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# Flat Periodic Representations of $\mathcal{U}_q(\mathcal{G})$

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**Abstract.** We give explicit expression of flat periodic representations, when they exist, of the quantum analogues of simple Lie algebras and their affine extensions for a parameter of deformation $q$ equal to a root of unity. By "flat periodic," we mean that these representations have no highest weight and that all the weights have multiplicity 1.

## 1. Introduction

In [1] and [2], it was proved that $\mathcal{U}_q(SU(2))$ have periodic representations (i.e. with injective action of generators) parametrized by 3 continuous parameters. In [3], this was extended to $\mathcal{U}_q(SU(3))$ and in [4, 5] to $\mathcal{U}_q(SU(n))$. In [6], De Concini and Kac studied the representations of $\mathcal{U}_q(\mathcal{G})$ at root of unity. They proved that their dimensions are bounded and that they were parametrized by $\dim \mathcal{G}$ complex continuous parameters. (We prefer the word periodic instead of cyclic, since cyclic has already its own meaning in the theory of modules.)

Periodic representations of $\mathcal{U}_q(SU(2))$ have proved their interest in the generalization of the chiral Potts model [7, 8]. Periodic representations of $\mathcal{U}_q(SU(3))$ are used for the same purpose in [9, 10], and this is extended to flat periodic representations of $\mathcal{U}_q(A_n^{(1)})$ in [11]. In [12], (flat) periodic representations of $\mathcal{U}_q(A_n^{(2)})$ and their intertwiners are related to the Boltzman weights of another statistical model, i.e. the Izergin-Korepin model.

In this paper, we consider "flat" periodic representations, i.e. for which all the weight spaces have dimension 1. If $q^m = 1$, the dimension of flat periodic representations of $\mathcal{U}_q(\mathcal{G})$ is $m^{\mathrm{rank}\,\mathcal{G}}$. Flat periodic representations are minimal periodic representations, in the sense that when they exist, their dimensions are the smallest possible. We will in the following give an explicit expression of flat periodic representations of $\mathcal{U}_q(\mathcal{G})$, for $\mathcal{G}$ a simple Lie algebra or an affine extended Lie algebra. We prove in fact that there is no flat periodic representation if the Dynkin diagram has a branching point, or a triple link or an extension on both sides of a double link, i.e. if $\mathcal{G}$ contains $D_4$, $G_2$ or $F_4$. In all the other cases, we give the action of the generators, the dimension and the number of parameters. In the particular case of $\mathcal{U}_q(SO(5))$, we find two solutions, one extending naturally to $\mathcal{U}_q(SU(2N+1))$ and the other to $\mathcal{U}_q(Sp(2N))$.

In Sect. 2, we define $\mathcal{U}_q(\mathcal{G})$ and prepare the construction. In Sect. 3, we look at the constraints due to each link of the Dynkin diagram. In Sect. 4, we gather these constraints to obtain flat periodic representations of $\mathcal{U}_q(\mathcal{G})$ for $\mathcal{G}$ a simple Lie algebra. In Sect. 5, we extend the procedure to affine Lie algebras.

## 2. Generalities

Let $\mathcal{G}$ be a complex simple Lie algebra or an affine type Lie algebra. The quantum analogue $\mathcal{U}_q(\mathcal{G})$ of the envelopping algebra of $\mathcal{G}$ is defined by the generators $(k_i^{\pm 1}, e_i, f_i)_{i\in I}$, where $I$ is the set of indices of the Dynkin diagram of $\mathcal{G}$ ($|I| = \operatorname{rank}\mathcal{G} = N$), and the relations [13]

$$\begin{cases} k_i k_i^{-1} = k_i^{-1} k_i = 1, \quad k_i k_j = k_j k_i, \\ k_i e_j k_i^{-1} = q_i^{a_{ij}} e_j, \\ k_i f_j k_i^{-1} = q_i^{-a_{ij}} f_j, \\ [e_i, f_j] = \delta_{ij} \dfrac{k_i^2 - k_i^{-2}}{q_i^2 - q_i^{-2}}, \\ \sum_{n=0}^{1-a_{ij}} (-1)^n \begin{bmatrix} 1-a_{ij} \\ n \end{bmatrix}_{q_i^2} e_i^{1-a_{ij}-n} e_j e_i^n = 0 \quad \text{for } i \neq j, \\ \sum_{n=0}^{1-a_{ij}} (-1)^n \begin{bmatrix} 1-a_{ij} \\ n \end{bmatrix}_{q_i^2} f_i^{1-a_{ij}-n} f_j f_i^n = 0 \quad \text{for } i \neq j \end{cases} \tag{2.1}$$

- where $A = (a_{ij})_{(i,j)\, in\, I^2}$ is the Cartan matrix of $\mathcal{G}$. In the following, we shall also denote $\mathcal{G} = \mathcal{G}(A)$. If $\{\alpha_i, i \in I\}$ is the set of simple noots, and $(.|.)$ the invariant bilinear form on the weight space, then $(\alpha_i|\alpha_j) \in Z$. (Let the shortest root be such that $(\alpha_s|\alpha_s) = 2$.) The Cartan matrix is $a_{ij} = 2\dfrac{(\alpha_i|\alpha_j)}{(\alpha_i|\alpha_i)}$.
- where $q_i = q^{(\alpha_i|\alpha_i)/2}$.
- where $\begin{bmatrix} m \\ n \end{bmatrix}_q = \dfrac{[m]_q!}{[n]_q!\,[m-n]_q!}$ with $[n]_q! = [n]_q [n-1]_q \ldots [2]_q [1]_q$ and $[0]_q = 1$, where by convention $[x]_q = \dfrac{q^x - q^{-x}}{q - q^{-1}}$.

In the following, we shall consider representations of the algebra structure only. Let us consider a finite dimensional simple module $M$ over $\mathcal{U}_q(\mathcal{G})$.

**Proposition.** *The generators $k_i$ are simultaneously diagonalizable on a simple module $M$.*

*Proof.* This is usual. See for example [13].

Let $q$ be a $m$-root of unity, i.e. we suppose that $m$ is the smallest integer such that $q^m = 1$. For simplicity, we restrict ourselves to odd values of $m$. (Generalization to even values is not difficult, but involves many different new specifications, in particular in the following proposition.)

**Proposition.** *As a consequence of the commutation relations, $(e_i)^m$, $(f_i)^m$ and $(k_i)^m$ belong to the center of $\mathcal{U}_q(\mathcal{G})$.*

We now restrict ourselves to **periodic** representation: we shall suppose in the following that the generators $e_i$ and $f_i$ are injective on $M$. Remark that this assumption is possible for $q$ a root of unity only. Such a representation is otherwise infinite dimensional.

For the following construction we suppose that the points of the Dynkin diagram of $\mathcal{G}$ can be labelled by indices $i \in I$ such that

- each $i$ is either even or odd,
- all the neighbours on the Dynkin diagram of an even index $i$ are odd and vice versa, i.e. $a_{ij} = 0$ unless $i$ is odd and $j$ is even or v.v.

*Remarks.* a) We do not suppose that the $i$'s build a subset of contiguous integers of $\mathbb{N}$. b) This restriction in fact only excludes the case of $\mathcal{G} = A_{2n}^{(1)}$ since all the tree-like diagrams satisfy the hypothesis. This case will actually be recovered afterwards.

Let $I$ denote the set of indices of the Dynkin diagram, with $|I| = N$. Let also $I_1$ be the subset of odd indices and $I_2$ the subset of even indices. Then

$$\begin{aligned} [f_{i_1}, e_{i_2}] &= 0 \quad \text{for } i_1 \in I_1, \quad i_2 \in I_2, \\ [f_{i_1}, f_{i'_1}] &= 0 \quad \text{for } i_1, i'_1 \in I_1, \\ [e_{i_2}, e_{i'_2}] &= 0 \quad \text{for } i_2, i'_2 \in I_2. \end{aligned} \tag{2.2}$$

Let $M$ be a periodic simple module. Since $(e_i)^m$ and $(f_i)^m$ are in the center of the algebra, they are scalar on $M$. So

$$\begin{aligned} (f_{i_1})^m \cdot v &= \alpha_{i_1}^m \cdot v \quad \forall i_1 \in I_1, \quad \forall v \in M, \\ (e_{i_2})^m \cdot v &= \alpha_{i_2}^m \cdot v \quad \forall i_2 \in I_2, \quad \forall v \in M, \end{aligned} \tag{2.3}$$

with $\alpha_{i_1} \neq 0$ and $\alpha_{i_2} \neq 0$.

Let $M_0$ be a common eigenspace of the $k_i$'s associated to the eigenvalues $q_i^{\mu_i}$. Let $P = \{p_i, i \in I\}$ with $p_i \in \mathbb{Z}$. Then

$$M_P \equiv \left( \prod_{i_1 \in I_1} f_{i_1}^{p_{i_1}} \right) \left( \prod_{i_2 \in I_2} e_{i_2}^{-p_{i_2}} \right) M_0 \tag{2.4}$$

is the common eigenspace of the $k_i$'s associated to the eigenvalues $k_i|_{M_P} = q_i^{\mu_i - \sum_{j \in I} a_{ij} p_j}$. It has the same dimension as $M_0$ since $f_{i_1}$ and $e_{i_2}$ are injective. Hence, if $P$ and $P'$ differ through $p'_i = p_i \pm m$, then $M_{P'} = M_P$. Furthermore if $A = (a_{ij})$ is invertible in $\mathbb{Z}/m\mathbb{Z}$, i.e. if $\det A$ and $m$ are coprime, then the set of integers $P = \{p_{i_{i \in I}}\}$ with $p_i$ defined modulo $m$ is in one to one correspondence with a common eigenspace of the $k_i$'s.

Let $\mathcal{G}$ be a simple Lie algebra. Then $\det A \neq 0$. More precisely,

$$\begin{aligned} &\det A(A_N) = N + 1, \\ &\det A(B_N) = \det A(C_N) = 2, \\ &\det A(D_N) = 4, \\ &\det A(E_6) = 3, \\ &\det A(E_7) = 2, \\ &\det A(E_8) = \det A(F_4) = \det A(G_2) = 1. \end{aligned} \tag{2.5}$$

The condition that $\det A(G)$ and $m$ are coprime is always fulfilled if $\mathcal{G}$ is not of type $A_N$ or $E_6$ (since $m$ is odd). The case $E_6$ will be excluded later. In the case of $A_N$, the condition is satisfied if $N+1$ and $m$ are coprime, but as in the case of $SU(3)$, the condition can be removed (see [3]) without lost of irreducibility, minimal representation remaining minimal, but not flat.

Let us suppose that $\det A(\mathcal{G})$ and $m$ are coprime: Then

$$M = \bigoplus_{\substack{P=\{p_{i_{i\in I}}\} \\ p_i = 0,\dots,m-1}} M_P . \tag{2.6}$$

Since $(f_{i_1})_{i_1\in I_1}$ and $(e_{i_2})_{i_2\in I_2}$ provide mutually commuting isomorphisms between the different $M_P$'s let us denote

$$M = \left[ \bigoplus_{\substack{P=\{p_i,\, i\in I\} \\ p_i = 0,\dots,m-1}} \mathbb{C}\,|P\rangle \right] \otimes \mathcal{M} \tag{2.7}$$

so that

$$\begin{cases} f_{i_1}|P\rangle \otimes |x\rangle = \alpha_{i_1} |\dots, p_{i_1}+1,\dots\rangle \otimes |x\rangle \\ e_{i_2}|P\rangle \otimes |x\rangle = \alpha_{i_2} |\dots, p_{i_2}-1,\dots\rangle \otimes |x\rangle \\ k_i|P\rangle \otimes |x\rangle = q_i^{\mu_i - \sum_{j\in I} a_{ij}p_j} |P\rangle \otimes |x\rangle . \end{cases} \tag{2.8}$$

From $[e_i, f_i] = \dfrac{k_i^2 - k_i^{-2}}{q_i^2 - q_i^{-2}}$, we get, for $i_1 \in I_1$ and $i_2 \in I_2$,

$$\begin{aligned} e_{i_1}|P\rangle \oplus |x\rangle = {} & \frac{1}{\alpha_{i_1}(q_{i_1}^2 - q_{i_1}^{-2})^2} |p_{i_1}-1\rangle \otimes \\ & \otimes \left[ -q_{i_1}^{2(\mu_{i_1} - \sum_{j\in I} a_{i_1 j}p_j + 1)} - q_{i_1}^{-2(\mu_{i_1} - \sum_{j\in I} a_{i_1 j}p_j + 1)} + \beta_{P_2}^{(i_1)} \right] |x\rangle \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} f_{i_2}|P\rangle \oplus |x\rangle = {} & \frac{1}{\alpha_{i_2}(q_{i_2}^2 - q_{i_2}^{2})^2} |p_{i_2}+1\rangle \otimes \\ & \otimes \left[ -q_{i_2}^{2(\mu_{i_2} - \sum_{j\in I} a_{i_2 j}p_j - 1)} - q_{i_2}^{-2(\mu_{i_2} - \sum_{j\in I} a_{i_2 j}p_j - 1)} + \beta_{P_1}^{(i_2)} \right] |x\rangle , \end{aligned} \tag{2.10}$$

where $\beta_{P_2}^{(i_1)}$ (respectively $\beta_{P_1}^{(i_2)}$) is an operator acting on $\mathcal{M}$, and whose expression only depends on $P_2 = \{p_{i_2};\, i_2 \in I_2\}$ (respectively on $P_1 = \{p_{i_1};\, i_1 \in I_1\}$).

**Proposition.** *The dependence of* $\beta_{P_2}^{(i_1)}$ on $P_2$ *is given by*

$$\beta_{P_2}^{(i_1)} = \sum_{\substack{l_{i_2} = -a_{i_2 i_1},\, -a_{i_2 i_1}+2,\dots,\, a_{i_2 i_1} \\ (i_2\in I_2)}} \beta_{\{l_{i_2}\}_{i_2\in I_2}}^{(i_1)} \prod_{j_2\in I_2} q_{j_2}^{2 l_{j_2} p_{j_2}} \tag{2.11}$$

*and* $\beta_{P_1}^{(i_2)}$ *depends in the same way on the indices* $p_{i_1}$,

$$\beta_{P_1}^{(i_2)} = \sum_{\substack{l_{i_1} = -a_{i_1 i_2},\, -a_{i_1 i_2}+2,\dots,\, a_{i_1 i_2} \\ (i_1\in I_1)}} \beta_{\{l_{i_1}\}_{i_1\in I_1}}^{(i_2)} \prod_{j_1\in I_1} q_{j_1}^{2 l_{j_1} p_{j_1}} , \tag{2.12}$$

*where* $\beta_{\{l_{i_2}\}_{i_2\in I_2}}^{(i_1)}$ *and* $\beta_{\{l_{i_1}\}_{i_1\in I_1}}^{(i_2)}$ *are constant operators on* $\mathcal{M}$, *the indices* $l_i$ *of which being the exponents of* $q_i^2$ *in* $\beta$.

*Proof.* The dependence of $\beta_{P_2}^{(i_1)}$ on $p_{i_2}$ is provided by the Serre relation involving one power of $e_{i_1}$ and $e_{i_2}$ to the power $(1 - a_{i_2 i_1})$:

$$\sum_{n=0}^{1-a_{i_2 i_1}} (-1)^n \begin{bmatrix} 1 - a_{i_2 i_1} \\ n \end{bmatrix}_{q_{i_2}^2} \beta^{(i_1)}_{\dots, p_{i_2}-n, \dots} = 0. \tag{2.13}$$

Since the roots of the polynomial

$$\sum_{n=0}^{1-a} (-1)^n \begin{bmatrix} 1 - a \\ n \end{bmatrix}_{q_i^2} x^n = 0$$

are $q_i^{-2a}, q_i^{-2(a-2)}, q_i^{-2(a-4)} \dots q_i^{2(a-2)}, q_i^{2a}$, then $\beta_{P_2}^{(i_1)}$ is a linear combination of $q_{i_2}^{2lp_{i_2}}$ with $l \in \{-a_{i_2 i_1}, -a_{i_2 i_1} + 2, \dots, a_{i_2 i_1}\}$ each coefficient being an operator on $\mathscr{M}$ independent of $p_{i_2}$. This proves the proposition.

In particular, if $a_{i_2 i_1} = 0$, then $\beta_{P_2}^{(i_1)}$ does not depend on $p_{i_2}$.

$\beta^{(i_1)}_{(l_{i_2})_{i_2 \in I_2}}$ and $\beta^{(i_2)}_{(l_{i_1})_{i_1 \in I_1}}$ generate an auxiliary algebra, as in [3, 5], whose defining relations are provided by the not already used Serre relations and by $[e_i, f_j] = 0$ for $i \neq j$.

In the following, we study flat periodic representations. Since the $e_i$'s and $f_i$'s are injective, the multiplicities of the weights, i.e. the dimension of the weight spaces $M_P$, are independent of $P$. For flat periodic representations, we suppose then that these multiplicities are 1, i.e. $\dim \mathscr{M} = 1$. The dimension of a flat periodic representation is hence $\dim M = m^N$. ($N = \operatorname{rank} \mathscr{G}$). Since $\dim \mathscr{M} = 1$, all the operators $\beta$ commute mutually.

In the particular case $\mathscr{G} = \mathscr{G}(A_1)$, i.e. $\mathscr{U}_q(SU(2))$, no further constraint appears. A flat periodic representation of $\mathscr{U}_q(SU(2))$ is characterized $\alpha = \alpha_1$, $\mu = \mu_1$ and $\beta = \beta^{(1)} \in \mathbb{C}$. All periodic irreducible representations of $\mathscr{U}_q(SU(2))$ are flat (of dimension $m$; the multiplicities of the weights are 1) (see [2]).

## 3. Constraints due to a Particular Link of the Dynkin Diagram

Let $\mathscr{G}$ be a simple Lie algebra with Cartan matrix $A$. We now focus on a particular link $(i_1, i_2)$ of the Dynkin diagram, i.e. consider the action of the subalgebra generated by $e_{i_1}, e_{i_2}, f_{i_1}, f_{i_2}, k_{i_1}$ and $k_{i_2}$ on a subspace of $M$ defined by fixed values of $p_j$'s for $j \neq i_1, i_2$. This subalgebra is either $\mathscr{U}_q(SU(3))$, (if $a_{i_1 i_2} = a_{i_2 i_1} = -1$) or $\mathscr{U}_q(SO(5))$, (if $a_{i_1 i_2} = -2$ and $a_{i_2 i_1} = -1$ or vice versa) or finally $\mathscr{U}_q(G_2)$, (if $a_{i_1 i_2} = -3$ and $a_{i_2 i_1} = -1$). We define effective $\mu'_{i_1}$ and $\mu'_{i_2}$ by

$$\begin{aligned} \mu'_{i_1} &= \mu_{i_1} - \sum_{\substack{j \neq i_1 \\ j \neq i_2}} a_{i_1 j} p_j, \\ \mu'_{i_2} &= \mu_{i_2} - \sum_{\substack{j \neq i_1 \\ j \neq i_2}} a_{i_2 j} p_j, \end{aligned} \tag{3.1}$$

so that the constant values of $p_j$ other than $p_{i_1}$ and $p_{i_2}$ are taken in account by replacing $\mu_{i_1}$, $\mu_{i_2}$ by the corresponding $\mu'_{i_1}$, $\mu'_{i_2}$. The dependence of $\beta^{(i_1)}$ (respectively $\beta^{(i_2)}$) on $p_{i_2}$ (respectively on $p_{i_1}$) will be made explicit in this section.

a) *First case:* subalgebra $\mathcal{U}_q(SU(3))$ $(q_{i_1} = q_{i_2})$.

The auxiliary algebra is defined by the generators $\beta^{(i_1)}_{\pm 1}$, $\beta^{(i_2)}_{\pm 1}$ (related, up to a normalization, to $u$, $u'$, $v$ and $v'$ of [3]) and the relations (simpler than in [3] since $\dim \mathcal{M} = 1$ so that the $\beta$'s are scalar):

$$\begin{cases} \beta^{(i_1)}_{+1}\,\beta^{(i_2)}_{+1} = q_{i_1}^{-2(\mu'_{i_1}+\mu'_{i_2})} \\ \beta^{(i_1)}_{-1}\,\beta^{(i_2)}_{-1} = q_{i_1}^{2(\mu'_{i_1}+\mu'_{i_2})} \\ \beta^{(i_1)}_{+1}\,\beta^{(i_1)}_{-1} = 1. \end{cases} \tag{3.2}$$

The dependence of $\mu'_{i_1} + \mu'_{i_2}$ on the $p_j$'s is then easy to distribute between $\beta^{(i_1)}$ and $\beta^{(i_2)}$, since for all $j \neq i_1, i_2$ we have $a_{i_1 j} a_{i_2 j} = 0$. So

$$\begin{cases} \beta^{(i_1)} = q_{i_1}^{2\left(p_{i_2} - \mu_{i_1} + \sum\limits_{\substack{j \neq i_1 \\ j \neq i_2}} a_{i_1 j} p_j + \lambda\right)} + q_{i_1}^{-2\left(p_{i_2} - \mu_{i_1} + \sum\limits_{\substack{j \neq i_1 \\ j \neq i_2}} a_{i_1 j} p_j + \lambda\right)} \\ \beta^{(i_2)} = q_{i_2}^{2\left(p_{i_1} - \mu_{i_2} + \sum\limits_{\substack{j \neq i_1 \\ j \neq i_2}} a_{i_2 j} p_j - \lambda\right)} + q_{i_2}^{-2\left(p_{i_1} - \mu_{i_2} + \sum\limits_{\substack{j \neq i_1 \\ j \neq i_2}} a_{i_2 j} p_j - \lambda\right)} \end{cases}, \tag{3.3}$$

where $\lambda$ is a complex parameter, independent of the $p_j$'s.

The dependence of $\beta^{(i_1)}$ on $p_j$'s other than $p_{i_2}$ is hence already known, and has to be compatible with the constraints provided by the other links of the Dynkin diagram.

b) *Second case:* subalgebra $\mathcal{U}_q(SO(5))$.

Let $a_{i_1 i_2} = -2$ and $a_{i_2 i_1} = -1$ so that $q_{i_2} = q_{i_1}^2$. Then

$$\begin{aligned} \beta^{(i_1)} &= \beta^{(i_1)}_{+1}\, q_{i_2}^{2p_{i_2}} + \beta^{(i_1)}_{-1}\, q_{i_2}^{-2p_{i_2}} \\ \beta^{(i_2)} &= \beta^{(i_2)}_{+2}\, q_{i_1}^{4p_{i_1}} + \beta^{(i_2)}_{0} + \beta^{(i_2)}_{-2}\, q_{i_1}^{-4p_{i_1}}. \end{aligned} \tag{3.4}$$

The relations satisfied by $\beta^{(i_1)}_{\pm 1}$, $\beta^{(i_2)}_{\pm 2}$ and $\beta^{(i_2)}_0$ are given in the appendix in the general case of non-commuting $\beta$'s ($\dim \mathcal{M} \geqq 1$). In the case of flat periodic representations ($\dim \mathcal{M} = 1$), they reduce to either

$$\begin{cases} \beta^{(i_2)}_0 = \beta^{(i_1)}_{+1} = \beta^{(i_1)}_{-1} = 0 \\ \beta^{(i_2)}_{+2} = -\, q_{i_2}^{-2\mu'_{i_1} - 2\mu'_{i_2}} \\ \beta^{(i_2)}_{-2} = -\, q_{i_2}^{2\mu'_{i_1} + 2\mu'_{i_2}} \end{cases} \tag{3.5}$$

or

$$\begin{cases} \beta^{i_2}_{+2} = \beta^{i_2}_{-2} = 0 \\ \beta^{i_2}_0 = \varepsilon(q_{i_2} + q_{i_2}^{-1}) \\ \beta^{i_1}_{+1} = \varepsilon q_{i_2}^{-\mu'_{i_1} - 2\mu'_{i_2}} \\ \beta^{i_1}_{-1} = \varepsilon q_{i_2}^{\mu'_{i_1} + 2\mu'_{i_2}}. \end{cases} \quad \text{with } \varepsilon = \pm 1 \tag{3.6}$$

We see that either $\beta^{(i_2)}$ or $\beta^{(i_1)}$ (not both) can depend on $p_j$'s related to other points of the Dynkin diagram. Furthermore, it depends, through $\mu'_{i_1}$ and $\mu'_{i_2}$ on $p_j$'s related to neighbouring points of both $i_1$ and $i_2$. As a consequence, quantum Lie algebras whose Dynkin diagrams have extensions on both sides of the double link $(i_1, i_2)$, i.e. which contain $\mathcal{U}_q(F_4)$ will not admit flat periodic representations.

c) *Third case:* subalgebra $\mathscr{G} = \mathscr{G}(G_2)$.
Let $a_{12} = -3$, $a_{21} = -1$ so that $q_2 = q_1^3$. Then

$$\begin{cases} \beta^{(1)} = q_2^2 \beta^{(1)}_{+1} + q_2^{-2} \beta^{(1)}_{-1} \\ \beta^{(2)} = q_1^6 \beta^{(2)}_{+3} + q_1^2 \beta^{(2)}_{+1} + q_1^{-2} \beta^{(2)}_{-1} + q_1^{-6} \beta^{(2)}_{-3} . \end{cases} \tag{3.7}$$

But the relations between $\beta$'s required by the further constraints cannot be satisfied for commuting $\beta$'s. So there is no periodic representation $\mathscr{U}_q(\mathscr{G}(G_2))$ of dimension $m^2$.

## 4. Extension to General Simple Lie Algebra

We follow the classification $A_N$, $B_N$, .... The action of $e_i$, $f_i$, $k_i$ is given by (2.8), (2.9), (2.10), the value of $\beta$ being made explicit in each case.

a) *First case:* $A_N$ ($\mathscr{U}_q(SU(N+1))$).
$N+1 = \det A_N$ and $m$ are supposed to be coprime. The results (3.3) of the study of $\mathscr{U}_q(SU(3))$ immediately lead to

$$\beta^{(i)} = q^{2(p_{i-1} - p_{i+1} + \lambda_i)} + q^{-2(p_{i-1} - p_{i+1} + \lambda_i)} \tag{4.1}$$

with

$$\lambda_{i+1} = \lambda_i - \mu_i - \mu_{i+1} \bmod \frac{m}{2} \tag{4.2}$$

and $p_i \equiv 0$ if $i$ is out of the Dynkin diagram.

Flat periodic representations of $\mathscr{U}_q(SU(N+1))$, of dimension $M^N$ have then $2N+1$ complex parameters $(\alpha_i, \mu_i)_{i=1\dots N}$ and $\lambda_1$.

After a change of normalization of the basis, they can be expressed as

$$\begin{aligned} k_i|P\rangle &= q^{\mu_i - 2p_i + p_{i-1} + p_{i+1}} |P\rangle , \\ f_i|P\rangle &= c_i \left( \left[ p_i - p_{i-1} - \frac{1}{2}(\mu_i + \lambda_i - 1) \right]_{q^2} \right. \\ &\quad \left. \cdot \left[ p_{i+1} - p_i - \frac{1}{2}(-\mu_i + \lambda_i - 1) - 1 \right]_{q^2} \right)^{1/2} |p_i + 1\rangle , \\ e_i|P\rangle &= \frac{1}{c_i} \left( \left[ p_i - p_{i-1} - \frac{1}{2}(\mu_i + \lambda_i - 1) - 1 \right]_{q^2} \right. \\ &\quad \left. \cdot \left[ p_{i+1} - p_i - \frac{1}{2}(-\mu_i + \lambda_i - 1) \right]_{q^2} \right)^{1/2} |p_i - 1\rangle , \end{aligned} \tag{4.3}$$

corresponding to the example of partially periodic representations given in [5] in terms of the Gelfand-Zetlin basis.

We have restricted ourselves to the case when $m$ and $N+1$ are coprime. If this is not so, a treatment analoguous to that of [3] (Sect. V) proves that the above relations also define the smallest periodic representation of $\mathscr{U}_q(\mathscr{G}(A_N))$. The weights are however degenerate in this case, so that the representation is still minimal, but not flat.

b) $B_N$: $\mathcal{U}_q(SO(2N+1))$.

We already know from the preceding section that the subalgebra $\mathcal{U}_q(SO(5))$ has two four-parameters flat representations. Let $\alpha_1$ be the short root ($i_1 = 1$ in the study of $\mathcal{U}_q(SO(5))$) and $\alpha_2 \ldots \alpha_N$ the others. $\beta^{(1)}$ cannot depend on $p_3$ (involved in $\mu'_2$) since $a_{13} = 0$. So the second solution (3.6) of $\mathcal{U}_q(SO(5))$ cannot be extended to one of $\mathcal{U}_q(SO(2N+1))$ for $N > 2$.

The first solution (3.5) coupled to the constraint (3.3) of $\mathcal{U}_q(SU(3))$ for the other links leads to

$$\begin{cases} \beta^{(1)} = 0 \\ \beta^{(i)} = -q^{4(p_{i-1}-p_{i+1}+\lambda_i)} - q^{-4(p_{i-1}-p_{i+1}+\lambda_i)} \end{cases} \quad i = 2, \ldots, N \tag{4.4}$$

with (4.2) and $\lambda_2 = -\mu_1 - \mu_2$. Flat periodic representations of $\mathcal{U}_q(SO(2N+1))$ have $2N$ complex parameters $(\alpha_i, \mu_i)_{i=1,\ldots,N}$ and also dimension $m^N$.

c) $C_N$: $\mathcal{U}_q(Sp(2N))$.

Only the second solution (3.6) of $\mathcal{U}_q(SO(5))$ generalizes to one of $C_N$ ($N > 2$). Let $a_0$ be the long root ($i_2 = 0$ in the study of $\mathcal{U}_q(SO(5))$) and $a_1 \ldots a_{N-1}$ the others. The flat periodic representation of $\mathcal{U}_q(Sp(2N))$ are given by

$$\begin{cases} \beta^{(0)} = \varepsilon(q^2 + q^{-2}) \\ \beta^{(1)} = \varepsilon(q^{2(2p_0-p_2+\lambda_1)} + q^{-2(2p_0-p_2+\lambda_1)}) \\ \beta^{(i)} = \varepsilon(q^{2(p_{i-1}-p_{i+1}+\lambda_i)} + q^{-2(p_{i-1}-p_{i+1}+\lambda_i)}) \end{cases} \quad i = 2, \ldots, N-1 \tag{4.5}$$

with (4.2) and $\lambda_1 = -2\mu_0 - \mu_1$, and have $2N$ complex parameters $(\alpha_i, \mu_i)_{i=0,\ldots,N-1}$.

d–e) $D_N$, ($\mathcal{U}_q(SO(2N))$) and $E$-type exceptional cases.

Let us focus on the $D_4$ ($\mathcal{U}_q(SO(8))$) subalgebra related to the neighbourhood of the branching point of the Dynkin diagram. Let $i_2 = 2$ be the label of the branching point, and 1, 3 and $*$ his three odd neighbours. The $\mathcal{U}_q(SU(3))$ subalgebra related to the points 1 and 2 of the Dynkin diagram provides the expression (3.3) for $\beta^2$:

$$\beta^{(2)} = q^{2(p_1-p_3-p_*-\lambda)} + q^{-2(p_1-p_3-p_*-\lambda)} \tag{4.6}$$

which is obviously inconsistent with that obtained from the subalgebras of type $\mathcal{U}_q(SU(3))$ corresponding to the links $(3,2)$ or $(*,2)$, i.e.

$$\begin{aligned} \beta^{(2)} &= q^{2(-p_1+p_3-p_*-\lambda')} + q^{-2(-p_1+p_3-p_*-\lambda')}, \\ \beta^{(2)} &= q^{2(-p_1-p_3+p_*-\lambda'')} + q^{-2(-p_1-p_3+p_*-\lambda'')}, \end{aligned} \tag{4.7}$$

since the symmetry between $p_1$, $p_3$ and $p_*$ is broken. So there is no flat periodic representation of $\mathcal{U}_q(SO(8))$ and hence no flat periodic representation of $(D_N)_{N \geq 4}$ and $(E_N)_{N=6,7,8}$.

f) $F_4$ case. As we remarked before, the two flat periodic representations of $\mathcal{U}_q(SO(5))$ cannot be generalized on both sides of the double line of the Dynkin diagram. $F_4$ has then no flat periodic representation.

g) $G_2$ was already excluded in the previous section.

## 5. Flat Periodic Representations of Quantum Analogues of Affine Type Lie Algebras

Let $\mathscr{G}$ be an affine extension of a simple Lie algebra and $A$ the Cartan matrix of $\mathscr{G}$. Then there exists [14] a unique vector $\delta = (\delta_i)_{i\in I}$ such that $A\delta = 0$ and the $\delta_i$ are positive integers, one among them being equal to 1. Let again $\mathbb{C}\,|0\rangle$ be the common eigenspace of $k_i$, $i \in I$ associated to the eigenvalues $q_i^{\mu_i}$ (unique up to a numerical factor since we consider flat periodic representations) and let $|P\rangle$ with $p_i = 0, \ldots, m-1$ for $i \in I$ be defined as before by the action of $(f_{i_1})_{i_1\in I_1}$ and $(e_{i_2})_{i_2\in I_2}$ on $|0\rangle$. Since

$$AP = AP' \Leftrightarrow P \equiv P' \bmod \delta \tag{5.1}$$

we then identify the states differing by an integer multiple of $\delta$, i.e. we consider the equivalence classes

$$|P) = \{|P'\rangle \quad \text{such that} \quad P' - P = 0 \bmod \delta\}. \tag{5.2}$$

**Proposition.** *The operator*

$$\mathscr{C} = \prod_{i\in I} k_i^{\delta_i} \in \mathscr{U}_q(\mathscr{G}) \tag{5.3}$$

*is in the center of the algebra $\mathscr{U}_q(\mathscr{G})$.*

*Proof.* $\mathscr{C}$ commutes with $k_j$; with $e_j$, we have, since $q_i^{a_{ij}} = q_j^{a_{ji}}$,

$$\mathscr{C} e_j = \prod_{i\in I} k_i^{\delta_i} e_j = \left(\prod_{i\in I} q_i^{\delta_i a_{ij}}\right) e_j \mathscr{C} = q_j^{\left(\sum_i a_{ji}\delta_i\right)} e_j \mathscr{C}.$$

*Remark.* In [14], the canonical central element is $c = \sum_{i\in I} \delta_i^V h_i$, where the system of coroots $\{h_i\}_{i\in I}$ generates the Cartan subalgebra, and where $(\delta_i^V = \frac{1}{2}(\alpha_i|\alpha_i)\delta_i)_{i\in I}$ is an eigenvector related to eigenvalue 0 of ${}^tA$, the transposed Cartan matrix. $k_i$ is in fact related to $h_i$ by $k_i = q_i^{h_i}$ so that $\mathscr{C} = q^c$.

In the following, the eigenvalue $q^{\sum_i \delta_i^V \mu_i}$ of $\mathscr{C}$ will be constrained by the application of Serre relations and $[e_i, f_j] = 0$ for $i \neq j$ on the module, so that, in particular,

$$\left[\sum_i \delta_i^V \mu_i\right]_{q^2} = 0, \tag{5.4}$$

corresponding to the vanishing of the central charge modulo $m$.

We now consider case by each affine Lie algebra, following the classification of [14]. In each case, we give the expression of $\delta$ and $\beta^{(i)}$. Since we already know that branching points in the Dynkin diagram forbids the existence of flat periodic representations, we do not have to consider the cases $B_n^{(1)}$, $D_n^{(1)}$, $E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)}$, and $A_{2n-1}^{(2)}$. The cases $G_2^{(1)}$ and $D_4^{(3)}$ are excluded since the triple link of their subdiagram $G_2$ does not allow flat periodic representations. $F_4^{(1)}$ and $E_6^{(2)}$ are also excluded since they contain $F_4$.

a) $A_1^{(1)}$. Here $\delta = |1, 1\rangle$

This new case of 2-points Dynkin diagram was not studied in Sect. 3. In this case the constraints reduce to

$$[\mu_1 + \mu_2]_{q^2} = 0, \tag{5.5}$$

i.e. $q^{2(\mu_1+\mu_2)} = \varepsilon$ with $\varepsilon = \pm 1$ and

$$\begin{cases} \beta^{(1)} = \beta_0^{(1)} \\ \beta^{(2)} = \beta_0^{(2)} = \varepsilon \beta_0^{(1)}. \end{cases} \tag{5.6}$$

The dimension is $m$ and the independent parameters are $\alpha_1$, $\alpha_2$, $\mu_1$ and $\beta_0^{(1)}$. (This case is in fact a particular expression of the following.)

The flat periodic representations of $\mathcal{U}_q(\mathcal{G}(A_1^{(1)}))$ are also described in [12], where their intertwiner is given, and is related to the chiral Potts model.

b) $A_n^{(1)}$ $(n > 1)$, $n$ odd; here $|\delta\rangle = |1, 1, \ldots, 1\rangle$.

Flat periodic representations of $\mathcal{G}(A_n^{(1)})$ are still given by (4.3) with $i = 1, \ldots, n+1$. The constraint (4.2) on $l_i$ has to be periodic in $i$, so that

$$[\sum \mu_i]_{q^2} = 0 \tag{5.7}$$

corresponding to the vanishing of the central charge modulo $m$. The dimension is $m^n$ and there are $2(n+1)$ independent parameters: $(\alpha_i)_{i=1,\ldots,n+1}$, $(\mu_i)_{i=1,\ldots,n}$, $\lambda_1$.

In the first section, we excluded the case when $n$ is even since this forbids the partition of $I$ into $I_1$ and $I_2$, subsets of odd and even indices, such that among two neighbouring points of the Dynkin diagram, one belongs to $I_1$ and the other to $I_2$. However with the choice of basis of (4.3), the symmetry between odd and even indices is recovered and this construction of flat periodic modules also holds for $A_n^{(1)}$ with even $n$.

Flat periodic representations of $\mathcal{U}_q(\mathcal{G}(A_n^{(1)}))$ were used in [11] to generalize the chiral Potts model.

c) $C_n^{(1)}$ $n \geqq 2$. Here $|\delta\rangle = |1, 2, 2, \ldots, 2, 1\rangle$.

In this case, the second solution (3.6) of $\mathcal{U}_q(SO(5))$ is used for subdiagrams $(0, 1)$ and $(n-1, n)$, whereas the constraint (3.3) provided by $\mathcal{U}_q(SU(3))$ holds for the other links,

$$\begin{cases} \beta^{(0)} = \beta^{(n+1)} = \varepsilon(q^2 + q^{-2}) \\ \beta^{(i)} = \varepsilon(q^{2(a_{ii-1}p_{i-1} - a_{ii+1}p_{i+1} + \lambda_i)} + q^{-2(a_{ii-1}p_{i-1} - a_{ii+1}p_{i+1} + \lambda_i)}) \end{cases} \tag{5.8}$$

with (4.2), $\lambda_1 = -2\mu_0 - \mu_1$ and $\lambda_n = -2\mu_{n+1} - \mu_n$.

As a consequence

$$[\sum \mu_i]_{q^2} = 0. \tag{5.9}$$

d) $A_2^{(2)}$.

In this new case of two point Dynkin diagram not studied in Sect. 3, we have $a_{12} = -4$, $a_{21} = -1$ and $|\delta\rangle = |2, 1\rangle$.

The constraints are

$$\begin{cases} \beta^{(1)} = 0 \\ \beta^{(2)} = -\varepsilon(q^4 + q^{-4}), \end{cases} \tag{5.10}$$

where $\varepsilon = q^{2(2\mu_1 + 4\mu_2)}$ and $\varepsilon^2 = 1$ so that

$$[2\mu_1 + 4\mu_2]_{q^2} = 0. \tag{5.11}$$

The dimension is $m$ and the parameters are $\alpha_1$, $\alpha_2$, $\mu_1$.

Flat periodic representations of $\mathcal{U}_q(\mathcal{G}(A_2^{(2)}))$ have already been derived in [12] with their intertwiners.

e) $A_{2n}^{(2)}$ with $n \geqq 2$. Here $|\delta\rangle = |2, 2, \ldots, 2, 1\rangle$.

We have to extend the first solution (3.5) of $\mathcal{U}_q(SO(5))$ from subdiagram (1,2), and the second solution (3.6) from subdiagram $(2n-1, 2n)$, using $\mathcal{U}_q(SU(3))$ constraints (3.3) in between. Hence

$$\begin{cases} \beta^{(1)} = 0 \\ \beta^{(i)} = \varepsilon\left(q^{4(p_{i-1} - a_{ii+1} p_{i+1} + \lambda_i)} + q^{-4(p_{i-1} - a_{ii+1} p_{i+1} + \lambda_i)}\right) \quad \text{for } i = 2, \ldots, 2n-1 \\ \beta^{(2n)} = \varepsilon(q^4 + q^{-4}) \end{cases} \tag{5.12}$$

with

$$\begin{cases} \lambda_2 = \frac{m}{8} - \mu_1 - \mu_2 \mod \frac{m}{4} \\ \lambda_{i+1} = \lambda_i - \mu_i - \mu_{i+1} \mod \frac{m}{4} \\ \lambda_{2n-1} = \mu_{2n-1} + 2\mu_{2n} \mod \frac{m}{4} \end{cases} \tag{5.13}$$

so that

$$\left[ 2\mu_1 + 4 \sum_{i=2}^{2n} \mu_i \right]_{q^2} = 0. \tag{5.14}$$

The dimension is $m^{2n-1}$ and the parameters are $(\alpha_1)_{i=1,2n}$ and $(\mu_i)_{i=1,2n-1}$.

f) $D_{n+1}^{(2)}$ with $n \geqq 2$. Here $|\delta\rangle = |1, 1, \ldots, 1\rangle$.

We use the first solution (3.5) of $\mathcal{U}_q(SO(5))$ on both edges of the Dynkin diagram, and that (3.3) of $\mathcal{U}_q(SU(3))$ in the middle, so that

$$\begin{cases} \beta^{(1)} = \beta^{(n+1)} = 0 \\ \beta^{(i)} = q^{4(p_{i-1} - p_{i+1} + \lambda_i)} + q^{-4(p_{i-1} - p_{i+1} + \lambda_i)} \end{cases} \quad \text{for } i = 2, \ldots, n \tag{5.15}$$

with

$$\begin{cases} \lambda_2 = \frac{m}{8} - \mu_1 - \mu_2 \mod \frac{m}{4} \\ \lambda_{i+1} = \lambda_i - \mu_i - \mu_{i+1} \mod \frac{m}{4} \\ \lambda_n = \frac{m}{8} + \mu_n + \mu_{n+1} \mod \frac{m}{4} \end{cases} \tag{5.16}$$

so that

$$\left[ \mu_1 + 2 \sum_{i=2}^{n} \mu_i + \mu_{n+1} \right]_{q^2} = 0. \tag{5.17}$$

The dimension is $m^n$ and the independent parameters are $(\alpha_i)_{i=1,n+1}$, $(\mu_i)_{i=1,n}$.

## 6. Conclusion

Several methods have been used for the study of explicit expression of representations of quantized envelopping algebras at root of unity. Inclusion into a Weyl algebra led to the expression of periodic representations of $\mathcal{U}_q(sl(n+1,\mathbb{C}))$ in [4]. The use of the auxiliary algebra led in [3] to all the periodic representations of $\mathcal{U}_q(SU(3))$, and here to flat periodic representations of $\mathcal{U}_q(\mathcal{G})$. The Gelfand-Zetlin basis was used in [5] and [15], providing explicit expression of periodic and partially periodic representations of $\mathcal{U}_q(SU(N))$ and $\mathcal{U}_q(IU(N))$. However, explicit expressions of most general representations of $\mathcal{U}_q(\mathcal{G})$ for $q^m = 1$ do not exist now.

Another step toward comprehension of the case $q^m = 1$ will be the search for a $R$-matrix, intertwiner of representations, and related to Boltzman weights of statistical models.

## Appendix

We present here the relations satisfied by the operators $\beta^{(i_1)}_{\pm 1}$, $\beta^{(i_2)}_{\pm 2}$ and $\beta^{(i_2)}_0$ generating the auxiliary algebra related to $\mathcal{U}_q(SO(5))$, in the general case of non-commuting $\beta$'s ($\dim \mathcal{M} \geqq 1$):

$$q_{i_2}^2 \beta^{(i_2)}_{+2} \beta^{(i_2)}_0 - q_{i_2}^{-2} \beta^{(i_2)}_0 \beta^{(i_2)}_{+2} = 0, \tag{A.1}$$

$$q_{i_2}^2 \beta^{(i_2)}_0 \beta^{(i_2)}_{-2} - q_{i_2}^{-2} \beta^{(i_2)}_{-2} \beta^{(i_2)}_0 = 0, \tag{A.2}$$

$$q_{i_2}^2 \beta^{(i_2)}_{+2} \beta^{(i_2)}_{-2} - q_{i_2}^{-2} \beta^{(i_2)}_{-2} \beta^{(i_2)}_{+2} = (q_{i_2}^2 - q_{i_2}^{-2}) \left[ 1 - \frac{1}{(q_{i_2} + q_{i_2}^{-1})^2} \beta^{(i_2)^2}_0 \right], \tag{A.3}$$

$$q_{i_2}^2 \beta^{(i_1)}_{+1} \beta^{(i_2)}_{+2} - q_{i_2}^{-2} \beta^{(i_2)}_{+2} \beta^{(i_1)}_{+1} = 0, \tag{A.4}$$

$$q_{i_2}^2 \beta^{(i_2)}_{-2} \beta^{(i_1)}_{-1} - q_{i_2}^{-2} \beta^{(i_1)}_{-1} \beta^{(i_2)}_{-2} = 0, \tag{A.5}$$

$$(q_{i_2} - q_{i_2}^{-1})\, q_{i_2}^{-\mu_1} \beta^{(i_2)}_0 - (q_{i_2}^2 - q_{i_2}^{-2})\, q_{i_2}^{2\mu_2} \beta^{(i_1)}_{+1} + [\beta^{(i_1)}_{-1}, \beta^{(i_2)}_{+2}] = 0, \tag{A.6}$$

$$(q_{i_2} - q_{i_2}^{-1})\, q_{i_2}^{\mu_1} \beta^{(i_2)}_0 - (q_{i_2}^2 - q_{i_2}^{-2})\, q_{i_2}^{-2\mu_2} \beta^{(i_1)}_{-1} - [\beta^{(i_1)}_{+1}, \beta^{(i_2)}_{-2}] = 0, \tag{A.7}$$

$$q_{i_2} \beta^{(i_1)}_{+1} \beta^{(i_2)}_0 - q_{i_2}^{-1} \beta^{(i_2)}_0 \beta^{(i_1)}_{+1} = (q_{i_2}^2 - q_{i_2}^{-2}) [q_{i_2}^{-\mu_1 - 2\mu_2} + q_{i_2}^{\mu_1} \beta^{(i_2)}_{+2}], \tag{A.8}$$

$$q_{i_2} \beta^{(i_2)}_0 \beta^{(i_1)}_{-1} - q_{i_2}^{-1} \beta^{(i_1)}_{-1} \beta^{(i_2)}_0 = (q_{i_2}^2 - q_{i_2}^{-2}) [q_{i_2}^{\mu_1 + 2\mu_2} + q_{i_2}^{-\mu_1} \beta^{(i_2)}_{-2}], \tag{A.9}$$

$$\begin{aligned} &q_{i_2}^3 \beta^{(i_1)}_{-1} \beta^{(i_1)^2}_{+1} - (q_{i_2} + q_{i_2}^{-1}) \beta^{(i_1)}_{+1} \beta^{(i_1)}_{-1} \beta^{(i_1)}_{+1} + q_{i_2}^{-3} \beta^{(i_1)^2}_{+1} \beta^{(i_1)}_{-1} \\ &\quad - (q_{i_2}^2 - q_{i_2}^{-2})(q_{i_2} - q_{i_2}^{-1}) \beta^{(i_1)}_{+1} = 0, \end{aligned} \tag{A.10}$$

$$\begin{aligned} &q_{i_2}^3 \beta^{(i_1)^2}_{-1} \beta^{(i_1)}_{+1} - (q_{i_2} + q_{i_2}^{-1}) \beta^{(i_1)}_{-1} \beta^{(i_1)}_{+1} \beta^{(i_1)}_{-1} + q_{i_2}^{-3} \beta^{(i_1)}_{+1} \beta^{(i_1)^2}_{-1} \\ &\quad - (q_{i_2}^2 - q_{i_2}^{-2})(q_{i_2} - q_{i_2}^{-1}) \beta^{(i_1)}_{-1} = 0. \end{aligned} \tag{A.11}$$

This paper is restricted to flat periodic representation, but we briefly present in this appendix the following result as an example of the more general scope of the formalism of the auxiliary algebra. The following $1 \leqq k \leqq m$ dimensional irreducible representations of the auxiliary algebra (A.1–11) generalize (3.6)

$$\begin{aligned} \beta_0^{(i_2)} &= c(q_{i_2} + q_{i_2}^{-1})\, \mathrm{diag}(q_{i_2}^{4i})_{i=0,\dots,k-1}, \\ (\beta_{+2}^{(i_2)})_{i+1,i} &= K^{-1}(c\,q_{i_2}^{4i+2} - \lambda^{-1}), \\ (\beta_{-2}^{(i_2)})_{i,i+1} &= K(c\,q_{i_2}^{4i+2} - \lambda), \\ \beta_{\pm 1}^{(i_1)} &= (q_{i_2} + q_{i_2}^{-1})\,(q_{i_2}^{\mp\mu_{i_1}\mp 2\mu_{i_2}} - q_{i_2}^{\pm\mu_{i_1}\mp 2}\, \beta_{\pm 2}^{(i_2)})\,(\beta_0^{(i_2)})^{-1}. \end{aligned} \tag{A.12}$$

They lead to $km^2$ dimensional representations of $\mathcal{U}_q(SO(5))$ with parameters $\mu_{i_{1,2}}$, $\alpha_{i_{1,2}}$, $c$, $K$ and $\lambda$. According to [6] however, there exist up to $m^2$ dimensional irreducible representations of it, leading to $m^4$ dimensional irreducible representation of $\mathcal{U}_q(SO(5))$.

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