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Isosystolic Inequalities and the Topological Expansion for Random Surface and Matrix Models

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Abstract. Using the isosystolic inequalities on Riemann surfaces, we prove that for many random surface or matrix models the radius of convergence of the perturbative series at fixed genus is independent of the genus. This result applies for instance to the dynamically triangulated random surface model in any dimension or to many matrix models with regular propagators in the superrenormalizable domain, for instance $\lambda\phi^3$ in dimension $d<6$, $(\lambda\phi^4+\sqrt{\lambda}\phi^3)$ in dimension $d<4$, and various other $P(\phi)_2$ models (in particular all those containing an odd power of ϕ). We hope that this result is a first step towards a more rigorous understanding of the genus dependence of surface models or of quantum gravity coupled with matter fields.

I. Introduction

The topological expansion in field theory goes back to the proposal by 't Hooft to study gauge theory with gauge group $SU(N)$ by means of a $1/N$ expansion. This is because the order g in this $1/N$ expansion is in perturbation theory the sum of all Feynman graphs of genus g [1]. Topological expansions of this kind can be written in general for quantum field theories in which the field is an N by N matrix field.

Each order in the $1/N$ expansion is a full perturbative series in its own right, with Feynman graphs equipped with the regular propagator $(p^2+1)^{-1}$. The n^{th} order of this series is made of relatively few graphs: for instance in one of the simplest models, namely the matrix model with $\mathrm{Tr}\,\phi^4$ interaction (which for simplicity we call the ϕ^4 model) there are at most K^n graphs of order n and fixed genus (in sharp contrast with the ordinary ϕ^4 perturbative series, in which there are about $K^n n!$ graphs at order n) [2]. In the series of papers [3–5] the matrix models in 0 or 1 dimension of space time were beautifully analyzed. The asymptotic number of planar graphs with n vertices as n gets large was in particular computed, hence the optimal value of the constant K for planar graphs was found.

If we combine this result with the uniform bounds on convergent renormalized Feynman graphs in the superrenormalizable case ([6, 7] and references therein) it is clear that the planar series for superrenormalizable matrix models have a finite radius of convergence, and this statement is also true for the series at any fixed genus g, (although the fact that this is a rigorous theorem was perhaps never explicitly stated in the literature up to now). This is no longer true for just renormalizable situations like ϕ_4^4 in which renormalons are expected to prevent the planar series from having a finite radius of convergence; however even in this case the planar series can be resummed by Borel summation if the model is asymptotically free (this is the case for the "wrong sign" ϕ_4^4 model) [7–9].

Interest in matrix models was revived more recently for several reasons. They provide a clue to the discretization of random surface models [10–12]. A model of particular interest in this context is the discretized version of Polyakov's action for a bosonic string embedded in d dimensions. If the string world-sheet is discretized with triangles, the genus g series is made of the same graphs as the ϕ^3 matrix models, but computed with exponential propagators e^{-p^2} rather than the regular propagators $(p^2+1)^{-1}$ [10–12]. This model is obviously well defined order by order in any dimension. We call it the dynamically triangulated random surface model in dimension d or DTRS [13]. Similarly there are "dynamically quadrangulated" models corresponding to a ϕ^4 interaction and so on.

At least in the superrenormalizable domain where there is only a finite number of mass renormalizations these DTRS models may have the same infrared behavior as the ordinary matrix models with the regular Feynman propagators [13]. This remark has been exploited in dimension 1, where the regular model with Feynman propagators can be solved to a large extent [3].

Since string or superstring theory is nothing but two dimensional gravity coupled to some particular set of conformal fields, any summation of the topological expansion for matrix models should shed light on the non-perturbative aspects of string theory.

Physically one is not so much interested in the radius of convergence of the fixed genus series as in the strength of the corresponding singularity. More precisely it is expected that the genus g series behave at large n like $K^n n^{-3+\gamma_{str}}$, where K is model dependent, and the exponent γ_{str}, called the string susceptibility, is a physically important quantity which should be universal at least in some sense. In dimension d out of the interval [1, 25] it is expected that γ_{str} depends linearly on the genus, with the formula

$$\gamma_{str}=2+(1-g)(1/12)\left[d-25-\sqrt{(d-1)(d-25)}\right] \tag{I.1}$$

[14–16]. (The borderline cases $d=1$ or $d=25$ are special in many respects.)

Recently a non-perturbative summation over the genus was in some sense achieved for matrix models in dimension 0 or 1 [17–19]. Indeed in this case not only the leading behavior in K^n with exact value of K but also the subleading behavior in $n^{-3+\gamma_{str}}$ of the genus g series can be computed. The formula (I.1) can be checked rigorously because the matrix model can be studied very precisely by means of orthogonal polynomials [4–5]. What was shown is that if one sends (according to a certain scaling relation) both N to infinity and λ (the coupling constant) to the critical value $1/K$, where all the fixed genus series simultaneously diverge, a certain resummation occurs. More precisely in this limit a certain global differential equation can be written for the sum of the topological expansion [17–19].

This method is clearly limited to $d \leqq 1$. This is because if we decompose a hermitian matrix into a unitary and a diagonal piece only in $d \leqq 1$ does the unitary piece decouple; only the diagonal piece of the matrix remains in the action and the model is therefore in a certain sense reduced to a vector rather than a matrix model. This is why a single matrix model or a one dimensional finite chain of matrix models can be solved in detail through orthogonal polynomials.

In more than one dimension, plaquettes or Wilson loops begin to exist; accordingly the unitary piece of the matrix becomes important and the model has the intricacies of a true gauge field theory; it seems hopeless to expect a solution as detailed analytically as in 0 or 1 dimension.

Curiously from the point of view of conformal field theory, the central charge $c=1$ is also a borderline case. It is only for $c \leqq 1$ (or $c \geqq 25$ than there is a precise conjecture for the exponent $\gamma_{\rm str}$ of quantum gravity coupled with matter of central charge c namely formula (I.1) (with d, the dimension, replaced by c).

The region $1<c<25$ or $1<d<25$ (which is the physically most interesting one at least for gauge or string theory) remains therefore totally mysterious both from the conformal field or the matrix model point of view. In this paper we propose to examine the objects which are clearly defined above dimension 1 in the matrix models, namely the renormalized Feynman amplitudes themselves. As a first step we prove that the leading behavior of all the fixed genus series is independent of the genus, by a method which applies to many models, in particular the DTRS in any dimension and many regular matrix models in the superrenormalizable domain.

Although we are still far from being able to resum the topological expansion in these dimensions higher than one, we hope that refining the method of this paper might lead to the study of the subleading behavior (presumably it might be necessary to gather more semi-rigorous knowledge in a first stage; numerical simulations might also be useful). The subleading behavior should be model independent at least up to some degree and would give directly the critical exponent of the corresponding discretized random surface models embedded in d dimensions (the case $d=3$ is therefore of particular physical interest). Moreover if the genus dependence of this critical index $\gamma_{\rm str}$ is given by a regular formula (such as the linear formula (I.1) for $c<1$ or $c>25$), then some kind of universal scaling might also exist in these models and perhaps a non-perturbative resummation of all the genera also happens when $N\to\infty$ and $\lambda\to 1/K$, the critical value common to all genera.

Let us conclude with some remarks. First our method also applies to non-integer dimensions, for instance to the ϕ^3 matrix model in dimension d with $\mathrm{Re}\, d<6$, in which we use the ordinary analytic continuation in dimension of Feynman amplitudes [20]. Ordinary functional integrals in constructive field theory have up to now always resisted rigorous dimensional interpolation, in contrast with individual amplitudes. Hopefully this may not be true for a future non-perturbative formulation of surface models.

Our second remark concerns the borderline case of just renormalizable models. Here the first quantity of interest is no longer the radius of convergence of the fixed genus series, which is expected to be 0. However it is certainly possible to extend the construction of the planar wrong sign ϕ_4^4 to any fixed genus; the only technical difficulty has to do with the fact that the renormalization group flow for the coupling constant must remain the same as in the planar case. At fixed genus the number of coupling constant counterterms of nontrivial genus remains indeed bounded, hence these counterterms cannot generate a nontrivial flow. Therefore we expect in the asymptotically free case every fixed genus series to be Borel summable and we expect that the Borel radius of convergence is universal, corresponding to the position of the first *planar* renormalon. What would be the analogue of a scaling behavior in this case is not clear to us.

Beyond the superrenormalizable domain, there is an infinite number of divergent operators. Because of these ultraviolet divergences it is presumably no longer reasonable to expect the matrix model to have anything in common with the DTRS model of two dimensional gravity coupled to a d dimensional matter field. This explains perhaps why nothing particular seems to happen in the matrix model at dimension 25, in contrast with conformal field theory.

II. The Matrix Model and its Topological Expansion

A matrix model is a model in which the field ϕ is an N times N matrix valued field. Let us consider a given polynomial $P(x) = \sum_{k=3}^{p} a_k x^k$. We define a matrix model in dimension d with interaction corresponding to this polynomial by the formal functional integral

$$dv = \frac{1}{Z}\exp - \int_{\mathbb{R}^d} \sum_{k=3}^{p} a_k \frac{\lambda^{(k-2)/2}}{k!\,N^{(k-2)/2}} \mathrm{Tr}(\phi)^k - (1/2)\left[\int_{\mathbb{R}^d} \mathrm{Tr}(\phi)^2 + \int_{\mathbb{R}^d} \mathrm{Tr}(\partial_\mu \phi)^2\right] D\phi\,. \tag{II.1}$$

where $D\phi$ is a product of independent formal Lebesgue measures for each independent component of ϕ. The constants a_k can be taken arbitrary but they play no role in what follows and we can therefore set them equal to 1 for simplicity.

The Feynman rules of this model are best understood by considering that since propagators carry a matrix index they should be represented by double lines (one for each matrix index) or ribbons. Remark that the vertices in this theory contain a trace, hence have an important cyclic symmetry (see Fig. 1). (Note that for simplicity for the tadpole of Fig. 3 and for Figs. 7–8 we use simple lines instead of double lines.)

There are various versions of this model: the field ϕ could be real symmetric, in which case the index dependence of the two point function $\langle \phi_{i,j}, \phi_{i',j'} \rangle$ is $\delta_{i,i'}\delta_{j,j'} + \delta_{i,j'}\delta_{j,i'}$ and one ends up with ribbons which cover an unoriented surface; or, most of the time, the field ϕ is chosen hermitian, so that $\langle \phi_{i,j}, \phi_{i',j'} \rangle$ is $\delta_{i,j'}\delta_{j,i'}$. This rule can be pictured by drawing ribbons with oriented borders; the corresponding arrows allow a consistent choice of a normal to the surface covered by the graph, and one ends up with a theory of oriented surfaces (see Fig. 2).

The reader might ask whether at this stage one should not restrict oneself to semibounded polynomials in order that a regularized version of the model makes

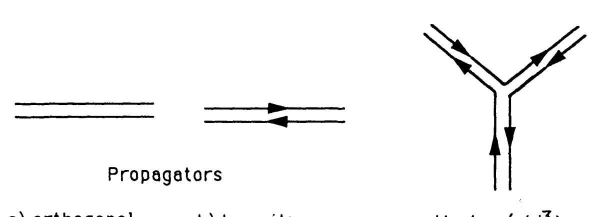

Fig. 1

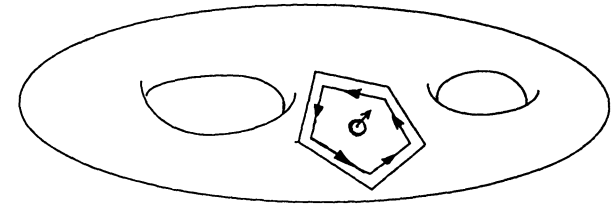

Fig. 2

sense at finite N. This would rule out the ϕ^3 interaction since it is unbounded below. However the usual stability requirements do not apply to planar or fixed genus series (this is why planar ϕ_4^4 with "wrong sign" can in fact be constructed). The planar or fixed genus ϕ^3 model not only makes sense, it corresponds to the study of surfaces by triangulations, so it is in fact a very natural model in this context. Therefore we do not want to impose semi-boundedness of P.

However if one is really interested in applying the $1/N$ expansion scheme to a model at finite N, stability becomes important; one can consider the particular model $(\lambda\phi^4+\sqrt{\lambda}\phi^3)$ in which the measure is:

$$dv=\frac{1}{Z}\exp-\frac{\lambda}{4!N}\int \mathrm{Tr}(\phi)^4-\frac{\sqrt{\lambda}}{3!\sqrt{N}}\int \mathrm{Tr}(\phi)^3-(1/2)\left[\int \mathrm{Tr}(\phi)^2+\int \mathrm{Tr}(\partial_\mu\phi)^2\right]D\phi\,, \tag{II.2}$$

This model is well defined at finite N and all the results of this paper apply to it. However we warn the reader that at the moment the method of this paper does not apply to models with purely even powers of the field such as the pure ϕ^4 model, because their graphs have some additional topological invariants that do not appear in theories with at least one odd interaction (see below). In these theories for the moment our method applies only to the topologically trivial piece of the perturbative expansion.

Let us return to the general Feynman rules of model (II.1). To each closed loop (of single lines) or each closed border of the ribbons there corresponds a sum over possible values of the index flowing through this loop, hence there is a factor N per such loop. There is also a factor $(\lambda/N)^{(k-2)/2}$ per vertex of type k. If the graph has E external legs and v_k vertices of type k, with $\sum_k v_k=v$ its number of internal lines is $l=\sum_k (k/2)v_k-E/2$, and its order in λ is $n=\sum_k v_k(k-2)/2$, which is an integer for E even. The simplest case is for connected vacuum graphs in which $E=0$. Let us fill every closed index loop by a flat "face" and call f the number of such faces. We generate in this way a compact surface of genus g with Euler's relation $f-l+v=2-2g$ which here takes the form $f-\sum_k (k-2)v_k/2=2-2g$. The dependence in N being $N^{f-\sum_k(k-2)v_k/2}$, we obtain that the overall factor for a connected vacuum ϕ^4 graph of genus g is N^{2-2g}. Hence the $1/N$ expansion is a topological expansion.

We consider the quantity

$$N^{-2}\log Z=\sum_g N^{-2g}\sum_n \lambda^n a_n(g) \tag{II.3}$$

in which

$$a_n(g)=\sum_G I_G \tag{II.4}$$

is the sum over all connected vacuum graphs at genus g and order n of the corresponding amplitude I_G. This amplitude is given by the ordinary formula for Feynman integrals. It is interesting to remark that in the definition of the amplitudes I_G we can forget that the graph G is made of double lines: the formula is the same as for a one component ordinary $P(\phi)$ model. However the symmetry factor, or in other words the exact set of graphs entering in (II.4), is not the same as in the ordinary theory and must be computed using the cyclic rule for the vertex and the double lines for the propagators, with or without arrows (hermitian or real symmetric case).

In a superrenormalizable theory at dimension 2 or above, some of the amplitudes I_G are divergent and must be replaced in the ususal way by renormalized amplitudes I_G^R. The renormalization is defined by subtraction at 0 external momentum of the divergent diagrams which in the superrenormalizable domain are just a finite number of one particle irreducible mass diagrams.

Let us consider the radius of convergence of the series $a_n(g)$ defined by the usual formula

$$\varrho(g) = \limsup_{n\to\infty} |a_n(g)|^{1/n} . \tag{II.5}$$

Our main result is

Theorem 1. *The radius of convergence $\varrho(g)$ of the perturbative series at genus g is in fact independent of g for the following models:*

– *The $P(\phi)_2$ model where P contains at least one odd power.*

– *The ϕ_d^3 model with $d<6$ and the $(\lambda\phi^4+\sqrt{\lambda}\phi^3)$ model defined by* (II.2) *in dimension $d<4$.*

– *The discretized string model in any dimension defined by the same graphs as the $P(\phi)$ model but with exponential propagators, where P contains at least one odd power. The* DTRS *model corresponding to $P(\phi)=\phi^3$ is an interesting particular case.*

In the course of proving Theorem 1 we obtain a more precise estimate (see below). However since this estimate is presumably still far from optimal, we prefer to present our result in the form of Theorem 1.

We will give a complete proof of the theorem in the case of ϕ_2^3 which is the simplest case to which our method applies, and give rather complete indications for the $(\lambda\phi^4+\sqrt{\lambda}\phi^3)_{2,3}$. The case of exponential propagators is clearly easier and will be treated briefly. Other models are left to the reader.

Our method requires three ingredients: isosystolic inequalities, bounds to relate amplitudes of graphs which are roughly similar, and some surgery rules to raise or lower the genus. The first ingredient is completely general, the second is completely general in superrenormalizable situations, but may require a sophisticated phase space analysis as in [26]. Unfortunately the third ingredient, our surgery rules, is not completely general up to now, and this is why we have restrictions to interactions with at least one odd power. For purely even interactions there may be some topological obstructions to "local surgery" to lower or raise the genus. This is why in this case we can treat only the topologically trivial piece of these theories, in which these obstructions disappear.

Theorem 1 applies also to the radius of convergence of every fixed genus series with a fixed number of external legs E; if we call $a_n^E(g)$ the corresponding n^{th} order and define $\varrho^E(g) = \limsup_{n\to\infty} |a_n^E(g)|^{1/n}$ we have $\varrho^E(g) = \varrho^E(0) = \varrho^0(0)$ for any g and E. This can be verified by the same method as for vacuum series and is left to the reader.

III. Proof of the Main Theorem

We shall prove that the convergence radius $\varrho(g+1)$ is equal to $\varrho(g)$ for any g. Hence for all g it will be equal to the planar radius $\varrho(0)$. Our strategy is to construct two mappings ψ and χ which associate respectively to a graph of genus g a graph of genus $g+1$ and to a graph of genus $g+1$ a graph of genus g, in such a way that two associated graphs have about the same order (i.e. number of vertices) and are similar except for a percentage of their lines which gets smaller and smaller as this order increases. We must also check that not too many graphs project themselves on the same graph in the mappings ψ and χ, and finally we need theorems which tell us that the amplitudes of two graphs roughly similar are roughly equal. With all these ingredients we can conclude that the convergence radius of the two series (at genus g and $g+1$) is the same.

We specialize for simplicity to the case of ϕ_2^3. Since amplitudes of graphs with tadpoles vanish after renormalization in two dimensions, we can limit the definition of the mappings ψ and χ to graphs without tadpoles (Fig. 3) but we must be careful that the image of ψ and χ is also made of graphs without tadpoles (such graphs are called Wick-ordered graphs or W-graphs for short).

The easy part of the correspondence is the mapping ψ (to raise the genus by one unit). Indeed to any connected vacuum ϕ^3 graph G of genus g and order n without tadpoles we can associate a connected vacuum ϕ^3 graph G' of genus $g+1$ and order $n+4$, again without tadpoles, by simply inserting the two-point graph G_0 of Fig. 3 on any of the lines of G. In this way a correspondence $G \to G' = \psi(G)$ is defined. The inverse image of a graph G' of genus $g+1$ is made of at most p graphs of genus g, where p is the total number of subgraphs in G' isomorphic to G_0. It is easy to check that $p \leqq g+1$. With a theorem relating the amplitude of G' to the amplitude of G the inequality $\varrho(g) \leqq \varrho(g+1)$ follows.

The difficult part of the problem is to find a correspondence which lowers the genus by one unit. Let us consider a connected vacuum ϕ^3 graph G of genus $g+1$ with n vertices (n even), hence $3n/2$ lines and $(n/2)-2g$ faces, which has no tadpoles (otherwise recall that $I_G^R = 0$). The dual graph G^* defines a Riemannian metric on the compact orientable surface of genus $g+1$ in the usual way [10]: the dual graph defines a realization of the surface as a simplex whose faces are triangles since G is a ϕ^3 graph. If we take each face to be a regular flat equilateral triangle of area 1 and side length $2/3^{1/4}$, the corresponding metric γ is singular with curvature concen-

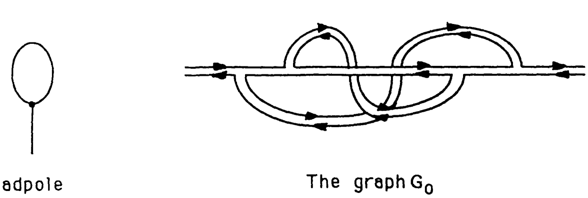

Fig. 3

trated at the vertices where strictly more or strictly less than six triangles meet. (The fact that the metric is singular might seem uncomfortable for the application of the results below, but in fact it is not; we can always smooth out the singularities, then pass to the limit.)

The total area of the surface for this metric γ is the number of faces of G^*, hence is n. We want to associate to G a graph $\chi(G)$ of genus g by cutting one of the "handles" of G. But in order not to change too much the amplitude of G, we want to do this by cutting the smallest possible number of lines of G. This is the same as finding the shortest noncontractible path on the dual graph G^*, since cutting a line of G is the same as making an elementary step on G^*. Since G^* defines the metric γ on the surface of genus $g+1$ and each edge of G^* has the same length (namely $2/3^{1/4}$) for this metric, we are lead to the problem of finding the shortest nontrivial closed loop for this metric. This problem is known in mathematics under the name of the isosystolic problem. Let $S(\gamma)$ be such a shortest nontrivial loop and $s(\gamma)$ be its length. The loop $S(\gamma)$ is not necessarily made of lines of G^*, hence cannot be directly interpreted as a path on G^*. However in each triangle which is a face of G^* it reduces to a straight line joining two points A and B on the border of the triangle, since on each elementary triangle the metric γ is the flat ordinary metric. We can join A to B by a path staying on the border of the triangle which is at worst twice as long. Therefore we can find a nontrivial closed path on G^* of length at most $2s(\gamma)$.

The maximal value of $s(\gamma)$ when γ (hence G) varies is controlled in terms of $A=n$, the total area of the surface with metric g by the so-called isosystolic inequality:

Theorem III.1. *For every genus there exists a constant* $K(g)$ *such that*

$$s(\gamma) \leqq K(g) A^{1/2}. \tag{III.1}$$

The optimal value of $K(g)$ for general g is not known. Theorem III.1 was proved first in [21, 22]. Loewner proved that

$$K(1) = \sqrt{2/\sqrt{3}}. \tag{III.2}$$

There is a uniform bound on $K(g)$, namely [25]

$$K(g) \leqq 2/\sqrt{3}. \tag{III.3}$$

Furthermore $K(g)$ decreases at large g. In [25] it is proved that for every $\theta < 1$ there exists a constant c_0 such that:

$$K(g) \leqq c_0 g^{-\theta}. \tag{III.4}$$

From (III.4) it is tempting to speculate that the true asymptotic behavior of $K(g)$ might be something like $g^{-1} \log g$, but up to now this true behavior is not known.

For a review on isosystolic inequalities and related topics we refer to [23–25]. Here to illustrate the problem with an example let us simply give the proof of (III.2), which is elementary.

By reparametrization, a conformal transformation and modular invariance, we know that we can write any metric on the torus $\mathbb{R}^2/\mathbb{Z}^2$ as

$$\gamma = e^{\phi} |dx + \tau dy|^2, \tag{III.5}$$

where ϕ is a conformal factor and τ, the modulus, can be chosen in the fundamental modular region: $\operatorname{Im} \tau > 0$, $|\tau| \geqq 1$, $-1/2 \leqq \operatorname{Re} \tau \leqq 1/2$. Each closed loop at a fixed height $0 \leqq y \leqq 1$ is longer or equal to the systolic length $s(\gamma)$. Therefore, using the

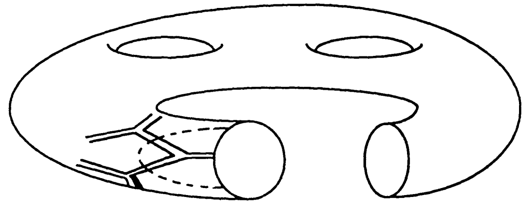

Fig. 4

Schwarz inequality:

$$s(\gamma) \leqq \int_0^1 \int_0^1 dx dy e^{\Phi/2} \leqq \left(\int_0^1 \int_0^1 dx dy e^{\Phi} \right)^{1/2} . \tag{III.6}$$

The area of the torus is nothing but $A = \int_0^1 \int_0^1 dx dy e^{\Phi} \operatorname{Im} \tau$, hence $s(\gamma) \leqq \sqrt{A/\operatorname{Im} \tau}$ and the worst case is for $\operatorname{Im} \tau$ minimal, hence for the "hexagonal torus" in which $\operatorname{Im} \tau = \sqrt{3}/2$. This concludes the proof of (III.2). Remark that the worst torus corresponds to the closest packing of disks in the plane; this fact has interesting higher dimensional analogues [24].

We want in fact to break an even number of lines and to contract them again in order to compare a vacuum graph of genus $g+1$ to a vacuum graph of genus g. For ϕ^3 graphs we can require that L is even. Indeed either L is even or it is odd, in which case we choose one particular vertex hooked to a line l_0 among the L lines. If the cut is changed so as to pass through the other side of the vertex (as shown in Fig. 4) the parity of L changes (but not the homotopy class of the cut). L is changed in $L \pm 1$. This trick is only possible for vertices with an odd number of lines and this is why our proof at the moment does not cover purely even theories such as ϕ^4. We hope that this restriction is only technical and can be overcome in the future.

Remark that by cutting one non-contractible cycle on a genus $g+1$ graph we obtain a graph which is still connected. Using the simple bound (III.3) together with these remarks we obtain that to any vacuum connected ϕ^3 graph G of genus $g+1$ and order n we can associate a connected ϕ^3 graph G' of genus g with $E=2L$ external legs by cutting an *even* number $L=2k$ of internal lines of G, which is at most $(3^{1/4}/2)2s(\gamma)+1$ [1]. By (III.3) this proves that $L \leqq 2\sqrt{n}$ (at least if $n>4$ which we can assume from now on).

Now we can form from the graph G' a connected vacuum ϕ^3 graph G'' of genus g, which furthermore has no tadpoles, in the following way. There is a cyclic ordering of the $L=2k$ half lines on one side of the cut as $l_1, \ldots, l_{2k}, l_{2k+1} = l_1$. We want to pair these half lines together and repeat the operation on the other side of the cut to obtain a vacuum graph of genus g (see Fig. 5). We could simply contract l_1 with l_2, l_3 with l_4 and so on, but this process could create tadpoles. This is because some pair of consecutive lines can be hooked to the same vertex; such lines are called doublets, the other ones are called singlets. To avoid tadpoles we call $I = \{i_1 < i_2 < i_h\}$ the set of indices in $[1, 2k]$ such that l_i and l_{i+1} are hooked to the same vertex, hence form a doublet. When $i_{j+1} - i_j$ is odd there is an even number of singlets between the two consecutive doublets of indices j and $j+1$. We contract

[1] It could be less, in particular because when the path of length at most $2s(\gamma)$ contains consecutive pieces of edges of a triangle in reverse direction we can contract these pieces

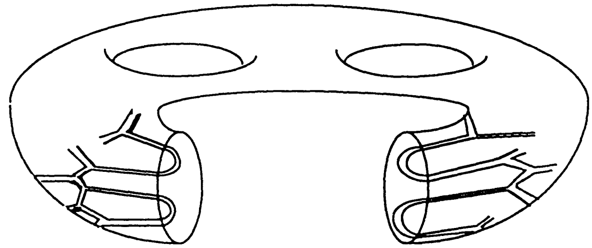

Fig. 5

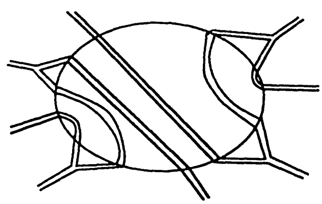

Fig. 6

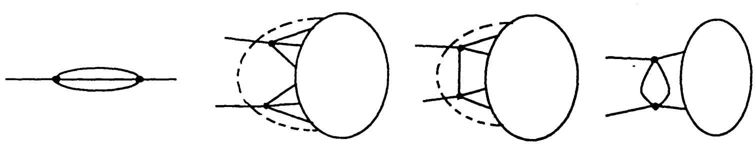

Fig. 7

them together in their natural consecutive order. Similarly when there is an odd number of singlets between two consecutive doublets, we contract them in the natural order except one, the last one. In this way we are reduced to the situation of doublets separated only by one singlet or nothing. We can contract together the ends of two consecutive doublets separated by nothing and consider the two remaining ends as new singlets and continue in this way until we arrived at a remaining set of doublets all separated by singlets. They must be therefore in even number and are then contracted according to Fig. 6.

The crucial point is that this rule of contractions is "planar" (i.e. the lines created in this way can be drawn on the disk filling the circle of the cut without any crossing). Therefore the graph $G''=\chi(G)$ obtained at the end of this construction is of genus exactly g, as required, and has no tadpoles since G had no tadpoles and no one was created.

In the case of $(\lambda\phi^4+\sqrt{\lambda}\phi^3)_3$ it is not sufficient that our surgery rules do not create any tadpole; we want also that they do not create any divergent subgraph, hence any "blobs" B (see Fig. 7). This can be ensured because the only case where a blob could form are pictured in Fig. 7. In the two first cases a and b the cut is not optimal, as shown. Hence only case c can occur. But the two half lines not to be contracted are then exactly similar to a doublet of the preceding type, and we have just shown that we can avoid the contraction of the two lines of such a doublet.

For more complicated models such as ϕ^4 models in non-integer dimension d close to 4 there are more and more subgraphs to avoid but they are all two point subgraphs; hence the argument of Fig. 7 generalizes: the only case where the cut cannot be improved gives a doublet, whose contraction can be avoided. Hence our surgery rules can be adapted to all superrenormalizable situations.

Finally we want to check, in the correspondence defined above, that not too many different G's of genus $g+1$ can be associated to the same given G'' of genus g and order n. The maximal number of such different G's associated to a given G'' of order n, called $F(n, g)$, is certainly bounded by the number of sets of L lines of G (with $L \leqq 2\sqrt{n}$) times the total number of possibilities to contract the $2L$ half lines together. Hence this number can be bounded, for $n \geqq 50$ (where $\sqrt{n/2}+1 \leqq \sqrt{n}-1$) in the following way:

$$F(n,g) \leqq \sum_{L=0}^{2\sqrt{n}} \frac{l!}{L!(l-L)!}(2L)!! \leqq \sum_{L=0}^{2\sqrt{n}} (2l)^L \leqq (3n)^{2\sqrt{n}+1}, \tag{III.7}$$

where $l = 3n/2$ and $2L!! \equiv (2L-1)(2L-3)\dots 3.1$. Remark that the bound is in fact uniform in the genus g, and that it is of the form $(1+\varepsilon(n))^n$ with $\lim_{n\to\infty} \varepsilon(n) = 0$, hence it will have no effect on the radius of convergence.

Altogether we have proved:

Theorem III.2. *To any vacuum connected ϕ^3 W-graph G of genus $g+1$ and order n we can associate a connected vacuum ϕ^3 W-graph G'' of genus g and order n by cutting L internal lines of G and contracting them again differently, where L satisfies the inequality*

$$L \leqq 2\sqrt{n}. \tag{III.8}$$

Furthermore the corresponding mapping $G \to \chi(G) = G''$ is such that for any G'', $\chi^{-1}(G'')$ has at most $(3n)^{2\sqrt{n}+1}$ elements.

We need now theorems which tell us that breaking a line or branching a new line or inserting a G_0 subgraph in a graph is something which does not change too much the associated amplitude. Remark that the amplitudes being the same as for ordinary one component ϕ^3 graphs, we can forget the double lines and all the topological subtleties for this particular problem. The machinery for this kind of result was developed in [26] and it extends clearly to any bosonic superrenormalizable theory. We are going to adapt the results of [26] to our purpose, and from now on a certain familiarity with [26] is therefore assumed. We give a precise theorem for ϕ_3^4, which is a "difficult" superrenormalizable case and is precisely the model treated in [26]; but the same method really applies to any bosonic superrenormalizable theory; in particular it applies much more easily to ϕ_2^3.

In ϕ_3^4, apart from the vacuum graphs with 1, 2 or 3 vertices and the tadpole, the only divergent graph is the blob of Fig. 7. A graph without tadpoles and blobs is called a CC graph (connected, convergent). (This notion extends to ϕ_d^4 but the list of two point divergent graphs increases as one approaches $d=4$.) We start with a rather easy bound which does not require any multiscale analysis in the style of [26].

Theorem III.3. *Let G be a* CC *ϕ^4 graph with n vertices and $E \geqq 2$ external lines. We consider a graph $\bar{G}$ obtained from G by contracting together two external lines of G. There exists a strictly positive constant C_1 such that if $\bar{G}$ is also* CC, *the corresponding amplitudes at zero external momenta satisfy:*

$$I_G \leqq C_1 \cdot n \cdot I_{\bar{G}}. \tag{III.9}$$

Furthermore we have a similar inequality in the DTRS *model; for any connected graphs G and $\bar{G}$ such as above (but no longer necessarily* CC *graphs), if we define J_G*

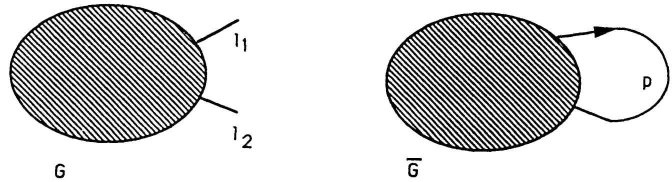

Fig. 8

and $J_{\bar{G}}$ as the amplitudes computed with exponential propagators instead of regular Feynman propagators we have

$$J_G \leqq C_1 \cdot n^{d/2} \cdot J_{\bar{G}}. \tag{III.10}$$

Proof. We start with the more difficult case of Feynman propagators. Using the α-parametric representation we have (with a mass equal to unity for simplicity):

$$I_G = \int_0^\infty \prod_l d\alpha_l \, e^{-\sum_l \alpha_l} \frac{1}{U(\alpha)^{d/2}}, \tag{III.11}$$

$$I_{\bar{G}} = \int \frac{d^d p}{p^2+1} I_G(p^2) = \int \frac{d^d p}{p^2+1} \int_0^\infty \prod_l d\alpha_l \, e^{-\sum_l \alpha_l - \frac{V_G(\alpha)}{U_G(\alpha)} p^2} \frac{1}{U_G(\alpha)^{d/2}}, \tag{III.12}$$

where $I_G(p^2)$ is the amplitude of G with all external momenta put to 0 except for a momentum p entering through one and exiting through the other of the two external lines l_1 and l_2 contracted to form $\bar{G}$ (see Fig. 8).

The polynomials U_G and V_G are the usual Symanzik polynomials whose definition can be found, e.g. in [7]. Let us put $r = p^2$. We have (where $CC'\dots$ stand for unessential constants which depend only on d):

$$I_{\bar{G}} \geqq C \int_0^1 I_G(r) r^{(d-2/2)} dr \geqq C' \int_0^\infty \prod_l d\alpha_l \, e^{-\sum_l \alpha_l} \frac{1}{V_G(\alpha)^{d/2}}. \tag{III.13}$$

The right-hand side of (III.12) is similar to I_G but with V substituted to U. Recall that U is a sum over the spanning trees T of G of the product of the α parameters of the lines *not* in T and V is a sum of similar products but performed over the two-trees T' (i.e. spanning trees minus one line) which cut G into two pieces, one containing l_1 and the other containing l_2. Since every two tree is obtained from a spanning tree by cutting a line (possibly in a non-unique way) we have

$$V_G(\alpha) \leqq \sum_l \alpha_l \cdot U_G(\alpha), \tag{III.14}$$

and therefore, using the homogeneity property of U:

$$\begin{aligned} I_G &= \int_0^\infty d\lambda \int_0^\infty \prod_l d\alpha_l \, e^{-\sum_l \alpha_l} \delta\Big(\sum_l \alpha_l - \lambda\Big) \frac{1}{U_G(\alpha)^{d/2}} \\ &= \int_0^\infty d\lambda \lambda^{\frac{4-d}{2} n + \frac{d-2}{2} E - d - 1} e^{-\lambda} \int_0^\infty \prod_l d\alpha_l \delta\Big(\sum_l \alpha_l - 1\Big) \frac{1}{U_G(\alpha)^{d/2}} \\ &= \Gamma\left(\frac{4-d}{2} n + \frac{d-2}{2} E - d\right) \int_0^\infty \prod_l d\alpha_l \delta\Big(\sum_l \alpha_l - 1\Big) \frac{1}{U_G(\alpha)^{d/2}}. \end{aligned} \tag{III.15}$$

Similarly by (III.13–14):

$$I_{\bar{G}} \geqq \Gamma\left(\frac{4-d}{2}n+\frac{d-2}{2}E-d-1\right)\int_0^\infty \prod_l d\alpha_l\, \delta\left(\sum_l \alpha_l - 1\right)\frac{1}{U_G(\alpha)^{d/2}}, \quad \text{(III.16)}$$

hence (since $E \leqq 4n$) we obtain the theorem, namely

$$I_G \leqq C_1 \cdot n \cdot I_{\bar{G}} \quad \text{(III.17)}$$

for some d-dependent constant C_1 simply because $\Gamma(an)/\Gamma(an-1)=an$.

We turn now to the easier case of graphs with exponential propagators. Their amplitude J_G is then explicitly given by the formula

$$J_G = T(G)^{-d/2}, \quad \text{(III.18)}$$

where $T(G)$ is the number of spanning trees in G. But consider a graph G with $l(G)$ lines and let $\bar{G}=G\cup\{l_0\}$, hence the line l_0 is added to G by contracting two half lines l_1 and l_2 in G. Any spanning tree of $\bar{G}$ containing l_0 is obtained from a two tree of G separating l_1 from l_2, and such a two tree is it self obtained from a spanning tree of G by cutting a line. Therefore $T(\bar{G}) \leqq l(G)T(G)$, and for $d \geqq 0$ this gives $J_G \leqq l(G)^{d/2} J_{\bar{G}}$. Since the number of lines is linear in the number of vertices for any polynomial model we obtain (III.10).

We want now theorems which tell us that when we add lines to a graph or when we insert a G_0 subgraph (see Fig. 3) we do not increase too much its amplitude. Such theorems are more delicate because there is a potential ultraviolet problem: the integral over d^dp in (III.9) does not converge for $d \geqq 2$ unless some internal convergence is extracted out of G. It is this kind of problem which were solved in [26] using a phase space analysis.

We introduce an exponential cutoff on each propagator, i.e.; we define

$$C_\kappa(x,y) \equiv \frac{1}{(4\pi)^{d/2}}\int_\kappa^\infty \frac{d\alpha}{\alpha^{d/2}} e^{-\alpha-|x-y|^2/(4\alpha)}. \quad \text{(III.19)}$$

The amplitude $I_{G,\kappa}$ with cutoff κ is similar to the amplitude without cutoff, but where every propagator $\frac{1}{(2\pi)^d}\frac{1}{p^2+1}$ is replaced by $\hat{C}_\kappa(p)=\frac{1}{(2\pi)^d}\frac{1}{p^2+1}e^{-\kappa^{-1}(p^2+1)}$.

Theorem III.4. *Let ε be a small constant. For any* CC *graph G of order n of a superrenormalizable theory (such as ϕ_3^4) there exists some (model-dependent) constant C such that if we define the cutoff $\kappa_n = n^\varepsilon$, we have*

$$I_G \leqq e^{Cn/(\log n)^{1/4}} I_{G,\kappa_n}. \quad \text{(III.20)}$$

This theorem is a simple consequence of the more detailed bounds (3.3) and (3.34–35) in [26]. In particular it was proved in [26, Theorem III.1, p. 66] that if for any subgraph $F \subset G$ we define I_{G,F,κ_n} as the amplitude with cutoff κ_n on the lines of $G-F$ and no cutoff on the lines of F, there exists a constant K with

$$I_{G,F,\kappa_n} \leqq K^{l(F)} I_{G,\kappa_n}. \quad \text{(III.21)}$$

This bound is quite non-trivial and requires a phase space analysis. With easier methods such as the ones of [6, 7] some simpler bounds (also proved in [26]) state that the amplitude $I^{\kappa_n}_{G,F}$ defined with propagators C_{κ_n} for the lines of $G-F$ and $C-C_{\kappa_n}$ for the lines of F is bounded by:

$$I^{\kappa_n}_{G,F} \leqq K^{-l(F)\varepsilon\log n} I_{G,\kappa_n}. \quad \text{(III.22)}$$

We can then write $I_G = \sum_{F \subset G} I^{\kappa_n}_{G,F}$ and distinguish two contributions, when $l(F) \leqq n/\sqrt{\log n}$, and when $l(F) > n/\sqrt{\log n}$. In the first case we use (III.21) and obtain

$$\begin{aligned} I^1_G &= \sum_{F \subset G,\, l(F) \leqq n/\sqrt{\log n}} I^{\kappa_n}_{G,F} \\ &\leqq \sum_{F \subset G,\, l(F) \leqq n/\sqrt{\log n}} I_{G,F,\kappa_n} \\ &\leqq \sum_{k=0}^{n/\sqrt{\log n}} K^k \frac{(2n)\dots(2n-k+1)}{k!} I_{G,\kappa_n} \\ &\leqq (2eK)^{n/\sqrt{\log n}} e^{\frac{n}{2\sqrt{\log n}} \log\log n} I_{G,\kappa_n}. \end{aligned} \tag{III.23}$$

In the second case we use (III.22) (and $l(G)=2n$) to obtain:

$$\begin{aligned} I^2_G &= \sum_{F \subset G,\, l(F) > n/\sqrt{\log n}} I^{\kappa_n}_{G,F} \\ &\leqq 2^{2n} K^{-n\varepsilon\sqrt{\log n}} I_{G,\kappa_n}. \end{aligned} \tag{III.24}$$

Combining (III.23) and (III.24) yields (III.20).

Using these bounds we can compare the amplitudes for a graph G of genus $g+1$ and its image $\chi(G) = G''$, using the intermediate graph G' in which L lines of G have been cut. We have

$$I_G \leqq e^{Cn/(\log n)^{1/4}} I_{G,\kappa_n} \leqq e^{Cn/(\log n)^{1/4}} \quad (C_2 n^\varepsilon)^L I_{G'}, \tag{III.25}$$

where C_2 is some constant. This is because $C_{\kappa_n}(x,y) \leqq C_2 \kappa_n = C_2 n^\varepsilon$ in dimension 3 $\left(\text{since } \int \frac{d^dp}{p^2+1} \text{ is linearly divergent in dimension 3; similar inequalities of course also exist in other dimensions and lead to similar conclusions in the superrenormalizable case}\right)$. Finally using Theorem III.3 we have

$$I_{G'} \leqq (C_1 n)^L I_{G''}. \tag{III.26}$$

Gathering (III.25–26) with Theorem III.2 which bounds L and the number of elements in the inverse image $\chi^{-1}(G'')$ we obtain finally for some constant C':

$$a_n(g+1) \leqq e^{C'n/(\log n)^{1/4}} \quad a_n(g), \tag{III.27}$$

hence $\varrho(g+1) \leqq \varrho(g)$.

To obtain the bound in the other direction, we use the correspondence ψ and again Theorem III.4 to show that the insertion of a subgraph G_0 does not reduce too much the amplitude of a graph G. More precisely if $G' = \psi(G)$ is obtained from G by such an insertion on a line l_0, the amplitude for G' is equal to the amplitude of G in which the line l_0 has a propagator

$$\bar{C}(p) \equiv \frac{1}{(p^2+1)^2} \int d^3k d^3k' \frac{1}{(p+k)^2+1} \frac{1}{(k^2+1)^2} \frac{1}{(k+k')^2+1} \frac{1}{k'^2+1} \tag{III.28}$$

instead of the regular propagator $C(p) = \frac{1}{p^2+1}$. But it is easy to check on (III.28) that for some constant K:

$$\bar{C}(p) \geqq K \cdot \frac{1}{(p^2+1)^3}. \tag{III.29}$$

Now if $p^2 \geqq n^{1+2\varepsilon}$ we have for some constant K',

$$C_{\kappa_n}(p) \leqq K' e^{-n^{1+\varepsilon}} \frac{1}{p^2+1}. \tag{III.30}$$

Therefore for n large enough the corresponding region of integration contributes almost nothing to I_{G,κ_n}. More precisely if we define $I_{G,\kappa_n,l_0\mathrm{low}}$ as the same integral as I_{G,κ_n}, but with the restriction that the momentum p of the line l_0 satisfies $p^2 \leqq n^{1+2\varepsilon}$ (i.e. we have now an additional *sharp* momentum cutoff on the line l_0, we have for some constant K_1, using the general method for uniform bounds on Feynman amplitudes [6, 7]:

$$I_{G,\kappa_n} - I_{G,\kappa_n,l_0\mathrm{low}} \leqq K' K_1^n e^{-n^{1+\varepsilon}}. \tag{III.31}$$

Furthermore we have ([26], Lemma III.2 or [27]), for some constant K_2:

$$I_{G,\kappa_n} \geqq K_2^n. \tag{III.32}$$

Therefore we have for n large enough (combining (III.31) and (III.32)):

$$I_{G,\kappa_n} \leqq 2 I_{G,\kappa_n,l_0\mathrm{low}} \leqq 2K \cdot n^{2+2\varepsilon} I_{G',\kappa_n} \leqq 2K n^{2+2\varepsilon} I_{G'}, \tag{III.33}$$

and using (III.20) we obtain

$$I_G \leqq 2K \cdot n^{2+2\varepsilon} e^{Cn/(\log n)^{1/4}} I_{G'}. \tag{III.34}$$

This together with the bound on the number of elements in $\psi^{-1}(G')$ (which we recall is $g+1$) completes the proof that for some constant C':

$$a_n(g) \leqq e^{C'n/(\log n)^{1/4}} a_{n+2}(g+1), \tag{III.35}$$

hence we obtain the converse inequality $\varrho(g) \leqq \varrho(g+1)$. This achieves the proof that $\varrho(g) = \varrho(g+1)$.

In the case of exponential propagators we do not need any ultraviolet machinery such as Theorem III.4. The exponential propagator is directly bounded by 1, so adding a line always decreases the amplitude. The converse, deleting a line is controlled by the last part of Theorem III.3. This is enough for the treatment of the DTRS models in any dimension.

We want now to discuss the case of purely even theories such as ϕ^4. In such a theory since the dual graph has faces with even number of sides, the parity of the length of any non-contractible loop in the dual graph is conserved under homotopy. In more pedantic terms for each graph we have a representation of the homology group of the surface into $\mathbb{Z}/2\mathbb{Z}$. If a loop such as the systolic loop that we want to break has odd parity, there is nothing to do to change this by local modifications. A graph of genus 1 such as the one of Fig. 9 has no genus 0

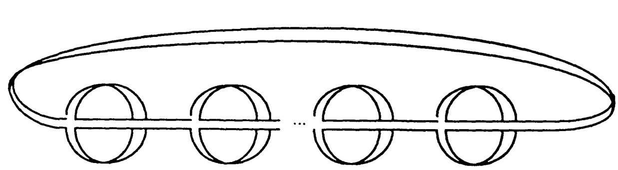

Fig. 9

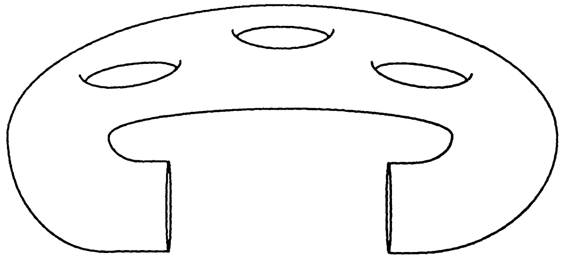

Fig. 10

counterpart which looks like him up to a small number of local modifications. Hence in this case our method fails because our surgery rule requires an even number of lines to be broken. Let us define the "trivial sector" of the theory as made of the graphs for which the representation above is trivial (hence such that each dual graph has even number of lines along *any* non-contractible loop. Then for this trivial "sector" our surgery rules apply and we can still conclude that the radius of convergence is independent of the genus.

There is a way to obtain Feynman rules which automatically restrict us to this sector. For instance we can consider a model with two hermitian matrix fields ψ^1 and ψ^2, such that the propagators are

$$\begin{gathered}\langle\psi^1,\psi^1\rangle=\langle\psi^2,\psi^2\rangle=0;\\ \int e^{ip(x-y)}dy\langle\psi^1_{i,j}(x),\psi^2_{i'j'}(y)\rangle=\frac{1}{p^2+1}\,\delta_{i,j'}\delta_{j,i'}\,.\end{gathered}\tag{III.36}$$

These rules correspond to a gaussian measure which is not positive, hence the associated functional integration is purely formal. But if we add an even interaction such as $\sum_{k=2}^{p} a_k \operatorname{Tr}(\psi^1\psi^2)^k$ the vertices have lines with alternating indices 1 and 2 and the lines must connect a field of type 1 to a field of type 2. Hence it is easy to check that the set of graphs obtained in this way is such that every non-contractible loop crosses an even number of lines (on the dual graph every such loop has an even number of lines). Hence for this model, our surgery rules apply and the radius of convergence is again independent of the genus (in the superrenormalizable domain).

One could perhaps speculate that the distinction made in this paper between purely even models and other models may have something in common with the recent discussions over the "doubling of solutions" for even models (see [29–31]).

We end up with a few remarks on the rôle of isosystolic inequalities and some open questions.

Isosystolic inequalities hold on a continuous Riemann surface, and the natural framework for them seems to be therefore the continuum. It may be possible to use them to prove direct bounds between string amplitudes at various genera. However remark that if we cut a continuum Riemann surface along an isosystolic loop and glue each side of the cut together to lower the genus we have to "fold" at some point (see Fig. 10). We think that there might be an associated regularization problem. The interest of the matrix models is that they provide precisely a well defined regularization of such problems. We hope that it is the interplay between the ideas from the mathematically well developed continuum theory of Riemann surfaces and the simple and physically attractive discretization provided by the matrix models which may be most fruitful.

Another remark is that at genus one the isosystolic inequality is saturated in the case of the most symmetric torus (the one with the largest discrete symmetry group), which corresponds to an orbifold point in moduli space. It is reasonable to expect this phenomenon to generalize to higher genus: presumably the worst isosystolic inequalities are obtained for the most symmetric Riemann surfaces with g handles, hence for orbifold points in moduli space, but the description of these surfaces is impaired by our poor understanding of moduli space at high genus. Nevertheless it is a reasonable speculation that the contribution of such symmetric points may dominate the string functional integral at large genus [28].

Let us conclude with a brief remark concerning membrane models (models of manifolds with dimension greater than two). For Riemannian manifolds of more than two dimensions there is a well developed theory of isosystolic inequalities; in particular for d-dimensional torus the saturated inequalities also occur for very symmetric cases, corresponding to closest periodic packing of spheres in d dimensions. Membrane models might be discretized by using variables with d indices ($d=2$ corresponds to matrices), and the corresponding isosystolic inequalities might also be useful in that context.

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