Unitarization of a Singular Representation of $SO(p, q)$

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Abstract. A geometric construction of a certain singular unitary representation of $SO_e(p,q)$, with $p + q$ even is given. The representation is realized geometrically as the kernel of a $SO_e(p,q)$-invariant operator on a space of sections over a homogeneous space for $SO_e(p,q)$. The K-structure of these representations is elucidated and we demonstrate their unitarity by explicitly writing down an $\mathfrak{so}(p,q)$-invariant positive definite hermitian form. Finally, we demonstrate that the annihilator in $\mathscr{U}[\mathfrak{g}]$ of this representation is the Joseph ideal, which is the maximal primitive ideal associated with the minimal coadjoint orbit.

1. Introduction

The irreducible unitary representations of a semisimple Lie group G fall into two basic classes; the tempered representations which enter the Plancheral decomposition of $L^2(G)$, and the "singular representations" which form the complement of the tempered representations in the full unitary dual of G. There are fairly uniform geometric constructions of the tempered representations that associate these representations with certain orbits of semisimple elements is the dual of the Lie algebra of G.

There is no such uniform scheme for constructing the singular unitary representations. A good geometric construction seems to be the procedure of Rawnsley, Schmid and Wolf ([R–S–W]) which uses indefinite harmonic theory to unitarize Dolbeault cohomology. However, the procedure works only in a narrow setting; it associates most of the unitary highest weight modules to elliptic coadjoint orbits. Other singular representations have geometric realizations. For example, the metaplectic representation is constructed by a quantization procedure known as the Kostant–Sternberg–Blattner method of moving polarizations (see [B]). There are also constructions using Howe's dual pair picture (e.g., [M]), and there are constructions using twister techniques (e.g. [E–P–W] and [N1]). Even so, most of the success in constructing singular representations has been limited to singular highest weight representations.

Singular representations are also of particular interest to physicists. For in field theories based on a semisimple spacetime symmetry group, the most appropriate analogs of massless particles have always been singular representations. Indeed, in such theories, the important property of gauge invariance, which is inherent to Poincaré-invariant field theories involving massless particles with spin ≥ 1, can be generalized in such a way as to include particles of all spin; even sub-massless, preon-like particles (see e.g., [A–D–F–S] and [F–F]).

This property of gauge invariance is related to the fact that singular representations have low Gelfand–Kirillov dimension. Let (π, V) be an irreducible representation of a Lie algebra $\mathfrak{g}$ on a vector space V, let v be any vector in V, and let $\mathcal{U}^n[\mathfrak{g}]$ denote the subspace of the universal enveloping algebra of $\mathfrak{g}$ consisting of products of at most n elements of $\mathfrak{g}$. The Gelfand–Kirillov dimension ([Vo]) measures the "size" of a representation (π, V) in terms of the growth rate of $\dim(U^n[\mathfrak{g}] \cdot v)$ as $n \to \infty$. The fact that a representation has low Gelfand–Kirillov dimension may be regarded as an indication that the representation has some "missing states" in the same sense as massless particles with spin ≥ 1 lack "longitudinal states." And, like longitudinal photons, these missing states tend to reappear in field theoretical realizations as pure gauge solutions to the field equations. The fact that the Gelfand–Kirillov dimension of a singular representation is small may also be regarded as an indication that the annihilator of the representation in $\mathcal{U}[\mathfrak{g}]$ is rather large.

In this paper we formulate a geometric construction of a certain unitary representation of $SO_e(p,q)$; with $p + q$ even and greater than 5. As we will see, the representation constructed is a small representation of $SO_e(p,q)$ in the sense that (i) it has the lowest possible Gelfand–Kirillov dimension, and (ii) its annihilator in the universal enveloping algebra of $\mathfrak{so}(p,q)$ is the Joseph ideal ([J]), the (maximal) primitive ideal associated with the minimal coadjoint orbit. This construction includes two highest weight representations of $SO_e(2,q)$ which are not amenable to the Rawnsley–Schmid–Wolf construction ([Z, N2]), and which are also important physically since they correspond to scalar particles in a conformal field theory over a q-dimensional spacetime. Indeed, the unitary structure of our construction may be regarded as a generalization of the usual Klein–Gordon inner product for scalar field theories. Also included in this construction is the distinguished spherical representation of $SO_e(4,4)$ discussed by Kostant ([K, K2]). However, in general, the representations constructed here are neither highest weight nor spherical representations.

Our construction is geometric in the sense that the representations are realized within the kernel of a G-invariant differential operator $\square'$ on a space of L^2 sections on a homogeneous space for G. Most importantly, the Hilbert space inner product is given explicitly in terms of operators $\mathcal{D}_+$ and $\mathcal{D}_-$, where $\square' = \mathcal{D}_+ \mathcal{D}_-$. Part of the point is that our construction is elementary in the sense that none of the fancy machinery of modern representation theory is used. Indeed, the geometric setting of this construction can be traced back to Dirac ([D]).

II. The Differential Operator $\square'$

Let $\mathbb{R}^{p,q}$ be $\mathbb{R}^{p+q}$ endowed with the quadratic form $Q(y) = \sum\limits_{i=1}^{p} y_i{}^2 - \sum\limits_{j=1}^{q} y_{p+j}{}^2$, where the y_k are the coordinates of $y \in \mathbb{R}^{p+q}$ with respect to some fixed basis. Let

$C = \{y \in \mathbb{R}^{p,q} : Q(y) = 0\}$ and $C^* = C \setminus \{0\}$. We denote the projectivized cone by X; it is obtained from C^* by identifying all points lying on the same line through the origin. X is a connected $(p + q - 2)$-dimensional subvariety of $(p + q - 1)$-dimensional real projective space.

The orthogonal group $O(p,q)$ is the group of linear transformations of $\mathbb{R}^{p,q}$ preserving Q. We assume throughout that $2 \leq p \leq q$ and $2 < q$. $O(p,q)$ has four connected components; we let $G = SO_e(p,q)$ denote the component containing the identity.

$$\sigma(g) = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} g \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$$

is a Cartan involution, so the elements of G fixed by σ form a maximal compact subgroup K of G. K is isomorphic to $SO(p) \times SO(q)$. G acts transitively on C^* and X.

Both the Laplacian

$$\square = \sum_{i=1}^{p} \left(\frac{\partial}{\partial y_i} \right)^2 - \sum_{j=1}^{q} \left(\frac{\partial}{\partial y_{p+j}} \right)^2$$

and the Euler operator

$$\mathscr{E} = \sum_{k=1}^{p+q} y_k \frac{\partial}{\partial y_k}$$

are G-invariant operators on $C^\infty(\mathbb{R}^{p,q})$. One would like to restrict $\square$ to C^*; for, as a general principle, the kernel of such an operator would have a chance to be an irreducible representation. However, there is no natural way to restrict $\square$ to C^*. We will show that $\square$ nevertheless determines an invariant operator $\square'$ on functions which are homogeneous of a certain degree.

Let $C^\infty(C^*, d)$ be the space of C^∞ functions on C^* which are homogeneous of degree $d \in \mathbb{Z}$; i.e., $f(\lambda y) = \lambda^d f(y)$ for all $\lambda \in \mathbb{R} - \{0\}$ and all $y \in \mathbb{R}^{p,q}$. Left translation by G preserves $C^\infty(C^*, d)$. Since $\mathscr{E}$ is a vector field on $\mathbb{R}^{p,q}$ that is tangent to C^*, $\mathscr{E}$ restricts to a well-defined differential operator on C^*. $\mathscr{E}f = df$, for all $f \in C^\infty(C^*, d)$.

Proposition. *For $p + q \in 2\mathbb{N}$, and $d = 2 - \dfrac{p+q}{2}$, there is a well defined operator*

$$\square' : C^\infty(C^*, d) \to C^\infty(C^*, d-2).$$

Proof. Let $f \in C^\infty(C^*, d)$ and let F be any C^∞ homogeneous extension of f to a conical neighborhood of C^*. We claim that

$$\square' f = \square F|_{C^*}$$

is independent of the extension F when $d = 2 - \dfrac{p+q}{2}$.

Adopting spherical coordinates $\{r, \theta_1, \ldots, \theta_{p-1}; \rho, \varphi_1, \ldots, \varphi_{q-1}\}$ on $\mathbb{R}^{p,q} \approx \mathbb{R}^p \times \mathbb{R}^q$, one finds

$$\mathscr{E} = r \frac{\partial}{\partial r} + \rho \frac{\partial}{\partial \rho}$$

and

$$\Box = \left(1 - \frac{r^2}{\rho^2}\right)\left(\frac{\partial}{\partial r}\right)^2 + \left(\frac{p-1}{r^2} + \frac{q-3}{\rho^2}\right)r\frac{\partial}{\partial r} + 2\frac{r}{\rho^2}\frac{\partial}{\partial r}\mathscr{E}$$

$$- \frac{1}{\rho^2}(\mathscr{E} + q - 2)\mathscr{E} - \frac{1}{\rho^2}(\Omega_{SO(p)} - \Omega_{SO(q)}),$$

where $\Omega_{SO(p)}$ (respectively, $\Omega_{SO(q)}$) is the differential operator on $\mathbb{R}^p$ (respectively, $\mathbb{R}^q$) corresponding to the Casimir operator of $SO(p)$ (respectively, $SO(q)$). When F is homogeneous of degree d, we have $\mathscr{E}F = dF$. Upon restriction to the cone (setting $r = \rho$), we obtain

$$\Box F|_{C^*} = (p + q - 4 + 2d)r\frac{\partial F}{\partial r}\bigg|_{C^*}$$

$$- \frac{1}{r^2}((d + q - 2)d - \Omega_{SO(p)} + \Omega_{SO(q)})F|_{C^*}.$$

If we now set $d = 2 - \dfrac{p+q}{2}$, the term involving the derivative with respect to r vanishes. Thus, when $d = 2 - \dfrac{p+q}{2}$,

$$\Box'f \equiv \Box F|_{C^*} = \left(-\Omega_{SO(p)} + \Omega_{SO(q)} - \left(\frac{p-2}{2}\right)^2 + \left(\frac{q-2}{2}\right)^2\right)F|_{C^*} \qquad (1)$$

is independent of the way we choose F. ∎

Henceforth, we shall always assume that $p+q$ is even and $d = 2 - \dfrac{p+q}{2}$.

We will now determine the K-types of Ker $\Box'$; i.e., we decompose the representation of K on Ker $\Box'$ (in fact, on all of $C^\infty(C^*, d)$) into irreducibles.

Let S^{p-1} and S^{q-1} be the unit spheres inside $\mathbb{R}^p$ and $\mathbb{R}^q$. $S^{p-1} \times S^{q-1} \subset C^*$ and a function in $C^\infty(C^*, d)$ is determined completely by its values on $S^{p-1} \times S^{q-1}$ (because every point of C^* is on a line through a point of $S^{p-1} \times S^{q-1}$). Thus,

$$C^\infty(C^*, d) \approx \{\phi \in C^\infty(S^{p-1} \times S^{q-1})|\phi(-y) = (-1)^d\phi(y)\}.$$

Consider $SO(n)$, the special orthogonal group. It is well known (see, e.g., [Vi], Chapter IX, Sect. 2) that

$$C^\infty(S^{n-1})_{K\text{-finite}} \approx L^2(S^{n-1})_{K\text{-finite}} \approx \begin{cases} \bigoplus_{a\in\mathbb{Z}^+} V_a, & n > 2, \\ \bigoplus_{a\in\mathbb{Z}} V_a, & n = 2, \end{cases}$$

where V_a is the irreducible representation of $SO(n)$ corresponding to homogeneous harmonic polynomials on $\mathbb{R}^n$ of degree a and "K-finite" means the functions whose K-translates span a finite dimensional subspace. Imposing the additional condition $\phi(-y) = (-1)^d\phi(y)$ we arrive at the following fact: for $p + q$ even and $d = 2 - \dfrac{p+q}{2}$,

$$
C^\infty(C^*,d)_{K\text{-finite}} \approx
\begin{cases}
\displaystyle\bigoplus_{\substack{a,b\in\mathbf{Z}^+ \\ a+b=d\bmod 2}} V_{a,b}, & 2 < p \le q, \\[2em]
\displaystyle\bigoplus_{\substack{a\in\mathbf{Z} \\ b\in\mathbf{Z}^+ \\ a+b=d\bmod 2}} V_{a,b}, & 2 = p < q,
\end{cases}
$$

where $V_{a,b} = V_a \otimes V_b$ (considered as an irreducible representation of $SO(p) \times SO(q)$).
We are now ready to determine $\mathscr{H}_K = (\operatorname{Ker} \square')_{K\text{-finite}}$. The Casimir operator $\Omega_{so(n)}$
acts on V_a by $(a^2 + (n-2)a)I$. Thus, if $\phi \in V_{a,b}$, as in (1), then

$$
-\square'\phi = (a^2 + (p-2)a) - (b^2 + (q-2)b) + \left(\frac{p+2}{2}\right)^2 - \left(\frac{q+2}{2}\right)^2
$$

$$
= \left(a + \frac{p}{2} - 1\right)^2 - \left(b + \frac{q}{2} - 1\right)^2.
$$

Thus,

$$
V_{a,b} \subset \mathscr{H}_K \iff a + \frac{p}{2} - 1 = \pm\left(b + \frac{q}{2} - 1\right),
$$

and we have

Proposition. *For $p+q$ even,*

$$
\mathscr{H}_K \approx
\begin{cases}
\displaystyle\bigoplus_{\substack{a+p/2=b+q/2 \\ b\in\mathbf{Z}^+}} V_{a,b}, & 2 < p \le q, \\[2em]
\displaystyle\bigoplus_{\substack{a=\pm(b+q/2-1) \\ b\in\mathbf{Z}^+}} V_{a,b}, & 2 = p < q.
\end{cases}
\qquad\blacksquare
$$

III. The Invariant Inner Product

In order to have a unitary representation we need a Hilbert space $\mathscr{H}$ with a
G-invariant inner product. Here $\mathscr{H}$ will be the completion of $\mathscr{H}_K$ with respect to
the positive definite hermitian form defined below. Most of our work is directed
towards proving the invariance of this form.

Consider the $(n-1)$-sphere S^{n-1} and the Casimir $\Omega_{SO(n)}$ acting on S^{n-1}. Set

$$
\mathscr{D}_n = \left(\Omega_{SO(n)} + \frac{(n-2)}{4}\right)^{1/2} \quad \text{on } L^2(S^{n-1})_{K\text{-finite}};
$$

more precisely, on the K-type V_a, $\mathscr{D}_n$ is $\left(a + \frac{n}{2} - 1\right)$. Note that $\square' = (\mathscr{D}_p + \mathscr{D}_q)(\mathscr{D}_p - \mathscr{D}_q)$
on $C^\infty(C^*,d)_{K\text{-finite}}$, where $\mathscr{D}_p$ (respectively $\mathscr{D}_q$) acts on S^{p-1} (respectively S^{q-1}). Let

$$
(\phi, \psi) = \int_{S^{p-1}\times S^{q-1}} \phi\bar{\psi}\,d\omega,
$$

where $d\omega$ is a fixed K-invariant measure on $S^{p-1} \times S^{q-1}$.

Definition. $\langle \phi, \psi \rangle = ((\mathscr{D}_p + \mathscr{D}_q)\phi, \psi) - (\phi, (\mathscr{D}_p - \mathscr{D}_q)\psi)$ *for* ϕ, ψ *in* $\mathscr{H}_K$.

Remark. For $\phi \in V_{a,b} \subseteq \mathscr{H}_K$, $(\mathscr{D}_p \pm \mathscr{D}_q)\phi = \left(\left(a + \frac{p}{2} - 1 \right) \pm \left(b + \frac{q}{2} - 1 \right) \right)\phi$. Thus if $p \neq 2$, $\mathscr{H}_K = \operatorname{Ker}(\mathscr{D}_p - \mathscr{D}_q)$ and the form $\langle , \rangle$ becomes

$$\langle \phi, \psi \rangle = ((\mathscr{D}_p + \mathscr{D}_q)\phi, \psi).$$

When $p = 2$ we have

$$(\mathscr{D}_p + \mathscr{D}_q)\phi = \begin{cases} 2a\phi, & \text{for } a > 0 \\ 0, & \text{for } a < 0, \end{cases}$$

$$(\mathscr{D}_p - \mathscr{D}_q)\phi = \begin{cases} 0, & \text{for } a > 0 \\ -2a\phi, & \text{for } a < 0, \end{cases}$$

and $\mathscr{H}_K = \operatorname{Ker}(\mathscr{D}_p + \mathscr{D}_q) \oplus \operatorname{Ker}(\mathscr{D}_p - \mathscr{D}_q)$. Thus $\langle , \rangle$ is positive definite.

Theorem. $\langle , \rangle$ *is a positive definite invariant hermitian form on* $\mathscr{H}_K$.

By "invariant" we mean invariant under K and for all $X \in \mathfrak{so}(p,q)$, the (real) Lie algebra of G,

$$\langle X\phi, \psi \rangle + \langle \phi, X\psi \rangle = 0. \tag{2}$$

Proof. As $\mathscr{D}_p \pm \mathscr{D}_q$ are scalars on each K-type, it is clear that $\langle , \rangle$ is K-invariant. We may write $\mathfrak{so}(p,q) = \mathfrak{k} + \mathfrak{p}$; the Cartan decomposition, where $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{p}$ is its orthogonal complement with respect to the Killing form:

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & B \\ {}^t B & 0 \end{pmatrix} : B \in M_{p \times q}(\mathbb{R}) \right\}.$$

The isotropy representation of K on $\mathfrak{p}$ is irreducible, therefore we need to check (2) for just one element of $\mathfrak{p}$. We choose

$$L = \begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 \\ \hline 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & 0 & & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \end{pmatrix}.$$

In spherical coordinates, the action of L on $C^\infty(S^{p-1} \times S^{q-1})$ is given by

$$L = d\cos(\theta_{p-1})\cos(\phi_{q-1}) - \cos(\theta_{p-1})\sin(\phi_{q-1})\frac{\partial}{\partial \phi_{q-1}} - \cos(\phi_{q-1})\sin(\theta_{p-1})\frac{\partial}{\partial \theta_{p-1}}.$$

We write $L = dL_0 + L_1$ where

$$L_0 = \cos(\theta_{p-1})\cos(\phi_{q-1})$$

and

$$L_1 = \cos(\theta_{p-1})\sin(\phi_{q-1})\frac{\partial}{\partial \phi_{q-1}} + \cos(\phi_{q-1})\sin(\theta_{p-1})\frac{\partial}{\partial \theta_{p-1}}.$$

One can easily check

$$(\phi, L\psi) = -(L\phi, \psi) + 2(L_0\phi, \psi). \tag{3}$$

In order to confirm (2) we shall write down an explicit basis for each K type $V_{a,b}$ in terms of spherical functions. Let

$$A = (a_1, a_2, \ldots, a_{p-2}), \quad a \le a_1 \le a_2 \le \cdots \le a_{p-3} \le |a_{p-2}|,$$
$$B = (b_1, b_2, \ldots, b_{q-2}), \quad b \le b_1 \le b_2 \le \cdots \le b_{q-3} \le |b_{q-2}|,$$

and set

$$\psi_{A,B} = P_{a,a_1}^{(p-1)/2}(\cos(\theta_{p-1}))P_{a_1,a_2}^{(p-3)/2}(\cos(\theta_{p-2}))\cdots P_{a_{p-3},a_{p-2}}^{1/2}(\cos(\theta_2))e^{ia_{p-2}\theta_1}$$
$$\cdot P_{b,b_1}^{(q-1)/2}(\cos(\phi_{q-1}))P_{b_1,b_2}^{(q-3)/2}(\cos(\phi_{q-2}))\cdots P_{b_{q-3},b_{q-2}}^{1/2}(\cos(\phi_2))e^{ib_{q-2}\phi_1},$$

where

$$P_{n,m}^{\lambda}(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}C_n^{\lambda}(x)$$

are the "associated spherical functions" and C_n^{λ} is the Gegenbauer polynomial

$$C_n^{\lambda}(x) \sum_{r=0}^{[n/2]}(-1)^r\frac{\Gamma(n-r+\lambda)}{\Gamma(\lambda)r!(n-2r)!}(2x)^{n-2r}$$

(see, [Vi], Chapter IX, Sect. 3).

Lemma.

(a) $\cos\theta P_{n,m}^{\lambda}(\cos\theta) = \dfrac{1}{2\lambda+2n}[(n-m+1)P_{n+1,m}^{\lambda}(\cos\theta)$

$$+ (2\lambda+n+m-1)P_{n-1,m}^{\lambda}(\cos\theta)].$$

(b) $-\sin\theta\dfrac{d}{d\theta}P_{n,m}^{\lambda}(\cos\theta) = \dfrac{1}{2\lambda+2n}[-n(n-m+1)P_{n+1,m}^{\lambda}(\cos\theta)$

$$+ (2\lambda+n)(n+m+2\lambda-1)P_{n-1,m}^{\lambda}(\cos\theta)].$$

(c) $\displaystyle\int_0^{\pi}P_{n,m}^{\lambda}(\cos\theta)P_{n',m}^{\lambda}(\cos\theta)(\sin\theta)^{2\lambda-1}d\theta = 0 \quad for \quad n \ne n'.$

Each of these identities can be derived from properties of Gegenbauer polynomials.

Note that when $p=2$ the form of $\psi_{A,B}$ is slightly different. For now we assume that $p \ne 2$; we will come back to the case $p=2$ later. To make the notation a little easier we let $A^{\pm} = (a \pm 1, a_1, a_2, \ldots, a_{p-2})$, and similarly for $B^{\pm}$. Furthermore, we understand $\psi_{A^{\pm},B^{\pm}}$ to be zero if $A^{\pm}$ or $B^{\pm}$ is not allowable (recall $a \ge a_1 \ge \cdots$ etc. must hold).

Using the lemma we get:

$$L\psi_{A,B} = C_{+,+}\psi_{A^+,B^+} + C_{+,-}\psi_{A^+,B^-} + C_{-,+}\psi_{A^-,B^+} + C_{-,-}\psi_{A^-,B^-}, \tag{4}$$

where

$$C_{+,+} = \frac{-(a-a_1+1)(b+b_1+1)}{2\left(b+\frac{q}{2}-1\right)},$$

$$C_{+,-} = \frac{(a-a_1+1)(b+b_1+q-3)\left(-\left(a+\frac{p}{2}-1\right)+\left(b+\frac{q}{2}-1\right)\right)}{4\left(a+\frac{p}{2}-1\right)\left(b+\frac{q}{2}-1\right)},$$

$$C_{-,+} = \frac{(a+a_1+p-3)(b+b_1+1)\left(\left(a+\frac{p}{2}-1\right)-\left(b+\frac{q}{2}-1\right)\right)}{4\left(a+\frac{p}{2}-1\right)\left(b+\frac{q}{2}-1\right)},$$

$$C_{-,-} = \frac{(a+a_1+p-3)(b+b_1+q-3)}{2\left(b+\frac{q}{2}-1\right)}.$$

Note that for $V_{a,b} \subset \mathscr{H}_K$ we have $C_{+,-}, C_{-,+} = 0$. On easily confirms, using (4) and the fact that $(\phi_{A^\pm B^\pm}, \phi_{A'B'}) \neq 0$ requires $a' = a \pm 1$,

$$\langle L\phi_{A,B}\psi_{A',B'}\rangle + \langle \phi_{A,B}, L\phi_{A',B'}\rangle$$

$$= ((\mathscr{D}_p + \mathscr{D}_q)L\phi_{A,B}\psi_{A',B'}) + ((\mathscr{D}_p + \mathscr{D}_q)\phi_{A,B}, L\psi_{A',B'})$$

$$= 2\left(a'+\frac{p}{2}-1\right)(L\phi_{A,B}, \psi_{A',B'}) + 2\left(a+\frac{p}{2}-1\right)(\phi_{A,B}, L\psi_{A',B'})$$

$$= 0.$$

When $p = 2$,

$$\psi_{A,B} = e^{ia\theta}P_{b,b_1}^{(q-2)/2}(\cos(\phi_{q-1}))\cdots P_{b_q-3,b_q-2}^{1/2}(\cos(\phi_2))e^{ib_q-2\phi_1}$$

and

$$L\psi_{A,B} = \frac{1}{4\left(b+\frac{q}{2}-1\right)}\left[-(b-b_1+1)\left(a+b+\frac{q}{2}-1\right)\psi_{A^+,B^+}\right.$$

$$+ (b+b_1+q-3)\left(-a+b+\frac{q}{2}-1\right)\psi_{A^+,B^-}$$

$$+ (b-b_1+1)\left(a-b-\frac{q}{2}+1\right)\psi_{A^-,B^+}$$

$$\left. + (b+b_1+q-3)\left(a+b+\frac{q}{2}-1\right)\psi_{A^-,B^-}\right]. \tag{5}$$

As before one confirms:

$$\langle L\psi_{A,B}, \psi_{A',B'}\rangle + \langle \psi_{A,B}, L\psi_{A',B'}\rangle = 0. \quad \blacksquare$$

IV. Reducibility

In this section we give a few more facts about the representations appearing in $C^\infty(C^*, d)$ $\left(d = 2 - \dfrac{p+q}{2} \text{ still}\right)$.

Proposition. *When $p > 2$, $\mathcal{H}_K$ is an irreducible $(\mathfrak{g}, K)$-module (hence $\mathcal{H}$ is an irreducible unitary representation of G). When $p = 2$, $\mathcal{H}_K = \mathcal{H}_K^+ \oplus \mathcal{H}_K^-$ with*

$$\mathcal{H}_K^{\pm} = \bigoplus_{\substack{b \in \mathbb{Z}^+ \\ a = \pm(b + (q/2) - 1)}} V_{a,b}.$$

$\mathcal{H}_K^+$ *and* $\mathcal{H}_K^-$ *are irreducible $(\mathfrak{g}. K)$-modules.*

Proof. When $p \neq 2$ this follows from Eq. (4). Note that for $V_{a,b} \subset \mathcal{H}_K, C_{+-} = 0 = C_{-+}$ and no C_{++} or C_{--} is zero. Thus, one can get from any $V_{a,b}$ in $\mathcal{H}_K$ to any other K-type in $\mathcal{H}_K$ by applying L. The situation for $p = 2$ is similarly restricted by Eq. (5), which has the form

$$L\psi_{A,B} = C\psi_{A^+,B^+} + D\psi_{A^-,B^-}, \quad a > 0,$$
$$L\psi_{A,B} = E\psi_{A^+,B^-} + F\psi_{A^-,B^+}, \quad a < 0,$$

where the coefficients C, D, E, F are never zero. Of course, $\psi_{A^\pm,B^-} = 0$ when $b = 0$. Thus, all of the $V_{a,b} \subset \mathcal{H}_K$ with $a > 0$ are obtained by applying L to $V_{(q/2)-1,0}$ and no $V_{a,b}$ with $a < 0$ can be obtained in this way. Therefore, $\mathcal{H}_K^+$ is irreducible and invariant. Similarly for $\mathcal{H}_K^-$. qed.

We may also study the larger representation $V = C^\infty(C^*, d)$. This is a degenerate principal series representation. In Sect. 2 we determined the K-types, Eqs. (4) and (5) give the $\mathfrak{g}$-action.

Case I: $p \neq 2$. $V/\mathcal{H}_K$ is the direct sum of two irreducible representations. This follows from the same reasoning as above; none of the coefficients in Eq. (4) are zero unless $V_{a,b} \subset \mathcal{H}_K$. Therefore, within each of the two regions separated by the line $a + \dfrac{p}{2} = b + \dfrac{q}{2}$, L can take any K-type to any other K-type. Furthermore, one cannot cross the line $a + \dfrac{p}{2} - 1 = b + \dfrac{q}{2} - 1$ by applying L (or $\mathfrak{g}$). Thus the K-types of the irreducible constituents are $\left\{ V_{a,b} : a + \dfrac{p}{2} - 1 > b + \dfrac{q}{2} - 1 \right\}$ and $\left\{ V_{a,b} : a + \dfrac{p}{2} - 1 < b + \dfrac{q}{2} - 1 \right\}$. Each of these irreducible constituents is unitarizable. To see this, we first note that the K-invariant inner product $(,)$ actually provides a G-invariant pairing between $C^\infty(C^*, d)$ and $C^\infty(C^*, d - 2)$ (see [Kn], p. 273). Since $\Box'$ is a G-invariant operator mapping $C^\infty(C^*, d)$ to $C^\infty(C^*, d - 2)$, we can give $C^\infty(C^*, d)$ an invariant hermitian form by defining

$$\langle\!\langle \phi, \psi \rangle\!\rangle = (\Box'\phi, \psi).$$

This form is positive (respectively, negative) definite on $V_{a,b}$, $a + \dfrac{p}{2} - 1 > b + \dfrac{q}{2} - 1$,

$\left(\text{respectively, } a + \dfrac{p}{2} - 1 < b + \dfrac{q}{2} - 1\right)$, because if $\phi \in V_{a,b}$ we have

$$\langle\!\langle \phi, \phi \rangle\!\rangle = \left(\left(a + \frac{p}{2} - 1\right)^2 - \left(b + \frac{q}{2} - 1\right)^2\right)(\phi, \phi).$$

Note that $\mathscr{H}_K$ is the null space of $\langle\!\langle\,,\,\rangle\!\rangle$.

Case II: $p = 2$. $C^\infty(C^*, d)\mathscr{H}_K$ has three irreducible constituents. The K-types of these are the $V_{a,b}$ with

$$|a| < b + \frac{q}{2} - 1, \quad a > b + \frac{q}{2} - 1, \quad -a > b + \frac{q}{2} - 1.$$

The hermitian form $\langle\!\langle\,,\,\rangle\!\rangle$ defined above is negative on the first and positive on the other two. ∎

V. The Annihilator

Let $\mathscr{U}^n[\mathfrak{g}]$ be the elements of degree $\leq n$ in the universal enveloping algebra of a simple complex Lie algebra $\mathfrak{g}$ not of type A_n, and let $gr\,\mathscr{U}[\mathfrak{g}] = \bigoplus\limits_{n=0}^{\infty} \mathscr{U}^n[\mathfrak{g}]/\mathscr{U}^{n-1}[\mathfrak{g}]$. We note that $gr\,\mathscr{U}[\mathfrak{g}] \approx S[\mathfrak{g}]$, the symmetric algebra of $\mathfrak{g}$, which may also be identified with $P(\mathfrak{g}^*)$, the ring of polynomials on $\mathfrak{g}^*$. There is also a function $gr: \mathscr{U}[\mathfrak{g}] \to gr(\mathscr{U}[\mathfrak{g}])$ defined as follows: for $u \in \mathscr{U}^n[\mathfrak{g}]$ let $gr(u)$ be the image of u in $\mathscr{U}^n[\mathfrak{g}]/\mathscr{U}^{n-1}[\mathfrak{g}]$. If $\mathscr{J}$ is an ideal in $\mathscr{U}[\mathfrak{g}]$, then $I = gr(\mathscr{J})$ may be identified with an ideal in $P(\mathfrak{g}^*)$. The characteristic variety of $\mathscr{J}$ is the zero locus of I in $\mathfrak{g}^*$. We will identify the ideal in $\mathscr{U}[\mathfrak{g}]$ which annihilates $\mathscr{H}_K$ and see that its characteristic variety is the closure of the minimal coadjoint orbit in $\mathfrak{g}^*$. It is in this sense that $\mathscr{H}_K$ is associated to the minimal nilpotent orbit.

   More precisely, we will see that the annihilator of $\mathscr{H}_K$ is the Joseph ideal. This is an ideal constructed by Joseph ([J]) as an attempt to formulate a quantization procedure for non-polarizable orbits. It is the unique primitive ideal ($=$ annihilator of an irreducible representation) whose characteristic variety is the closure of the minimal nilpotent orbit. This ideal is maximal and completely prime. We will use the characterization of the Joseph ideal given below. This is due to Garfinkle ([G]).

   Let $S^2[\mathfrak{g}]$ be the space of homogeneous elements of degree 2 in the symmetric algebra of $\mathfrak{g}$ and let β be the highest root of $\mathfrak{g}$ with respect to some positive root system. Under the adjoint action of $\mathfrak{g}$, $S^2[\mathfrak{g}]$ decomposes into a direct sum of the form

$$S^2[\mathfrak{g}] = F_{2\beta} \oplus E_0 \oplus E_1 \oplus \cdots \oplus E_k$$
$$\equiv F_{2\beta} \oplus E,$$

where $F_{2\beta}$ is the irreducible finite dimensional representation of $\mathfrak{g}$ with highest weight 2β, E_0 is the 1-dimensional submodule consisting of scalar multiples of the Casimir operator, and the E_i, $i = 1, \ldots, k$, are the remaining irreducible summands of $S^2[\mathfrak{g}]$. ($F_{2\beta}$ and E_0 always appear in the decomposition of $S^2[\mathfrak{g}]$.)

Theorem. (*Garfinkle*) *Suppose $\mathscr{J}$ is any ideal in $\mathscr{U}[\mathfrak{g}]$ of infinite codimension. Let $I = gr(\mathscr{J})$. Then $\mathscr{J}$ is the Joseph ideal if and only if*

$$I \cap S^2 = E.$$

Our purpose now is to show that the annihilator of $\mathscr{H}$ is the Joseph ideal. Let $\mathfrak{g}$ again denote the complexification of $\mathfrak{so}(p,q)$ and let (π, W) be the natural representation of $\mathfrak{g}$ on $\mathbb{R}^{p,q}$.

We recall that all finite dimensional representations of $GL(W)$ can be realized within the tensor algebra $\mathfrak{T}(W)$ of W. In fact, the irreducible finite dimensional representations of $GL(W)$ correspond precisely to the tensors with fixed Young symmetry (see [W]). This correspondence not only allows one to parameterize the irreducible finite dimensional representations of $GL(W)$ by Young diagrams, it also allows one to compute their tensor products from a knowledge of the tensor products of representations of the symmetric group S_n; $n = p + q$.

The finite dimensional representations of $O(p,q)$ can also be realized within the tensor algebra $\mathfrak{T}(W)$; however, in this case, irreducible representations correspond to traceless tensors with fixed Young symmetry. To reduce further to $SO(p,q)$ one must take into account the behavior of a traceless tensor under reflections. However, for our purpose it will be sufficient to decompose the tensor product of two copies of the adjoint representation with respect to $O(p,q)$. (The adjoint representation of $SO(p,q)$ extends uniquely to an irreducible representation of $O(p,q)$.)

The adjoint representation of $\mathfrak{g}$ is equivalent to the representation of $\mathfrak{g}$ on $\Lambda^2(W)$, which we denote in terms of Young diagrams by

$$\boxed{\overset{\square}{\square}}\ .$$

Using techniques described in [H], it is easy to show that, as a representation of $O(p,q)$, $\mathfrak{g} \otimes \mathfrak{g}$ decomposes as

$$\boxed{\boxminus\boxminus} \oplus \boxed{\square\square} \oplus 1 \oplus \overset{\square}{\underset{\square}{\square}} \oplus \boxed{\square\square\atop\square} \oplus \overset{\square}{\square}\ .$$

The first four terms are also the summands of $S^2[\mathfrak{g}] \subset \mathfrak{T}[\mathfrak{g}]$. In particular,

$$\boxed{\boxminus\boxminus}$$

corresponds to the irreducible representation of $SO(p,q)$ with highest weight 2β and dimension $\dfrac{n(n^3 - 7n - 6)}{12}$, while the 1 represents the trivial representation. Upon restriction to $SO(p,q)$, the representation

$$\boxed{\square\square}$$

is an irreducible representation of dimension $\dfrac{(n+2)(n-1)}{2}$. The representation denoted by

$$\overset{\square}{\underset{\underset{\square}{\square}}{\square}}$$

is irreducible and of dimension $\dfrac{n(n-1)(n-2)(n-3)}{24}$; except when $n = 8$, in which

case it is a direct sum of two, irreducible, mutually contragredient representations of dimension 35. Thus,

$$E = 1 \oplus \square\square \oplus \begin{array}{c}\square\\\square\\\square\end{array}.$$

Let us now denote by g_{ij} the components of the metric tensor of $\mathbb{R}^{p,q}$ and by g^{ij} the components of its matrix inverse. Let $\{M_{ij} = -M_{ji};\ 1 \le i < j \le p+q\}$ be the basis for $\mathfrak{so}(p,q)$ in which the commutation relations take the form

$$[M_{ij}, M_{kl}] = g_{il}M_{jk} - g_{ik}M_{jl} - g_{jl}M_{ik} + g_{jk}M_{il}.$$

(M_{ij} corresponds to the generator of a (pseudo-) rotation in the $i-j$ plane of $\mathbb{R}^{p,q}$.) The element

$$Q = \sum_{i,j,k,l} g^{ij}g^{kl}M_{ik} \otimes M_{jl}$$

then corresponds to $-2\Omega_{SO(p,q)}$ regarded as an element of $S^2[\mathfrak{g}]$. The subspace of $S^2[\mathfrak{g}]$ corresponding to the representation

$$\square\square$$

is spanned by elements of the form

$$S_{ij} = \sum_{k,l} g^{kl}M_{ik} \otimes M_{lj} - \frac{2}{p+q}g_{ij}Q,$$

$1 \le i \le j \le p+q$; and the subspace of $S^2[\mathfrak{g}]$ corresponding to the representation

$$\begin{array}{c}\square\\\square\\\square\end{array}$$

is spanned by elements of the form

$$S_{ijkl} = M_{ij} \otimes M_{kl} - M_{ik} \otimes M_{jl} + M_{il} \otimes M_{jk} + M_{jk} \otimes M_{il} - M_{jl} \otimes M_{ik} + M_{kl} \otimes M_{ij},$$

$1 \le i < j < k < l \le p+q$. We thus have

$$E_0 = \mathbb{C} \cdot Q,$$
$$E_1 = \text{span}\{S_{ij};\ 1 \le i \le j \le p+q\},$$
$$E_2 = \text{span}\{S_{ijkl};\ 1 \le i < j < k < l \le p+q\}.$$

Now let $\mathcal{J} = \text{ann}(\mathcal{H}_K)$, $I = gr(\mathcal{J})$. We aim to show that E_0, E_1 and E_2 are in $I \cap S^2[\mathfrak{g}]$.

Now, on $C^\infty[C^*, d]$, $\mathfrak{so}(p,q)$ acts by

$$M_{ij} \to \hat{M}_{ij} = \sum_{k=1}^{p+q}\left(y_i g_{jk}\frac{\partial}{\partial y_k} - y_j g_{ik}\frac{\partial}{\partial y_k}\right).$$

As is easy to check, the operators

$$\hat{E}_{ijkl} = \hat{M}_{ij}\hat{M}_{kl} - \hat{M}_{ik}\hat{M}_{jl} + \hat{M}_{il}\hat{M}_{jk} + \hat{M}_{jk}\hat{M}_{il} - \hat{M}_{jl}\hat{M}_{ik} + \hat{M}_{kl}\hat{M}_{ij}$$

vanish identically on $C^\infty[C^*, d]$, while

$$\hat{\Omega}_{SO(p,q)} = -\frac{1}{2}\sum_{i,j,k,l} g^{ij}g^{kl}\hat{M}_{ik}\hat{M}_{jl}$$

acts by the scalar $\lambda_0 = 2\left(\left(\dfrac{(p+q)^2}{2}\right)^2 - p - q\right)$. Thus,

$$E_{ijkl} = gr(\hat{E}_{ijkl}) \subset I \cap S^2[\mathfrak{g}],$$

so $E_2 \subset I \cap S^2[\mathfrak{g}]$. And

$$Q = gr(\Omega_{SO(p,q)} - \lambda_0) \subset I \cap S^2[\mathfrak{g}],$$

so $E_0 \subset I \cap S^2[\mathfrak{g}]$.

Because $\mathscr{J}$ is the annihilator of a representation, $I \cap S^2[\mathfrak{g}]$ is $\mathfrak{g}$-invariant. Therefore, since E_1 corresponds to an irreducible representation of $\mathfrak{g}$, to show that $E_1 \subset I \cap S^2[\mathfrak{g}]$, it suffices to exhibit one element of E_1 lying in $I \cap S^2[\mathfrak{g}]$. Now from (1), it is evident that

$$\Xi = gr\left(\Omega_{SO(p)} - \Omega_{SO(q)} + \left(\frac{p-2}{2}\right)^2 - \left(\frac{q-2}{2}\right)^2\right) \in I \cap S^2[\mathfrak{g}].$$

On the other hand, a simple calculation verifies $\sum\limits_{i=1}^{n} E_{ii} = gr(\Xi)$. Thus, $E_1 \subset I \cap S^2[\mathfrak{g}]$.

Finally, we note Eqs. (4) and (5) imply $gr(L^k) \notin I$ $k = 1, 2, \dots$. From this observation, it follows that, (i) the annihilator of $\mathscr{H}_K$ must have infinite codimension in $\mathscr{U}[\mathfrak{g}]$ and (ii) $F_{2\beta} \notin I \cap S^2[\mathfrak{g}]$. Thus, we have proved:

Theorem. *The annihilator of $\mathscr{H}_K$ in $\mathscr{U}[\mathfrak{g}]$ is the Joseph ideal.*   ■

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