have thus proved:

Proposition 2.4. *A necessary and sufficient condition for the generalized b–c system to have a nonzero two point function, in the sense of Definition 2.1, is for the quotient vector space*

$$H^0(M \times M, \mathcal{M}_{\alpha\beta})/H^0(M \times M, \mathcal{F}_{\alpha\beta}) \tag{2.5}$$

to have strictly positive dimension, where the denominator in (2.5) *denotes the image under the injective map i of* (2.4).

We can now state Theorem 2.2 in the equivalent form:

Theorem 2.5. *Necessary and sufficient conditions for the quotient vector space* (2.5) *to have strictly positive dimension are that* $p=q=g-1$, $\beta = K \otimes \alpha^{-1}$ *and that* α *lies in the complement of the canonical theta divisor* Θ *in* $\mathrm{Pic}^{g-1}(M)$. *If these conditions hold, the dimension of the quotient vector space* (2.5) *is one.*

Proof. To understand what has to be proved, note that (2.4) is a part of the long cohomology exact sequence

$$\begin{aligned}0 \to H^0(M\times M, \mathscr{F}_{\alpha\beta}) &\xrightarrow{i} H^0(M\times M, \mathscr{M}_{\alpha\beta}) \\ \to H^0(\Delta, \mathscr{M}_{\alpha\beta}|\Delta) &\xrightarrow{\delta} H^1(M\times M, \mathscr{F}_{\alpha\beta}) \to \dots .\end{aligned} \tag{2.6}$$

It is clear from (2.6) that the quotient vector space (2.5) will have dimension zero if and only if the *connecting homomorphism* δ in (2.6) is injective. The investigation of such a question is somewhat delicate, in general. The following key lemma enables us to resolve this problem.

Lemma 2.6. *Let $\xi \in \mathrm{Pic}^r(M)$, $\eta \in \mathrm{Pic}^s(M)$ (with r and s arbitrary integers). Let $d = \dim H^0(M, \xi)$. Denote $p_1^*(\eta)\otimes p_2^*(\xi)$ by $\mathscr{F}_{\xi\eta}$ and $\mathscr{F}_{\xi\eta}\otimes\mathcal{O}(\Delta)$ by $\mathscr{M}_{\xi\eta}$. Then $H^0(M\times M, \mathscr{F}_{\xi\eta})$ and $H^0(M\times M, \mathscr{M}_{\xi\eta})$ are isomorphic (under i) and hence have the same dimension if either* (a) *$r < g-1$ and d is arbitrary, or* (b) *if $r = g-1$ and $d \neq 0$. In both cases s is arbitrary. The roles of ξ and η can, of course, be interchanged.*

Proof. We shall find it convenient to rearrange the two cases (a) and (b) of the statement of the lemma to (i) $d=0$ and $r<g-1$, and (ii) $d\neq 0$ and $r \leqq g-1$.

Case (i): *$d=0$ and $r<g-1$.* If $d=0$ then, by the Künneth formula, $H^0(M\times M, \mathscr{F}_{\xi\eta}) = 0$, and hence we have to prove that $H^0(M\times M, \mathscr{M}_{\xi\eta}) = 0$. The following simple argument enables us to deal with this case rapidly.

Assume the contrary and let σ be a nonzero element of $H^0(M\times M, \mathscr{M}_{\xi\eta})$. Restricting σ to $\{P\}\times M$ for general $P\in M$, we get a nonzero holomorphic section of $\xi\otimes\mathcal{O}(P)$. However, it is easily seen from the Riemann-Roch theorem that for general P we have $h^0(M, \xi\otimes\mathcal{O}(P)) = 0$ if $\deg(\xi) < g-1$. This settles case (i).

Case (ii): *$d\neq 0$ and $r\leqq g-1$.* This case is considerably more subtle. We now have

$$h^0(M\times M, \mathscr{F}_{\xi\eta}) = d\times h^0(M,\eta)$$

by the Künneth formula and so we have to prove that

$$h^0(M\times M, \mathscr{M}_{\xi\eta}) = d\times h^0(M,\eta). \tag{2.7}$$

Now note that p_1, the canonical projection from $M\times M$ to its first factor, has associated with it the *direct image functor* p_{1*} which is left exact [19]. Applying p_{1*} to the short exact sequence (2.3) (with α, β replaced by ξ, η) we get the associated long exact sequence [19] of *higher direct image sheaves*

$$0\to p_{1*}(\mathscr{F}_{\xi\eta}) \to p_{1*}(\mathscr{M}_{\xi\eta}) \to p_{1*}(\mathscr{M}_{\xi\eta}|\Delta) \to R^1p_{1*}(\mathscr{F}_{\xi\eta}) \to R^1p_{1*}(\mathscr{M}_{\xi\eta}) \to \dots . \tag{2.8}$$

Since p_1 is proper, surjective and with range a curve, it follows that the first three terms of (2.8) are vector bundles over M. Computing the first few terms (using the *projection formula* [19]) we get

$$0\to \eta\otimes\mathbb{C}^d \to p_{1*}(\mathscr{M}_{\xi\eta}) \xrightarrow{\delta} \eta\otimes\xi\otimes K^{-1} \to \dots . \tag{2.9}$$

Now the first term of (2.9) is a trivial rank d vector bundle and the third term has rank one. If we can prove that the second term in (2.9) has rank d, this will mean that the image of δ in (2.9) is the zero sheaf, and so

$$p_{1*}(\mathscr{M}_{\xi\eta}) = \eta\otimes\mathbb{C}^d .$$

We are then through, since

$$\begin{aligned} h^0(M\times M,\mathcal{M}_{\xi\eta}) &= h^0(M,p_{1*}(\mathcal{M}_{\xi\eta}))\\ &= h^0(M,\eta\otimes\mathbb{C}^d)\\ &= d\times h^0(M,\eta)\,, \end{aligned}$$

which is what we set out to prove.

To prove that $p_{1*}(\mathcal{M}_{\xi\eta})$ indeed has rank d, note that as a consequence of *Grothendieck's semicontinuity theorem* (see Theorem 12.11 (a) of [19]), we have that if P denotes a generic point of M,

$$\begin{aligned} \operatorname{rank} p_{1*}(\mathcal{M}_{\xi\eta}) &= \dim H^0(p_1^{-1}(P),\mathcal{M}_{\xi\eta}|p_1^{-1}(P))\\ &= \dim H^0(\{P\}\times M,\mathcal{M}_{\xi\eta}|\{P\}\times M)\\ &= \dim H^0(M,\xi\otimes\mathcal{O}(P))\,. \end{aligned}$$

It is now an elementary exercise to show, using the Riemann-Roch theorem, that for generic P we have $\dim H^0(M,\xi\otimes\mathcal{O}(P))=d$. This completes the proof of Lemma 2.6.

Completion of the Proof of Theorems 2.2 and 2.5. If we put $\xi=\alpha, \eta=\beta, r=p, s=q$ in Lemma 2.6, where $p+q=2g-2$, we see immediately that if $p\neq q$ then either p or q is less than $g-1$. Lemma 2.6(a) then shows that the two point function is identically zero. Hence we must have $p=q=g-1$. Lemma 2.6(b) then shows that the two point function is identically zero unless both α and β lie in the complement of Θ in $\mathrm{Pic}^{g-1}(M)$. What remains to be proved is that a further necessary condition for the two point function to be not identically zero is that $\beta=K\otimes\alpha^{-1}$, and that, once this is satisfied, it is then also unique. The last step, viz. uniqueness, is actually Theorem 5.8 of [11], which holds for all $g\geqq 0$, but a different proof will emerge out of the present argument.

Consider the exact sequence (2.8) with $\xi=\alpha, \eta=\beta$ and $\alpha,\beta\in\mathrm{Pic}^{g-1}(M)-\Theta$. It is easy to check that, since α is in the complement of Θ, we have

$$p_{1*}(\mathcal{F}_{\alpha\beta})=0=R^1p_{1*}(\mathcal{F}_{\alpha\beta})\,. \tag{2.10}$$

Hence we get

$$p_{1*}(\mathcal{M}_{\alpha\beta})=\beta\otimes\alpha\otimes K^{-1}\,. \tag{2.11}$$

Now, under our conditions on α and β, the denominator of (2.5) is zero dimensional. Hence, to have a nonzero two point function, it is sufficient that $\mathcal{M}_{\alpha\beta}$ should have a nonzero holomorphic section. But

$$\begin{aligned} H^0(M\times M,\mathcal{M}_{\alpha\beta}) &= H^0(M,p_{1*}(\mathcal{M}_{\alpha\beta}))\\ &= H^0(M,\beta\otimes\alpha\otimes K^{-1})\,, \end{aligned} \tag{2.12}$$

by (2.11). The line bundle $\beta\otimes\alpha\otimes K^{-1}$ is of zero degree and so has a nonzero holomorphic section if and only if it is the trivial line bundle. In that case $\beta=K\otimes\alpha^{-1}$ and

$$h^0(M\times M,\mathcal{M}_{\alpha\beta})=h^0(M,\mathcal{O}_M)=1\,. \tag{2.13}$$

This completes the proof of Theorem 2.5 and hence of Theorem 2.2.

Remark 2.7. It is easy to check that the hypotheses of cases (a) and (b) of Lemma 2.6 are optimal. As a result we need only assume $p+q\leqq 2g-2$, but no further weakening of the hypotheses of Theorems 2.2 and 2.5 is possible if we want to retain the conclusions.

We have shown in Theorem 6.5 of [11] that, when the conditions of Theorem 2.2 are satisfied, the two point function is given by the *Szegö kernel* $S_\alpha(Q,P)$ for $g \geqq 2$. In Theorem A.3 of Appendix A and Theorem B.6 of Appendix B of the present paper we have computed the two point function for genus 0 and 1, respectively. We shall continue to use the same symbol $S_\alpha(Q,P)$ for these cases as well, just as we have agreed to denote the prime form by $E(Q,P)$ for all genera. The reader can put the appropriate expression according to the genus.

Remark 2.8. It is interesting to consider the case of the fermionic b–c system with *odd spin structure* (see Verlinde-Verlinde [2]), where we must necessarily have $g \geqq 1$. By Theorem 2.2 this system does not have a two point function in the sense of Definition 2.1, i.e. given by the OPE (1.2). Nevertheless, we can show that this system does have a 4-point and higher $2n$-point functions, defined uniquely by the OPE's (1.2) and (A.11), just as in the no zero mode case [11, 13]. In this way we can give a direct proof of a theta function identity, which was obtained by Fay [15] as a corollary to his famous identity.

3. "Twist Structures" on M

A generalization of the b–c system that has been studied in the literature is when we are given a distinguished set of points on M around which the b field and c field have a given (rational) monodromy. Such problems arise when there are so-called *twist fields* in the problem, a special case being that of *spin fields*, or when dealing with *orbifolds*. Some of the relevant references, which are known to us, are [6, 7, 21].

We shall show in this section how our methods can be generalized to deal rigorously with the most general problem of this kind. We give a complete solution to this problem for the b–c system. We thereby not only give a precise meaning to formal expressions appearing in the literature, but also give a rigorous proof that certain correlation functions which are usually not considered, or ruled out on heuristic grounds, indeed do not occur. Our analysis is based, as in Sect. 2 and throughout this paper, on the OPE (1.2) and maximal analyticity.

We define a *twist structure* on M to be an assignment of N_+ positive rational numbers $\mu_i (1 \leqq i \leqq N_+)$ to N_+ distinct distinguished points $x_1, \dots, x_{N_+}$ of M and N_- negative rational numbers $(-\nu_j)(1 \leqq j \leqq N_-)$ to N_- points $y_1, \dots, y_{N_-}$ of M (where the y's are distinct from each other as well as from the x's) such that

$$\sum_{i=1}^{N_+} \mu_i - \sum_{j=1}^{N_-} \nu_j = l, \tag{3.1}$$

where l is a (positive or negative) integer called the *total twist* (cf. Zucchini [21]). The b field and c field are required to have the following behaviour in the neighbourhood of these points:

$$\begin{aligned} b(z) &\sim (z-x_i)^{-\mu_i} \quad (1 \leqq i \leqq N_+) \\ &\sim (z-y_j)^{\nu_j} \quad (1 \leqq j \leqq N_-) \end{aligned}, \tag{3.2a}$$

$$\begin{aligned} c(z) &\sim (z-x_i)^{\mu_i} \quad (1 \leqq i \leqq N_+) \\ &\sim (z-y_j)^{-\nu_j} \quad (1 \leqq j \leqq N_-) \end{aligned}. \tag{3.2b}$$

As in the case of the basic OPE (1.2), Eqs. (3.2a, b) must be taken to refer to the behaviour of the correlation functions $C(m, n)$ in the neighbourhood of such points. We shall regard (3.2a, b) as being, along with (1.2), the basic defining relations of our system.

The problem we shall now consider is that of obtaining the two point function $\langle b(Q)c(P)\rangle$ in the presence of a given twist structure, our system being defined by (1.2) and (3.2a, b). It will be evident to the reader that the problem is not posed precisely. All sources of ambiguity will be spelled out as we proceed.

If we want maximum generality, we should start once again with the generalised b–c system of Sect. 2. This would make our exposition tedious and is unnecessary since our aim is to make contact with the existing literature. One possible use of greater generality would be if a treatment of the spin $(1-J)$, J system were required in which zero modes were to be eliminated by a twist structure, rather than by poles as is conventional [14]. The modifications that would have to be made to the discussion below are obvious (put $l=(2J-1)(g-1)$ in Theorem 3.2 below).

We shall, therefore, take c (respectively b) to be a section of α (respectively $K\otimes\alpha^{-1}$), where $\deg(\alpha)=g-1$. We put no further restrictions on α at present. We expect the two point function $\langle b(Q)c(P)\rangle$ to be, in some sense, a *multi-valued section* of $p_1^*(K\otimes\alpha^{-1})\otimes p_2^*(\alpha)$, and so we must determine a *covering space* $\tilde{M}$ from the given data on which we can interpret it as a *meromorphic* section of a line bundle over $\tilde{M}\times\tilde{M}$.

In the following we shall write $E(z, x)$ for the unique holomorphic section of the line bundle $\mathcal{O}(x)$ over M, with divisor x, for any $x\in M$. This is consistent with Definition 2.3 of the prime form when we freeze the second variable. It is then easy to see that our problem of making precise the behaviour of the two point function near the x_i, y_j, as given by (3.2a, b), is really one of making sense of the formal expression

$$\xi(z)\equiv\prod_{i=1}^{N_+}(E(z,x_i))^{\mu_i}\Big/\prod_{j=1}^{N_-}(E(z,y_j))^{\nu_j}. \tag{3.3}$$

For if (3.3) were defined, $\langle b(Q)c(P)\rangle E(Q,P)\xi(Q)/\xi(P)$ would be holomorphic in each variable and we could hope to use algebraic geometry methods to count the number of such sections.

Now, we can certainly write $\mu_i=p_i/d\,(1\leqq i\leqq N_+)$, $\nu_j=q_j/d\,(1\leqq j\leqq N_-)$, where the p_i, q_j, d are positive integers. Then $\xi(z)$ is the "d^{th} root" of

$$\prod_{i=1}^{N_+}(E(z,x_i))^{p_i}\Big/\prod_{j=1}^{N_-}(E(z,y_j))^{q_j}, \tag{3.4}$$

which is the canonical meromorphic section of the line bundle associated to the divisor

$$\sum_{i=1}^{N_+}p_ix_i-\sum_{j=1}^{N_-}q_jy_j. \tag{3.5}$$

Our problem is to construct a covering space $\tilde{M}$ of M on which the d^{th} root of (3.4) can be interpreted as a meromorphic section of a line bundle pulled up from M. Our solution to this problem is based on the following lemma, which we have adapted from Lemma 1 of [22], where the basic idea of the construction is attributed to Mumford.

Lemma 3.1. *Let ζ be a holomorphic line bundle over a compact, connected Riemann surface M and D an effective (i.e. positive) divisor on M such that $\zeta^d = \mathcal{O}(D)$ for some positive integer d. Let σ denote the canonical holomorphic section of $\mathcal{O}(D)$ with divisor D. Then there is a d-fold cyclic covering $\pi: \tilde{M} \to M$, ramified precisely over the support of D, such that $\pi^*(\zeta)$ admits a holomorphic section τ satisfying*

$$\tau^d = \pi^*(\sigma) \text{ in } H^0(\tilde{M}, \pi^*(\mathcal{O}(D))). \tag{3.6}$$

Let $D = \sum_1^k m_i P_i$, where the P_i are distinct points of M and the m_i are positive integers. Then $\tilde{M}$ is irreducible if and only if the greatest common divisor of $(d, m_i; 1 \leqq i \leqq k)$ is unity and nonsingular if and only if $m_i = 1$ for $1 \leqq i \leqq k$.

Proof. We explain the construction of $\tilde{M}$ in some detail, since this method may be useful in other problems of mathematical physics and [22] is rather brief.

Let proj : $\zeta \to M$ be the projection map for the line bundle ζ. We can also look at proj as a surjective holomorphic map from the *total space* $|\zeta|$ of ζ, which is two dimensional. We shall construct $\tilde{M}$ as a subvariety of the total space $|\zeta|$. A point $z \in |\zeta|$ is a pair (x, v), where $x \in M$ and v belongs to the fibre of ζ over x. Thus $z^d = (x, v^{\otimes d})$ and proj$(z) = x$. We define $\tilde{M}$ to be the subvariety of the total space of ζ defined by

$$\tilde{M} = \{(x, v) \in |\zeta| \,|\, (x, v^{\otimes d}) = (x, \sigma(x))\}. \tag{3.7}$$

We define π to be the restriction of proj to $\tilde{M}$.

By the interpretation of proj as a surjective holomorphic map from the total space of ζ to M, proj can be used to pull back line bundles over M to $|\zeta|$. Thus proj$^*(\zeta)$ is the pullback of the line bundle ζ over M to the total space $|\zeta|$. The fibre of proj$^*(\zeta)$ over $z = (x, v) \in |\zeta|$ is the fibre of ζ at proj$(z) = x$. Since v belongs to the fibre of ζ over x, we see that proj$^*(\zeta)$ has a *tautological section*, viz. $z = (x, v) \to v$. Let τ denote the restriction of this tautological section to $\tilde{M}$. Then τ is a holomorphic section of the line bundle $\pi^*(\zeta)$ over $\tilde{M}$. By (3.7), τ^d maps $z = (x, v) \in \tilde{M}$ to $v^{\otimes d} = \pi^*(\sigma(x))$. Thus $\tau^d = \pi^*(\sigma)$.

Clearly π is ramified at $z \in \tilde{M}$ such that $\tau^d(z) = 0$. From (3.7) we see that such z lie over the support of D. The nonsingularity condition follows by inspection from (3.7) by writing the local equation in the neighbourhood of a point of ramification. The irreducibility criterion follows from the same local equations and standard Galois theory. This completes the proof of Lemma 3.1.

Lemma 3.1 is not directly applicable to our situation since (3.5) is not a positive divisor. We can, however, add a suitable positive divisor to (3.5) and subtract it later. The freedom in choosing the divisor to be added to (3.5) introduces a certain arbitrariness in the construction of $\tilde{M}$, which is intrinsic to the problem since physics only gives us the nonpositive divisor (3.5). Since $\tilde{M}$ has merely an auxiliary role, this is of no importance. So choose positive integers n_j such that $n_j - v_j$ is positive $(1 \leqq j \leqq N_-)$ and consider the effective divisor

$$D = \sum_{i=1}^{N_+} p_i x_i + \sum_{j=1}^{N_-} (n_j d - q_j) y_j. \tag{3.8}$$

In view of Lemma 3.1, if we want $\tilde{M}$ to be irreducible we must require the g.c.d. of d and the integer coefficients in (3.8) to be unity. While irreducibility is not strictly essential, it is reasonable to say that, if this is not satisfied, then the original problem has been badly posed. The fact that by Lemma 3.1 the covering space $\tilde{M}$

will be a singular curve unless the coefficients appearing in (3.8) are all unity is of no importance: $\tilde{M}$ can always be replaced by its canonical normalization if desired, but this step is not necessary to interpret our formula for the two point function. In the case of *spin fields* each $p_i=1$, each $q_j=1$, $d=2$ and we can choose each $n_j=1$. Then the integer coefficients in (3.8) are all unity and $\tilde{M}$ is nonsingular as well as irreducible.

The line bundle $\mathcal{O}(D)$ has a canonical holomorphic section $\sigma(z)$ with divisor D, which can be written as a product of prime forms:

$$\sigma(z)=\prod_{i=1}^{N_+}(E(z,x_i))^{p_i}\prod_{j=1}^{N_-}(E(z,y_j))^{(n_jd-q_j)}. \tag{3.9}$$

Now note that

$$\deg(\mathcal{O}(D))=d\left(l+\sum_{j=1}^{N_-}n_j\right), \tag{3.10}$$

i.e. a multiple of d. Hence, by the divisibility of the group $\mathrm{Pic}^0(M)$ for $g\geqq 1$ [24] and the fact that $\mathrm{Pic}(M)=\mathbb{Z}$ for $g=0$, we can find a line bundle ζ such that

$$\zeta^d=\mathcal{O}(D). \tag{3.11}$$

In fact, there are d^{2g} such line bundles and so we have to make a choice if $g\geqq 1$. The choice of ζ plays no role in the present abstract discussion, but should be kept in mind when interpreting the final formula for the two point function. It reflects the fact that our original problem was not well posed and so there is an ambiguity in the answer.

We now have the data, viz. $(\zeta, d, \mathcal{O}(D), \sigma)$, to apply Lemma 3.1. We can thus give a rigorous meaning to

$$\prod_{i=1}^{N_+}(E(z,x_i))^{\mu_i}\prod_{j=1}^{N_-}(E(z,y_j))^{(n_j-\nu_j)} \tag{3.12}$$

as a multi-valued section of ζ, which is properly defined as a holomorphic section of the pullback of ζ to the covering space $\tilde{M}$. Now, defining

$$\gamma=\zeta\otimes\mathcal{O}\left(-\sum_{j=1}^{N_-}n_jy_j\right), \tag{3.13}$$

we see that we have given a precise meaning to $\xi(z)$ of (3.3) as a multi-valued section of γ, which is properly defined as a meromorphic section of $\pi^*(\gamma)$ over $\tilde{M}$. Note that

$$\deg(\gamma)=l\equiv\text{“total twist.”} \tag{3.14}$$

Then from the OPE (1.2), (3.2) and maximal analyticity, we see that

$$\langle b(Q)c(P)\rangle E(Q,P)\xi(Q)/\xi(P) \tag{3.15}$$

is the pullback to $\tilde{M}\times\tilde{M}$ of a holomorphic section of the following line bundle over $M\times M$:

$$\begin{aligned}&p_1^*(K\otimes\alpha^{-1})\otimes p_2^*(\alpha)\otimes\mathcal{O}(\Delta)\otimes p_1^*(\gamma)\otimes p_2^*(\gamma^{-1})\\&=p_1^*(K\otimes\tilde{\alpha}^{-1})\otimes p_2^*(\tilde{\alpha})\otimes\mathcal{O}(\Delta),\ \text{where}\ \tilde{\alpha}=\alpha\otimes\gamma^{-1}.\end{aligned} \tag{3.16}$$

Since

$$\deg(\tilde{\alpha})+\deg(K\otimes\tilde{\alpha}^{-1})=g-1-l+g-1+l=2g-2, \tag{3.17}$$

we see that we are dealing with the generalized b–c system of Sect. 2. Thus the generalized system we discussed in Sect. 2 is precisely necessary to deal with the ordinary b–c system in the presence of a twist structure. We can then apply Theorem 2.2 to conclude that to have a nonzero two point function we must have $\tilde{\alpha} \in \mathrm{Pic}^{g-1}(M) - \Theta$. As a consequence *the total twist l of the twist structure must be zero.* We have thus proved:

Theorem 3.2. *Given a twist structure on M defined by the data (D, γ, l), the twisted spin $\frac{1}{2}$ b–c system over M (i.e. c is a section of $\alpha \in \mathrm{Pic}^{g-1}(M)$, b of $K \otimes \alpha^{-1}$) has a nonzero two point function if and only if the total twist l is zero and $\tilde{\alpha} \equiv \alpha \otimes \gamma^{-1}$ lies in the complement of the theta divisor Θ. When these conditions are satisfied the two point function is unique and given by*

$$\frac{\xi(P)}{\xi(Q)} S_{\tilde{\alpha}}(Q, P). \tag{3.18}$$

In (3.18) *the Szegö kernel has to be interpreted for $g=0$ and 1 as the two point function of Appendix A and Appendix B respectively, with corresponding expressions for the prime form in the definition of $\xi(z)$ in* (3.3).

Atick and Sen have computed the correlation function of the b–c system in the presence of spin fields $S^+(x_i)$, $S^-(y_j)$ over the torus [6] and for higher genus [7] using the stress tensor method. Formula (3.18) agrees with their results for correlators containing pairs of spin fields $S^+(x_i)S^-(y_i)$ in both cases, but we are also able to show that other correlators containing unequal numbers of the two kinds of spin fields must vanish. This was expected, but not proved, by them. In the genus 0 case, the Ramond sector two point function written down by Dixon et al. [21] is a symmetrised form of (3.18) in the $g=0$ case with spin fields at 0 and ∞.

The methods of this section can be easily combined with those of Sect. 4 to obtain higher point functions in the presence of a twist structure (see Remark 4.5). We can also consider the fermionic b–c system as discussed by us in [11–14] in the presence of a twist structure of zero total twist. It is easy to see that the theorem proved there [11–14] that the $2n$-point function is a determinant of two point functions remains valid in the presence of such a twist structure. This leads to an identity which reduces, after cancelling common factors, to Fay's identity for $g \geqq 2$ and to Cauchy's and Frobenius' identities for $g=0$ and 1, respectively.

4. The b–c System with Arbitrary Statistics

In this section we shall discuss the general correlation function $C(m,n)$, which was written symbolically in (1.1) as

$$C(m,n) \equiv \langle b(Q_1) \dots b(Q_m) c(P_1) \dots c(P_n) \rangle .$$

Let $\{M_i; 1 \leqq i \leqq m+n\}$ be $m+n$ copies of M and form the product manifold $M^{m+n} \equiv \prod_{i=1}^{m+n} M_i$. Let

$$\begin{aligned} p_i : M^{m+n} &\to M \\ (z_1, \dots, z_i, \dots, z_{m+n}) &\to z_i \end{aligned} \tag{4.1}$$

denote the projection to the i^{th} factor. Then, if we consider the (twisted) spin $\frac{1}{2}$ b–c system of Theorem 2.2, $C(m,n)$ is a meromorphic section of the holomorphic line bundle

$$\mathscr{F}_\alpha(m,n) \equiv p_1^*(K \otimes \alpha^{-1}) \otimes \ldots \otimes p_m^*(K \otimes \alpha^{-1}) \otimes p_{m+1}^*(\alpha) \otimes \ldots \otimes p_{m+n}^*(\alpha) \quad (4.2)$$

over M^{m+n}, with $\alpha \in \mathrm{Pic}^{g-1}(M) - \Theta$. The poles of $C(m,n)$ occur at coincident points of certain of its arguments, viz. when $z_i = z_j$ for some i, j such that $1 \leqq i \leqq m$, $m+1 \leqq j \leqq m+n$. Denoting by Δ_{ij} the diagonal of $M_i \times M_j$, we see that the equation $z_i = z_j$ in M^{m+n} defines the divisor (in this case a subvariety)

$$D_{ij} \equiv \mathrm{pr}_{ij}^{-1}(\Delta_{ij}), \quad (4.3)$$

where pr_{ij} denotes the projection

$$\begin{aligned} \mathrm{pr}_{ij}: M^{m+n} &\to M_i \times M_j \\ (z_1, \ldots, z_i, \ldots, z_j, \ldots, z_{m+n}) &\to (z_i, z_j). \end{aligned} \quad (4.4)$$

Giving the divisor D_{ij} is equivalent to giving a holomorphic line bundle $\mathcal{O}(D_{ij})$ on M^{m+n} along with a canonical holomorphic section having D_{ij} as its zero divisor. Since $\mathcal{O}(D_{ij})$ is the pullback to M^{m+n} by pr_{ij} of the line bundle $\mathcal{O}(\Delta_{ij})$ on $M_i \times M_j$, it is consistent with Definition 2.3 to refer to this holomorphic section as the prime form and to denote it by $E(z_i, z_j)$.

The higher order correlation functions of the b–c system depend both on the analyticity properties determined by the OPE and on the specification of particle type, or *statistics*. Thus the correlation functions of the fermionic b–c system and the β–γ system must be different beyond their two point functions, even though both are governed by the same OPE (1.2).

It seems natural to separate these two aspects of the problem. Our approach will be, therefore, to first determine the vector space of all the meromorphic sections of $\mathscr{F}_\alpha(m,n)$ which are allowed by the OPE (1.2), which we shall denote by $V(m,n)$, and to see then how the specification of particle type picks out a particular correlation function $C(m,n)$ for the system in question. Not every element of $V(m,n)$ need correspond to the correlation function of a particular b–c system, since our specification of particle type may be more restrictive than analyticity considerations permit.

We must now express the constraints imposed by the OPE (1.2) and maximal analyticity as precise postulates that must be satisfied by an arbitrary element v of $V(m,n)$. These constraints should be expressed as statements regarding *holomorphic* sections of line bundles in order to be able to use algebraic geometry methods. Now a correlation function $C(m,n) \in V(m,n)$ has a polar divisor

$$D_p(m,n) \equiv \sum_{\substack{1 \leqq i \leqq m \\ m+1 \leqq j \leqq m+n}} D_{ij} \quad (4.5a)$$

by the OPE and maximal analyticity. The canonical section of $\mathcal{O}(D_p(m,n))$ with divisor $D_p(m,n)$ will be denoted by $E_p(m,n)$, where

$$E_p(m,n) \equiv \prod_{\substack{1 \leqq i \leqq m \\ m+1 \leqq j \leqq m+n}} E(z_i, z_j). \quad (4.5b)$$

Thus we must require that $vE_p(m,n)$ be a holomorphic section of $\mathscr{F}_\alpha(m,n) \otimes \mathcal{O}(D_p(m,n))$. Since $\mathscr{F}_\alpha(m,n)$ has no nonzero holomorphic sections, this condition means that if $v \neq 0$ the polar divisor of v is nonempty and contained in $D_p(m,n)$.

The OPE (1.2) imposes a further constraint, which comes from the fact that (1.2) is an operator relation and the I on the right-hand side is the identity operator.

Consider an arbitrary nonzero element $v \in V(m,n)$ and let D_{ij} belong to its polar divisor. Note that

$$D_{ij} = \Delta_{ij} \times M^{m+n-2}, \qquad M^{m+n-2} \equiv \prod_{\substack{k=1 \\ k \neq i,j}}^{m+n} M_k. \tag{4.6}$$

Then the OPE (1.2) says that

$$vE(z_i, z_j)|M^{m+n-2} \in V(m-1, n-1). \tag{4.7}$$

We shall now formulate our postulates for $V(m,n)$. It turns out to be notationally convenient to state them either for $m \geqq n$ or $m \leqq n$. We shall choose to state them for $m \geqq n$. This is of no consequence since it turns out that there are only two cases to consider, when $m \neq n$ and $m = n$, and whatever we prove for $m > n$ applies immediately for $m < n$.

Let $\mathfrak{S}_m$ denote the group of permutations of $\{1, \ldots, m\}$, where $m \geqq n$. For each $\sigma \in \mathfrak{S}_m$ we define the divisor

$$D_\sigma(m,n) \equiv \sum_{i=1}^{n} D_{\sigma(i), m+i}. \tag{4.8}$$

Thus $D_\sigma(m,n) \subset D_p(m,n)$ for each $\sigma \in \mathfrak{S}_m$. The line bundle $\mathcal{O}(D_\sigma(m,n))$ on M^{m+n} has a canonical holomorphic section $E_\sigma(m,n)$, where

$$E_\sigma(m,n) \equiv \prod_{i=1}^{n} E(z_{\sigma(i)}, z_{m+i}). \tag{4.9}$$

We shall denote the intersection of the D_{ij} appearing on the right-hand side of (4.8) by $\cap D_\sigma(m,n)$; note that for $n > 1$ it is not of codimension one in M^{m+n} and hence is not a divisor of M^{m+n}.

We state our postulates as follows. Every element $v \in V(m,n)$ must satisfy:

$\mathscr{P}1$) *v is a meromorphic section of $\mathscr{F}_\alpha(m,n)$ such that $vE_p(m,n)$ is a holomorphic section of $\mathscr{F}_\alpha(m,n) \otimes \mathcal{O}(D_p(m,n))$.*

$\mathscr{P}2$) *For each $\sigma \in \mathfrak{S}_m$ the restriction of $vE_\sigma(m,n)$ to $\cap D_\sigma(m,n)$ is an element of the vector space*

$$U_\sigma(m,n) \equiv H^0(\cap D_\sigma(m,n), \mathscr{F}_\alpha(m,n) \otimes \mathcal{O}(D_\sigma(m,n)) | \cap D_\sigma(m,n)). \tag{4.10}$$

Moreover, the only element for which this restriction is the zero vector of $U_\sigma(m,n)$ for every $\sigma \in \mathfrak{S}_m$ is the zero vector of $V(m,n)$.

Postulate $\mathscr{P}1$ is an evident requirement that we have already discussed, while postulate $\mathscr{P}2$ is simply condition (4.7) applied repeatedly until all possible poles are removed. Note that condition (4.7) does not prescribe the lower order correlation functions of the system when statistics are specified: in Theorem 4.6 below that is indeed satisfied, while that is not necessarily the case for the examples of generalized statistics discussed following Theorem 4.6 below.

We must now determine the dimension of $V(m,n)$.

Proposition 4.1. *If $m \neq n$, $\dim V(m,n) = 0$.*

Proof. For each $\sigma \in \mathfrak{S}_m (m > n)$,

$$\begin{aligned} vE_\sigma(m,n) | \cap D_\sigma(m,n) &\in \mathscr{F}_\alpha(m,n) \otimes \mathcal{O}(D_\sigma(m,n)) | \cap D_\sigma(m,n) \\ &= p^*_{\sigma(n+1)}(K \otimes \alpha^{-1}) \otimes \ldots \otimes p^*_{\sigma(m)}(K \otimes \alpha^{-1}) | \Delta_{\sigma(1), m+1} \times \ldots \times \Delta_{\sigma(n), m+n} \\ &\quad \times M_{\sigma(n+1)} \times \ldots \times M_{\sigma(m)}, \end{aligned} \tag{4.11}$$

and so by the Künneth formula

$$U_\sigma(m,n)=0\,.$$

The proposition follows from the second part of $\mathscr{P}2$.

Proposition 4.2. *Let $m=n$. Then there exist linear maps i, r and $j\equiv\bigoplus_{\sigma\in\mathfrak{S}_n} j_\sigma$ such that the following is a commutative diagram of linear isomorphisms:*

$$\begin{tikzcd} & W(n,n) \arrow[dl, "i"'] \arrow[dr, "j"] & \\ V(n,n) \arrow[rr, "r"] & & U(n,n) \end{tikzcd}$$

where $U(n,n)\equiv\bigoplus_{\sigma\in\mathfrak{S}_n} U_\sigma(n,n)$, $W(n,n)\equiv\bigoplus_{\sigma\in\mathfrak{S}_n} W_\sigma(n,n)$, with $U_\sigma(n,n)$ as defined in (4.10),

$$W_\sigma(n,n)\equiv H^0(M^{2n},\mathscr{F}_\alpha(n,n)\otimes\mathcal{O}(D_\sigma(n,n)))\,, \tag{4.12}$$

and $j_\sigma: W_\sigma(n,n)\to U_\sigma(n,n)$.

Proof. The map j_σ is defined by restriction of each element of $W_\sigma(n,n)$ to $\cap D_\sigma(n,n)$. To prove that it is a linear isomorphism, observe that, by the Künneth formula,

$$\begin{aligned} W_\sigma(n,n) &= \bigotimes_{i=1}^{n} H^0(M_{\sigma(i)}\times M_{n+i}, p^*_{\sigma(i)}(K\otimes\alpha^{-1})\otimes p^*_{n+i}(\alpha)\otimes\mathcal{O}(\Delta_{\sigma(i),n+i})) \\ &= \bigotimes_{i=1}^{n} H^0(\Delta_{\sigma(i),n+i}, p^*_{\sigma(i)}(K\otimes\alpha^{-1})\otimes p^*_{n+i}(\alpha)\otimes\mathcal{O}(\Delta_{\sigma(i),n+i})|\Delta_{\sigma(i),n+i}) \quad (4.13) \\ &= U_\sigma(n,n)\,, \end{aligned}$$

where to get (4.13) we used Proposition 5.5 of [11], which is easily seen to hold for all genera $g\geqq 0$. This proves that $j\equiv\bigoplus_{\sigma\in\mathfrak{S}_n} j_\sigma$ is a linear isomorphism.

The map r is defined by

$$\begin{aligned} r: V(n,n)&\to U(n,n) \\ v&\to \bigoplus_{\sigma\in\mathfrak{S}_n} vE_\sigma(n,n)|\cap D_\sigma(n,n)\,. \end{aligned} \tag{4.14}$$

To define the map i, take an arbitrary element $w=\bigoplus_{\sigma\in\mathfrak{S}_n} w_\sigma\in W(n,n)$ and define

$$\begin{aligned} i: W(n,n)&\to V(n,n) \\ w=\bigoplus_{\sigma\in\mathfrak{S}_n} w_\sigma &\to \sum_{\sigma\in\mathfrak{S}_n} w_\sigma/E_\sigma(n,n)\,. \end{aligned} \tag{4.15}$$

It is clear that the right-hand side of (4.15) satisfies $\mathscr{P}1$. The fact that it also satisfies $\mathscr{P}2$ and hence lies in $V(n,n)$ emerges from the proof that the diagram commutes, which we now give. Take $0\neq w\in W(n,n)$, then

$$i(w)=\sum_{\sigma\in\mathfrak{S}_n} w_\sigma/E_\sigma(n,n)\,,$$

$$r(i(w))=\sum_{\sigma,\sigma'\in\mathfrak{S}_n}\frac{w_\sigma}{E_\sigma(n,n)}E_{\sigma'}(n,n)|\cap D_{\sigma'}(n,n)=\sum_{\sigma\in\mathfrak{S}_n} w_\sigma|\cap D_\sigma(n,n)=j(w)\,.$$

We must now prove that i and r are isomorphisms. The very definition of i shows that it is injective, while the fact that r is injective is the second part of $\mathscr{P}2$. To show that r is surjective, take a nonzero element $u \in U(n,n)$. Then by the isomorphism j we can find a nonzero element $w \in W(n,n)$ such that $j(w)=u$. Since i is injective, $i(w)$ is a nonzero element of $V(n,n)$ and, since the diagram commutes, $r(i(w))=u$. Hence r is surjective. Since j and r are isomorphisms, i is also an isomorphism. This completes the proof of Proposition 4.2.

Theorem 4.3. *The dimension of the vector space $V(n,n)$ is $n!$*

Proof. By Proposition 4.2 we have

$$\dim V(n,n) = \dim U(n,n) = \dim W(n,n),$$

where $W(n,n) = \bigoplus_{\sigma \in \mathfrak{S}_n} W_\sigma(n,n)$. Using (4.13) and the conclusion of Theorem 2.2 (more explicitly stated in Theorem 5.8 of [11] which holds for all $g \geqq 0$) we see that

$$\dim W_\sigma(n,n) = 1.$$

Hence

$$\dim W(n,n) = \sum_{\sigma \in \mathfrak{S}_n} 1 = n!,$$

which proves the theorem.

The isomorphism i between $W(n,n)$ and $V(n,n)$ also gives us:

Theorem 4.4. *A basis for $V(n,n)$ is provided by products of Szegö kernels*

$$\left\{ \prod_{i=1}^{n} S_\alpha(Q_{\sigma(i)}, P_i) \middle| \sigma \in \mathfrak{S}_n \right\},$$

where we interpret $S_\alpha(Q,P)$ for $g=0$ by Theorem A.3 and for $g=1$ by Theorem B.6.

Proof. An immediate consequence of the definition of the isomorphism i, Eq. (4.13), Proposition 6.1 of [11] for $g \geqq 2$, Theorem A.3 for $g=0$, Theorem B.6 for $g=1$.

Remark 4.5. The above analysis extends immediately to, e.g., the spin $(1-J)$, J system by simply replacing the Szegö kernel $S_\alpha(Q,P)$ in Theorem 4.4 by the corresponding two point function written down in [14]. If there is also a twist structure with total twist zero, simply multiply the two point function by $\xi(P)/\xi(Q)$ as in (3.18).

A b–c system will be completely specified if we can associate to it a unique set of correlation functions $\{C(n,n); n \in \mathbb{N}\}$. Thus for each $n \in \mathbb{N}$ we must pick out a one dimensional subspace of $V(n,n)$ for a given b–c system. We can do this by specifying the particle type or *statistics.*

To study the possibilities, we must define an action of the symmetric group $\mathfrak{S}_n$ on $V(n,n)$. We can define the action on a basis element given by Theorem 4.4 and extend by linearity. Since we have two kinds of particles, viz. the b fields labelled by $Q_1, Q_2, \dots$ and the c fields labelled by $P_1, P_2, \dots$ we can define a natural action of $\mathfrak{S}_n$ on the labels $\{1, \dots, n\}$ of $Q_1, \dots, Q_n$ and of $P_1, \dots, P_n$ separately. Thus $V(n,n)$ is an $\mathfrak{S}_n \times \mathfrak{S}_n$-module. Considered as an $\mathfrak{S}_n$-module, $V(n,n)$ is simply the regular representation of $\mathfrak{S}_n$. Then we know that $V(n,n)$ is a multiplicity free direct sum of irreducible $\mathfrak{S}_n \times \mathfrak{S}_n$ modules $V_\lambda \otimes V_\lambda$, where V_λ is the irreducible representation of $\mathfrak{S}_n$ corresponding to the partition λ of n. The only one dimensional representations are the alternating and symmetric representations, corresponding to Fermi and Bose statistics, respectively. We thus get:

Theorem 4.6. (a) *In the case of Fermi statistics the $2n$-point function $\langle b(Q_1)\dots b(Q_n)\,c(P_1)\dots c(P_n)\rangle$ is antisymmetric in the Q-variables and separately in the P-variables and, as a result,*

$$\langle b(Q_1)\dots b(Q_n)c(P_1)\dots c(P_n)\rangle = \det(S_\alpha(Q_i,P_j))|_{i,j=1}^n, \tag{4.16}$$

where det *denotes the determinant.*

(b) *In the case of Bose statistics, where it is conventional to write β for b and γ for c, the $2n$-point function $\langle \beta(Q_1)\dots\beta(Q_n)\gamma(P_1)\dots\gamma(P_n)\rangle$ is symmetric in the Q-variables and separately in the P-variables and, as a result,*

$$\langle \beta(Q_1)\dots\beta(Q_n)\gamma(P_1)\dots\gamma(P_n)\rangle = \mathrm{perm}(S_\alpha(Q_i,P_j))|_{i,j=1}^n, \tag{4.17}$$

where perm *denotes the permanent.*

While Theorem 4.6 exhausts the possibilities of choosing one dimensional subspaces of $V(n,n)$ by the usual notions of the connection between permutation symmetry and particle statistics, we can investigate whether other possibilities exist under less restrictive conditions. A condition which appears quite natural is to simply demand the invariance of the $2n$-point function $\langle b(Q_1)\dots b(Q_n)c(P_1)\dots c(P_n)\rangle$ when the Q's and P's are simultaneously permuted in like fashion, i.e.

$$\langle b(Q_1)\dots b(Q_n)c(P_1)\dots c(P_n)\rangle = \langle b(Q_{\sigma(1)})\dots b(Q_{\sigma(n)})c(P_{\sigma(1)})\dots c(P_{\sigma(n)})\rangle \tag{4.18}$$

for each $\sigma\in\mathfrak{S}_n$. This clearly means that we must restrict the $\mathfrak{S}_n\times\mathfrak{S}_n$ representation in $V(n,n)$ to the diagonally embedded subgroup $\mathfrak{S}_n$. It is obvious that the regular representation $V(n,n)$ decomposes so as to give a one dimensional representation of $\mathfrak{S}_n$ for each conjugacy class of $\mathfrak{S}_n$, i.e. for each partition λ of n. We thus obtain:

Theorem 4.7. *For each partition λ of n, there is a unique $2n$-point function, satisfying postulates $\mathscr{P}1$, $\mathscr{P}2$ and the invariance condition* (4.18), *given by*

$$\langle b(Q_1)\dots b(Q_n)c(P_1)\dots c(P_n)\rangle_\lambda = \mathrm{imm}_\lambda(S_\alpha(Q_i,P_j))|_{i,j=1}^n, \tag{4.19}$$

where imm_λ *denotes the immanant. This is defined* [23] *for an $n\times n$ matrix (A_{ij}) by*

$$\mathrm{imm}_\lambda(A_{ij})|_{i,j=1}^n \equiv \sum_{\sigma\in\mathfrak{S}_n}\chi_\lambda(\sigma)\prod_{i=1}^n A_{\sigma(i),i}, \tag{4.20}$$

where χ_λ is the character corresponding to the irreducible representation of $\mathfrak{S}_n$ labelled by the partition λ of n.

Theorem 4.7 includes the case of Fermi and Bose statistics discussed in Theorem 4.6, while including other cases. Other possibilities, not covered by Theorem 4.7, abound, since the $\mathfrak{S}_n\times\mathfrak{S}_n$-module $V(n,n)$ is, of course, a $G\times G$-module for any subgroup G of $\mathfrak{S}_n$. In that case simply put $A_{ij}=S_\alpha(Q_i,P_j)$ and restrict the sum in (4.20) to $\sigma\in G$. For example, we could consider $G=\mathbb{Z}_n$, viewed as the group of n cyclic permutations of $\{1,\dots,n\}$. Then the coefficients in (4.20) are $1,\omega,\dots,\omega^{n-1}$, where ω is a primitive n^{th} root of unity and the sum in (4.20) is over the n cyclic permutations. It is in this sense that the term "arbitrary statistics" in the title is justified.

Appendix A: The Case $g=0$

In this appendix we discuss the concept of the prime form, the two point function of the b–c system and the full fermionic b–c system (discussed earlier for $g \geqq 2$ in [11–13]) for the case when M is a compact, connected Riemann surface of genus zero. Our analysis of the fermionic b–c system in this appendix leads to a proof of the genus zero analogue of Fay's identity, viz. an identity for rational functions due to Cauchy [16].

We first consider the concept of the prime form for genus zero. A discussion of this has been given by Mumford [20], but his treatment is not suitable for our needs. We shall give a completely independent discussion, in which we follow Definition 2.3, thereby extending the treatment of the prime form that we gave in [11] for $g \geqq 2$.

Since $g=0$ we can, and from this point shall, identify M with the *complex projective line* $\mathbb{P}^1$. Then, as is well known [19], M has an open covering by two copies of the affine line $\mathbb{A}^1$: $U_0 = \mathbb{A}^1$, with affine coordinate x, and $U_\infty = \mathbb{A}^1$, with affine coordinate y, which are glued along open subsets $V_0 = \{x \in U_0 - (0)\}$ and $V_\infty = \{y \in U_\infty - (0)\}$ by the map

$$g_{\infty 0} : V_0 \ni x \to 1/x \in V_\infty . \tag{A.1}$$

We are interested in line bundles over $\mathbb{P}^1$. As is well known [18, 19], $\mathrm{Pic}(\mathbb{P}^1) = \mathbb{Z}$, so that a line bundle is determined up to isomorphism by its degree or *Chern class*. From the Riemann-Roch theorem (or directly) we see that the line bundle of degree 1 has a two dimensional space of holomorphic sections. We choose a special basis $\{\mathscr{H}, \mathscr{I}\}$ for the space of sections, which we now define:

$$\begin{aligned} \text{for } x \in U_0, \quad & \mathscr{H}(x) = x, \quad \mathscr{I}(x) = 1, \\ y \in U_\infty, \quad & \mathscr{H}(y) = 1, \quad \mathscr{I}(y) = y. \end{aligned} \tag{A.2}$$

The transition function $g_{\infty 0}$ was defined in (A.1).

In the genus zero case the line bundle $\mathscr{F}_\alpha(1,1) \equiv p_1^*(K \otimes \alpha^{-1}) \otimes p_2^*(\alpha) = p_1^*(\alpha) \otimes p_2^*(\alpha)$, since $\deg(K) = -2$, $\deg(\alpha) = -1$ and so $\alpha = K \otimes \alpha^{-1}$.

Proposition A.1. $\mathscr{O}(\Delta) = p_1^*(\alpha^{-1}) \otimes p_2^*(\alpha^{-1})$.

Proof. An easy application of the *seesaw principle* [24, 25]. Alternatively, note that $\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{Z} \times \mathbb{Z}$ and hence the statement of the proposition is simply the obvious equality

$$(1,1) = (1,0) + (0,1)$$

in $\mathbb{Z} \times \mathbb{Z}$.

From Proposition A.1 and (2.2) we get the short exact sequence

$$O \to \mathscr{O}_{M \times M} \to p_1^*(\alpha^{-1}) \otimes p_2^*(\alpha^{-1}) \to \alpha^{-2} \to O . \tag{A.3}$$

Passing to cohomology,

$$O \to \mathbb{C} \to H^0(M \times M, p_1^*(\alpha^{-1}) \otimes p_2^*(\alpha^{-1})) \xrightarrow{\delta} H^0(M \times M, \alpha^{-2}) \to \dots . \tag{A.4}$$

By the Künneth formula the four dimensional space $H^0(M \times M, p_1^*(\alpha^{-1}) \otimes p_2^*(\alpha^{-1}))$ is spanned by vectors of the form $p_1^*(u_i) \otimes p_2^*(u_j)$, where the u_i $(i=1,2)$ are a basis of

$H^0(M, \alpha^{-1})$. Under δ we have

$$p_1^*(u_i) \otimes p_2^*(u_j) \xrightarrow{\delta} u_i \otimes u_j . \tag{A.5}$$

Since α^{-1} is of degree 1, we can use $\{\mathscr{H}, \mathscr{I}\}$ as the basis. From Definition 2.3, we see that the prime form is simply the kernel of the map δ, which is one dimensional as we can see from (A.4). It is obvious from (A.5) that the kernel of δ is spanned by

$$p_1^*(\mathscr{H}) \otimes p_2^*(\mathscr{I}) - p_1^*(\mathscr{I}) \otimes p_2^*(\mathscr{H}) .$$

We have thus proved:

Theorem A.2. *The unique holomorphic section of $\mathcal{O}(\Delta)$ with divisor Δ, called the prime form, is given for genus zero by*

$$E(Q, P) \equiv p_1^*(\mathscr{H}) \otimes p_2^*(\mathscr{I}) - p_1^*(\mathscr{I}) \otimes p_2^*(\mathscr{H}) . \tag{A.6}$$

From (A.2) *we have:*

$$E(Q, P)|_{U_0 \times U_0} = (Q - P) . \tag{A.7}$$

Note that our result is consistent with Mumford's definition [20].

By the OPE (1.2), the two point function of the b–c system for $g = 0$ is the meromorphic section of $\mathscr{F}_\alpha(1,1) \equiv p_1^*(\alpha) \otimes p_2^*(\alpha)$ with polar divisor Δ. Note that since α has negative degree, $\mathscr{F}_\alpha(1,1)$ has no holomorphic sections and so the two point function is defined by a holomorphic section of $\mathscr{F}_\alpha(1,1) \otimes \mathcal{O}(\Delta)$. However, by Proposition A.1, $\mathcal{O}(\Delta) = \mathscr{F}_\alpha(1,1)^{-1}$, and so

$$\mathscr{F}_\alpha(1,1) \otimes \mathcal{O}(\Delta) = \mathcal{O}_{M \times M} , \tag{A.8}$$

where the right-hand side is the trivial line bundle on $M \times M$. Hence there is a one dimensional space of holomorphic sections, viz. $\mathbb{C}$. We have thus proved:

Theorem A.3. *The two point function $\langle b(Q) c(P) \rangle$ is unique and given by*

$$\langle b(Q) c(P) \rangle = 1/E(Q, P) , \tag{A.9}$$

where (A.9) *is to be interpreted in the sense of* (A.8). *Moreover, from* (A.7),

$$\langle b(Q) c(P) \rangle|_{U_0 \times U_0} = \frac{1}{(Q - P)} . \tag{A.10}$$

In [11–13] we discussed the *fermionic* b–c system by adding to the OPE (1.2) the additional OPE's

$$\begin{aligned} b(Q) b(P) &\sim O(Q - P) , \\ c(Q) c(P) &\sim O(Q - P) . \end{aligned} \tag{A.11}$$

Then the correlation functions $C(m, n)$ are defined by a holomorphic section of

$$\mathscr{M}_\alpha(m, n) \equiv \mathscr{F}_\alpha(m, n) \otimes \mathcal{O}(-D(m, n)) , \tag{A.12}$$

where $\mathscr{F}_\alpha(m, n)$ is the line bundle on M^{m+n} defined in (4.2), while

$$D(m, n) = D_z(m, n) - D_p(m, n) , \tag{A.13a}$$

$$D_z(m, n) = \sum{}' D_{ij} + \sum{}'' D_{ij} , \tag{A.13b}$$

where $\sum'$ (respectively $\sum''$) runs over $1 \leqq i < j \leqq m$ (respectively $m+1 \leqq i < j \leqq m+n$), and $D_p(m,n)$ was defined in (4.5a).

Theorem A.4. *Let M be a compact, connected Riemann surface of any genus $g \geqq 0$ and let $\mathcal{M}_\alpha(m,n)$ be as defined above. Then*

(i) *if* $m \neq n$, $\dim H^0(M^{m+n}, \mathcal{M}_\alpha(m,n)) = 0$,

(ii) *if* $m = n$, $\dim H^0(M^{2n}, \mathcal{M}_\alpha(n,n)) = 1$.

Proof. This a restatement for $g \geqq 0$ of Theorems 5.1, 5.5 in [13], whose proofs are easily seen to hold without any change for all genera $g \geqq 0$.

Thus in the $g=0$ case under discussion, we see that for the fermionic b–c system we have $C(m,n)=0$ for $m \neq n$, while the $2n$-point function $C(n,n)$ is unique (after normalisation). As we argued earlier [11–13], this implies that $C(n,n)$ is the determinant of its two point functions, i.e. Wick's theorem holds. We showed earlier [11, 13] that for $g \geqq 2$ this gives Fay's identity [15]. To see what we get for $g=0$, we must be able to write down $C(n,n)$ explicitly.

Proposition A.5. *For $g=0$, $\mathcal{M}_\alpha(n,n)$ is the trivial line bundle on M^{2n}.*

Proof. This can, of course, be proved by the seesaw principle [24, 25]. An amusing alternative way to prove it is to note that for $g=0$, $\mathrm{Pic}(M^{2n}) = \mathbb{Z}^{2n}$ and then note that as elements of $\mathbb{Z}^{2n}$ we have

$$\mathcal{F}_\alpha(n,n) = (-1, \ldots, -1),$$
$$\mathcal{O}(D_p(n,n)) = (n, \ldots, n),$$
$$\mathcal{O}(D_z(n,n)) = (n-1, \ldots, n-1),$$

while the trivial line bundle on M^{2n} is, of course, the neutral element of $\mathbb{Z}^{2n}$. Then the statement of the proposition is a simple equality in $\mathbb{Z}^{2n}$.

Proposition A.5 shows that the $2n$-point function $C(n,n)$ is simply the meromorphic section of $\mathcal{O}(D(n,n))$ with divisor $D(n,n)$. This can easily be written down in terms of the prime form. Writing this on the domain $(U_0)^{2n}$, we find:

$$C(n,n) = \frac{\prod_{1 \leqq i < j \leqq n} (Q_i - Q_j)(P_j - P_i)}{\prod_{1 \leqq i, j \leqq n} (Q_i - P_j)}. \tag{A.14}$$

However, the determinant of two point functions

$$\det(\langle b(Q_i) c(P_j) \rangle)|_{i,j=1}^{n}$$

is a meromorphic section of $\mathcal{F}_\alpha(n,n)$ satisfying the same properties as $C(n,n)$. By the uniqueness theorem, we must have (also on $(U_0)^{2n}$):

$$C(n,n) = \mathrm{const} \times \det\left(\frac{1}{(Q_i - P_j)}\right)\Bigg|_{i,j=1}^{n}. \tag{A.15}$$

It is easily seen that the constant in (A.15) is unity and so we have proved:

Theorem A.6 (Cauchy's identity [16]).

$$\frac{\prod_{1 \leqq i < j \leqq n} (Q_i - Q_j)(P_j - P_i)}{\prod_{1 \leqq i, j \leqq n} (Q_i - P_j)} = \det\left(\frac{1}{Q_i - P_j}\right)\Bigg|_{i,j=1}^{n}. \tag{A.16}$$

Appendix B: The Case $g=1$

In this appendix we discuss the concept of the prime form, the two point function of the b–c system (or *Szegö kernel*), and the full fermionic b–c system for the case when M is a compact, connected Riemann surface of genus $g=1$. Our analysis of the fermionic b–c system leads, via a proof of the validity of the Wick representation of the $2n$-point function, to the $g=1$ case of Fay's identity, which seems to be originally due to Frobenius [17].

For $g=1$, M and $\mathrm{Pic}^0(M)$ are isomorphic under the Abel map. We shall, therefore, identify the two when convenient. The holomorphic cotangent bundle K is now the trivial line bundle on M. The canonical theta divisor Θ is now a subset of $\mathrm{Pic}^0(M)$ consisting of just one element, viz. the neutral element of the group $\mathrm{Pic}^0(M)$ since it corresponds to the trivial line bundle on M.

By Definition 2.3 the prime form is the holomorphic section of $\mathcal{O}(\Delta)$ with divisor Δ. In fact $\mathcal{O}(\Delta)$ has a one dimensional space of holomorphic sections for $g \geqq 1$. This was proved in Proposition 5.3 of [11] for $g \geqq 2$. The proof for $g=1$ is more subtle.

Proposition B.1. *Let M be a compact, connected Riemann surface of genus one. The line bundle $\mathcal{O}(\Delta)$ on $M \times M$ has a one dimensional space of holomorphic sections.*

Proof. We start with the exact sequence (2.2) and note that K_Δ is the trivial line bundle on Δ. We now apply the left exact direct image functor p_{1*} on (2.2) to get the long exact sequence of higher direct image sheaves

$$O \to \mathcal{O}_M \to p_{1*}(\mathcal{O}(\Delta)) \xrightarrow{j} \mathcal{O}_M \to R^1 p_1^*(\mathcal{O}_{M\times M}) \to \dots . \tag{B.1}$$

For the same reason as for (2.8), (B.1) is an exact sequence of vector bundles. As in the proof of Lemma 2.6(ii), we conclude from Grothendieck's semicontinuity theorem that for a generic $P \in M$,

$$\begin{aligned} \operatorname{rank} p_{1*}(\mathcal{O}(\Delta)) &= \dim H^0(p_1^{-1}(P), \mathcal{O}(\Delta)|p_1^{-1}(P)) \\ &= \dim H^0(\{P\}\times M, \mathcal{O}(\Delta)|\{P\}\times M) \\ &= \dim H^0(M, \mathcal{O}(P)) = 1 . \end{aligned}$$

Thus $p_{1*}(\mathcal{O}(\Delta))$ in (B.1) is a line bundle sandwiched in an exact sequence between two copies of the trivial line bundle on M. This is only possible if the image of the map j in (B.1) is the zero sheaf, i.e. if

$$p_{1*}(\mathcal{O}(\Delta)) = \mathcal{O}_M . \tag{B.2}$$

Then

$$\begin{aligned} \dim H^0(M\times M, \mathcal{O}(\Delta)) &= \dim H^0(M, p_{1*}(\mathcal{O}(\Delta))) \\ &= \dim H^0(M, \mathcal{O}_M) = 1 . \end{aligned}$$

This proves the proposition.

Proposition B.2. *Let ξ be an arbitrary element of* $\mathrm{Pic}^0(M)$ *and let ϕ_ξ^1 denote the map*

$$\begin{aligned} \phi_\xi^1 : M\times M &\to \mathrm{Pic}^0(M) , \\ (Q,P) &\to \mathcal{O}(Q-P)\otimes \xi . \end{aligned}$$

Then,

$$\mathcal{M}_\xi(1,1) \equiv p_1^*(\xi^{-1}) \otimes p_2^*(\xi) \otimes \mathcal{O}(\Delta) = \phi_\xi^{1*}(\mathcal{O}(\Theta)) . \tag{B.3}$$

Proof. A straightforward application of the seesaw principle [24, 25].

Corollary B.3. *Let ϕ_0^1 denote the map*

$$\phi_0^1 : M \times M \to \mathrm{Pic}^0(M),$$
$$(Q, P) \to \mathcal{O}(Q-P).$$

Then,

$$\mathcal{O}(\Delta) = \phi_0^{1*}(\mathcal{O}(\Theta)). \tag{B.4}$$

Proof. Put ξ as the trivial line bundle in (B.3).

Theorem B.4. *The prime form $E(Q, P)$ can be identified with a theta function:*

$$E(Q, P) = \frac{\theta_1(Q-P)}{\theta_1'(0)}. \tag{B.5}$$

Proof. The theta function convention followed in (B.5) is that of [26]. The argument of θ_1 in (B.5) makes sense because of the isomorphism between M and $\mathrm{Pic}^0(M)$ referred to earlier. The result follows from Proposition B.1, Eq. (B.4) and the fact that the line bundle $\mathcal{O}(\Theta)$ over $\mathrm{Pic}^0(M)$ can be trivialised over the covering space $\mathbb{C}$ of $\mathrm{Pic}^0(M)$ and its unique holomorphic section can be identified with the θ-function, for which we refer to [18]. The fact that θ_1 appears in (B.5) is because Θ is identified with the zero element of the group $\mathrm{Pic}^0(M)$.

Remark B.5. Note that the expression for the prime form $E(Q, P)$ in Theorem B.4 is the same as the one written down by Hejhal [27] starting from a different definition, viz. Klein's prime form. It also agrees with the one used in the physics literature, e.g. see [6].

By definition the two point function is a meromorphic section of $p_1^*(K \otimes \alpha^{-1}) \otimes p_2^*(\alpha)$ with only a simple pole on the diagonal, where α lies in the complement of Θ. Since Θ is the neutral element of $\mathrm{Pic}^0(M)$, the only constraint on α is that $\alpha \neq 0$. Thus the two point function is defined by the unique holomorphic section of $\mathcal{M}_\alpha(1, 1)$, uniqueness being a consequence of Theorem 2.2. This section can be identified with a translate of θ_1 by $-\alpha$ by Proposition B.2. Then using Theorem B.4 we get

Theorem B.6. *The two point function $\langle b(Q)c(P)\rangle$ is given by the Szegö kernel: for $\alpha \neq 0$,*

$$\langle b(Q)c(P)\rangle = S_\alpha(Q, P) \equiv \frac{\theta_1(Q-P-\alpha)}{\theta_1(-\alpha)E(Q, P)}, \tag{B.6}$$

where $E(Q, P)$ is given by (B.5).

Remark B.7. The expression (B.6) is consistent with Hejhal's definition of the Szegö kernel for θ_2, θ_3, θ_4 [27]. Note that we have used the isomorphism of M and $\mathrm{Pic}^0(M)$ under the Abel map in writing (B.6).

Let us now consider the fermionic b–c system on M. A brief summary of our earlier work [11–13] is given in Appendix A after Theorem A.3. Thus in the case of $g=1$ as well, $C(m, n) = 0$ for $m \neq n$ while $C(n, n)$ is unique. Consequently Wick's theorem holds, so that $C(n, n)$ can be expressed as a determinant of two point functions. To get an identity, we must identify the unique holomorphic section of $\mathcal{M}_\alpha(n, n)$, which we did in Appendix A for $g=0$ and in [11, 13] for $g \geqq 2$.

Proposition B.8. *Let ξ be any element of* $\mathrm{Pic}^0(M)$ *and let ϕ_ξ^n denote the map*

$$\phi_\xi^n : M^{2n} \to \mathrm{Pic}^0(M),$$
$$(Q_1, \dots, Q_n, P_1, \dots, P_n) \to \mathcal{O}\left(\sum_1^n Q_i - \sum_1^n P_i\right) \otimes \xi .$$

Then

$$\mathcal{M}_\xi(n,n) = \phi_\xi^{n*}(\mathcal{O}(\Theta)). \tag{B.7}$$

Proof. A straightforward application of the seesaw principle [24, 25].

When $\xi = \alpha \in \mathrm{Pic}^0(M) - \Theta$, then Proposition B.8 and Theorem A.4(ii) show that $\mathcal{M}_\alpha(n,n)$ has the unique holomorphic section $\theta_1\left(\sum_1^n Q_i - \sum_1^n P_i - \alpha\right)$. Thus

$$C(n,n) = \frac{\theta_1\left(\sum_1^n Q_i - \sum_1^n P_i - \alpha\right)}{\theta_1(-\alpha)} \frac{\prod_{1 \leqq i < j \leqq n} E(Q_i, Q_j) E(P_j, P_i)}{\prod_{1 \leqq i, j \leqq n} E(Q_i, P_j)}. \tag{B.8}$$

But we have shown that $C(n,n)$ is also a determinant of two point functions, so that

$$C(n,n) = \text{const} \times \det \left(\frac{\theta_1(Q_i - P_j - \alpha)}{\theta_1(-\alpha)} \frac{1}{E(Q_i, P_j)} \right) \Bigg|_{i,j=1}^n . \tag{B.9}$$

Comparing (B.8) and (B.9), it is easy to see that the constant is unity and so we have proved:

Theorem B.9 (Frobenius' identity [17]). *For* $\alpha \neq 0$,

$$\frac{\theta_1\left(\sum_1^n Q_i - \sum_1^n P_i + \alpha\right)}{\theta_1(\alpha)} \frac{\prod_{1 \leqq i < j \leqq n} \theta_1(Q_i - Q_j)\theta_1(P_j - P_i)}{\prod_{1 \leqq i, j \leqq n} \theta_1(Q_i - P_j)}$$
$$= \det \left(\frac{\theta_1(Q_i - P_j + \alpha)}{\theta_1(Q_i - P_j)\theta_1(\alpha)} \right) \Bigg|_{i,j=1}^n . \tag{B.10}$$

Remark B.10. In our notation, what Frobenius [17] actually writes (after correcting a small error in [17] in going from Eq. (11) to Eq. (12)) is the identity:

$$\sigma(\alpha)^{n-1} \sigma\left(\alpha + \sum_1^n (Q_i + P_i)\right) \frac{\prod_{1 \leqq i < j \leqq n} \sigma(Q_i - Q_j)\sigma(P_i - P_j)}{\prod_{1 \leqq i, j \leqq n} \sigma(Q_i + P_j)}$$
$$= \det \left(\frac{\sigma(Q_i + P_j + \alpha)}{\sigma(Q_i + P_j)} \right) \Bigg|_{i,j=1}^n , \tag{B.11}$$

where σ is the Weierstrass function [26]. Using the well known relation between σ and θ_1 [26], it is easy to see that (B.11) is equivalent to (B.10).

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