Commun. Math. Phys. 139, 433–440 (1991)

Communications in
Mathematical
Physics

Higher-Order Polarizations on the Virasoro Group and Anomalies*

Victor Aldaya[1] **and Jose Navarro-Salas**[2]

[1] Departamento de Física Teórica y del Cosmos, Facultad de Ciencias, Universidad de Granada, Granada, Spain, and IFIC (Centro Mixto Universidad de Valencia-C.S.I.C.), Spain
[2] Departamento de Física Teórica, Facultad de Fisicas, Universidad de Valencia, Burjassot E-46100-Valencia, Spain, and IFIC (Centro Mixto Universidad de Valencia-C.S.I.C.), Spain

Received April 23, 1990

Abstract. In a previous paper the authors showed that the space of (first order) polarized functions on the Virasoro group is not, in general, irreducible. The full reduction was explicitly achieved by taking the orbit of the enveloping algebra through the vacuum. This additional step provided the proper quantization in the "strong-coupling" domain $0 < c \leqq 1$. In this paper we introduce the concept of "higher order polarization" as a generalization of that of polarization. We prove that the imposing of the additional (higher-order) polarization conditions is equivalent to the taking of the above-mentioned orbit. This demonstrates that the generalized (higher-order) polarization conditions suffice to obtain the irreducible Hilbert spaces. We also discuss the need for higher order polarizations in terms of anomalies.

1. Introduction

The standard geometric methods for quantizing a phase space (symplectic manifold) make use of the concept of polarization. Generally speaking, given a symplectic manifold (M, ω) the corresponding quantum Hilbert space is the space of sections of a line bundle (whose first Chern class is $[\omega]$) once the polarization conditions are imposed. A polarization P is an isotropic differential system ($\omega(X, X') = 0$ if $X', X \in P$) and the polarization conditions are

$$\nabla_X \psi = 0 \quad \text{if} \quad X \in P, \tag{1.1}$$

where ψ are sections of the line bundle and ∇ is a covariant derivative whose curvature is given by the symplectic form. The conditions (1.1) are always first-

* Research partially supported by the Comision Interministerial de Ciencia y Tecnologia (CICYT)

order differential equations. From the physical point of view the polarization is required in order that the "wave functions" do not depend on both momentum and position variables. Otherwise the wave functions would violate the uncertainty principle and would provide, in addition, a highly reducible representation of the algebra of observables. Typical simplectic manifolds for which there is a general way for quantizing them are cotangent bundles and Kähler manifolds. The latter case has a special significance in different branches. If a Kähler manifold M is the quotient of a compact simple group G (or the central extension of the loop group $L\tilde{G}$) by its maximal torus T, the holomorphic sections of a line bundle over M carry the irreducible unitary representations of G (or $L\tilde{G}$) via the Borel–Weil–Bott Theorem [1]. This result arises as a particular case in the coadjoint orbit method [2]. On the other hand Kähler geometry also arises in the standard quantization of field theories; in this case the complex structure J of the phase space comes from the Fourier decomposition of the field into positive and negative frequencies [3].

In reference [4] we showed that the standard (first-order) polarization equations (1.1) do not provide, in general, enough conditions to obtain the irreducible representation of the Virasoro group. This fact was verified for what would naively be (see Sect. 3) the quantization of the orbit $\mathrm{diff}\, S^1/S^1$ in the strong-coupling domain $0 < c \leqq 1$ $(h \geqq 0)$. The proper quantization was achieved by working on the Virasoro group manifold and by taking the orbit of the enveloping algebra through the vacuum, which turned out to be irreducible. This mechanism provided a way out of the problem formulated by Witten [5] of quantizing the non-Kähler orbits $\mathrm{diff}\, S^1/SL^{(r)}(2,\mathbb{R})$, $r = 2, 3, \ldots$ ($SL^{(r)}(2,\mathbb{R})$ generated by $\langle L_{-r}, L_0, L_r \rangle$). In fact, the representations initially thought of as associated with the orbits $\mathrm{diff}\, S^1/SL^{(r)}(2,\mathbb{R})$ – those having null vectors at level r – came from the proper quantization of the ($\mathrm{diff}\, S^1/S^1$ orbit of the) Virasoro group.

The aim of this paper is to prove that the full reduction can equivalently be accomplished by imposing additional (non-standard) "higher-order" polarization conditions. "Higher-order polarizations" are subalgebras of the left enveloping algebra rather than subalgebras of the Virasoro left algebra and thus will be imposed by means of higher-order differential operators. A vector field X of a standard polarization defines a first-order operator (i.e. a derivation) by means of the covariant derivative ∇_X. The concept of a "higher-order polarization" will be introduced in a natural way in the context of a "group approach to quantization" which had been already used in reference [4] (see also [6]).

In ref. [7] a third way of obtaining all the irreducible representations of the Virasoro group is proposed. It is based on the geometric quantization (and/or path integral approach) although it is formulated on the so-called, model space.

2. Higher-Order Polarizations on the Virasoro Group

The idea of the "group approach to quantization" [4] when applied to the Virasoro group is to perform the quantization from the group manifold itself instead of its coadjoint orbits [5]. One starts with the complex functions on the Virasoro group $\mathscr{F}(\mathrm{diff}\, \tilde{S}^1, \mathbb{C})$. A naive quantization (or prequantization) is achieved from the right-invariant vector fields $\tilde{X}^{\mathrm{R}}$ acting on $\mathscr{F}(\mathrm{diff}\, \tilde{S}^1, \mathbb{C})$. Of course this representation is highly reducible. Non-trivial operators commuting with the representation can be found; in fact they are the left-invariant vector fields $\tilde{X}^L$, which commute with the representation $\tilde{X}^{\mathbb{R}}$ (as a consequence of the relation $[\tilde{X}^L, \tilde{X}^{\mathbb{R}}] = 0$). In order to reduce the representation we look for a maximal set of compatible conditions on $\mathscr{F}(\widetilde{\mathrm{diff}\, S^1}, \mathbb{C})$ thus trivializing the action of the left-invariant vector fields $\tilde{X}^L$. More precisely, we impose the maximum number of compatible (first-order differential) equations of the form $\tilde{X}^L\psi = 0$, i.e. we impose a (first-order) polarization (the group analogue of Eqs. (1.1) on the phase space).

As we have already mentioned, the space of (first order) polarized functions (the analogue of holomorphic sections in Kähler quantization) on the Virasoro group is not, in general, irreducible although it does not contain null vector states. The full reduction is achieved by taking a well defined invariant subspace $\mathscr{H}_{(c,h)}$, i.e. the orbit of the right enveloping Virasoro algebra through the vacuum $|0\rangle$,

$$\mathscr{H}_{(c,h)} \equiv \langle \tilde{X}^{\mathbb{R}}_{l_{-n_j}} \cdots \tilde{X}^{\mathbb{R}}_{l_{-n_1}} |0\rangle \rangle \quad n_k > 0, \quad j = 1, 2, \dots . \tag{2.1}$$

The irreducible carrier space $\mathscr{H}_{(c,h)}$ is a proper subspace of the space of (first-order) polarized functions for those values of c and h for which the Kac determinant is zero [8]. Of course this fact does not hold for compact simple groups (finite-dimensional Lie groups and loop groups) because of the Borel–Weil–Bott Theorem [1]. In these cases the (first-order) polarization conditions are already sufficiently restrictive so as to lead to the irreducible Hilbert space. In the next section we will reinterpret the requirement of imposing higher-order polarization conditions in terms of anomalies.

Following the above given argument for introducing the first order polarization we can argue that any element in the left enveloping algebra commutes with the representation ($[\tilde{X}^L \cdots \tilde{X}^L, \tilde{X}^{\mathbb{R}}] = 0$). So we can further reduce the representation by imposing additional (higher-order) differential equations of the form.

$$\Sigma c_{n_1, \dots, n_k} \tilde{X}^L_{l_{n_1}} \cdots \tilde{X}^L_{l_{n_k}} \psi = 0. \tag{2.2}$$

The comments above make natural the introduction of the concept of higher order polarization.

Definition 1. *A first-order polarization* [4] *P on the Virasoro group is a maximal left-subalgebra of the Lie algebra containing a subalgebra of the kernel, $\mathscr{G}_\Theta$, of the central extension* $\widetilde{\mathrm{diff}\, S}^1 \to \mathrm{diff}\, S^1$, *and excluding the central generator Ξ.*

The subscript Θ stands for the connection 1-form on the principal bundle $\widetilde{\mathrm{diff}\, S^1} \to \mathrm{diff}\, S^1$, whose curvature $d\Theta$ is the symplectic form on $\mathrm{diff}\, S^1/\mathscr{G}_\Theta$ and $\mathscr{G}_\Theta$ generates the characteristic module of Θ. The actual structure of $\mathscr{G}_\Theta$ depends on the particular values of c and c' (or $h = (c - c')/24$) in the Lie algebra:

$$[\tilde{X}^L_{l_n}, \tilde{X}^L_{l_m}] = -i(n-m)\tilde{X}^L_{l_{n+m}} - \frac{i}{12}(cn^3 - c'n)\delta_{n,-m}\Xi. \tag{2.3}$$

Definition 2. *A higher-order polarization P^{HO} on the Virasoro group is a maximal subalgebra of the left-enveloping Virasoro algebra containing a first-order polarization and excluding the central generator.*

First order polarizations were classified in ref. [4] and are essentially determined by

$$\langle \tilde{X}^L_{l_{n \leq 0}} \rangle \quad \text{if} \quad c \neq c' \quad (h \neq 0), \tag{2.4a}$$

$$\langle \tilde{X}^L_{l_{n \leq 1}} \rangle \quad \text{if} \quad c = c' \quad (h = 0). \tag{2.4b}$$

The structure of a higher-order polarization is given by the following proposition.

Proposition 1. *A higher order polarization is a left ideal of the left enveloping algebra, $U\tilde{V}$, of the Virasoro algebra $\tilde{V}$ generated by a basis of a given first-order polarization as well as a set of higher-order operators*

$$\tilde{Z}_N^{L(c,h)} \equiv \sum_{\substack{i_1+\cdots+i_p=N \\ 1\leqq i_1\leqq i_2\leqq\cdots\leqq i_p}} \lambda(c,h)_{i_1,\ldots,i_p} \tilde{X}^L_{l_{i_1}} \cdots \tilde{X}^L_{l_{i_p}}, \tag{2.5}$$

whose parallel combinations $Z_{-N}(c,h) \equiv \sum \lambda(c,h)_{i_1,\ldots,i_p} L_{-i_1} \cdots L_{-i_p}$ in the standard Verma module approach [8] create null states of level N (which are not induced from null states at a previous level) from the vacuum (highest-weight vector).

The concrete values of c and h for the existence of those singular higher-order operators (2.5) are given by the well known Kac formula [8]: $h = 1/48\ (13-c)$ $(k^2+s^2) \pm (c^2-26c+25)(k^2-s^2) - 24ks - 2 + 2c$, where k and s are positive integers such that $k \cdot s \leqq N$.

The *proof* of Proposition 1 is as follows. We consider only the first-order polarization P (2.4a) because the polarization (2.4b) could be considered as a particular higher-order polarization. Since $P^{\mathrm{HO}} \supset P$, any operator in the left ideal $U\tilde{V}P$ is in P^{HO}. By definition, none of the first-order generators $\tilde{X}^L_{l_{m>0}}$ is in P^{HO}, so that only linear combinations of higher order operators $\tilde{X}^L_{l_{i_1}} \cdots \tilde{X}^L_{l_{i_j}} \tilde{X}^L_{l_{m>0}}$ would be in P^{HO}. We can restrict our analysis, without any loss of generality, to those operators for which $i_1 \cdots i_j > 0$, otherwise any operator $\tilde{X}^L_{l_{i_k}}$ with negative index $i_k \leqq 0$ could be moved (by commutation) to the first place from the right, thus providing one term in $U\tilde{V}P$ and the other ones with only positive indices. We arrive this way at the general form (2.5) and we have to prove now that the coefficients λ's characterize the null vector states of the standard Verma module associated with c and h.

Let us consider the set of all operators $\tilde{O}^L_N(c,h)$ of the form (2.5), not necessarily associated with null vectors, contained in the polarization P^{HO}. The vectors $O_{-N}(c,h)|0\rangle$, where $O_{-N}(c,h)$ are the Verma module analogues of $\tilde{O}^L_N(c,h)$, generate a submodule of the Verma module. In fact $L_{n>0}O_{-N}(c,h)|0\rangle = [L_{n>0}, O_{-N}(c,h)]|0\rangle$ because $L_{n>0}|0\rangle = 0$, and the commutator $[L_{n>0}, O_{-N}(c,h)]$ of P^{HO}, can only give rise to either $O_{-N+n}(c,h)$ or an element of $U\tilde{V}P$ which again annihilates $|0\rangle$. This ensures that $O_{-N}(c,h)|0\rangle$ are null vectors since, as it is well known [8], any submodule of a Verma module is made out of null vectors.

Having proved that any higher operator of the general form (2.5) that belongs to P^{HO} is in correspondence with a null vector of the standard Verma module, the only thing which remains to be proved is that all null operators (of the form (2.5)) are in P^{HO}. In fact, any null operator $Z_{-N}(c,h)$ in the Verma module approach verifies, by construction, $[\{L_{n\geqq 0}\}, Z_{-N}(c,h)] \subseteqq U\tilde{V}\{L_{n\geqq 0}\}$, which when translated to our scheme, means that $[P, \tilde{Z}^L_N(c,h)] \subseteqq U\tilde{V}P$. It is straightforward to realize that the operators $\tilde{Z}^L_N(c,h)$ also close when we consider them with the rest of P^{HO} which, at most, consists of null operators and of $U\tilde{V}P$. ■

The next task now is to find the space of higher-order polarized functions, i.e., complex functions on $\widetilde{\mathrm{diff}}\,S^1$ satisfying:

$$\Xi\psi = i\psi$$
$$X\psi = 0 \quad \forall X \in P^{\mathrm{HO}}. \tag{2.6}$$

Proposition 2. *The space of higher-order polarized functions coincides with* $\mathscr{H}_{(c,h)}$.

Proof. The non-trivial point to be proved is that any function in $\mathscr{H}_{(c,h)}$ satisfies the higher-order polarization equations (2.6). Moreover, given the structure of the subspace $\mathscr{H}_{(c,h)}$, the problem can be reduced to checking that the vacuum wave function $|0\rangle \equiv \zeta W$ (see ref. [4]) satisfies (2.6). In fact, any operator $X \in P^{\mathrm{HO}}$ belongs to the left enveloping algebra and therefore commutes with any operator in the right enveloping algebra.

In ref. [4] it was proved that $\mathscr{H}_{(c.h)}$ does not contain any null vector state and hence it follows that the functions

$$\sum_{\substack{i_1+\cdots+i_p=N\\ 1\leqq i_1\leqq i_2\cdots\leqq i_p}} \lambda(c,h)_{i_1,\ldots,i_p}\tilde{X}^R_{l_{-i_p}}\cdots\tilde{X}^R_{l_{-i_p}}\zeta W, \tag{2.7}$$

where the coefficients λ's are those of Proposition 1, are identically zero!. Making now use of the general diffeomorphism $g\to g^{-1}$, defined for any Lie group, we can interchange the roles of the left and right generators. We could thus think of $\tilde{X}^{\mathbb{R}}_{l_{n\geqq 0}}\zeta W=0$ as if the wave function of the vacuum, ζW, were a (right) first-order polarized function. The orbit of the left Virasoro algebra through the vacuum can then be seen as an irreducible carrier space for the Virasoro group and, accordingly, the functions

$$\sum_{\substack{i_1+\cdots+i_p=N\\ 1\leqq i_1\leqq\cdots\leqq i_p}} \lambda(c,h)_{i_1,\ldots,i_p}\tilde{X}^L_{l_{i_1}}\cdots\tilde{X}^L_{l_{i_p}}\zeta W, \tag{2.8}$$

where the coefficients λ's are again those of Proposition 1, are identically zero. This establishes that the vacuum ζW is a higher-order polarized function, thus demonstrating the first part of the proposition.

We have thus proved that the space of higher-order polarized functions contains the orbit $\mathscr{H}_{(c,h)}$. The only thing which remains to be done is to observe that, in the finite dimensional space of first-order polarized functions of a given level m, $F_{(m)}=\{|l_m\rangle,|l_{m-1},l_1\rangle,|l_{m-2},l_2\rangle,|l_{m-2},l_1,l_1\rangle,\ldots\}$, the number of independent linear algebraic equations coming from the higher-order polarization is, at least, as big as the number of independent null vectors of level m in the standard Verma module. On the other hand, the dimension of the subspace of level m of $\mathscr{H}_{(c,h)}$ is equal to the number of first-order polarized functions minus the number of null vectors, of level m. Since the orbit $\mathscr{H}_{(c,h)}$ is already a solution of the higher order polarization conditions it exhausts all the solutions. ■

3. Relation with Anomalies

This section is devoted to the interpretation of the need for higher-order polarizations in quantizing physical systems in terms of anomalies. In standard terminology, and roughly speaking, a physical system is said to be anomalous if the classical symmetry is modified in the quantization process (see refs. [3, 9–12]). In our quantization approach we say that the system is anomalous when there is no (first-order) polarization which contains the entire characteristic subalgebra $\mathscr{G}_\Theta$, i.e., the subalgebra associated with those parameters for whom classical Noether invariants can be written in terms of the basic ones (co-ordinates of the classical phase space).

In these cases polarizations containing only a proper subalgebra of $\mathscr{G}_\Theta$ allow us to continue the quantization procedure. The price paid for using these *non-full* polarizations is, in general, the non-irreducibility of the space of (first order) polarized functions on the group and a further step leading to the irreducible representation is required. A partial solution had been found for systems for which a vacuum can be naturally defined. The orbit of the right enveloping algebra through the vacuum provides the irreducible subspace [4].

The generalization of the concept of polarization to higher orders opens the possibility of finding (higher-order) polarizations leading directly to the irreducible quantum representations. Those higher-order polarizations are especially suitable for anomalous systems without a natural vacuum (i.e. a highest-weight-like vector). At the same time we can apply our new approach also to the systems endowed with a natural vacuum. In this case the new approach brings out very clearly the anomalous character of the theory.

Let us illustrate the algebraic meaning of anomalies in the present approach and, in particular, the connection with higher-order polarizations, with the example of the bosonic string in Minkowski space. The underlying symmetry for this dynamical system is essentially characterized by the (centrally extended) semi-direct product diff S^1 ⓢ (Loops on $\mathbb{R}^{1,d-1}$) whose Lie algebra is given by

$$\begin{aligned}
&[\tilde{X}^L_{\alpha^\mu_n}, \tilde{X}^L_{\alpha^\nu_m}] = n\delta_{n,-m}\Xi, \quad \Xi \equiv \text{central generator},\\
&[\tilde{X}^L_{l_n}, \tilde{X}^L_{\alpha^\mu_m}] = -mi\tilde{X}^L_{\alpha^\mu_{n+m}},\\
&[\tilde{X}^L_{l_n}, \tilde{X}^L_{l_m}] = -i(n-m)\tilde{X}^L_{l_{n+m}} - \frac{i}{12}(cn^3 - c'n)\delta_{n,-m}\Xi. \qquad (3.1)
\end{aligned}$$

It is well known that for the classical values $c = 0 = c'$ the space, (diff S^1 ⓢ Loops on $\mathbb{R}^{1,d-1}$)/Diff S^1 cannot be quantized without violating the symmetry (3.1). The reason for this obstruction is that the group diff S^1 does not preserve the polarization $\{\tilde{X}^L_{\alpha^\mu_{n<0}}\}$ of the Loop space. In other words, the generators $\langle \tilde{X}^L_{l_m}, \tilde{X}^L_{\alpha^\mu_{n\leq 0}}, m\in Z\rangle$ do not close a (first-order polarization) subalgebra. We have to consider only *half* of the Virasoro generators (half the algebra $\mathscr{G}_\Phi$) to close a first-order polarization subalgebra $P = \langle \tilde{X}^L_{l_{m\leq 0}}, \tilde{X}^L_{\alpha^\mu_{n\leq 0}}, m\in Z\rangle$, even though *all* the Virasoro generators have classical Noether invariants which can be given in terms of the basic ones, i.e. those of α^μ_n, $n\in Z$. The wave functions polarized with the subalgebra P provide an irreducible representation where, unfortunately, the operators associated with the Virasoro parameters $\tilde{X}^{\mathbb{R}}_{l_m}$ cannot be realized as functions of $\tilde{X}^{\mathbb{R}}_{\alpha^\mu_n}$. This fact is a manifestation of the anomalous character of the theory.

A natural way of "incorporating" all the Virasoro generators in a "polarization" is by allowing those higher-order operators (of the left enveloping algebra) whose leading terms are linear precisely in the generators $\tilde{X}^L_{l_{m>0}}$ to appear in it. However such a higher order polarization P^{HO} exists *only* for $c = c' = d$ (= dimension of Minkowski space), thus modifying the classical values ($c = 0 = c'$). P^{HO} is a left ideal of the left enveloping algebra of (3.1) generated by the first order polarization $P = \langle \tilde{X}^L_{l_{m\leq 0}}, \tilde{X}^L_{\alpha^\mu_{n\leq 0}}\rangle$ and the higher-order operators $\tilde{Z}^L_{N>0}$

$$\tilde{Z}^L_N = \tilde{X}^L_{l_N} - 1/2 \sum_{\substack{n+m=N\\ 1\leq n,m}} \eta^{\mu\nu}\tilde{X}^L_{\alpha^\mu_m}\tilde{X}^L_{\alpha^\nu_n}. \qquad (3.2)$$

Let us note in passing that the expression (3.2) is closely related to the Sugawara construction of the Virasoro algebra [13]. The arguments of Sect. 2 applies to this particular case and it can be shown that the analogous states $Z_{-N}|0\rangle$ in the Verma module (see the analogous notation in Sect. 2) are null vectors. The higher-order polarized functions also provide the irreducible carrier space for the central extension of the group diffS^1 ⓢ(Loops on $\mathbb{R}^{1,d-1}$). This space coincides again with the orbit of the right enveloping algebra through the vacuum. The Virasoro anomaly is then related to the requirement of higher-order polarizations. In ref. [14] the critical values of the string theory appear, in the context of BRST supergroups, as those for which all the Virasoro generators can be solved in terms of the basic ones. Nevertheless, the basic wave functions $|0\rangle$, $\tilde{X}^{\mathbb{R}}_{\alpha^\mu_n}|0\rangle$ depend on both α^μ_m and l_k parameters.

In the case given above the anomaly was associated with commutators between non-symplectic generators and this is the usual situation (chiral anomaly, Faddeev's anomaly, etc.). The quantization of the Virasoro group itself presents, nevertheless, an anomaly attached to the properly symplectic generators and for this reason the standard geometric methods run into grave difficulties [5]. We will now discuss this anomaly in just the same manner as in the previous example. The structure of the characteristic subalgebras $\mathscr{G}_\Theta$ of the Virasoro group has already been discussed in ref. [4]. When the constants c, c' satisfy

$$c'/c = r^2, \quad r = 2, 3, 4, \ldots, \tag{3.3}$$

the characteristic subalgebras $\mathscr{G}^{(r)}_\Theta$ are $sL^{(r)}(2,\mathbb{R})$ and the corresponding classical phase spaces (which are non-Kähler coadjoint orbits) are given by

$$\mathrm{diff}\, S^1/SL^{(r)}(2,\mathbb{R}). \tag{3.4}$$

The subalgebras $\mathscr{G}^{(r>1)}_\Theta$ cannot be enlarged to a (first-order) polarization – and this is how the anomaly appears – so that we are forced to consider a non-full polarization containing only a subalgebra of $\mathscr{G}^{(r)}_\Theta$, actually $\langle \tilde{X}^L_{l_{-r}}, \tilde{X}^L_{l_0}\rangle$. The only allowed polarization turns out to be $P \equiv \langle \tilde{X}^L_{l_{n\leqq 0}}\rangle$ which is independent of the specific value r. Like in the previous example the wave functions polarized with that P carry an irreducible representation for the *classical values* (3.3) of r. Once again, this representation suffers from the drawback that none of the operators $\tilde{X}^R_{l_{\pm r}}$ can be written in terms of the symplectic ones $\tilde{X}^R_{l_{n\neq \pm r,0}}$. The problem can also be solved trying to "incorporate" the generator $\tilde{X}^L_{l_{+r}}$ in the polarization conditions. This is accomplished by means of a higher-order polarization operator whose leading terms is linear in $\tilde{X}^L_{l_{+r}}$. However, this is only possible for some values of c and c' – *the quantum values* – different from the classical ones (3.3). The classification of these higher-order polarizations has been given in Sect. 2.

We see from the analysis given above that there are two important features which deserve further comments. First of all, the polarization P does not depend on the value r which characterizes the classical phase space and, what is more, it can be used for the case $r^2 \neq \mathbb{N}^2$ (diffS^1/S^1). The second point to be noted is that the quantum modification of the values of c and c', initially associated with a phase space in (3.4), could be seen as classical values associated with another classical phase space (diffS^1/S^1). Therefore we must conclude that, in general, quantization is not properly associated with individual orbits in the group but rather with the whole of the group. This conclusion is also supported by the fact (see Sect. 2 and ref. [4]) that even the quantization of the orbit diffS^1/S^1, which admits a first-order full polarization, requires a higher-order polarization (or requires the taking of the orbit of the left enveloping algebra through the vacuum) to obtain the irreducible Hilbert space in the strong-coupling domain $0 < c \leqq 1$.

Finally we would like to add that the notion of higher-order polarization introduced in this paper could be extended to the standard geometric quantization provided that the underlying "classical manifold" is presymplectic rather than symplectic.

References

1. Pressley, A., Segal, G.: Loop Groups. Oxford: Calendon Press 1986
2. Souriau, J. M.: Structure des Systemes dynamiques, Paris: Dunod 1970; Kirillov, A. A.: Elements of the theory of representation. Berlin, Heidelberg, New York: Springer 1975
3. Bowick, M. J., Rajeev, S. G.: Anomalies as curvature in complex geometry. Nucl. Phys. **B296,** 1007 (1987)
4. Aldaya V., Navarro-Salas J.: Quantization on the Virasoro group. Commun. Math. Phys. **126**, 575 (1990)
5. Witten, E.: Coadjoint orbits of the Virasoro group. Commun. Math. Phys. **114**, 1 (1988)
6. Aldaya, V., Navarro-Salas, J., Ramirez, A.: Algebraic quantization on a group and nonabelian constraints. Commun. Math. Phys. **121**, 541 (1989)
7. Alekseev, A., Shatashvili, S.: From Geometric Quantization to Conformal Field Theory. Commun. Math. Phys. **128**, 197 (1990); La, H., Nelson, P., Schwartz, A. S.: Virasoro Model Space, UPR-0415 IASSNS-HEP-90-25
8. Kac, V. G.: Lecture Notes in Physics, Vol. **94**, p. 441. Berlin, Heidelberg, New York: Springer 1982; Feigin, B. L., Fuchs, D. B.: Funct. Anal. Appl. **16**, 114 (1982)
9. Treiman, S., Jackiw, R., Zumino, B., Witten, E.: Current algebra and anomalies, Princeton, NJ: Princeton University Press and Singapore: World Scientific 1985
10. Nelson, P., Alvarez-Gaume', L.: Hamiltonian interpretation of anomalies. Commun. Math. Phys. **99**, 103 (1985)
11. Alvarez-Gaumé, L.: Introduction to Anomalies. In: Fundamental Problems of Gauge Field Theory, NATO ASI Series B: Phys. Vol. **141**. New York: Plenum Press 1986
12. Mickelsson, J.: Current algebras and Groups. New York: Plenum Press 1989
13. Goddard, P., Olive, D.: Kac–Moody and Virasoro algebras in relation to Quantum Physics. Int. J. Mod. Phys. **A1**, 303 (1986)
14. Aldaya, V., Loll, R., Navarro-Salas, J.: BRST Supergroups and Quantization. Phys. Lett **B225**, 340 (1989)

Communicated by N. Yu. Reshetikhin