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$(Z_N \times)^{n-1}$ Generalization of the Chiral Potts Model

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Abstract. We show that the R-matrix which intertwines two n-by-N^{n-1} state cyclic L-operators related with a generalization of $U_q(sl(n))$ algebra can be considered as a Boltzmann weight of four-spin box for a lattice model with two-spin interaction just as the R-matrix of the checkerboard chiral Potts model. The rapidity variables lie on the algebraic curve of the genus $g = N^{2(n-1)}((n-1)N-n)+1$ defined by $2n-3$ independent moduli. This curve is a natural generalization of the curve which appeared in the chiral Potts model. Factorization properties of the L-operator and its connection to the SOS models are also discussed.

0. Introduction

As it is observed in [1] the chiral Potts model [2–4] can be considered as a part of some new algebraic structure related to the six vertex R-matrix. In particular, the high genus algebraic relations among the Boltzmann weights of the chiral Potts model arise as a condition of the existence of an intertwining operator for two different representations of some quadratic Hopf algebra [5–7] which generalizes the $U_q(sl(2))$ algebra.

It is natural to make an attempt to find new solvable lattice models whose Boltzmann weights obey high genus algebraic relations generalizing the results of [1] for the case of other R-matrices.

This program for the case of the three state R-matrix of [8] which is related to the $U_q(sl(3))$ algebra at $q^{2N}=1$ has been partially realized in [9, 10].

In the present paper we extend the result of [9, 10]. We construct an n-by-$N^{(n-1)}$ state cyclic L-operator related with an n-state R-matrix of [8] and find explicitly the corresponding $N^{(n-1)}$-state R-matrix. This result is described below.

Consider an oriented square lattice $\mathscr{L}$ and its medial lattice $\mathscr{L}'$ (shown in Fig. 1 by solid and dashed lines, respectively). The oriented vertical (horizontal) lines of $\mathscr{L}'$ carry rapidity variables p, p' (q, q') in alternating order (note that the orientations of rapidity lines shown by open arrows alternate, too). The edges of

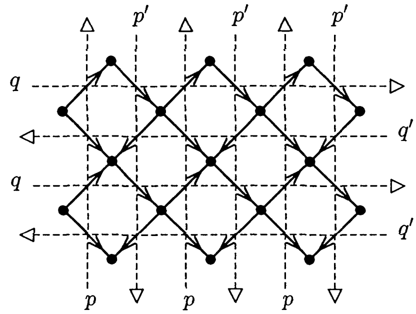


Fig. 1

the lattice $\mathscr{L}$ are oriented in such a way that all the $NW-SE$ edges have the same ($NW-SE$) direction while the $NE-SW$ edges are oriented in a checkerboard order.

Each rapidity variable p is represented by n 2-vectors $(h_\alpha^+(p), h_\alpha^-(p))$, $\alpha=1,\dots,n$, $n\geqq 2$ which specify a point of the algebraic curve defined by the relations

$$\begin{pmatrix} h_\alpha^+(p)^N \\ h_\alpha^-(p)^N \end{pmatrix} = K_{\alpha\beta} \begin{pmatrix} h_\beta^+(p)^N \\ h_\beta^-(p)^N \end{pmatrix}, \quad \forall \alpha, \beta \in Z_n, \tag{0.1}$$

where $K_{\alpha\beta}$ are 2×2 complex matrices of moduli with the unit determinant satisfying the relations

$$K_{\alpha\alpha} = K_{\alpha\beta} K_{\beta\gamma} K_{\gamma\alpha} = 1\,. \tag{0.2}$$

There are $2n-3$ independent moduli, since the variables $h_\alpha^\pm(p)$ entering (0.1) are defined up to a gauge transformation

$$\begin{pmatrix} h_\alpha^+(p) \\ h_\alpha^-(p) \end{pmatrix} \to U_\alpha \begin{pmatrix} h_\alpha^+(p) \\ h_\alpha^-(p) \end{pmatrix}, \quad K_{\alpha\beta} \to U_\alpha^N K_{\alpha\beta} U_\beta^{-N},$$

where $U_\alpha = \mathrm{diag}(u_\alpha, u_\alpha^{-1})$ which does not affect the Boltzmann weights [see Eq. (0.5), below]. This curve is a natural generalization of the curve which appeared in the chiral Potts model [4]. Using the Riemann-Hurwitz formula one can calculate the genus g of the curve (0.1) in a generic case:

$$g = N^{2(n-1)}((n-1)N-n)+1\,.$$

On each site of the lattice $\mathscr{L}$ place $(n-1)$ Z_N-spins which are described by a local variable

$$m = (m_1, \dots, m_n)\,, \quad m_\alpha = 0, 1, \dots, N-1\,, \quad \alpha = 1, \dots, n\,, \quad N \geqq 2 \tag{0.3a}$$

with the identification:

$$k \sim m \Leftrightarrow k_\alpha - m_\alpha = k_\beta - m_\beta \pmod{N}\text{:} \quad \forall \alpha, \beta = 1, \dots, n\,. \tag{0.3b}$$

Then there are only two kinds of neighbouring local state pairs depending on the relative orientation of the dashed and solid lines as indicated in Fig. 2, with states l and m, and Boltzmann weights $\overline{W}_{pq}(l,m)$ and $(\overline{W}_{qp}(l,m))^{-1}$ on the edges of $\mathscr{L}$. The arrow from l to m indicates that the argument is (l,m) rather than (m,l).

It is convenient to denote by Z_n (Z_N) the set of integer residues *modulo* n (N). Besides, the set of local variables with identification (0.3b) will be denoted as AZ_N^n.

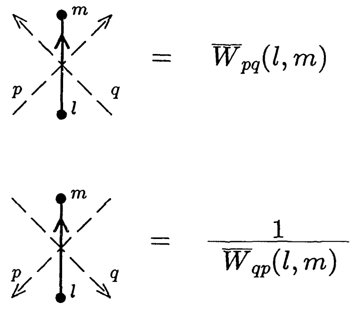


Fig. 2

To write down $\overline{W}_{pq}(l,m)$ introduce the function $g_{pq}(l,m)$, $l,m \in AZ_N^n$ which satisfies the relations

$$g_{pq}(k,m) = g_{pq}(k,l)g_{pq}(l,m)\,, \qquad \forall k,l,m \in AZ_N^n\,, \tag{0.4a}$$

$$g_{pq}(m,m) = 1\,, \qquad \forall m \in AZ_N^n\,. \tag{0.4b}$$

Then it is unambiguously defined by the following relations:

$$g_{pq}(m, m+\delta_\alpha) = \frac{h_{\alpha-1}^+(p)h_{\alpha-1}^-(q) - h_{\alpha-1}^-(p)h_{\alpha-1}^+(q)\omega^{m_{\alpha-1,\alpha}}}{h_\alpha^+(p)h_\alpha^-(q) - h_\alpha^-(p)h_\alpha^+(q)\omega^{1+m_{\alpha,\alpha+1}}}\,, \tag{0.5}$$

the symbol δ_α means a unit vector in the α^{th} direction, i.e., all its components vanish except the value 1 in the α^{th} place; $\omega = \exp(2\pi i/N)$; $m_{\alpha,\beta} \equiv m_\alpha - m_\beta$.

The Boltzmann weight $\overline{W}_{pq}(l,m)$ has the form

$$\frac{\overline{W}_{pq}(l,m)}{\overline{W}_{pq}(0,0)} = \omega^{Q(l,m)} g_{pq}(0, l-m)\,, \tag{0.6}$$

where

$$Q(l,m) = \sum_{\alpha \in Z_n} m_{\alpha-1,\alpha}(l_\alpha - m_\alpha)\,. \tag{0.7}$$

One can show that

$$\sum_l \overline{W}_{pq}(k,l)\overline{W}_{qp}(l,m) = \bar{\delta}_{k,m}\Phi_{pq}\,, \tag{0.8}$$

where

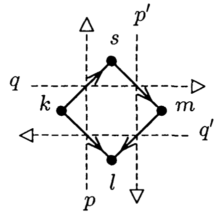
$$\bar{\delta}_{k,m} \equiv \begin{cases} 1\,, & k = m \pmod N\,; \\ 0\,, & \text{otherwise}\,, \end{cases} \tag{0.9}$$

$$\Phi_{pq} = N^{n-1}\overline{W}_{pq}(0,0)\overline{W}_{qp}(0,0)\frac{t(p)^N - t(q)^N}{t(p) - t(q)} \prod_{\alpha \in Z_n} \frac{x_\alpha(p) - x_\alpha(q)}{x_\alpha(p)^N - x_\alpha(q)^N}\,, \tag{0.10}$$

$$x_\alpha(p) \equiv h_\alpha^-(p)/h_\alpha^+(p)\,, \qquad t(p) \equiv \prod_{\alpha \in Z_n} x_\alpha(p)\,. \tag{0.11}$$

The above-mentioned R-matrix which intertwines two L-operators is just the Boltzmann weight of an elementary box of the lattice $\mathscr{L}$ shown in Fig. 3:

$$\langle k, l | S(q, q'; p, p') | m, s \rangle = \frac{\overline{W}_{qp}(k,s)\overline{W}_{p'q}(s,m)\overline{W}_{q'p'}(m,l)}{\overline{W}_{q'p}(k,l)}\,. \tag{0.12}$$



$$= \quad S_{k,l}^{m,s}(q,q';p,p')$$

Fig. 3

It is $(Z_N\times)^{n-1}$-invariant in the sense that

$$\begin{aligned}&\langle k+\delta_\alpha, l+\delta_\alpha|S(q,q';p,p')|m+\delta_\alpha, s+\delta_\alpha\rangle\\&=\langle k,l|S(q,q';p,p')|m,s\rangle\,,\qquad \alpha\in Z_n\,,\end{aligned}\tag{0.13}$$

which makes our lattice model $(Z_N\times)^{n-1}$-symmetric. This R-matrix satisfies the following inversion relation:

$$S_{12}(p,q';q,q')S_{21}(q,q';p,p')=\Phi_{pq}\Phi_{p'q'}I_{12}\,,\tag{0.14}$$

where I_{12} is the identity matrix. Besides, R-matrix (0.12) should satisfy the Yang-Baxter equation

$$\begin{aligned}&S_{12}(p,p';q,q')S_{13}(p,p';r,r')S_{23}(q,q';r,r')\\&=S_{23}(q,q';r,r')S_{13}(p,p';r,r')S_{12}(p,p';q,q')\,.\end{aligned}\tag{0.15}$$

We verified this equation numerically at $n=3$, $N=2$. Up to the moment we have not yet proved this equation analytically and claim it as a conjecture.

We notice that a particular form of the R-matrix just described has been obtained in [10] for the case $n=3$. We also remark that in the trigonometric limit $K_{\alpha\beta}\to 1$ (identity matrix) the R-matrix (0.12) at $n=3$ and $N=2$ is equivalent to that found in [11]. In the same limit we expect the 9-state R-matrix found in [12] to be equivalent to the R-matrix (0.12) at $n=N=3$.

1. The L-Operator

In this section we construct the particular cyclic (i.e. with no highest weight vector) L-operator related with the n-state R-matrix of [8].

Let $L(x)$ be an operator in $C^n\otimes C^M$, satisfying the following Yang-Baxter equation (YBE) represented in Figs. 4 and 5:

$$\begin{aligned}&\sum_{\alpha'',\beta'',l}R_{\alpha\alpha'',\beta\beta''}(x/y)L_{\alpha''\alpha',kl}(x)L_{\beta''\beta',lm}(y)\\&=\sum_{\alpha'',\beta'',l}L_{\beta\beta'',kl}(y)L_{\alpha\alpha'',lm}(x)R_{\alpha''\alpha',\beta''\beta'}(x/y)\,,\end{aligned}\tag{1.1}$$

where $L_{\alpha\beta,kl}(x)$, $\alpha,\beta=1,\dots,n$, $k,l=1,\dots,M$, denote the matrix elements of $L(x)$ and $R_{\alpha\beta,\gamma\delta}(x)$ is defined explicitly as [8]:

$$\begin{aligned}&R_{\alpha\beta,\gamma\delta}(x)=R_{\alpha\beta,\gamma\delta}(x,q,\varrho)\\&=\delta_{\alpha\beta}\delta_{\gamma\delta}\delta_{\alpha\gamma}(q-1)(x+x^{-1}q^{-1})+\delta_{\alpha\beta}\delta_{\gamma\delta}\varrho_{\alpha\gamma}(x-x^{-1})+\delta_{\alpha\delta}\delta_{\beta\gamma}\sigma_{\alpha\beta}\,,\end{aligned}\tag{1.2}$$

where $\delta_{\alpha\beta}$ is the Kroneker symbol, $\varrho_{\alpha\beta}$ are nonzero complex parameters such that

$$\varrho_{\alpha\alpha}=\varrho_{\alpha\beta}\varrho_{\beta\alpha}=1\,,\qquad \forall\alpha,\beta=1,\dots,n\tag{1.3}$$

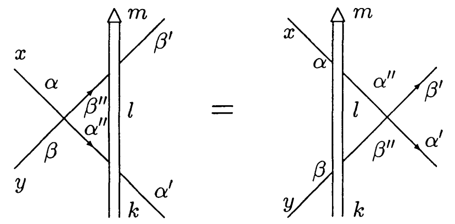


Fig. 4

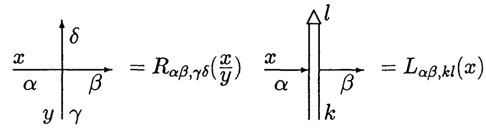


Fig. 5

and

$$\sigma_{\alpha\beta} \equiv \begin{cases} 0, & \alpha=\beta; \\ (q-q^{-1})x, & \alpha<\beta; \\ (q-q^{-1})x^{-1}, & \alpha>\beta. \end{cases} \tag{1.4}$$

Here x is a variable, while $q, \varrho_{\alpha\beta}$ are considered as constants.

Note, that the R-matrix has the following explicit dependence on the variable x:

$$R(x)=xR^{+}+x^{-1}R^{-}, \tag{1.5}$$

where R^{+} and R^{-} are independent of x. If one searches for an L-operator of a similar form

$$L(x)=xL^{+}+x^{-1}L^{-}, \tag{1.6}$$

where L^{+} (L^{-}) is independent of x, then using (1.2)–(1.6) one can reduce Eq. (1.1) to the following relations:

$$R_{12}^{-}L_1^{\pm}L_2^{\pm}=L_2^{\pm}L_1^{\pm}R_{12}^{-}, \qquad R_{12}^{-}L_1^{-}L_2^{+}=L_2^{+}L_1^{-}R_{12}^{-}, \tag{1.7}$$

where we use the standard matrix notations. Hereafter, we assume that L^{+} (L^{-}) has an upper (lower) triangular form as a matrix in C^n. Relations (1.7) can be written explicitly as

$$[L_{\alpha\alpha}^{\pm}, L_{\beta\beta}^{\pm}]=[L_{\alpha\alpha}^{+}, L_{\beta\beta}^{-}]=0, \tag{1.8a}$$

$$L_{\alpha\alpha}^{\pm}L_{\beta\gamma}=q^{\mp\delta_{\alpha\beta}\pm\delta_{\alpha\gamma}}\varrho_{\beta\alpha}\varrho_{\alpha\gamma}L_{\beta\gamma}L_{\alpha\alpha}^{\pm}, \tag{1.8b}$$

$$L_{\alpha\beta}L_{\beta\alpha}\varrho_{\alpha\beta}-L_{\beta\alpha}L_{\alpha\beta}\varrho_{\beta\alpha}=(q-q^{-1})(L_{\alpha\alpha}^{+}L_{\beta\beta}^{-}-L_{\beta\beta}^{+}L_{\alpha\alpha}^{-}), \tag{1.8c}$$

$$L_{\alpha\beta}L_{\alpha\gamma}=q^{-\varepsilon_{\alpha\beta\gamma}}\varrho_{\beta\gamma}L_{\alpha\gamma}L_{\alpha\beta}, \quad n>2, \tag{1.8d}$$

$$L_{\alpha\gamma}L_{\beta\gamma}=q^{-\varepsilon_{\alpha\beta\gamma}}\varrho_{\beta\alpha}L_{\beta\gamma}L_{\alpha\gamma}, \quad n>2, \tag{1.8e}$$

$$L_{\alpha\beta}L_{\beta\gamma}\varrho_{\alpha\beta}-L_{\beta\gamma}L_{\alpha\beta}\varrho_{\beta\gamma}=-\varepsilon_{\alpha\beta\gamma}(q-q^{-1})L_{\beta\beta}^{\varepsilon_{\alpha\beta\gamma}}L_{\alpha\gamma}, \quad n>2, \tag{1.8f}$$

$$L_{\alpha\beta}L_{\gamma\delta}\varrho_{\alpha\gamma}-L_{\gamma\delta}L_{\alpha\beta}\varrho_{\beta\delta}=(q^{\varepsilon_{\alpha\beta\gamma}}-q^{\varepsilon_{\alpha\beta\gamma}})L_{\gamma\beta}L_{\alpha\delta}, \quad n>3, \tag{1.8g}$$

where $\alpha, \beta, \gamma, \delta$ in (1.8d)–(1.8g) do not coincide and $\varepsilon_{\alpha\beta\gamma}$ is the antisymmetric in α, β, γ symbol such that

$$\varepsilon_{\alpha\beta\gamma}=1, \quad \text{if} \quad \alpha<\beta<\gamma, \tag{1.9}$$

and

$$L_{\alpha\beta} \equiv \begin{cases} L^{+}_{\alpha\beta}, & \alpha<\beta; \\ L^{-}_{\alpha\beta}, & \alpha>\beta. \end{cases} \tag{1.10}$$

These relations can be considered as a definition of some quadratic Hopf algebra [5–7] with co-multiplication $\Delta L^{\pm}_{\alpha\beta} \equiv \sum_{\gamma} L^{\pm}_{\alpha\gamma} \otimes L^{\pm}_{\gamma\beta}$, which generalizes the $U_q(sl(n))$ algebra [7].

In this paper we restrict ourselves to a special irreducible cyclic representation of this algebra which gives a factorizable (in the sense that will be explained in the next section) L-operator. Such a representation exists provided that

$$q^2 \equiv \omega \equiv \exp(2\pi i/N), \tag{1.11a}$$

$$\varrho_{\alpha\beta} = q^{\varepsilon_{\alpha\beta}} \omega^{s_\alpha - s_\beta}, \tag{1.11b}$$

with $N \geqq 2$, s_α being some arbitrary integers and

$$\varepsilon_{\alpha\beta} \equiv \begin{cases} 1, & \alpha>\beta; \\ 0, & \alpha=\beta; \\ -1, & \alpha<\beta. \end{cases} \tag{1.12}$$

To write down the corresponding formulae for our L-operator, let us define the operators X_α, $Z_{\alpha\beta}$, $\alpha, \beta = 1, \ldots, n$ by the following relations:

$$[X_\alpha, X_\beta] = [Z_{\alpha\beta}, Z_{\gamma\delta}] = 0, \tag{1.13a}$$

$$X_\alpha Z_{\beta\gamma} = \omega^{\delta_{\alpha\beta} - \delta_{\alpha\gamma}} Z_{\beta\gamma} X_\alpha, \tag{1.13b}$$

$$Z_{\alpha\alpha} = Z_{\alpha\beta} Z_{\beta\gamma} Z_{\gamma\alpha} = X_1 X_2 \ldots X_n = 1, \tag{1.13c}$$

$$X_\alpha^N = Z_{\alpha\beta}^N = 1. \tag{1.13d}$$

Then we have

$$L^{+}_{\alpha\alpha} = \xi^{-1} v^{+}_{\alpha} Y_\alpha, \qquad L^{-}_{\alpha\alpha} = \xi v^{-}_{\alpha-1} Y_{\alpha-1}, \tag{1.14a}$$

$$L_{\alpha\beta} = \xi^{\varepsilon_{\alpha\beta}} t_\alpha^{-1} t_\beta Z_{\alpha\beta} (v^{+}_{\beta} Y_\beta + v^{-}_{\beta-1} Y_{\beta-1}), \qquad \alpha \neq \beta, \tag{1.14b}$$

where the indices α, β run over $1, \ldots, n$ *modulo n* in the sense that zero is identified with n (so that, e.g., $\varepsilon_{0,1} = \varepsilon_{n,1} = +1$),

$$Y_\alpha = Y_n \prod_{\beta=\alpha+1}^{n} X_\beta, \qquad \alpha = 1, \ldots, n-1, \qquad Y_n = \prod_{\beta=1}^{n} X_\beta^{s_\beta} \tag{1.15}$$

and $\xi, v^{\pm}_{\alpha}, t_\alpha$ are non-zero complex parameters.

The operators $X_\alpha, Z_{\alpha\beta}$ satisfying (1.13) can be realized in C^M with $M = N^{(n-1)}$ as follows:

$$\langle l | X_\alpha | m \rangle = \bar{\delta}_{l, m - \delta_\alpha}, \qquad \alpha = 1, \ldots, n, \tag{1.16a}$$

$$\langle l | Z_{\alpha\beta} | m \rangle = \omega^{m_\alpha - m_\beta} \bar{\delta}_{l,m}, \qquad \alpha, \beta = 1, \ldots, n. \tag{1.16b}$$

Here we use Dirac's notations for bra- and ket-vectors with n component indices ($m \equiv (m_1, \ldots, m_n)$, $m_\alpha = 0, \ldots, N-1 \pmod N$) identifying their values by the following rule:

$$k \sim m \Leftrightarrow k_\alpha - m_\alpha = k_\beta - m_\beta, \quad (\operatorname{mod} N) \quad \forall \alpha, \beta = 1, \ldots, n'. \tag{1.17}$$

The symbol δ_α means the unit vector in the α^{th} direction, i.e., all its components vanish except the value 1 in the α^{th} place and

$$\bar{\delta}_{l,m} \equiv \begin{cases} 1, & l=m \pmod N; \\ 0, & \text{otherwise}. \end{cases} \tag{1.18}$$

We will find it convenient to use the following notation:

$$m_{\alpha,\beta} \equiv m_\alpha - m_\beta, \qquad \alpha,\beta=1,\dots,n. \tag{1.19}$$

2. Factorization of L-Operator and Cyclic SOS Model

In this section we consider factorization properties of the L-operator (1.14) and discuss its connection with a particular case of the cyclic $sl(n)$ SOS model of [13].

The L-operator (1.14) factorizes in the following sense. Let $E_{\alpha\beta}$, $\alpha,\beta=1,\dots,n$ be a basis of n-by-n matrices with the elements

$$(E_{\alpha\beta})_{\gamma\delta} = \delta_{\alpha\gamma}\delta_{\beta\delta}, \qquad \alpha,\beta,\gamma,\delta=1,\dots,n. \tag{2.1}$$

With the aid of $E_{\alpha\beta}$'s the L-operator can be written in a matrix form:

$$L(x) = \sum_{\alpha,\beta=1}^{n} (xL^+_{\alpha\beta} + x^{-1}L^-_{\alpha\beta})E_{\alpha\beta}. \tag{2.2}$$

Substituting (1.14) in (2.2) and collecting the coefficients of Y_α we rewrite (2.2) as

$$L(x) \equiv \sum_{\alpha=1}^{n} L^{(\alpha)}(x) Y_\alpha, \tag{2.3}$$

where

$$\begin{aligned} L^{(\alpha)}(x) = {} & \sum_{\beta=1}^{n} (\xi/x)^{\varepsilon^-_{\beta\alpha}} t_\beta^{-1} t_\alpha v_\alpha^+ Z_{\beta\alpha} E_{\beta\alpha} \\ & + \sum_{\beta=1}^{n} (x/\xi)^{\varepsilon^-_{\alpha+1,\beta}} t_\beta^{-1} t_{\alpha+1} v_\alpha^- Z_{\beta,\alpha+1} E_{\beta,\alpha+1}, \end{aligned} \tag{2.4}$$

where

$$\varepsilon^-_{\alpha\beta} \equiv \begin{cases} 1, & \alpha>\beta; \\ -1, & \alpha \leqq \beta. \end{cases} \tag{2.5}$$

Note that the matrices $E_{\alpha\beta}$ can be written in a product form:

$$E_{\alpha\beta} = E_\alpha E^t_\beta, \tag{2.6}$$

with E_α being an n-component vector with elements

$$(E_\alpha)_\beta = \delta_{\alpha\beta}, \qquad \beta=1,\dots,n \tag{2.7}$$

which is thought to be a column $n \times 1$ matrix and E^t_α is the corresponding row $1 \times n$ matrix.

Now by using of (2.6) formula (2.4) takes the following factorized form:

$$\begin{aligned} L^{(\alpha)}(x) = {} & \left\{ \sum_{\beta=1}^{n} (\xi/x)^{\varepsilon^-_{\beta\alpha}} t_\beta^{-1} Z_{\beta\alpha} E_\beta \right\} \\ & \times \{ t_\alpha v^+_\alpha E^t_\alpha + (\xi/x)^{2\delta_{n\alpha}} t_{\alpha+1} v^-_\alpha Z_{\alpha,\alpha+1} E^t_{\alpha+1} \}. \end{aligned} \tag{2.8}$$

Formula (2.8) means that

$$\operatorname{rank} L^{(\alpha)}(x) = 1\,, \qquad \alpha = 1, \ldots, n\,, \tag{2.9}$$

where $L^{(\alpha)}(x)$ is considered as an $n \times n$ matrix.

Motivated by this observation we will define below new objects having a natural graphical representation.

By Z_n (Z_N) we will denote the set of integer residues *modulo* n (N). For Z_n we also fix the order of the elements: $1 < 2 < \ldots < (n-1) < n \equiv 0$. Besides, the set of integer n-component vectors with identification (1.17), introduced in Sect. 1, will be denoted as AZ_N^n.

Define the vectors $e_\alpha \in AZ_N^n$, $\alpha \in Z_n$ by

$$Y_\alpha |m\rangle = |m - e_\alpha\rangle\,, \tag{2.10}$$

where the operators Y_α are defined by (1.15). Consider the following "three spin interaction" weights $(l, m \in AZ_N^n,\ \alpha, \beta \in Z_n)$:

$$\psi_{l,\alpha}^{m}(x,h) = \begin{cases} \omega^{\langle v_\beta, l\rangle + l_{\beta,\alpha}} \left(x^{\delta_{n\beta}} h_\beta^+ \delta_{\beta\alpha} - x^{-\delta_{n\beta}} h_\beta^- \delta_{\beta,\alpha-1}\right), & \text{if} \quad m = l + e_\beta\,; \\ 0\,, & \text{otherwise}\,, \end{cases} \tag{2.11}$$

where ω is defined in (1.11a); $h \equiv \{h_\alpha^\pm\}$, $\alpha \in Z_n$, and x are non-zero complex parameters; v_α, $\alpha \in Z_n$ are some linear forms on index vectors which will be defined in Sect. 3.

Now, introduce "inverse" weights $\bar\psi_{l,\alpha}^{m}(x,h)$ defined by any of two relations

$$\sum_{m \in AZ_N^n} \bar\psi_{l,\alpha}^{m}(x,h)\, \psi_{l,\beta}^{m}(x,h) = \delta_{\alpha\beta}\,, \tag{2.12a}$$

$$\sum_{\alpha \in Z_n} \psi_{k,\alpha}^{l}(x,h)\, \bar\psi_{k,\alpha}^{m}(x,h) = \bar\delta_{l,m}\,, \tag{2.12b}$$

each of them is a consequence of the other (see Figs. 6 and 7 for graphical representation of these relations). Such inverse weights exist provided

$$\Delta(x,h) \equiv x \prod_{\alpha \in Z_n} h_\alpha^+ - x^{-1} \prod_{\alpha \in Z_n} h_\alpha^- \neq 0\,. \tag{2.13}$$

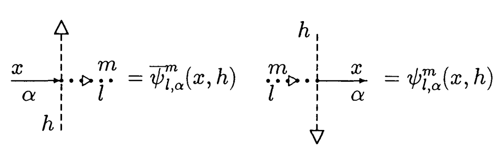


Fig. 6

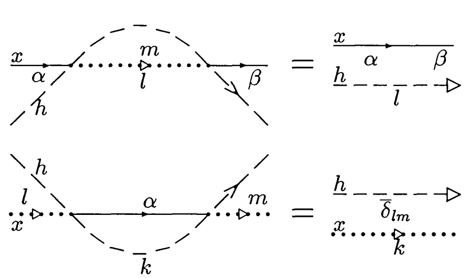
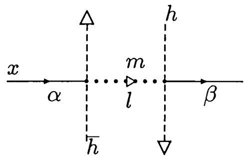


Fig. 7



$$= \mathscr{L}_{\alpha\beta,lm}(x,\bar{h},h)$$

Fig. 8

Explicitly, we have

$$\bar{\psi}^m_{l,\alpha}(x,h)=\begin{cases}\dfrac{\omega^{-\langle v_\beta,l\rangle+l_{\alpha,\beta}}}{\Delta(x,h)}\prod\limits_{\gamma=\beta+1}^{\alpha-1}(x^{\delta_{n\gamma}}h^+_\gamma)\prod\limits_{\gamma=\alpha}^{\beta-1}(x^{-\delta_{n\gamma}}h^-_\gamma), & m=l+e_\beta;\\ 0, & \text{otherwise},\end{cases} \tag{2.14}$$

where the products are taken in Z_n.

Consider the following expression (Fig. 8):

$$\mathscr{L}_{\alpha\beta,lm}(x,\bar{h},h)=\bar{\psi}^m_{l,\alpha}(x,\bar{h})\psi^m_{l,\beta}(x,h), \tag{2.15}$$

where $\bar{h}$ denotes another independent set of the parameters $h^\pm_\alpha$.

Now we will show that the h's and $\bar{h}$'s can be chosen in such a way that

$$L_{\alpha\beta}(x)=\Delta(x,\bar{h})\mathscr{L}_{\alpha\beta}(x,\bar{h},h). \tag{2.16}$$

Taking matrix elements of (2.3) and using (2.8) and (2.15) we rewrite (2.16) as

$$r_\beta(x,l)\omega^{l_{\beta,\alpha}}\{t_\beta v^+_\beta\delta_{\alpha\beta}+(\xi/x)^{2\delta_{n\beta}}t_{\beta+1}v^-_\beta\delta_{\beta,\alpha-1}\}=\psi^{l+e_\beta}_{l,\alpha}(x,h), \tag{2.17a}$$

$$r_\beta(x,l)^{-1}\omega^{l_{\alpha,\beta}}(\xi/x)^{\varepsilon_{\bar{\alpha}\beta}}t_\alpha^{-1}=\Delta(x,\bar{h})\bar{\psi}^{l+e_\beta}_{l,\alpha}(x,\bar{h}), \tag{2.17b}$$

where $r_\beta(x,l)$, $\beta\in Z_n$, $l\in AZ^n_N$ are some nonzero parameters which cancel in the L-operator. Comparing (2.17) with (2.11) and (2.14) and picking out terms with $\delta_{\beta\alpha}$ and $\delta_{\beta,\alpha-1}$ in (2.17a) we obtain

$$r_\beta(x,l)=\omega^{\langle v_\beta,l\rangle}r_\beta(x), \tag{2.18a}$$

$$r_\beta(x)t_\beta v^+_\beta=x^{\delta_{n\beta}}h^+_\beta, \tag{2.18b}$$

$$r_\beta(x)(\xi/x)^{2\delta_{n\beta}}t_{\beta+1}v^-_\beta=-x^{-\delta_{n\beta}}h^-_\beta, \tag{2.18c}$$

$$\frac{x}{\xi t_\alpha r_\alpha(x)}=\prod_{\beta\neq\alpha}(x^{\delta_{n\beta}}\bar{h}^+_\beta), \tag{2.18d}$$

$$\frac{(x/\xi)^{\varepsilon_{\alpha\beta}}}{t_\alpha r_\beta(x)}=\prod_{\gamma=\beta+1}^{\alpha-1}(x^{\delta_{n\gamma}}\bar{h}^+_\gamma)\prod_{\gamma=\alpha}^{\beta-1}(x^{-\delta_{n\gamma}}\bar{h}^-_\gamma),\quad \alpha\neq\beta, \tag{2.18e}$$

where $r_\beta(x)$, $\beta\in Z_n$ do not depend on $l\in AZ^n_N$.

Excluding t_α from (2.18e) and (2.18d) we have

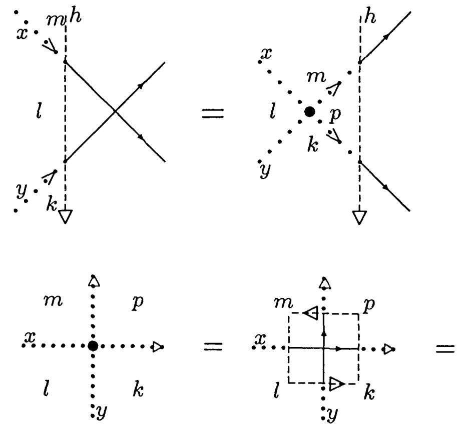
$$(\xi/x)^{\varepsilon_{\alpha\beta}+1}\frac{r_\alpha(x)}{r_\beta(x)}=\prod_{\gamma=\alpha}^{\beta-1}(x^{-\delta_{n\gamma}}\bar{h}^-_\gamma)\Big/\prod_{\gamma=\alpha+1}^{\beta}(x^{\delta_{n\gamma}}\bar{h}^+_\gamma),\quad \alpha\neq\beta. \tag{2.19}$$

Now multiplying (2.19) by itself with $(\alpha\leftrightarrow\beta)$ we have

$$\xi^2=\prod_{\alpha\in Z_n}(\bar{h}^-_\alpha/\bar{h}^+_\alpha). \tag{2.20}$$

Solution of (2.19) with respect to $r_\alpha(x)$ has the form

$$r_\alpha(x)=c\bar{h}^+_\alpha x^{\delta_{n\alpha}}\prod_{\gamma=\alpha}^{n}(\bar{h}^-_\gamma/\bar{h}^+_\gamma), \tag{2.21}$$



$= \quad W_{SOS}(k,l,m,p|\frac{x}{y})$

Fig. 9

where c is some constant parameter which is not fixed by Eq. (2.18). For convenience let us choose

$$c^{-1}=\xi \prod_{\alpha\in Z_n} \bar{h}_\alpha^+ . \tag{2.22}$$

Then from (2.18d) we calculate

$$t_\alpha=\prod_{\beta=\alpha}^{n} (\bar{h}_\beta^+/\bar{h}_\beta^-), \tag{2.23}$$

and from (2.18b) and (2.18c)

$$v_\alpha^\pm=\pm\xi^{\pm 1} h_\alpha^\pm \prod_{\beta\neq\alpha} \bar{h}_\beta^\pm . \tag{2.24}$$

Thus we have proved equality (2.16). Since the L-operator (2.2) satisfies the YBE (1.1) then due to (2.16) we can conclude that $\mathscr{L}(x,\bar{h},h)$ satisfies the same YBE,

$$R_{12}(x/y)\mathscr{L}_1(x,\bar{h},h)\mathscr{L}_2(y,\bar{h},h)=\mathscr{L}_2(y,\bar{h},h)\mathscr{L}_1(x,\bar{h},h)R_{12}(x/y). \tag{2.25}$$

Multiplying this equation from the left by a vector $(\psi_l^m(x,\bar{h})\otimes\psi_k^l(y,\bar{h}))^t\in C^n\otimes C^n$, $k,l,m\in AZ_N^n$, we obtain (see Fig. 9 for graphical representation)

$$\begin{aligned}&(\psi_l^m(x,h)\otimes\psi_k^l(y,h))^t R_{12}(x/y)\\ &\quad=\sum_{p\in AZ_N^n} W_{\mathrm{SOS}}(k,l,m,p\,|\,x/y)(\psi_k^p(x,h)\otimes\psi_p^m(y,h))^t ,\end{aligned} \tag{2.26}$$

where

$$\begin{aligned}&W_{\mathrm{SOS}}(k,l,m,p\,|\,x/y)\\ &\quad=(\psi_l^m(x,\bar{h})\otimes\psi_k^l(y,\bar{h}))^t R_{12}(x/y)\bar{\psi}_k^p(x,\bar{h})\otimes\bar{\psi}_p^m(y,\bar{h}).\end{aligned} \tag{2.27}$$

[Since the left-hand side of (2.26) is independent of $\bar{h}$, $W_{\mathrm{SOS}}(l,m,n,k\,|\,x)$ does not depend on $\bar{h}$.]. Similarly, one can prove that (Fig. 10)

$$\begin{aligned}&R_{12}(x/y)\bar{\psi}_k^l(x,\bar{h})\otimes\bar{\psi}_l^m(y,\bar{h})\\ &\quad=\sum_{p\in AZ_N^n} W_{\mathrm{SOS}}(k,p,m,l\,|\,x/y)\bar{\psi}_p^m(x,\bar{h})\otimes\bar{\psi}_k^p(y,\bar{h}).\end{aligned} \tag{2.28}$$

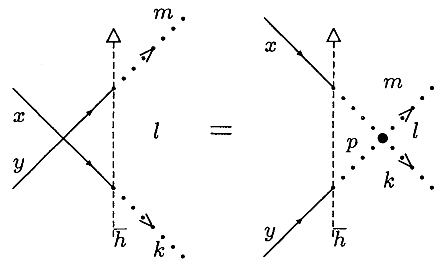


Fig. 10

Thus we have proved the first part of the following general statement:

Lemma. *If the L-operator, satisfying the YBE* (2.25), *can be decomposed in the form* (2.15) *with the three-spin weights* $\psi, \bar{\psi}$ *satisfying* (2.12), *then there exist the weights* W_{SOS} *satisfying* (2.26), (2.28). *Conversely, if there exist an R-matrix,* W_{SOS} *weights and vectors* ψ *satisfying Eqs.* (2.26), (2.13) *then the L-operator of the form* (2.15) *satisfies the YBE* (2.25).

The proof of the converse statement is similar. Explicit calculation gives the following formulae for the nonzero elements of W_{SOS}:

$$W_{\mathrm{SOS}}(k, k+e_\alpha, k+2e_\alpha, k+e_\alpha \,|\, x) = xq - x^{-1}q^{-1}, \tag{2.29a}$$

$$\begin{aligned} &W_{\mathrm{SOS}}(k, k+e_\alpha, k+e_\alpha+e_\beta, k+e_\beta \,|\, x) \\ &\quad = (x-x^{-1})\varrho_{\beta\alpha}\omega^{\langle \nu_\beta, e_\alpha\rangle - \langle \nu_\alpha, e_\beta\rangle}, \qquad \alpha \neq \beta, \end{aligned} \tag{2.29b}$$

$$W_{\mathrm{SOS}}(k, k+e_\alpha, k+e_\alpha+e_\beta, k+e_\alpha \,|\, x) = (q-q^{-1})x^{\varepsilon_{\alpha\beta}+\delta_{n\beta}-\delta_{n\alpha}}, \qquad \alpha \neq \beta. \tag{2.29c}$$

Note that W_{SOS} weights (2.29) correspond to a particular trigonometric case of the cyclic SOS model of [13] related with elliptic n-state R-matrix [14]. We can handle the elliptic case as well. In fact, applying our lemma to Eqs. (3.4) and (3.6) of [13] we obtain a cyclic L-operator related with the above elliptic R-matrix. These subjects will be discussed in a separate paper.

3. Construction of the Generalized Chiral Potts Model

In this section we calculate Boltzmann weights of the generalized chiral Potts model with the help of some linear equation (which, in fact, is a variant of the YBE) and determine the algebraic curve which constrains the rapidity variables.

First, we define one more set of "three spin interaction" weights ($l, m \in AZ_N^n$, $\alpha, \beta \in Z_n$)

$$\phi_{l,\alpha}^m(x,h) \equiv \begin{cases} \psi_{m,\alpha}^{m+e_\beta}(x^{-1}, \sigma(h)), & l = m - e_\beta; \\ 0, & \text{otherwise}, \end{cases} \tag{3.1a}$$

where the transformed set $\sigma(h)$ is given by

$$\sigma(h)_\alpha^\pm = h_\alpha^\mp, \qquad \alpha \in Z_n. \tag{3.2}$$

There is a graphical representation for these weights shown in Fig. 11. One can introduce also corresponding "inverse" weights $\bar{\phi}_{l,\alpha}^m(x,h)$ by any of two relations

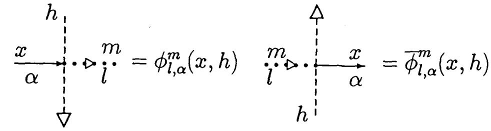


Fig. 11

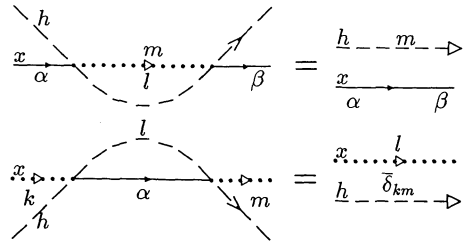


Fig. 12

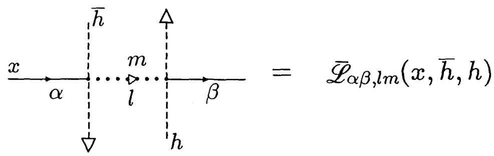


Fig. 13

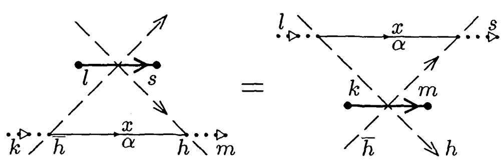


Fig. 14

(see Fig. 12):

$$\sum_{l\in AZ_N^n} \phi^{m}_{l,\alpha}(x,h)\bar{\phi}^{m}_{l,\beta}(x,h)=\delta_{\alpha\beta}\,, \tag{3.3a}$$

$$\sum_{\alpha\in Z_n} \bar{\phi}^{l}_{k,\alpha}(x,h)\phi^{l}_{m,\alpha}(x,h)=\bar{\delta}_{k,m} \tag{3.3b}$$

just as in Sect. 2. Explicit formula for $\bar{\phi}^{m}_{l,\beta}(x,h)$ has the form:

$$\bar{\phi}^{m}_{l,\alpha}(x,h)\equiv\begin{cases}\bar{\psi}^{m+e_\beta}_{m,\alpha}(x^{-1},\sigma(h)), & l=m-e_\beta\,;\\ 0, & \text{otherwise},\end{cases} \tag{3.1b}$$

Using these objects the other factorized L-operator can be constructed (Fig. 13)

$$\mathscr{L}_{\alpha\beta,lm}(x,\bar{h},h)=\phi^{m}_{l,\alpha}(x,\bar{h})\bar{\phi}^{m}_{l,\beta}(x,h)\,, \tag{3.4}$$

which satisfies the YBE (2.25), where the $R(x)$ is defined by (1.2) with $\varrho_{\alpha\beta}$ replaced by $\varrho_{\beta\alpha}$. Obviously, there are counterparts of Eqs. (2.26)–(2.28) with the corresponding W_{SOS} weights.

Now let us consider the following linear equation on the set of unknowns $\bar{W}_{h,\bar{h}}(l,m)$, $l,m\in AZ_N^n$, which was invented by Korepin and Tarasov [15] (see also [16]) for the case of the chiral Potts model (Fig. 14):

$$\begin{aligned}&\bar{W}_{h,\bar{h}}(l,s)\sum_{\alpha\in Z_n}\bar{\phi}^{l}_{k,\alpha}(x,\bar{h})\phi^{s}_{m,\alpha}(x,h)\\ &\quad=\bar{W}_{h,\bar{h}}(k,m)\sum_{\alpha\in Z_n}\psi^{l}_{k,\alpha}(x,h)\bar{\psi}^{s}_{m,\alpha}(x,\bar{h})\,.\end{aligned} \tag{3.5a}$$

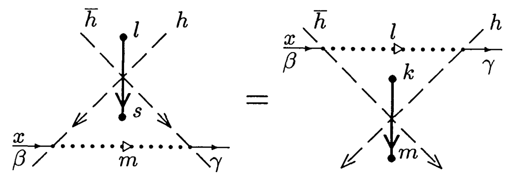


Fig. 15

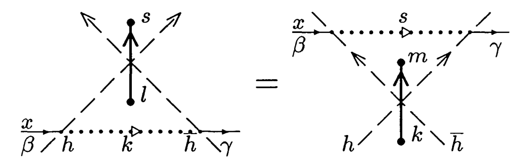


Fig. 16

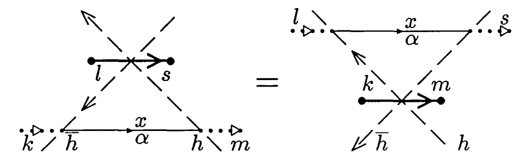


Fig. 17

Equation (3.5a) can be written in other forms by using inversion relations (2.12) and (3.3). Indeed, multiplying (3.5a) from the left by $\phi^{l}_{k,\beta}(x,\bar{h})$, from the right by $\psi^{s}_{m,\gamma}(x,\bar{h})$ and summing over k and s we obtain (Fig. 15),

$$\sum_{s\in AZ_N^n} \bar{W}_{h,\bar{h}}(l,s)\phi^{s}_{m,\beta}(x,h)\psi^{s}_{m,\gamma}(x,\bar{h}) = \sum_{k\in AZ_N^n} \phi^{l}_{k,\beta}(x,\bar{h})\psi^{l}_{k,\gamma}(x,h)\bar{W}_{h,\bar{h}}(k,m)\,. \tag{3.5b}$$

On the other hand, multiplying (3.5a) from the left by $\bar{\psi}^{l}_{k,\beta}(x,h)$, from the right by $\bar{\phi}^{s}_{m,\gamma}(x,h)$ and summing over l and m we obtain one more form of (3.5a) (Fig. 16):

$$\sum_{l\in AZ_N^n} \bar{\psi}^{l}_{k,\beta}(x,h)\bar{\phi}^{l}_{k,\gamma}(x,\bar{h})\bar{W}_{h,\bar{h}}(l,s) = \sum_{m\in AZ_N^n} \bar{W}_{h,\bar{h}}(k,m)\bar{\psi}^{s}_{m,\beta}(x,\bar{h})\bar{\phi}^{s}_{m,\gamma}(x,h)\,. \tag{3.5c}$$

It is useful to introduce the graphical representation for $(\bar{W}_{\bar{h},h}(l,m))^{-1}$ $(p\to h, q\to \bar{h})$ as in Fig. 2b. Then the same Eq. (3.5a) with $h\leftrightarrow\bar{h}$ can be represented as in Fig. 17.

Substitution of (2.11), (2.14), and (3.1) into (3.5a) at $l=k+e_\alpha$, $s=m+e_\beta$ gives explicitly

$$\frac{\bar{W}_{h\bar{h}}(k+e_\alpha, m+e_\alpha)}{\bar{W}_{h\bar{h}}(k,m)} = \omega^{2\langle v_\alpha, k-m\rangle + (k-m)_{\alpha,\alpha+1}}, \quad \alpha=\beta, \tag{3.6a}$$

$$\frac{\bar{W}_{h\bar{h}}(k+e_\alpha-e_\beta, m)}{\bar{W}_{h\bar{h}}(k,m)} = \omega^{\langle v_\alpha - v_\beta, 2k+e_\alpha\rangle + (k+m+e_\alpha)_{\alpha,\beta} - \langle e_\alpha - e_\beta, v_\beta\rangle} \times \frac{h^{+}_{\alpha}\bar{h}^{-}_{\alpha} - h^{-}_{\alpha}\bar{h}^{+}_{\alpha}\omega^{(k-m)_{\alpha,\alpha+1}}}{h^{+}_{\beta}\bar{h}^{-}_{\beta} - h^{-}_{\beta}\bar{h}^{+}_{\beta}\omega^{1+(k-m)_{\beta,\beta+1}}}, \quad \alpha\neq\beta. \tag{3.6b}$$

In (3.6b) we have used (3.6a) to move e_β from the second argument of $\bar{W}_{h\bar{h}}(k,m)$ to the first one.

To obtain the consistency condition between (3.6a) and (3.6b) let us divide (3.6b) by itself with replacements $k \to k+e_\gamma$, $m \to m+e_\gamma$:

$$\frac{\bar{W}_{h\bar{h}}(k+e_\alpha-e_\beta,m)}{\bar{W}_{h\bar{h}}(k,m)} \frac{\bar{W}_{h\bar{h}}(k+e_\gamma,m+e_\gamma)}{\bar{W}_{h\bar{h}}(k+e_\alpha-e_\beta+e_\gamma,m+e_\gamma)} = \omega^{-2\langle \nu_\alpha-\nu_\beta,e_\gamma\rangle-2(e_\gamma)_{\alpha,\beta}}. \tag{3.7}$$

In the left-hand side of (3.7) we can use (3.6a) twice and obtain

$$\omega^{-2\langle \nu_\gamma, e_\alpha-e_\beta\rangle-(e_\alpha-e_\beta)_{\gamma,\gamma+1}} = \omega^{-2\langle \nu_\alpha-\nu_\beta,e_\gamma\rangle-2(e_\gamma)_{\alpha,\beta}}. \tag{3.8}$$

Thus we have the following equations on linear forms ν_α:

$$\langle e_\gamma, \nu_\alpha-\nu_\beta\rangle - \langle e_\alpha-e_\beta, \nu_\gamma\rangle = \tfrac{1}{2}(e_\alpha-e_\beta)_{\gamma,\gamma+1} - (e_\gamma)_{\alpha,\beta}. \tag{3.9}$$

Solution of (3.9) is

$$\langle \nu_\alpha, e_\beta\rangle = \mu_{\alpha\beta} - \tfrac{1}{4}(e_\alpha+e_\beta)_{\alpha,\beta}, \tag{3.10}$$

where $\{\mu_{\alpha\beta}\}$ is an arbitrary set of numbers, symmetric in α, β:

$$\mu_{\alpha\beta} = \mu_{\beta\alpha}. \tag{3.11}$$

We choose

$$\mu_{\alpha\beta} = \tfrac{1}{4}(\delta_{\alpha\beta}-1). \tag{3.12}$$

The condition for $\bar{W}_{h,\bar{h}}(l,m)$ to be a finite dimensional $N^{n-1} \times N^{n-1}$ matrix has the form

$$\bar{W}_{h,\bar{h}}(l+Ne_\alpha,m) = \bar{W}_{h,\bar{h}}(l,m+Ne_\alpha) = \bar{W}_{h,\bar{h}}(l,m), \tag{3.13}$$

and imply the following equations

$$(h_\alpha^+\bar{h}_\alpha^-)^N - (h_\alpha^-\bar{h}_\alpha^+)^N = (h_\beta^+\bar{h}_\beta^-)^N - (h_\beta^-\bar{h}_\beta^+)^N, \quad \forall \alpha,\beta \in Z_n. \tag{3.14}$$

If we require that this condition should be satisfied at least for two different sets of values of h's while keeping $\bar{h}$'s fixed then it is not difficult to see that the manifold defined by (3.14) contains a direct product of two identical algebraic curves specified by the following equations:

$$\begin{pmatrix} h_\alpha^+(p)^N \\ h_\alpha^-(p)^N \end{pmatrix} = K_{\alpha\beta} \begin{pmatrix} h_\beta^+(p)^N \\ h_\beta^-(p)^N \end{pmatrix}, \quad \forall \alpha,\beta \in Z_n, \tag{3.15}$$

where the argument p of h's denotes a point of the curve and $K_{\alpha\beta}$ are 2×2 complex matrices of moduli with the unit determinant. Equations (3.15) with h's replaced by $\bar{h}$'s specify the other point q of this curve. The matrices $K_{\alpha\beta}$ satisfy the consistency relations (no summation)

$$K_{\alpha\alpha} = K_{\alpha\beta}K_{\beta\gamma}K_{\gamma\alpha} = 1, \quad \forall \alpha,\beta,\gamma \in Z_n. \tag{3.16}$$

There are $2n-3$ independent moduli, since a gauge transformation of the form

$$\begin{pmatrix} h_\alpha^+(p) \\ h_\alpha^-(p) \end{pmatrix} \to U_\alpha \begin{pmatrix} h_\alpha^+(p) \\ h_\alpha^-(p) \end{pmatrix}, \quad K_{\alpha\beta} \to U_\alpha^N K_{\alpha\beta} U_\beta^{-N}, \tag{3.17}$$

with matrices $U_\alpha = \mathrm{diag}(u_\alpha, u_\alpha^{-1})$ does not affect the $\bar{W}$'s. This curve is a natural generalization of the curve which appeared in the chiral Potts model [4]. Using the Riemann-Hurwitz formula one can calculate the genus g of curve (3.15) in a generic case:

$$g=N^{2(n-1)}((n-1)N-n)+1\,, \tag{3.18}$$

which at $n=2$ yields the genus of the curve which appeared in the chiral Potts model [4]. Hereafter we will write symbols $p, q, \ldots$ instead of $h, \bar{h}, \ldots$.

Now substituting (3.10) and (3.11) into (3.6) we obtain

$$\frac{\bar{W}_{pq}(l,m)}{\bar{W}_{pq}(0,0)}=\omega^{Q(l,m)}g_{pq}(0,l-m)\,, \tag{3.19}$$

where $g_{pq}(l,m)$ is uniquely defined by:

$$g_{pq}(k,k)=g_{pq}(k,l)g_{pq}(l,m)g_{pq}(m,k)=1\,, \tag{3.20a}$$

$$g_{pq}(k,k+e_\alpha-e_\beta)=\frac{h_\alpha^+(p)h_\alpha^-(q)-h_\alpha^-(p)h_\alpha^+(q)\omega^{k_{\alpha,\alpha+1}}}{h_\beta^+(p)h_\beta^-(q)-h_\beta^-(p)h_\beta^+(q)\omega^{1+k_{\beta,\beta+1}}} \tag{3.20b}$$

and $Q(l,m)$ satisfies the relations:

$$Q(l+e_\alpha-e_\beta,m)-Q(l,m)=m_{\alpha,\beta}\,, \tag{3.21a}$$

$$Q(l,m+e_\alpha-e_\beta)-Q(l,m)=(l-m)_{\beta+1,\alpha+1}+m_{\beta\alpha}+1-\delta_{\alpha\beta}\,. \tag{3.21b}$$

Explicit solution of (3.21) has the form

$$Q(l,m)=\sum_\alpha m_{\alpha-1,\alpha}(l_\alpha-m_\alpha)\,. \tag{3.22}$$

Now, applying consequently Eq. (3.5a) (Fig. 14), Eq. (3.5b) (Fig. 15), Eq. (3.5c) (Fig. 16), and again Eq. (3.5a) (Fig. 17) one can show that an R-matrix of the form (see Fig. 3)

$$\langle k,l|S(q,q';p,p')|m,s\rangle=\frac{\bar{W}_{qp}(k,s)\bar{W}_{p'q}(s,m)\bar{W}_{q'p'}(m,l)}{\bar{W}_{q'p}(k,l)} \tag{3.23}$$

intertwines two L-operators (2.15) with rapidity parameters (q,q') and (p,p') (see Fig. 18):

$$\begin{aligned}&\sum_\beta \mathscr{L}_{\alpha\beta}(x,q,q')\otimes\mathscr{L}_{\beta\gamma}(x,p,p')S(q,q';p,p')\\&\quad=\sum_\beta S(q,q';p,p')\mathscr{L}_{\beta\gamma}(x,q,q')\otimes\mathscr{L}_{\alpha\beta}(x,p,p')\,,\end{aligned} \tag{3.24}$$

where

$$\mathscr{L}_{\alpha\beta}(x,q,q')\equiv\mathscr{L}_{\alpha\beta}(x,h(q),h(q')) \tag{3.25}$$

and $\mathscr{L}_{\alpha\beta}(x,h(q),h(q'))$ is defined in (2.15) with $h\equiv h(q)$ and $\bar{h}\equiv h(q')$.

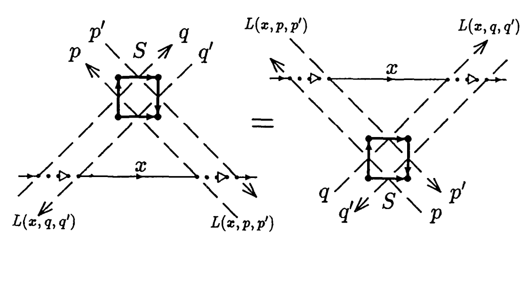


Fig. 18

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