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On The Origin of Integrability in Matrix Models

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Abstract. The matrix integrals involved in $2d$ lattice gravity are studied at finite N. The integrable systems that arise in the continuum theory are shown to result directly from the formulation of the matrix integrals in terms of orthogonal polynomials. The partition function proves to be a tau function of the Toda lattice hierarchy. The associated linear problem is equivalent to finding the polynomial basis which diagonalizes the partition function. The cases of one Hermitian matrix, one unitary matrix, and Hermitian matrix chains all fall within the Toda framework.

1. Introduction

Recently a great deal of progress has been made in the nonperturbative formulation of low-dimensional toy models of string theory – two-dimensional gravity coupled to $c < 1$ matter – by formulating the theory in terms of large-N matrix integrals [1, 2]. Soluble matrix integrals give the full partition function for string theory in simple backgrounds. One of the most remarkable features is that the continuum theory is governed by the KP hierarchy of commuting differential operators [2–4]. The Lax operators of KP provide different realizations of the Heisenberg algebra of the spectral parameter and its conjugate momentum [4], and the commuting KP flows parametrize the space of gravitational field theories for low c_{matter}. Continuum analyses suggest [3, 5] that the partition function is a tau function of the KP hierarchy.

These beautiful results raise many issues. One is to understand better the origin of the integrable systems that appear at the level of the matrix integrals. Exactly what is the connection of integrability to $2d$ gravity? Are there gravity theories for each integrable system having a thermodynamic limit? What is the precise nature of the tau function that appears? We would also like to connect the matrix model formulation to other approaches to $2d$ gravity and string theory. It is important to obtain lattice expressions for scaling operators in order to compare with

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continuum $2d$ field-theoretic analyses [6], and to understand how to find the continuum degrees of freedom in the matrix integral [7]. There are other lessons to be learned as well, regarding the nonperturbative nature of string [8], the spacetime geometry and symmetry of the models, the appearance of the $2d$ gravitational field as a (Euclidean) time parameter, the existence of an asymptotic expansion of the partition function in powers of the string coupling without any apparent string field theoretic graphical expansion. It is likely that to learn these lessons will require a better understanding of the matrix integrals themselves and not just the effective continuum theory.

We will show that as soon as one passes to the orthogonal polynomial formulation of the matrix problem, an integrable system governs changes in the matrix potential. This system is the Toda lattice hierarchy, and the times of the commuting flows are the coupling constants of the matrix potential. Fundamentally commutativity is simply well-definedness of correlation functions. The Hilbert space formulation in terms of orthogonal polynomials translates this into a classical integrable system. Thus we find that the differential operator formalism of the continuum is rooted in a discrete integrable matrix dynamics. The continuum dynamics is very closely reflected in that of the lattice theory, as is common in integrable field theories.

Our interest in the origin of integrability in these theories was stimulated by the appearance of Douglas' beautiful paper [4] on the subject. Shortly thereafter we discovered the relation to the Toda hierarchy, however the rather clean formulation of the present paper was not arrived at until recently. In the meantime, we have learned that E. Brezin; O. Alvarez and P. Windey; and E. Witten [9] are also aware of the connection between Toda flows and the one-matrix model; and also that an ITEP-Lebedev group [10] has obtained results similar but not identical to our own in both the one- and two-matrix cases. It is likely that many others have also come up with the idea.

2. Toda and the One Matrix Model

The solution to the one matrix model

$$\mathscr{Z} = \int \mathscr{D}\phi e^{-V(\phi)}, \quad V(\phi) = \sum t_k \operatorname{tr}\{\phi^k\} \tag{2.1}$$

proceeds by integration over the $U(N)$ angle variables of ϕ, with the result

$$\mathscr{Z}(t) = \int \prod_{i=1}^{N} d\lambda_i \Delta(\lambda) \exp\left(-\sum_{k,i} t_k \lambda_i^k\right) \Delta(\lambda). \tag{2.2}$$

Here the Jacobian $\Delta(\lambda) = \det(\psi_i(\lambda_j))$ is the antisymmetrized product of monic polynomials $\psi_n(\lambda) = \lambda^n + \cdots$. By choosing the ψ_n to be orthogonal with respect to the measure e^{-V},

$$h_n \delta_{nm} = \int d\lambda \psi_n(\lambda) e^{-V(\lambda)} \psi_m(\lambda), \tag{2.3}$$

the partition function is simply $\mathscr{Z} = \prod_{i=1}^{N} h_i$, and the problem reduces to finding a recursion relation for the h_i. This is readily achieved through consideration of matrix elements of λ and $d/d\lambda$ in the polynomial basis [11],

$$h_n Q_{nm} = \int \psi_n(\lambda) e^{-V(\lambda)} \lambda \psi_m(\lambda), \tag{2.4}$$

$$h_n P_{nm} = \int \psi_n(\lambda) e^{-V(\lambda)} \frac{d}{d\lambda} \psi_m(\lambda). \tag{2.5}$$

Considering different matrix elements one is led to

$$Q = \begin{pmatrix} b_1 & a_1 & & & 0 \\ 1 & b_2 & a_2 & & \\ & 1 & b_3 & a_3 & \\ 0 & & \ddots & \ddots & \ddots \end{pmatrix}. \tag{2.6}$$

Integration by parts in (2.5) yields

$$P = V'(Q)_+, \tag{2.7}$$

where

$$M = M_+ + M_0 + M_- \tag{2.8}$$

are the projections of a matrix M into its upper triangular, diagonal, and lower triangular parts, respectively. The matrix elements $P_{n,n} = 0$, $P_{n,n+1} = n$, when expressed in terms of Q, give recursion relations for $a_i = h_{i+1}/h_i$. In fact we see that these "string equations" may be rewritten

$$[V'(Q)_+, Q] = 1. \tag{2.9}$$

Our first observation is that $V'(Q)_+$ is a Lax conjugate of Q in the one-dimensional $GL(\infty)$ Toda hierarchy [12, 13], one of the simplest discrete integrable systems. The $SL(K)$ Toda chain consists of $2(K-1)$ dynamical variables a_k, b_k, $k = 1, \ldots, K-1$ the entries of a $K \times K$ Jacobi matrix of the form (2.6). The integrability results from the fact that there are $K-1$ quantities $\operatorname{tr}\{Q^k\}$ in involution with respect to the natural symplectic structure

$$\omega = \sum_{j=1}^{K} db_j \wedge \sum_{j \leqq i \leqq K-1} \frac{da_i}{a_i}.$$

These Hamiltonians and their canonically conjugate times are a set of action-angle variables for the system. To each of these conserved quantities is associated a Hamiltonian vector field

$$-\frac{\partial Q}{\partial t_k} = [Q_+^k, Q]. \tag{2.10}$$

The string equation is in fact compatible with Toda flow; taking the t_l derivative of Eq. (2.9) yields

$$\begin{aligned} &[lQ_+^{l-1}, Q] - \sum kt_k[[Q_+^l, Q^{k-1}]_+, Q] - \sum kt_k[Q_+^{k+1}, [Q_+^l, Q]] \\ &= [lQ_+^{l-1}, Q] - \sum kt_k[[Q_+^{k-1}, Q^l]_+, Q] - \sum kt_k[Q_+^l, [Q_+^{k-1}, Q]] \\ &= 0. \end{aligned} \tag{2.11}$$

The first equality expresses the commutativity of the Toda flows, the second employs the string equation.

The Toda system arises as a natural dynamical system on the coadjoint orbits of lower triangular matrices [12, 14–16]. The discussion here is taken from [15, 16]. That is, consider the factorization $G = AB$ of $GL(K)$ into $A =$ upper unipotent matrices and $B =$ lower triangular matrices with nonzero diagonal. The cotangent space $T^*(G)$ is a natural sympletic manifold with a left A-action and right B-action. The symplectic form is $\omega_\nu(\xi_1, \xi_2) = \mathrm{tr}\{\nu[\xi_1, \xi_2]\}$ for $\nu \in \mathbf{g}^*$, $\xi_1, \xi_2 \in \mathbf{g}$ (trace is the natural inner product). Let $\mathcal{O}$ be the orbit of a given point $y \in \mathbf{g}^*$ under the coadjoint action of $\mathbf{b}$, i.e.

$$\mathcal{O}_\mu = \{\nu \in \mathbf{g}^* \,|\, \nu = \mathrm{Ad}_b^* \mu \text{ for some } b \in B\}. \tag{2.12}$$

For us $\mathrm{ad}_b^* \mu = (b\mu b^{-1})_+$ for $(b \in \mathbf{b})$. In particular the coadjoint orbit of

$$\Sigma = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & \ddots \end{pmatrix} \tag{2.13}$$

under the action of B is

$$\begin{pmatrix} b_1 & a_1 & & \\ & b_2 & a_2 & \\ & & \ddots & \ddots \end{pmatrix} \tag{2.14}$$

and the coadjoint orbits generally are matrices whose first m upper diagonals are nonzero. Now consider any invariant Hamiltonian $\mathscr{H}$ on $T^*G \sim G \times \mathbf{g}^*$, i.e. $\mathrm{ad}^*_{(\delta\mathscr{H}/\delta\mu)}(\mu) = 0$. Then we have a dynamics consisting of geodesic motion on G

$$\begin{aligned} &\dot{\mu} = 0 & & \mu(t) = \mu_0 \\ &\dot{g} = T_e R_g \frac{\delta\mathscr{H}}{\delta\mu} & \Rightarrow\quad & g(t) = \exp\left(t \cdot \frac{\delta\mathscr{H}(\mu_0)}{\delta\mu}\right) \cdot g_0 \end{aligned}, \tag{2.15}$$

where T_e is the derivative at the identity, R_g right multiplication by g. The quantity μ_0 is the "momentum" of the geodesic flow; the invariant Hamiltonians are the invariant polynomials on $\mathbf{g}$. Divide this motion by the right B action as well as the part of the A action leaving fixed $\varepsilon = \sum e_{-\alpha}$, the sum of the negative simple roots (for $GL(K)$ ε is the matrix Σ^T with unit lower diagonal); this yields a dynamics on $\varepsilon + \mathbf{a}^\perp$ which is

$$\frac{d}{dt}(\varepsilon + \nu(t)) = \mathrm{ad}^*_{\Pi_{\mathbf{a}}(\delta\mathscr{H}(\varepsilon + \nu(t))/\delta\nu)}(\varepsilon + \nu(t)), \tag{2.16}$$

where $\nu(t)$ is in the span $\mathbf{a}^\perp \cong \mathbf{b}^*$ of the positive roots and the Cartan generators and $\Pi_{\mathbf{a}}$ is the projection onto $\mathbf{a}$. Equation (2.16) is essentially the Lax equation (2.10). Since $g(t) = a(t)b(t)$, the solution to the reduced system is given by

$$\varepsilon + \nu(t) = (\mathrm{Ad}_{a(t)})^*(\varepsilon + \nu_0). \tag{2.17}$$

Thus the prescription for solving the Toda equations is (1) Compute $g(t) = \exp\left(t\frac{\delta\mathscr{H}(\varepsilon + \nu_0)}{\delta\mu}\right)$; (2) Factor $g(t) = a(t)b(t)$; (3) Find $\varepsilon + \nu(t) = (\mathrm{Ad}_{a(t)})^*(\varepsilon + \nu_0)$. For

$G = GL(K)$, the desired factorization is

$$e^{-t_k Q^k} = W_+(t)H(t)W_1(t)$$
$$= \text{upper unipotent} \times \text{diagonal} \times \text{lower unipotent}, \tag{2.18}$$

$$Q(t) = W_+(t)^{-1}Q(0)W_+(t) = H(t)W_-(t)Q(0)W_-(t)^{-1}H(t)^{-1}. \tag{2.19}$$

Note that $W_+ = (W_-)^T$. Effectively one linearizes the problem by solving

$$\frac{\partial W_+}{\partial t_k} = Q_+^k W_+ .$$

W_+ defines a change of basis in $GL(K)$ which diagonalizes g. Now we have

$$h_i(t) = \frac{\Delta_{i+1}(t)}{\Delta_i(t)}, \tag{2.20}$$

where Δ_j is the determinant of the upper left $j \times j$ minor of $e^{-t_k Q^k}$ Note that the product of the first $N h_i$ is simply $\mathscr{Z} = \Delta_N$, and

$$a_i(t) = \frac{\Delta_{i+1}(t)\Delta_{i-1}(t)}{\Delta_i^2(t)} a_i(\hat{t})$$

is a discrete version of the string susceptibility. In the scaling limit it becomes the Painlevé potential.

How does all this relate to matrix models? Suppose we have solved the string equations (2.9) for some values of the couplings $t_k = \hat{t}_k$. Then the partition function at any nearby point in coupling space is given by [3]

$$\mathscr{Z}_N(t) = \langle \Psi_N(\hat{t}) | e^{-\sum(t_k - \hat{t}_k)\hat{Q}^k} | \Psi_N(\hat{t}) \rangle \equiv \tau(t). \tag{2.21}$$

Namely, the infinite-dimensional matrix $\hat{Q}$ is a point in the coadjoint orbit $\hat{Q} = \varepsilon + v_0 \in \mathcal{O}_\Sigma$ of $GL(\infty)$. The "wavefunction" $\Psi_N (= \Delta(\lambda)$ of (2.2)) is simply the highest weight state of the N-fold antisymmetric tensor representation. On the individual wavefunctions λ acts by the matrix Q. Q^k acts in the antisymmetric tensor representation as

$$(Q^k)_{N-AST} = \sum_{i=1}^{N} \overbrace{1 \otimes \cdots \otimes 1 \underbrace{\otimes Q^k \otimes}_{i} 1 \otimes \cdots \otimes 1}^{N},$$

so indeed a change in the potential $\exp(-\sum \delta t_k \lambda_i^k)$ acts on $\Delta(\lambda)$ as in (2.21). The factorization (2.18) is precisely the operation of rediagonalizing the inner product in terms of new monic polynomials $\psi_i(t,\lambda) = [W_-(t,\hat{t})]_{ij}\psi_j(\hat{t},\lambda)$, and the $h_i(t)$ are the ratios of the new norms to the old ones. $\tau(t)$ is a "tau-function" of the Toda hierarchy. One must be careful – a tau function was defined for this system also in [17] which is *not* the same as (2.21), basically because these authors work with a completely different set of L^2 functions. If we choose the potential $V(\hat{t})$ to be gaussian then the diagonalization (2.18) refers the system to the infinite set of functions on $\mathbf{R}^1$ spanned by Hermite polynomials, whereas in [17] the diagonalization refers to the doubly-infinite set of functions on the circle spanned by exponentials $e^{in\theta}$. These two systems do not appear to be related. When we refer to "the" tau function of the Toda hierarchy we will always mean (2.21).

In the standard tau-function formalism [17], a central role is played by the "wavefunction" $w(t,\lambda)=\tau\left(t_n-\frac{1}{n}\lambda^{-n}\right)$. Consider the characteristic polunomial

$$\begin{aligned} w_N(t,\lambda) &= \langle \Psi_N(t)|\det_N(\lambda-Q)|\Psi_N(t)\rangle \\ &= \lambda^N\langle \Psi_N(t)|\exp\left\{\left[\sum_n \frac{1}{n}\left(\frac{Q}{\lambda}\right)^n\right]\right\}|\Psi_N(t)\rangle = \lambda^N\tau\left(t_n-\frac{1}{n}\lambda^{-n}\right). \end{aligned} \tag{2.22}$$

It is the analogue in our context of the wavefunction; its zeros are the expectation values of the eigenvalues of the original matrix ϕ, moreover [11] $w_N(t,\lambda)$ is the N^{th} orthogonal polynomial $\psi_N(t,\lambda)$. Clearly solving for $w_N(t,\lambda)$ for all N is equivalent to determining $W_+(t,\lambda)$. It seems likely that one can adapt the structures of [17] to the present context and obtain a "Hirota form" of the flow equations. Basically one follows the same steps, replacing the matrix representative Σ of the spectral parameter λ in their basis $e^{in\theta}$ of functions on the circle by $Q(\hat{t})$, the one appropriate for our reference basis $\psi(\hat{t},\lambda)$ of functions on the line.

Finally, many of the interesting scaling theories involve matrix potentials for which the integral (2.2) does not exist. It has been proposed [18] to define the partition function by analytic continuation from regions where the theory exists. In the present context this means continuation of the Toda flows to complex time and perhaps complex Jacobi matrices as well. In the finite dimensional $SL(K)$ Toda chain some results appear to be known [16]. The solution for the entries of Q will generically develop poles in $t_k\in\mathbf{C}$ whenever $e^{-t_kQ^k}$ does not allow a W_+HW_- factorization. The geometry is apparently related to the cell decomposition of $G_{\mathbf{C}}/B_{\mathbf{C}}$, where B is the Borel subgroup. It would be interesting to understand the geometry in greater detail and extend the results to $GL(\infty)$.

3. Unitary Matrices and the Quaternionic Toda Hierarchy

A similar story transpires in the unitary one-matrix model

$$\mathscr{Z}=\int\mathscr{D}\phi\exp\{-V[\tfrac{1}{2}(\phi+\phi^\dagger)]-\bar{V}[\tfrac{1}{2}(\phi-\phi^\dagger)]\},\quad V(M)=\sum t_k\operatorname{tr}\{M^k\}. \tag{3.1}$$

Once again eliminating the $U(N)$ angular degrees of freedom,

$$\mathscr{Z}=\oint\prod_{a=1}^N dz_a\bar{\Delta}(z)\exp\left[-\sum_{k,a}(t_kz_{a+}^k+t_k^*z_{a-}^k)\right]\Delta(z). \tag{3.2}$$

Here $z_\pm=\left(z\pm\frac{1}{z}\right)$, $|z|=1$, and the Jacobian Δ is most conveniently organized in terms of the functions [19]

$$\begin{aligned} c_k &= z^{k+1/2}+z^{-k-1/2}+\sum_{j=0}^{k-1}a_{k,j}(z^{j+1/2}+z^{-j-1/2}), \\ s_k &= z^{k+1/2}-z^{-k-1/2}+\sum_{j=0}^{k-1}\bar{a}_{k,j}(z^{j+1/2}-z^{-j-1/2}). \end{aligned} \tag{3.3}$$

Then with $k\in\mathbf{Z}_+$ for $U(2N)$, $k\in\mathbf{Z}_+-\frac{1}{2}$ for $U(2N-1)$, we can represent the Jacobian

Δ as

$$\Delta = \det\begin{bmatrix} c_i(z_j) \\ s_i(z_j) \end{bmatrix} \quad \begin{matrix} i=0,\dots,N-1 \\ j=1,\dots,2N \text{ (or } 2N-1) \end{matrix}.$$

For $U(2N-1), s_0$ vanishes identically and is omitted. The polynomials (3.3) are to be chosen orthogonal with respect to the measure $d\mu = dz e^{-V-\bar{V}}$, i.e.

$$\int d\mu c_i c_j = \bar{h}_j \delta_{ij},$$
$$\int d\mu s_i s_j = h_j \delta_{ij},$$
$$\int d\mu c_i s_j = 0.$$

The advantage of this basis is that it realizes the relevant "canonical" commutation relations

$$[zd, z_\pm] = z_\mp, \quad [z_+, z_-] = 0, \quad z_+^2 - z_-^2 = 4 \tag{3.4}$$

in terms of tridiagonal quaternion matrices:

$$z_+\begin{pmatrix} c_k \\ s_k \end{pmatrix} = [\Sigma^T \otimes 1 + (b_+ \cdot \mathbb{1} \otimes 1 + \tilde{b}_+ \cdot \mathbb{1} \otimes \sigma_3) + (a_+ \cdot \Sigma \otimes 1 + \tilde{a}_+ \cdot \Sigma \otimes \sigma_3)]\begin{pmatrix} c_k \\ s_k \end{pmatrix},$$
$$z_-\begin{pmatrix} c_k \\ s_k \end{pmatrix} = [\Sigma^T \otimes \sigma_1 + (b_- \cdot \mathbb{1} \otimes \sigma_1 + \tilde{b}_- \cdot \mathbb{1} \otimes i\sigma_2) + (a_- \cdot \Sigma \otimes \sigma_1 + \tilde{a}_- \cdot \Sigma \otimes i\sigma_2)]\begin{pmatrix} c_k \\ s_k \end{pmatrix}. \tag{3.5}$$

The notation $a \cdot \Sigma$ means the matrix whose superdiagonal is $(a_1, a_2, \dots)$, $b \cdot \mathbb{1}$ is $\mathrm{diag}(b_1, b_2, \dots)$. Evaluation of various matrix elements of zd yields

$$zd = (k + \tfrac{1}{2})\mathbb{1} \otimes 1 + [V'(z_+)z_-]^+ + [\bar{V}'(z_-)z_+]^+.$$

The string equations are the matrix realization of (3.4), which give recursion relations for the a's and b's. The flows are generated by $(z_+^{l-1} z_-)^+$ for t_l and $(z_-^{l-1} z_+)^+$ for t_l^*, and the upper-triangular/lower-triangular factorization that solves for the partition function takes place in $GL(\infty, \mathbf{Q})$. Thus integrability is also manifest in the unitary one-matrix model. In the scaling limit one finds [20] that unitary matrix models produce a subset of the universality classes of the Hermitian one-matrix model, so perhaps the above is of somewhat less importance than the Toda system of Sect. 2, nevertheless it is interesting to see what types of integrable systems arise in the lattice models themselves. One advantage is that the twofold structure of wavefunctions observed by [21] is apparent here, but appears to be special to this particular model. We might also remark that no clear surface interpretation of these models has yet emerged, casting some doubt on whether the class of matrix models coincides with the space of string theories[1].

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[1] On the other hand, note that it does not seem essential that the asymptotic expansion of the partition function have such an interpretation; if, for instance, there were a matrix model for the superstring its asymptotic expansion could not be related to surfaces since the cosmological constant is not available as a probe of the theory. The diagrammatic vertices would glue together little pieces of "surface," but this surface need not be the string itself embedded in spacetime

4. The Two-Matrix Model and the 2*d* Toda Hierarchy

Finally we come to multimatrix integrals of the form

$$\mathscr{Z} = \int \prod_{m=0}^{M} \mathscr{D}\phi_m \exp[-\sum V_m(\phi_m) + \mathrm{tr}\{\phi_m\phi_{m+1}\}], \quad V(\phi) = \sum t_k^{(m)} \mathrm{tr}\{\phi_m^k\}. \tag{4.1}$$

It is possible to successively integrate over the $U(N)$ angular variables of each matrix [22], with the result

$$\mathscr{Z} = \int \prod_{m=0}^{M} \prod_{a=1}^{N} d\lambda_{a,m} \bar{\Delta}(\lambda_M) \exp\left[- \sum_{k,a,m} t_k^{(m)}(\lambda_{a,m})^k + \lambda_{a,m}\lambda_{a,m+1} \right] \Delta(\lambda_0). \tag{4.2}$$

We now have the freedom to choose the polynomials $\bar{\psi}$ in $\bar{\Delta}$ independently of the ψ in Δ, so as to diagonalize the quadratic form implicit in (4.2),

$$\begin{aligned} h_i\delta_{ij} &= \int \prod_{m=0}^{M} \prod_{a=1}^{N} d\lambda_a^{(m)} \bar{\psi}_i(\lambda_M) \exp\left[- \sum_{k,a,m} t_k^{(m)}(\lambda_{a,m})^k + \lambda_{a,m}\lambda_{a,m+1} \right] \psi_j(\lambda_0) \\ &\equiv \langle \bar{\psi}_i | \psi_j \rangle. \end{aligned} \tag{4.3}$$

As in the previous cases we define the coordinate matrices

$$\langle \bar{\psi}_i | \lambda_l | \psi_j \rangle = h_i Q_{ij}^{(l)} = \bar{Q}_{ij}^{(l)} h_j \tag{4.4}$$

depending on whether we act to the right or to the left. Similarly the derivative operator

$$\begin{aligned} h_i P_{ij}^{(l)} &\equiv \int \bar{\psi}_i(\lambda_M) \exp\left\{ \left[- \sum_{m=l}^{M} V_m(\lambda_m) + \lambda_m\lambda_{m+1} \right] \right\} \frac{\partial}{\partial \lambda_l} \\ &\quad \cdot \exp\left\{ \left[- \sum_{m=0}^{l-1} V_m(\lambda_m) + \lambda_m\lambda_{m+1} \right] \right\} \psi(\lambda_0) \\ &= h_i Q_{ij}^{(l-1)} \quad M \geqq l > 0 \\ &= h_i[-Q^{(l+1)} + V_l'(Q^{(l)})]_{ij} - \delta_{lM} \bar{P}_{ij}^{(M)} h_j \quad \text{all } l. \end{aligned} \tag{4.5}$$

In the last line $\bar{P}^{(M)}$ is the contribution from differentiating $\bar{\psi}$; it is a lower triangular matrix with $1, 2, 3, \ldots$ on the subdiagonal. Similarly $P^{(0)}$ is upper triangular with $1, 2, 3, \ldots$ on the superdiagonal; $\bar{Q}^{(M)}$ has unit superdiagonal and is otherwise lower triangular; and $Q^{(0)}$ has unit subdiagonal and is otherwise upper triangular. Given $P^{(0)}, Q^{(0)}$ satisfying $[P, Q] = 1$, the equations of motion (4.5) guarantee the commutation relations are satisfied for successive matrices along the chain until we reach $P^{(M)}, Q^{(M)}$; however, in contrast to the one-matrix model, $[P^{(0)}, Q^{(0)}] = 1$ alone is insufficient to determine the matrix elements of both operators. Instead one must evolve (4.5) to the other end of the chain, where the Heisenberg relations for $P^{(M)}, Q^{(M)}$ provide sufficient data to solve for the matrix elements recursively. Given one solution, we can perturb each of the matrix potentials to find nearby partition functions; as in the one-matrix model this can be expressed as a matrix element in $GL(\infty)$,

$$\begin{aligned} \mathscr{Z}_N(t) &= \langle \bar{\Psi}_N(\hat{t}) | \exp[-(t_k^{(M)} - \hat{t}_k^{(M)})(\hat{Q}^{(M)})^k] \exp[-(t_k^{(M-1)} - \hat{t}_k^{(M-1)})(\hat{Q}^{(M-1)})^k] \\ &\quad \cdots \exp[-(t_k^{(0)} - \hat{t}_k^{(0)})(\hat{Q}^{(0)})^k] | \Psi_N(\hat{t}) \rangle \\ &\equiv \tau(t). \end{aligned} \tag{4.6}$$

Our goal is to express (4.6) as an integrable system. For the general $(M+1)$-matrix integral this is rather difficult; however we will now show that in the case of the two-matrix model the flows in $t^{(0)} \equiv t$, $t^{(1)} \equiv \bar{t}$ are those of the two-dimensional Toda hierarchy [23, 17] (we believe the $M \geqq 2$ models to be governed by nonlocal conservation laws of the $2d$ Toda hierarchy). Equation (4.5) becomes

$$\begin{aligned} P^{(0)} + Q^{(1)} &= V_0'(Q^{(0)}), \\ H^{-1}\bar{P}^{(1)}H + Q^{(0)} &= V_1'(Q^{(1)}) \end{aligned} \tag{4.7}$$

for $H = \mathrm{diag}(h_i)$. Note also from (4.4) that since $Q^{(1)} = H^{-1}\bar{Q}^{(1)}H$, the Heisenberg algebras of the two pairs of matrices are analogous.

The two-dimensional Toda hierarchy is a set of flows on two matrices $L = (\Sigma^T + \text{upper triangular})$ and $M = H^{-1}(\Sigma + \text{lower triangular})H$; denoting $M_{\sim} \equiv M_0 + M_-$,

$$\begin{aligned} \partial_n L &= [L_+^n, L], & \bar{\partial}_n L &= [M_{\sim}^n, L], \\ \partial_n M &= [L_+^n, M], & \bar{\partial}_n M &= [M_{\sim}^n, M]. \end{aligned} \tag{4.8}$$

The mutual commutativity of these flows is expressed in the flatness conditions

$$\begin{aligned} \partial_n L_+^m - \partial_m L_+^n + [L_+^m, L_+^n] &= 0, \\ \bar{\partial}_n M_{\sim}^m - \bar{\partial}_m M_{\sim}^m + [M_{\sim}^m, M_{\sim}^n] &= 0, \\ \bar{\partial}_n L_+^m - \partial_m M_{\sim}^n + [L_+^m, M_{\sim}^n] &= 0. \end{aligned} \tag{4.9}$$

There is an associated linear problem; one can find an upper triangular matrix W with unit diagonal (the analogue of W_+ of the one-matrix case) and a lower triangular matrix V (the analogue of HW_- of the one-matrix case), such that

$$L(t,\bar{t}) = W(t,\bar{t})L(\hat{t},\hat{\bar{t}})W^{-1}(t,\bar{t}), \quad M(t,\bar{t}) = V(t,\bar{t})M(\hat{t},\hat{\bar{t}})V^{-1}(t,\bar{t}). \tag{4.10}$$

The flow equations for W, V are simply

$$\begin{aligned} \partial_n W &= L_+^n W, & \bar{\partial}_n W &= M_+^n W, \\ \partial_n V &= -L_{\sim}^n V, & \bar{\partial}_n V &= -M_{\sim}^n V. \end{aligned} \tag{4.11}$$

Clearly if we choose as initial conditions $L = Q^{(0)}(\hat{t},\hat{\bar{t}})$, $M = Q^{(1)}(\hat{t},\hat{\bar{t}})$ satisfying the string equations (4.7), then the flow (4.11) preserves the Heisenberg relations $[P^{(0)}, Q^{(0)}] = [Q^{(1)}, H^{-1}P^{(1)}H] = 1$. The derivatives (4.5) are related to the flow generators by considering separately the upper and lower triangular parts of (4.7). Thus the two-matrix model partition function is a tau function of the two-dimensional Toda hierarchy; the matrices W and V provide the desired change of basis to diagonalize the quadratic form at couplings $t, \psi_i(t) = V_{ij}\psi_j(\hat{t}), \bar{\psi}_i(t) = \bar{\psi}_j(\hat{t})W_{ji}^{-1}$; we have

$$\begin{aligned} \mathscr{Z} &= \langle \hat{\bar{\Psi}}_N | \exp[-(\bar{t}_k - \hat{\bar{t}}_k)Q_1^k] \exp[-(t_k - \hat{t}_k)Q_0^k] | \hat{\Psi}_N \rangle \\ &= \langle \hat{\bar{\Psi}}_N | W^{-1}(t,\bar{t})V(t,\bar{t}) | \hat{\Psi}_N \rangle \equiv \tau(t,\bar{t}). \end{aligned} \tag{4.12}$$

We conclude this section with a derivation of (4.12); the proof is directly adapted from section (1.2) of [17]. Define

$$\mathscr{W} = We^{-(\bar{t}-\hat{\bar{t}})\cdot\hat{M}}, \quad \mathscr{V} = Ve^{(t-\hat{t})\cdot\hat{L}}; \tag{4.13}$$

then it is straightforward to show that

$$\begin{aligned}\partial_n \mathscr{V} &= L_+^n \mathscr{V}, \quad \bar{\partial}_n \mathscr{V} = -M_\sim^n \mathscr{V},\\ \partial_n \mathscr{W} &= L_+^n \mathscr{W}, \quad \bar{\partial}_n \mathscr{W} = -M_\sim^n \mathscr{W}.\end{aligned} \tag{4.14}$$

Now note that

$$\begin{aligned}\partial_n \mathscr{V}\cdot\mathscr{V}^{-1} &= \partial_n \mathscr{W}\cdot\mathscr{W}^{-1} = L_+^n,\\ \bar{\partial}_n \mathscr{V}\cdot\mathscr{V}^{-1} &= \bar{\partial}_n \mathscr{W}\cdot\mathscr{W}^{-1} = -M_\sim^n.\end{aligned} \tag{4.15}$$

By induction, $\partial_{n_1}^{\alpha_1}\partial_{n_2}^{\alpha_2}\cdots\bar{\partial}_{m_1}^{\beta_1}\bar{\partial}_{m_2}^{\beta_2}\cdots \equiv \partial_{\vec{n}}^{\vec{\alpha}}\bar{\partial}_{\vec{m}}^{\vec{\beta}}$ acts as

$$\partial_{\vec{n}}^{\vec{\alpha}}\bar{\partial}_{\vec{m}}^{\vec{\beta}}\mathscr{V}\cdot\mathscr{V}^{-1} = \partial_{\vec{n}}^{\vec{\alpha}}\bar{\partial}_{\vec{m}}^{\vec{\beta}}\mathscr{W}\cdot\mathscr{W}^{-1}; \tag{4.16}$$

in other words $\mathscr{W}^{-1}\mathscr{V}(t,\bar{t}) = \mathscr{W}^{-1}\mathscr{V}(\hat{t},\hat{\bar{t}}) = A$, a constant matrix. Thus

$$W^{-1}V = e^{-(\bar{t}-\hat{\bar{t}})\cdot\hat{M}}Ae^{-(t-\hat{t})\cdot\hat{L}} \tag{4.17}$$

as advertised. This factorization is a matrix analogue of the Riemann–Hilbert problem [17].

5. Discussion

At the end of Sect. 2 we noted that the Toda tau function arising in matrix models differs significantly from that of standard free fermion formulations [17]. There do exist a collection of "fermions" [3] whose wavefunctions are the orthogonal polynomials ψ_i, whose ground state is the state in which the partition function is evaluated. But they are not really free in the usual sense; although all correlations can be written in terms of the two-point correlation, their two-point correlation [3] is quite different due to the different Hilbert space. In particular, there is no natural pairing of positive and negative frequency modes – states above and below the Fermi surface – as there is in the usual free fermion tau function. The "bottom" of the Fermi sea is quite different from the "top" of the sea of unfilled levels. We can define the partition function at couplings t given a solution to the string equation at $\hat{t}$, but there is no sensible $\hat{t}\to 0$ limit as there is in [17]. We believe the matrix integral at $\hat{t}$ plays the role of the "star operator" of recent continuum formulations [21, 5], perhaps in a somewhat more concrete realization.

There is also an analogy to the work of [24 and 25]; the reference theory $\hat{t}$ is arbitrary. The shift is accomplished by Eq. (2.19),

$$Q(t) = W_+(t,\hat{t})^{-1}Q(\hat{t})W_+(t,\hat{t}) = H(t,\hat{t})W_-(t,\hat{t})Q(\hat{t})W_-(t,\hat{t})^{-1}H(t,\hat{t})^{-1},$$

which allows us to rewrite the theory in terms of perturbations about couplings t. There is no "background independent" formulation, since the point with vanishing matrix potential is highly singular. Rather there is a kind of affine structure: the space of theories coupled to $2d$ gravity can, for low c, be formulated about any convenient point, with no preferred coordinate origin (except perhaps the trivial theory $c = -2$, but only because the calculations are simplest there). Our formulation of the space of minimal models coupled to gravity justifies the scenario of [25]. We have found a space (the Grassmannian $\mathrm{Gr}(N,\infty)$) in which the theory space is embedded linearly; we introduce perturbations by exponentiating local operators; the string equation can be thought of as a kind of renormalization group equation, showing how the theory flows if we change the mass scale (cosmological constant, N). Integrability means that the flow in the coupling space can be made linear; the nonlinearity enters when we try to eliminate $\hat{t}$ or change N. The linear embedding in $GL(\infty)$ helps explain why topological gravity is related to dynamical gravity. Topological gravity enumerates each point on the surface as a part of the moduli space of punctures. The fact that we can condense punctures to make physical surfaces of finite extent results from the possibility to exponentiate the "puncture operator" and flow to the dynamical gravity critical point.

The flows may also be linearized directly in the phase space of Jacobi matrices; the linearization is transparent in action-angle variables. The action variables are the eigenvalues (or their symmetric polynomials); the angle variables are the coordinates on the Jacobian of the spectral curve $\mathbf{C}-\{\lambda_i\}$. Moser [12] has given a description of these angle variables; for the $SL(K)$ Toda system, consider the matrix element of the resolvent

$$\begin{aligned} f(\lambda) &= (\lambda - Q)^{-1}_{K,K} = \frac{\psi_{K-1}(\lambda)}{\psi_K(\lambda)} \\ &\equiv \sum_{i=1}^{K} \frac{r_i^2}{\lambda-\lambda_i}. \end{aligned} \tag{5.1}$$

One can show that $\dfrac{d}{dt_k}\log r_i = \lambda_i^k$, thus the $\log r_i$ are the desired coordinates. The flow is isospectral; a_i, b_i are rational functions of r_i, λ_i. The isospectral nature of the flows is rather curious; it cannot survive the $K \to \infty$ limit and the analytic continuations that, for instance, take single cut distributions to multiple cut distributions of eigenvalues.

Much has been made recently of the fact that the continuum matrix model tau function satisfies the Virasoro constraints $L_n\tau = 0$, $n = -1, 0, 1, \ldots$. The same is true at the discrete level; the moments of the one-matrix loop equation [18]

$$V'(\lambda)\left\langle \operatorname{tr}\left[\frac{1}{\lambda-\phi}\right]\right\rangle = \left\langle \operatorname{tr}\left[\frac{1}{\lambda-\phi}\right]\operatorname{tr}\left[\frac{1}{\lambda-\phi}\right]\right\rangle - \text{subtractions} \tag{5.2}$$

are precisely the Virasoro constraints. We pick out L_n by contour integration around $|\lambda| = \infty$ against λ^{n-1}, expressing the result in terms of the t_k and their derivatives:

$$\left[\sum_{k=0}^{\infty} k t_k \frac{\partial}{\partial t_{k+n}} - \sum_{k=0}^{n} \frac{\partial}{\partial t_k}\frac{\partial}{\partial t_{n-k}}\right]\tau(t) \equiv L_n\tau = 0. \tag{5.3}$$

In other words (5.2) is the stress tensor $T(\lambda)$. The first two constraints L_{-1}, L_0 are the string equations $P_{N,N} = 0$, $P_{N,N+1} = N$. It would be interesting to extend this analysis to the loop equations of the multimatrix case.

There is also a connection to the work of [21], in that the string equations and flow equations may be thought of as the compatibility conditions for the triplet of operators

$$\lambda - Q, \quad \frac{d}{d\lambda} - V'(Q)_+, \quad \frac{d}{dt_k} - Q^k_+ \tag{5.4}$$

acting on the wavefunctions $\psi_i(t, \lambda)$. We have not so far been able to find evidence at the discrete level of the finite-dimensional vector bundle which plays such a crucial role in [21] and which seems to be related to the Dynkin diagram of the corresponding matter theory which is being coupled to gravity. One obvious place to look is the difference equation $Q\psi = \lambda\psi$; when Q is order r, where order is defined to be the number of nonvanishing diagonals, one needs r initial conditions to specify a solution.

In Sect. 4 we specialized to the two-matrix model mostly for reasons of simplicity. However this does not appear to impose any restriction on the class of minimal model critical points that can be studied. In the one-matrix model we are limited by the fact that Q is always a tridiagonal matrix, and so is related in the scaling limit to a second-order differential operator (the Toda flows scale to the KdV flows, the upper/lower triangular decomposition becomes the differential/integral splitting of pseudodifferential operators, etc.). In the two matrix model this restriction is lifted; the orders of $Q^{(1)}, Q^{(0)}$ are determined from (4.7) by the degree of V_0, V_1, respectively. These orders are the discrete analogues of the orders of the continuum differential operators $\mathscr{P}, \mathscr{Q}$ of [4]. We have enough parameters to tune the scaling limit so as to find any Lax pair of the KP hierarchy, since both Q and P may have as high an order as we like (or as low; we recover the one-matrix universality classes when one of the two matrices has gaussian potential). The ability to find any $c<1$ theory in the two-matrix model is rather reminiscent of quenched large-N systems of [26]. There the spacetime degrees of freedom were nontrivially embedded in the angle degrees of freedom of the matrices, and infinite spacetime properties reproduced on a hypercube of one lattice spacing. It appears here that the angle degrees of freedom may be effectively quenched; since all relevant operators of the discrete series are written in terms of the eigenvalue distribution, the M nontrivial $U(N)$ matrix probability distributions are irrelevant degrees of freedom and should be dynamically determined by the eigenvalue distribution.

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